Euclidean Greedy Drawings of Trees

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Abstract. Greedy embedding (or drawing) is a simple and efficient strategy to route messages in wireless sensor networks. For each source-destination pair of nodes s, t in a greedy embedding there is always a neighbor u of s that is closer to t according to some distance metric. The existence of Euclidean greedy embeddings in $\mathbb{R}^2$ is known for certain graph classes such as 3-connected planar graphs. We completely characterize the trees that admit a greedy embedding in $\mathbb{R}^2$. This answers a question by Angelini et al. (Graph Drawing 2009) and is a further step in characterizing the graphs that admit Euclidean greedy embeddings.

1 Introduction

Message routing in wireless ad-hoc and sensor networks cannot apply the same established global hierarchical routing schemes that are used, e.g., in the Internet Protocol. A family of alternative routing strategies in wireless networks known as *geographic routing* uses node locations as addresses instead. The *greedy routing* protocol simply passes a message at each node to a neighbor that is closer to the destination. Two problems with this approach are (i) that sensor nodes typically are not equipped with GPS receivers due to their cost and energy consumption and (ii) that even if nodes know their positions messages can get stuck at voids, where no node closer to the destination exists.

An elegant strategy to tackle these issues was proposed by Rao et al [13]. It maps nodes to virtual rather than geographic coordinates, on which the standard greedy routing is then performed. A mapping that always guarantees successful delivery is called a *greedy embedding* or *greedy drawing*.

The question about the existence of greedy embeddings for various metric spaces and classes of graphs has attracted a lot of interest. Papadimitriou and Ratajczak [12] conjectured that every 3-connected planar graph admits a greedy embedding into the Euclidean plane. Dhandapani [5] proved that every 3-connected planar triangulation has a greedy drawing. The conjecture by Papadimitriou and Ratajczak itself has been proved independently by Leighton and Moitra [10] and Angelini et al. [4]. Kleinberg [9] showed that every connected graph has a greedy embedding in the hyperbolic plane.

Since efficient use of storage and bandwidth are crucial in wireless sensor networks, virtual coordinates should require only few, i.e., $O(\log n)$, bits in order to keep message headers small. Greedy drawings with this property are called *succinct*. Eppstein and Goodrich proved the existence of succinct greedy drawings for 3-connected planar graphs in $\mathbb{R}^2$ [7], and Goodrich and Strash [8] showed it for any connected graph in the hyperbolic plane. Wang and He [15] used a custom distance metric and constructed convex, planar and succinct drawings for 3-connected planar graphs using Schnyder realizers [14].

It has been known that not all graphs admit a Euclidean greedy drawing in the plane, e.g., $K_{k,5k+1}$ ($k \geq 1$) has no such drawing [12], including the tree $K_{1,6}$. Leighton and Moitra [10] showed that a graph containing at least six *pairwise independent irreducible triples* (e.g., the complete binary tree containing 31 nodes) cannot have a greedy embedding. They used this fact to present a planar graph that admits a greedy embedding, although none of its spanning trees does. We show that there are trees with no greedy drawing that contain at most *five* such triples [11]. Further, some greedy-drawable trees have no succinct Euclidean greedy drawing [3].

Self-approaching drawings [1] are a subclass of greedy drawings with the additional constraint that for any pair of nodes there is a path ρ that is distance decreasing not just for the node sequence of ρ but for any triple of intermediate points on the edges of ρ. Alamdari et al. [1] gave a complete characterization of trees admitting self-approaching drawings. Since self-approaching drawings are greedy, all trees with a self-approaching drawing are greedy-drawable. However, there exist numerous trees that admit a greedy drawing, but no self-approaching one, and the characterization of those trees turns out to be quite complex.

Contributions. We give the first complete characterization of all trees that admit a greedy embedding in $\mathbb{R}^2$ with the Euclidean distance metric. This solves the corresponding open problem stated by Angelini et al. [2] and is a further step in characterizing the graphs that have greedy embeddings. We show that deciding whether T has a greedy drawing is equivalent to deciding whether there exists a valid angle assignment in a certain wheel polygon. This includes a non-linear constraint known as the *wheel condition* [6]. For most cases (all trees with maximum degree 4 and most trees with maximum degree 5) we are able to give an explicit solution to this problem, which provides a linear-time recognition algorithm. For trees with maximum degree 3 we give an alternative characterization by forbidden subtrees in the full version of this paper [11]. For some trees with one degree 5 node we resort to using non-linear solvers. For trees with nodes of degree ≥ 6 no greedy drawings exist.

Our proofs are constructive, however, we ignore the possibly exponential area requirements for our constructions. This is justified by the aforementioned result that some trees require exponential-size greedy drawings [2]. Due to space constraints several proofs are omitted; for details we refer to the full paper [11].

2 Preliminaries

In this section, we introduce the concept of the opening angle of a rooted subtree and present relations between opening angles that will be crucial for the characterization of greedy-drawable trees.

Let $T = (V, E)$ be a tree. A *straight-line drawing* Γ of T maps every node $v \in V$ to a point in the plane $\mathbb{R}^2$ and every edge $uv \in E$ to the line segment between its endpoints. We say that Γ is *greedy* if for every pair of nodes s, t there is a neighbor u of s with $|ut| < |st|$, where $|st|$ is the Euclidean distance between points s and t. To ease notation we identify nodes with points and edges with line segments. Furthermore we assume that all drawings are straight-line drawings.

It is known that for a greedy drawing Γ of T any subtree of T is represented in Γ by a greedy subdrawing [2]. We define the *axis* of an edge uv as its perpendicular bisector. Let h^u_{uv} denote the open half-plane bounded by the axis of uv and containing u. Let T^u_{uv} be the subtree of T containing u obtained from T by removing uv. Angelini et al. [2] showed that in a greedy drawing of T every subtree T^u_{uv} is contained in h^u_{uv}. The converse is also true.

