Augmenting the Connectivity of Planar and Geometric Graphs

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Abstract

In this paper we study some connectivity augmentation problems. We want to make planar graphs 2-vertex (or 2-edge) connected by adding edges such that the resulting graphs remain planar. We show that it is NP-hard to find a minimum-cardinality augmentation that makes a planar graph 2-edge connected. This was known for 2-vertex connectivity. We further show that both problems are hard in a geometric setting, even when restricted to trees. For the special case of convex geometric graphs we give efficient algorithms.

We also study the following related problem. Given a plane geometric graph G, two vertices s and t of G, and an integer k, how many edges have to be added to G such that G contains k edge- (or vertex-) disjoint s–t paths? For $k = 2$ we give optimal worst-case bounds; for $k = 3$ we characterize all cases that have a solution.

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1 Introduction

Augmenting a given graph to increase its connectivity is important, e.g., for securing communication networks against node and link failures. The planar version of the problem, where the augmentation has to preserve planarity, also has applications in graph drawing [4]. Recall that a graph is *k-vertex* (*k-edge*) *connected* if the removal of any subset of $k-1$ vertices (edges) does not make the graph disconnected. We consider the following two problems.

Planar 2-Vertex Connectivity Augmentation(PVCA): Given a connected planar graph $G=(V,E)$, find a smallest set E' of vertex pairs such that the graph $G'=(V,E\cup E')$ is planar and 2-vertex connected (biconnected).
Planar 2-Edge Connectivity Augmentation (PECA) is defined as PVCA, but with *2-vertex connected* replaced by *2-edge connected* (bridge-connected).

The corresponding problems without the planarity constraints have a long history, both for directed and undirected graphs. The unweighted cases can be solved in polynomial time, whereas the weighted versions are hard [2].

We also consider a geometric version of the above problems. Recall that a *geometric* graph is a graph where each vertex v corresponds to a point $\mu(v)$ in the plane and where each edge uv corresponds to the straight-line segment $\overline{\mu(u)\mu(v)}$. We are exclusively interested in geometric graphs that are *plane*, that is, whose edges intersect at most in their endpoints. Therefore, in this paper by geometric graph we always mean a plane geometric graph. Given such a graph G, we want to find a (small) set of vertex pairs such that adding the corresponding edges to G leaves G plane and augments its connectivity.

Abellanas et al. [1] have shown worst-case bounds for geometric PVCA and PECA. For geometric PVCA they show that $n-2$ edges are sometimes needed and are always sufficient. For geometric PECA they prove that $2n/3$ edges are sometimes needed and $6n/7$ edges are always sufficient. In the special case of plane geometric trees they show that $n/2$ edges are always sufficient.

Table 1 gives an overview about our results (full version in [6]) and what has been known previously about the complexity of PVCA and PECA.

problem	planar	outerplanar	geometric	convex
PVCA	NPC [4]	$O(n)$ [3]	NPC [5]	$O(n)$
PECA	NPC	$O(n)$ [3]	NPC [5]	$O(n)$
weighted PVCA	NPC	open	NPC	$O(n)$
weighted PECA	NPC	open	NPC	$O(n^2)$

Table 1
Complexity of PVCA and PECA.

2 Complexity results

In this section we show that PECA is NP-complete. This settles an open problem posed by Kant and Bodlaender [4]. Rappaport [5] has shown that geometric PVCA and PECA are NP-complete for non-connected graphs. We also show that geometric PVCA and geometric PECA are NP-complete even if restricted to trees. Both hardness proofs are by reduction from Planar3SAT, which is known to be NP-hard.

Theorem 2.1 PECA *is NP-complete.*

Theorem 2.2 *Let G be a plane geometric graph and $k > 0$ an integer. It is NP-complete to decide whether adding k edges suffices to make G bridge- or biconnected. This is true even if G is a tree with exactly $2k$ leaves.*

Once we have shown hardness of PECA or PVCA for connected geometric graphs it is easy to extend this to geometric trees. We iteratively replace an edge uv of the input graph that is contained in a cycle by the construction shown in Figure 1. Call the resulting tree T. It's not hard to see that an optimal augmentation of T induces an optimal augmentation of the input graph.

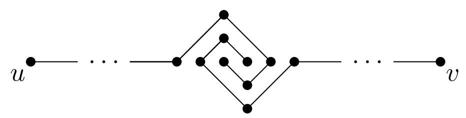

Fig. 1. Removing cycles in G.

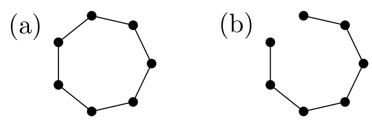

Fig. 2. A cycle (a) and a near-cycle (b).

3 Convex geometric graphs

In this section we consider the geometric version of PVCA and PECA in the special case that the input graph is a convex geometric graph. We call an edge *outer edge* if it belongs to the convex hull and *inner edge* otherwise.

Note that PVCA for a convex geometric graph G is trivial: G is biconnected iff it contains all edges of the convex hull. Thus we focus on PECA.

If a connected convex geometric graph does not contain an inner edge then it is either a *cycle* or a *near-cycle*, see Figure 2. We decompose an arbitrary convex geometric graph into cycles and near-cycles and use this decomposition to compute an edge set of minimum cardinality that bridge-connects the graph.

Theorem 3.1 PECA *for convex geometric graphs can be solved in linear (quadratic) time if non-edges have unit (positive) weights.*

4 s–t path augmentation

In this section we consider the following problems: Given a plane geometric graph $G = (S, E)$, two vertices $s \neq t$ of G, and an integer $k > 0$, find a smallest set E' of vertex pairs such that $G' = (S, E \cup E')$ is plane and contains k edge-disjoint s–t paths (s–t k-ECA). We also consider the corresponding problem for vertex-disjoint s–t paths (s–t k-VCA). We treat $k \in \{2, 3\}$.

How many edges are needed for an s–t path augmentation in the worst case? A zig-zagging s–t path whose vertices are in convex position (see also [1]) shows that sometimes $n-2$ edges are necessary for s–t 2-VCA and $n/2$ edges for s–t 2-ECA. Abellanas et al. [1] show that $n-2$ edges are always sufficient for PVCA and thus also for s–t 2-VCA. Now let's turn to s–t 2-ECA.

Theorem 4.1 *For a connected geometric graph with n vertices a solution to s–t 2-ECA has size at most $n/2$ and can be computed in linear time.*

Let G be a geometric graph and H its convex hull. Call an edge that connects two vertices of H but does not lie on H a *chord*. A chord uv is (s, t)*-separating* if s and t are vertices in different connected components of $G \setminus uv$.

Theorem 4.2 *If a connected geometric graph does not contain an (s, t)-separating chord, then s–t 3-VCA has a solution. It can be computed in $O(n^2)$ time.*

Theorem 4.3 *Let $G = (S, E)$ be a geometric graph and $s, t \in S$. Then s–t 3-ECA has a solution if G can be augmented such that s and t have degree ≥ 3.*

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