Lemma 1. *Let Γ be a drawing of T with $T^u_{uv} \subseteq h^u_{uv}$ $\forall uv \in E$. Then, Γ is greedy.*

Angelini et al. [2] further showed that greedy tree drawings are always planar and that in any greedy drawing of T the angle between two adjacent edges must be strictly greater than $60°$. Thus T cannot have a node of degree ≥ 6.

Let $\mathrm{ray}(u, \vec{uv})$ denote the ray with origin u and direction $\vec{uv}$. For $u, v \in V$, let $d_T(u, v)$ be the length of the u-v path in T. For vectors $\vec{ab}$, $\vec{cd}$, let $\angle_{\mathrm{ccw}}(\vec{ab}, \vec{cd})$ denote the counterclockwise turn (or turning angle) from $\vec{ab}$ to $\vec{cd}$.

Lemma 2 (Lemma 7 in [2]). *Consider two edges uv and wz in a greedy drawing of T, such that the path from u to w does not contain v and z. Then, the rays* $\mathrm{ray}(u, \vec{uv})$ *and* $\mathrm{ray}(w, \vec{wz})$ *diverge; see Fig. 1a.*

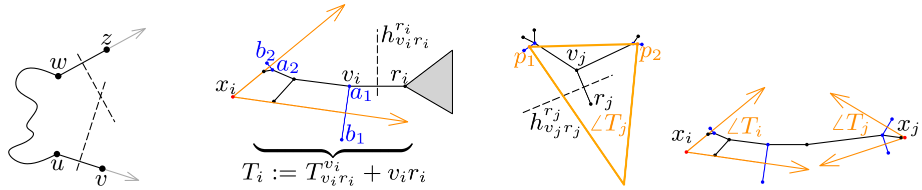
Lemma 3. *Let Γ be a greedy drawing of T, $v \in V$, $\deg(v) = 2$, $\mathrm{N}(v) = \{u, w\}$ the only two neighbors of v, and $T' = T - \{uv, vw\} + \{uw\}$. The drawing Γ' induced by replacing segments uv, vw by uw in Γ is also greedy.*

Next we generalize some concepts from Leighton and Moitra [10]. For $k = 3, 4, 5$, we define an *irreducible k-tuple* as a k-tuple of nodes $(b_1, \ldots, b_k)$ in a graph $G = (V, E)$, such that $\deg(b_1) = k$, $b_1b_2, b_1b_3, \ldots, b_1b_k \in E$ (we call these $k-1$ edges *branches* of the k-tuple) and removing any branch b_1b_j disconnects the graph. A k-tuple $(b_1, \ldots, b_k)$ and an l-tuple $(x_1, \ldots, x_l)$ are *independent*, if $\{b_1, \ldots, b_k\} \cap \{x_1, \ldots, x_l\} = \emptyset$, and deleting all the branches keeps b_1 and x_1 connected.

Let Γ be a greedy drawing of T. We shall consider subtrees $T_i = (V_i, E_i)$ of T, such that T_i has root r_i, $\deg(r_i) = 1$ in T_i and v_i is the neighbor of r_i in T_i. We define the *polytope of a rooted subtree* T_i as $\mathrm{polytope}(T_i) = \bigcap \{h^w_{uw} \mid uw \in E_i, uw \neq r_iv_i,\ d_T(w, r_i) < d_T(u, r_i)\}$.

Definition 1 (Extremal edges). *For $j = 1, 2$, let $a_jb_j \neq v_ir_i$ be an edge of T_i, $d_T(a_j, r_i) < d_T(b_j, r_i)$, such that $\angle_{\mathrm{ccw}}(\vec{v_ir_i}, \vec{a_jb_j})$ is maximum for $j = 1$ and minimum for $j = 2$. We call edges a_jb_j* extremal.

Note that by Lemma 2, $\mathrm{ray}(a_j, \vec{a_jb_j})$ and $\mathrm{ray}(v_i, \vec{v_ir_i})$ diverge. In the following two definitions, let $e_j = a_jb_j$, $j = 1, 2$ be the extremal edges of T_i.



(a) (b) (c) (d)

Fig. 1: (a) Sketch of Lemma 2. (b) Subtree T_i with opening angle $\angle T_i$ (orange), extremal edges $a_1 b_1$, $a_2 b_2$ (blue) and apex x_i (red). The subtree $T_{v_i r_i}^{r_i}$ (gray triangle) must be contained in the half-plane $h_{v_i r_i}^{r_i}$ and the cone $\angle T_i$. (c) Subtree T_j with closed angle $\angle T_j$ and boundary segment $p_1 p_2$. (d) Open angles of independent subtrees must contain apices of each other.

Definition 2 (Open angle). *Let $\angle_{\mathrm{ccw}}(\vec{a_1 b_1}, \vec{a_2 b_2}) > 180°$. Then,* polytope$(T_i)$ *is unbounded, and we say that T_i is drawn with an* open angle.

(a) If $a_1 b_1 \nsubseteq h_{a_2 b_2}^{b_2}$ and $a_2 b_2 \nsubseteq h_{a_1 b_1}^{b_1}$, define $\angle T_i = h_{a_1 b_1}^{a_1} \cap h_{a_2 b_2}^{a_2}$. Let x_i be the intersection of axis$(a_1 b_1)$ *and* axis$(a_2 b_2)$*. We set* apex$(\angle T_i) = x_i$*; see Fig. 1b.*

(b) If $a_j b_j \subseteq h_{a_k b_k}^{b_k}$ for $j = 1, k = 2$ or $j = 2, k = 1$, let $\angle T_i$ be the cone defined by moving the boundaries of $h_{a_1 b_1}^{a_1}$, $h_{a_2 b_2}^{a_2}$ to b_j ($b_k \in \angle T_i$), and apex$(\angle T_i) = b_j$*.*

We call $\angle T_i$ the opening angle *of T_i in Γ (orange in Fig. 1b). We write $|\angle T_i| = \alpha$, where α is the angle between the two rays of $\angle T_i$.*

Obviously, polytope$(T_i) \subseteq \angle T_i$ in (a). This is also true in (b) by Observation 1.

Observation 1 *Let h be an open half-plane and $p \notin h$. Let h' be the half-plane created by a parallel translation of the boundary of h' to p. Then, $h \subseteq h'$.*

Definition 3 (Closed and zero angle). *Let $\angle_{\mathrm{ccw}}(\vec{a_1 b_1}, \vec{a_2 b_2}) < 180°$ (or $= 180°$). Let $C_i = h_{a_1 b_1}^{a_1} \cap h_{a_2 b_2}^{a_2}$, and let p_j be the midpoint of e_j. We denote the part of C_i bounded by segment $p_1 p_2$ containing r by $\angle T_i$ and say that T_i is drawn with a* closed (or zero) angle*; see Fig. 1c. We write $|\angle T_i| < 0$ (or $= 0$).*

We say that two subtrees T_1, T_2 are *independent*, if $T_2 \setminus \{r_2\} \subseteq T_{v_1 r_1}^{r_1}$ and $T_1 \setminus \{r_1\} \subseteq T_{v_2 r_2}^{r_2}$. If T_1 and T_2 are independent, then $T_2 \setminus \{r_2\} \subseteq h_{v_1 r_1}^{r_1}$ and $T_1 \setminus \{r_1\} \subseteq h_{v_2 r_2}^{r_2}$ in Γ. Also, if $r_2 \notin T_{v_1 r_1}^{r_1}$, then $r_2 = v_1$.

Lemma 4. *Let T_i and T_j be independent, $|\angle T_i|, |\angle T_j| > 0$ in Γ. Then,* apex$(\angle T_i) \in \angle T_j$ *and* apex$(\angle T_j) \in \angle T_i$*.*

Lemma 5 (generalization of Claim 4 in [10]). *Let T_i, T_j be two independent subtrees. Then, either $|\angle T_i| > 0$ or $|\angle T_j| > 0$.*

We shall use the following lemma to provide a certificate of non-existence of a greedy drawing.

Lemma 6. *Let T_i, $i = 1, \ldots, d$ be pairwise independent subtrees, and $\alpha_i = |\angle T_i|$. Then, $\sum_{i=1,\ldots,d,\alpha_i>0} \alpha_i > (d-2)180°$.*

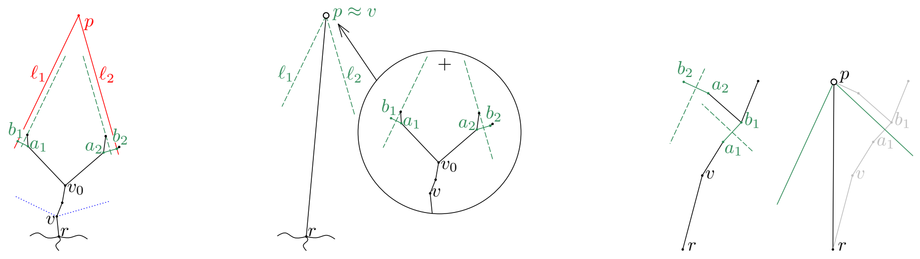


Fig. 2: Illustration of Lemma 8. (a) Greedy drawing Γ. Edges a_1b_1, a_2b_2 are extremal. Dotted blue: bounding cone of T'. (b) Greedy drawing Γ'. Subtree T^v_{rv} has been moved to a new point $p \notin V$ and drawn infinitesimally small. (c) Drawings Γ and Γ' for the case when a_1,b_1 lie on the r-b_2-path. Here, $p = b_2 \in V$.

Let T contain a set of n_k irreducible k-tuples, $k = 3, 4, 5$, that are all pairwise independent. Leighton and Moitra [10] showed that for $n_3 \geq 6$ no greedy drawing of T exists. We generalize this result slightly:

Lemma 7. *No greedy drawing of T exists if $n_3 + 2n_4 + 3n_5 \geq 6$.*

2.1 Shrinking lemma

We now present a lemma which is crucial for later proofs. Let the *bounding cone* of a subtree $T^v_{rv} + rv$ defined for an edge rv in a greedy drawing Γ of T be the cone with apex v and boundary rays $\mathrm{ray}(v, \vec{a_1b_1})$ and $\mathrm{ray}(v, \vec{a_2b_2})$ for extremal edges a_1b_1, a_2b_2 of $T^v_{rv} + rv$ that contains the drawing of T^v_{rv}.

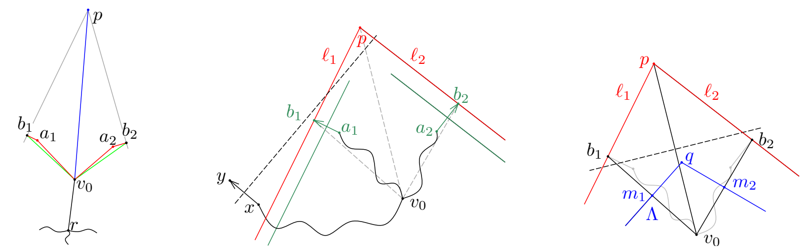
Lemma 8. *Let $T = (V, E)$ be a tree and $T' = T^v_{rv} + rv$, $rv \in E$, a subtree of T. Let Γ be a greedy drawing of T, such that $|\angle T'| > 0$. Then, there exists a point p in the bounding cone of T^v_{rv}, such that shrinking T^v_{rv} infinitesimally and moving it to p keeps the drawing greedy, and $|\angle T'|$ remains the same.*

Proof. Let $e_i = a_ib_i$, $i = 1, 2$, be the two extremal edges of T' in Γ, ρ_i the r-b_i-path, and $a_i \in \rho_i$; see Fig. 2 for a sketch. We distinguish two cases:

(1) Edge e_1 is not on ρ_2 and edge e_2 is not on ρ_1. Then, $\{a_1, b_1\} \subseteq h^{a_2}_{a_2b_2}$, and $\{a_2, b_2\} \subseteq h^{a_1}_{a_1b_1}$. Let ℓ_i be the line parallel to $\mathrm{axis}(e_i)$ through b_i and p the intersection of ℓ_1 and ℓ_2; see Fig. 2a. Let $v_0 \in V$ be the last common node of ρ_1 and ρ_2, and let η_i be the v_0-b_i-path in T, $i = 1, 2$.

We now define three intermediate drawings. Let Γ_1 be the drawing gained by replacing T' in Γ by the edge rv_0 and the two paths η_1 and η_2, and let $\Gamma_2 = \Gamma_1 - \eta_1 - \eta_2 + \{v_0b_1, v_0b_2\}$; see Fig. 3a. By Lemma 3, both Γ_1 and Γ_2 are greedy. Let $\Gamma_3 = \Gamma_2 - \{v_0b_1, v_0b_2\} + \{v_0p\}$. Let V_1 be the node set of T^r_{vr} with addition of v_0. Note that the nodes in V_1 have the same coordinates in Γ, Γ_1, Γ_2 and Γ_3. Further, since $v_0 \in h^{a_i}_{a_ib_i}$ for $i = 1, 2$, p lies inside the angle $\angle b_1v_0b_2 < 180°$.

We have to prove the greediness of Γ_3. Since $p \notin V$, it doesn't follow directly from Lemma 3. We first show that for an edge xy in Γ_3, $xy \neq v_0p$, where x is



(a) $\Gamma_1, \Gamma_2, \Gamma_3$ (b) (c)

Fig. 3: Proof of Lemma 8. (a): Intermediate drawings Γ_1 (black and red), Γ_2 (black and green) and Γ_3 (black and blue). (b): For an edge $xy \notin T^v_{rv}$, its axis doesn't cross v_0b_1, v_0b_2. It also doesn't cross v_0p due to Lemma 2. (c): It is $\Lambda \subseteq h^{v_0}_{v_0p}$.

closer to v_0 in T than y, it holds $p \in h^x_{xy}$. Edge xy is also contained in Γ_1. Nodes x, v_0 and a_i lie on the y-b_i-path in T, $i = 1, 2$. Hence, $\{v_0, a_1, a_2, b_1, b_2\} \subseteq \eta_1 \cup \eta_2 \subseteq h^x_{xy}$, therefore, axis$(xy)$ doesn't cross edges v_0b_1, v_0b_2. Now assume $p \notin h^x_{xy}$. Then, axis(xy) must cross v_0p, b_1p and b_2p (but not v_0b_i); see Fig. 3b. This is only possible if for some $i \in \{1, 2\}$, rays ray$(x, \vec{xy})$ and ray$(a_i, \vec{a_ib_i})$ are parallel or converge, which is a contradiction to Lemma 2.

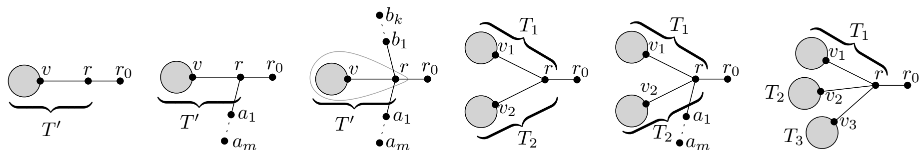
Next, we show that $V_1 \subseteq h^{v_0}_{pv_0}$. Without loss of generality, let v_0b_1 be directed upwards to the left and v_0b_2 upwards to the right. Note that a_1 lies to the right of v_0b_1 and a_2 to the left of v_0b_2 (otherwise, the edge a_ib_i would not be extremal in T'). Hence, $\angle v_0b_ip \geq 90°$. Let Λ be the opening angle of the subtree induced by edges $\{rv_0, v_0b_1, v_0b_2\}$ with root r in Γ_2 (blue in Fig. 3c). It is $\Lambda \subseteq h^{v_0}_{pv_0}$ (see [11]). Hence, $V_1 \subseteq \Lambda \subseteq h^{v_0}_{pv_0}$. This proves the greediness of Γ_3. Due to the extremality of a_1b_1, a_2b_2, p lies in the bounding cone of T'.

Removing v_0 and connecting r to p keeps the drawing greedy. Finally, we acquire Γ' by drawing the subtree T^v_{rv} of T infinitesimally small at p. Let C_1 be the cone $\angle T'$ in the original drawing Γ, and C_2 the cone bounded by ℓ_1 and ℓ_2, $a_i \in C_2$. By Observation 1, $C_1 \subseteq C_2$. Consider an edge e in T^v_{rv}, $e \notin \{e_1, e_2\}$ in Γ. Let ℓ be the line parallel to axis(e) through p. Due to the extremality of e_1, e_2, cone C_2 lies on one side of ℓ. Therefore, since $V_1 \subseteq C_2$, the drawing Γ' is greedy, and it is $\angle T' = C_2$. Since ℓ_i is parallel to axis(a_ib_i), $|\angle T'|$ in Γ' is as big as in Γ.

(2) Now assume a_1b_1 lies on ρ_2. Let Γ_4 be the drawing obtained by replacing T' in Γ by edge rb_2. By Lemma 3, Γ_4 is greedy. It is $b_2 \in h^{b_1}_{a_1b_1}$. Similar to (1), we acquire Γ' by drawing the subtree T^v_{rv} of T infinitesimally small at $p = b_2$. Then, $|\angle T'|$ remains the same as in Γ, see Fig. 2c. □

3 Opening angles of rooted trees

The main idea of our decision algorithm is to process the nodes of T bottom-up while calculating tight upper bounds on the maximum possible opening angles of the considered subtrees. If T contains a node of degree 5, it cannot be drawn



(a) case I (b) case II (c) case III (d) case IV (e) case V (f) case VII

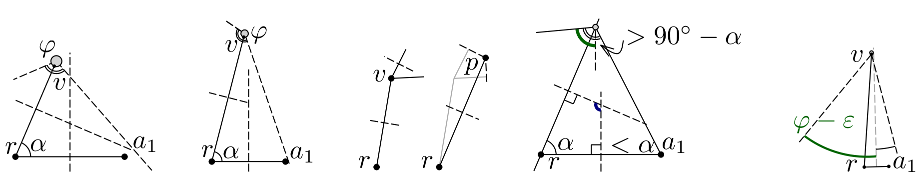
Fig. 4: (a)–(e): Possible cases when combining subtrees to maintain an open angle. Subtrees T_1, T_2 have opening angles $\in (90°, 120°)$. In case VII ((f)) or in case VI ($|\angle T_i| \leq 90°$ in IV or V for one $i \in \{1, 2\}$) no open angle is possible.

with an open angle, since each pair of consecutive edges forms an angle strictly greater than 60°. In this section, we consider trees with maximum degree 4.

If a subtree T' can be drawn with an open angle $\varphi - \varepsilon$ for any $\varepsilon > 0$, but not φ, we say that it has opening angle φ^- and write $|\angle T'| = \varphi^-$. For example, a triple has opening angle 120^- and a quadruple 60^-. We call a subtree *non-trivial* if it is not a single node or a simple path. Figure 4 shows possibilities to combine or extend non-trivial subtrees T', T_1, T_2. We shall now prove tight bounds on the possible opening angles for each construction. As we shall show later, only cases I–V are feasible for the resulting subtree to have an open angle. To compute the maximum opening angle of the combined subtree $\overline{T}$ in cases I–V, we use the following strategy. We show that applying Lemma 8 to T' does not decrease the opening angle of $\overline{T}$ in a drawing. Hence, it suffices to consider only drawings in which ${T'}^{v}_{rv}$ is shrunk to a point. We then obtain an upper bound by solving a linear maximization problem. Finally, we construct a drawing with an almost-optimal opening angle for $\overline{T}$ inductively using an almost-optimal construction for T'. We give a proof for case II, see [11] for the remaining cases.

Lemma 9. *Let T' be a subtree with $\angle T' = \varphi^-$, and consider the subtree $\overline{T} = T' + rr_0 + ra_1 + a_1a_2 + \ldots + a_{m-1}a_m$ in Fig. 4b. Then $|\angle\overline{T}| = (45° + \frac{\varphi}{2})^-$ if $\varphi > 90°$ (case (i)), and $|\angle\overline{T}| = \varphi^-$ if $\varphi \leq 90°$ (case (ii)).*

Proof. First, let $m = 1$. (i) Consider a greedy drawing Γ of $\overline{T}$. Let a_1r be drawn horizontally and v above it and to the left of axis(ra_1); see Fig. 5a,b,d. Due to Lemma 2, the right boundary of $\angle\overline{T}$ is formed by axis(ra_1). The left boundary is either formed by (1) the left boundary of $\angle T'$ (see Fig. 5a), or (2) by axis(rv)



(a) (b) (c) (d) (e)

Fig. 5: Optimal construction and tight upper bound for case II.

Table 1: Computing maximum opening angle of the combined subtree $\overline{T}$. Let $|\angle T_i| = \varphi_i^-$, $\varphi_i \geq \varphi_{i+1}$, and $|\angle T_i| = \varphi_i = 180°$ if T_i is a path.

case	φ_1	φ_2	φ_3	maximum $\lvert\angle\overline{T}\rvert$
I	$(0°, 180°]$	-	-	${\varphi_1}^-$
II.i	$180°$	$(90°, 120°]$	-	$(\frac{\varphi_2}{2} + 45°)^- \in (90°, 120°)$
II.ii	$180°$	$(0°, 60°]$	-	${\varphi_2}^- \in (0°, 60°)$
III	$180°$	$180°$	$(0°, 120°]$	$\frac{\varphi_3}{2}^- \in (0°, 60°)$
IV	$(90°, 120°]$	$(90°, 120°]$	-	$(\varphi_1 + \varphi_2 - 180°)^- \in (0°, 60°)$
V	$180°$	$(90°, 120°]$	$(90°, 120°]$	$(\frac{3}{4}\varphi_2 + \frac{1}{2}\varphi_3 - 112.5°)^- \in (0°, 60°)$
VI	$(0°, 120°]$	$(0°, 90°]$	-	$< 0°$
VII	$(0°, 120°]$	$(0°, 120°]$	$(0°, 120°]$	$< 0°$

(Fig. 5b). We apply Lemma 8 to ${T'}^v_{rv}$ in Γ and acquire Γ', in which ${T'}^v_{rv}$ is drawn infinitesimally small. In Γ', axis(ra_1) remains the right boundary of $\angle\overline{T}$. In case (1), the left boundary of $\angle\overline{T}$ is again formed by the left boundary of $\angle T'$, and $|\angle\overline{T}|$ remains the same. In case (2), the subtree ${T'}^v_{rv}$ must lie to the right of $\vec{rv}$ in Γ (since each edge in it is oriented clockwise relative to $\vec{rv}$), and so does the point p from Lemma 8. Thus, the edge rv is turned clockwise in Γ', and $|\angle\overline{T}|$ increases; see Fig. 5c. Thus, to acquire an upper bound for $|\angle\overline{T}|$ it suffices to only consider drawings in which ${T'}^v_{rv}$ is drawn infinitesimally small. Let $\alpha = \angle a_1 rv$. Then, for $\overline{\varphi} = |\angle\overline{T}|$ it holds: $\overline{\varphi} \leq 180° - \alpha$, $\overline{\varphi} < \varphi - 90° + \alpha$; see the blue and green angles in Fig. 5d. Thus, $\overline{\varphi}$ lies on the graph $f(\alpha) = 180° - \alpha$ or below it and strictly below the graph $g(\alpha) = \varphi - 90° + \alpha$. Maximizing over α gives $\overline{\varphi} < 45° + \frac{\varphi}{2}$. We can achieve $\overline{\varphi} = (45° + \frac{\varphi}{2})^-$ by choosing $\alpha = 135° - \frac{\varphi}{2} + \varepsilon'$ and drawing ${T'}^v_{rv}$ sufficiently small with $|\angle T'| = \varphi - \varepsilon$ for sufficiently small $\varepsilon, \varepsilon' > 0$.

(ii) Obviously, $|\angle T'| \geq |\angle\overline{T}|$. For the second part, see Fig. 5e. We choose $\angle a_1 rv = 90° - \frac{\varepsilon}{2}$ and draw ra_1long enough, such that its axis doesn't cross ${T'}^v_{rv}$. We rotate ${T'}^v_{rv}$ such that the right side of the opening angle $\angle T'$ and rv form an angle $\frac{3\varepsilon}{2}$. Then, the opening angle φ' of the drawing is defined by the left side of $\angle T'$ and the axis of ra_1 and is $\varphi - \varepsilon$.

For $m \geq 2$, draw $a_2, \ldots, a_m$ collinear with ra_1 and infinitesimally close to a_1. □

Tight upper bounds on opening angles of the combined subtree $\overline{T}$ for all possible cases are listed in Table 1. Note that no bounds in $(120°, 180°)$ and $(60°, 90°]$ appear. See [11] for the proofs of cases III–VII.

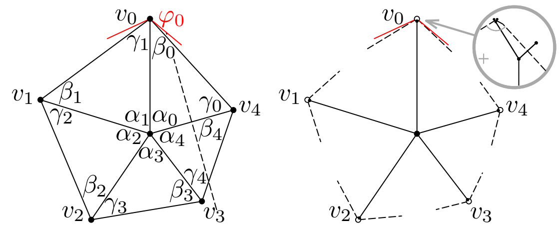
4 Arranging rooted subtrees with open angles

In this section, we consider the task of constructing a greedy drawing Γ of T by combining independent rooted subtrees with a common root. The following problem (restricted to $n \in \{3, 4, 5\}$) turns out to be fundamental in this context.

Problem 1. Given n angles $\varphi_0, \ldots, \varphi_{n-1} > 0°$, is there a convex n-gon P with corners $v_0, \ldots, v_{n-1}$ (in arbitrary order) with interior angles $\psi_i < \varphi_i$ for

maximize ε under:
$\varepsilon, \alpha_i, \beta_i, \gamma_i \in [0, 180]$,
$i = 0, \ldots, n-1$;
$\beta_i + \varepsilon \leq \alpha_i$, $\gamma_i + \varepsilon \leq \alpha_i$,
$\beta_i + \gamma_{i+1} + \varepsilon \leq \varphi_i$
$\alpha_i + \beta_i + \gamma_i = 180$,
$\alpha_0 + \ldots + \alpha_{n-1} = 360$,
$\prod_{i=0}^{n-1} \sin \beta_i = \prod_{i=0}^{n-1} \sin \gamma_i$

(a) optimization problem (*)



(b) (c)

Fig. 6: (a) Optimization problem (*); (b) sketch for (*). (c) Solving (*) lets us construct greedy drawings by placing sufficiently small drawings of subtrees at n-gon corners.

$i = 0, \ldots, n-1$, such that the star $K_{1,n}$ has a greedy drawing with root r inside P and leaves $v_0, \ldots, v_{n-1}$?

If Problem 1 has a solution we write $\{\varphi_0, \ldots, \varphi_{n-1}\} \in \mathcal{P}^n$. It can be solved using a series of optimization problems as in Fig. 6a (one for each fixed cyclic ordering of $(\varphi_1, \ldots, \varphi_n)$). The last constraint in (*) follows from applying the law of sines and is known as the wheel condition in the work of di Battista and Vismara [6].

Lemma 10. *It is $\{\varphi_0, \ldots, \varphi_{n-1}\} \in \mathcal{P}^n$ if and only if there exists a solution of (*) with $\varepsilon > 0$ for an ordering $(\varphi_0, \ldots, \varphi_{n-1})$.*

Deciding whether a solution of (*) with $\varepsilon > 0$ exists is in fact equivalent to deciding whether the wheel condition can be satisfied in the interior of a $2n-1$-dimensional simplex; see [11] for more details.

Theorem 1. *For $n = 3, 4, 5$, consider trees T_i, $i = 0, \ldots, n-1$ with root r, edge rv_i in T_i, $\deg(r) = 1$ in T_i, $T_i \cap T_j = \{r\}$ for $i \neq j$, such that each T_i has a drawing with opening angle at least $0 < \varphi_i - \varepsilon < 180°$ for any $\varepsilon > 0$. Then, tree $T = \bigcup_{i=0}^{n-1} T_i$ has a greedy drawing with $|\angle T_i| < \varphi_i$ for all $i = 0, \ldots, n-1$ if and only if $\{\varphi_0, \ldots, \varphi_{n-1}\} \in \mathcal{P}^n$.*

Proof. First, consider a drawing of $K_{1,n}$ with edges rv_i that solves $\mathcal{P}^n$, and, without loss of generality, let the angles be ordered such that $\psi_i := \angle v_{i-1} v_i v_{i+1} < \varphi_i$. We create a greedy drawing Γ of T by drawing $(T_i)_{rv_i}^{v_i}$ infinitesimally small at v_i with opening angle $\varphi_i - \varepsilon > \psi_i$ for a sufficiently small $\varepsilon > 0$ and orienting it such that $v_j \in \angle T_i$ for all $j \neq i$; see Fig. 6c.

Now assume a greedy drawing Γ_0 of T with $|\angle T_i| < \varphi_i$, $i = 0, \ldots, n-1$ exists. For one i, it might be $|\angle T_i| < 0$ in Γ_0. Then, there also exists a greedy drawing Γ, in which $0 < |\angle T_j| < \varphi_j$, $j = 0, \ldots, n-1$. By Lemma 5, the subtree $\overline{T} = \{rv_i\} + \bigcup_{j \neq i} T_j$ must have an open angle in Γ_0. We then obtain Γ by making the edge rv_i sufficiently long inside $\angle \overline{T}$ and drawing T_i with $|\angle T_i| > 0$, such that $\overline{T} \subseteq \angle T_i$ and $T_i \subseteq \angle \overline{T}$.

We apply Lemma 8 to T_0, then to $T_1, \ldots, T_{n-1}$ and obtain a greedy drawing Γ' of T with opening angles $\angle T_i$ of same size, such that each subtree $(T_i)_{rv_i}^{v_i}$ is

Table 2: Solving non-linear problem $\mathcal{P}^n$ explicitly. Let $\varphi_i \geq \varphi_{i+1}$, $\varphi_i \in (0°, 60°] \cup (90°, 120°] \cup \{180°\}$, $\sum_{i=0}^{n-1} \varphi_i > (n-2)180°$. See the full version for the proofs.

n	case	$\{\varphi_0, \ldots, \varphi_{n-1}\} \in \mathcal{P}^n$ iff
$3, 4$		always
5	$\varphi_0 = \ldots = \varphi_3 = 180°$	always
5	$\varphi_0 \leq 120°$	always
5	$\varphi_0 = \ldots = \varphi_2 = 180°$	$\varphi_3 + \varphi_4 > 120°$
5	$\varphi_0 = \varphi_1 = 180°$	$\varphi_2 + \varphi_3 + \varphi_4 > 240°$
5	$\varphi_0 = 180°, \varphi_1, \varphi_2, \varphi_3 \in (90°, 120°], \varphi_4 \leq 60°$	?
5	$\varphi_0 = 180°, \varphi_1, \ldots, \varphi_4 \in (90°, 120°]$	?

drawn infinitesimally small at v_i. For $n = 4, 5$, for each pair of consecutive edges rv_i, rv_j in Γ' the turn from rv_i to rv_j is less than 180°, so r lies inside the convex polygon with corners $v_0, \ldots, v_{n-1}$. Therefore, Γ' directly provides a solution of $\mathcal{P}^n$. For $n = 3$, v_1 might lie inside angle $\angle v_0 r v_2 \leq 180°$. However, since $\varphi_0 + \varphi_1 + \varphi_2 > 180°$, it is $\{\varphi_0, \varphi_1, \varphi_2\} \in \mathcal{P}^3$; see Table 2. □

Although Problem (*) is non-linear, we are almost always able to give tight conditions for the existence of the solution; see Table 2, which summarizes all possible cases. The last two cases for $n = 5$ are the only remaining ones to consider (for $\varphi_3 + \varphi_4 > 120°$, $\varphi_2 + \ldots + \varphi_4 > 240°$, $\varphi_1 + \ldots + \varphi_4 > 360°$). In practice, it is possible to either strictly prove $\{\varphi_0, \ldots, \varphi_4\} \notin \mathcal{P}^5$ or numerically construct a solution for many such sets of angles. If we drop the last constraint in (*), we acquire a linear program which has a constant number of variables and constraints and can be solved in $O(1)$ time. If it has no solution for any cyclic order of φ_i, neither has $\mathcal{P}^5$. For example, this is the case for $\{180°, 105°, 105°, 105°, 60°\}$. If this linear program has a solution, we can try to solve (*) using nonlinear programming solvers. However, if the non-linear solver finds no solution, we obviously have no guarantee that none exists. In [11], we present examples of trees for which we could prove the existence of a greedy drawing by solving $\mathcal{P}^5$ using MATLAB. Further, we formulate a sufficient condition for the first of the two above cases.

5 Recognition algorithm

Maximum degree 4. In this section we formulate Algorithm 1, which decides for a tree T with maximum degree 4 whether T has a greedy drawing. First, we describe a procedure to determine the tight upper bound for the opening angle of a given rooted subtree. Let $\mathrm{N}(v)$ denote the neighbors of $v \in V$ in T. After processing a node v, we set a flag $\mathrm{p}(v) = \text{true}$. Let $\mathrm{N_p}(v) = \{u \mid uv \in E, \mathrm{p}(u) = \text{true}\}$, and $\angle_{\text{optimal}}$ the new tight upper bound calculated according to Table 1.

Lemma 11. *Procedure getOpenAngle is correct and requires time $O(|V|)$.*

Proof. The algorithm processes tree nodes bottom-up. For $v \in V$, let π_v be the parent of v, $\deg(v) = d_v$, $T_v = T^v_{\pi_v v} + \pi_v v$ with root π_v. For a subtree with one

Procedure getOpenAngle(T,r)

Input : tree $T = (V, E)$, root $r \in V$, $d_r = 1$
Result: tight upper bound on $|\angle T|$, 0 if no open angle possible.
$\mathrm{p}(r) \leftarrow$ false
for $v \in V \setminus \{r\}$ **do**
  if $d_v = 5$ **then** **return** 0
  else if $d_v = 1$ **then**
    $\mathrm{p}(v) \leftarrow$ true
    $\angle(v) \leftarrow 180$
  else $\mathrm{p}(v) \leftarrow$ false
while $\exists v \in V : \neg\, \mathrm{p}(v)$ & $|\,\mathrm{N_p}(v)| = d_v - 1$
  if $\forall u \in \mathrm{N_p}(v) : \angle(u) = 180$ **then**
    $\angle(v) \leftarrow 180 - (d_v - 2) \cdot 60$
  else if *case I,...,V applicable* **then** $\angle(v) \leftarrow \angle_{\mathrm{optimal}}(\mathrm{N_p}(v))$
  else **return** 0
  $\mathrm{p}(v) \leftarrow$ true
return $\angle(v)$ for $\{v\} = \mathrm{N}(r)$

Algorithm 1: hasGreedyDrawing(T)

Input : tree $T = (V, E)$, max deg 4
Result: whether T has a greedy drawing
for $v \in V$ **do**
  if $d_v = 1$ **then**
    $\mathrm{p}(v) \leftarrow$ true; $\angle(v) \leftarrow 180$
  else $\mathrm{p}(v) \leftarrow$ false
while $\exists v \in V : \neg\, \mathrm{p}(v)$ & $|\,\mathrm{N_p}(v)| \geq d_v - 1$
  if $|\,\mathrm{N_p}(v)| = d_v$ **then**
    return $\sum_{u, uv \in E} \angle(u) > (d_v - 2)180$
  else if $\forall u \in \mathrm{N_p}(v) : \angle(u) = 180$ **then**
    $\angle(v) \leftarrow 180 - (d_v - 2) \cdot 60$
  else if *case I,...,V applicable* **then**
    $\angle(v) \leftarrow \angle_{\mathrm{optimal}}(\mathrm{N_p}(v))$
  else
    $w \leftarrow \mathrm{N}(v) - \mathrm{N_p}(v)$
    $\angle(w) \leftarrow getOpenAngle(T^w_{vw} + vw, v)$
    return $\angle(w) > 0$ & $\sum_{u, uv \in E} \angle(u) > (d_v - 2)180$
  $\mathrm{p}(v) \leftarrow$ true

or two nodes, define its opening angle as 180°. We prove the following invariant for the *while* loop: For each $v \in V$ with $\mathrm{p}(v) =$ true, $\angle(v) > 0$ stores a tight upper bound for the opening angle in a greedy drawing of T_v.

The invariant holds for all leaves of T after the initialization. The first *if*-statement inside the *while* body ensures that if all nodes in T_v except v have degree 1 or 2, then $\angle(v) = 180$ if $d_v = 1, 2$ in T, $\angle(v) = 120$ if $d_v = 3$ and $\angle(v) = 60$ if $d_v = 4$. Now consider the first *else* clause inside the *while* loop. Assume $\mathrm{p}(v) =$ false, $|\,\mathrm{N_p}(v)| = d_v - 1$ and the invariant holds for all subtrees T_u, $u \in \mathrm{N_p}(v)$. If one of the cases I–V can be applied to v and subtrees T_u, then, after the current loop, $\angle(v) > 0$ stores the tight upper bound for the opening angle in a greedy drawing of T_v; see Table 1. Otherwise, we have case VI or VII, and T_v cannot be drawn with an open angle. Each node v is processed in $O(d_v)$, and if for $u \in \mathrm{N}(v) - \mathrm{N_p}(v)$, it holds $|\,\mathrm{N_p}(u)| \geq d_u - 1$ after processing v, we put u in a queue. Hence the running time is $O(|V|)$. □

Algorithm 1 also requires $O(|V|)$ time and is similar to Procedure getOpenAngle, except that T now does not have a distinguished root. We proceed from the leaves of T inwards, until we reach some "central" node v with neighbors $\{u_1, \ldots, u_{d_v}\}$, such that a greedy drawing of T exists only if all tight upper bounds φ_i on $|\angle(T^{u_i}_{vu_i} + vu_i)|$ are positive. Then, we report true if and only if $\sum_{i=1}^{d_v} \varphi_i > (d_v - 2)180°$. See [11] for the formal correctness proof.

Maximum degree 5 and above. If T contains a node v with $\deg(v) \geq 6$, no greedy drawing exists. Also, a greedy-drawable tree can have at most one node of degree 5 by Lemma 7, otherwise, there are two independent 5-tuples. For unique $r \in V$, $\deg(r) = 5$, consider the five rooted subtrees $T_0, \ldots, T_4$ attached to it and the tight upper bounds φ_i on $|\angle T_i|$. If $\sigma = \sum_{i=0}^{4} \varphi_i \leq 540°$, T cannot be drawn greedily. The converse, however, does not hold. By Theorem 1, a greedy drawing exists if and only if $\{\varphi_0, \ldots, \varphi_4\} \in \mathcal{P}^5$. To decide whether $\{\varphi_0, \ldots, \varphi_4\} \in \mathcal{P}^5$, we apply the conditions from Table 2. If in the remaining case $\varphi_0 = 180°$, $\varphi_1, \ldots, \varphi_4 \leq 120°$ (i) the sufficient condition does not apply, (ii) the linear relaxation of Problem (*) has a solution, but (iii) the non-linear solver finds none, we report *uncertain*. The full formulation of this algorithm as well as uncertain examples can be found in [11].

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