A Sublinear Bound on the Page Number of Upward Planar Graphs

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Abstract

The page number of a directed acyclic graph G is the minimum k for which there is a topological ordering of G and a k-coloring of the edges such that no two edges of the same color cross, i.e., have alternating endpoints along the topological ordering. We address the long-standing open problem asking for the largest page number among all upward planar graphs. We improve the best known lower bound to 5 and present the first asymptotic improvement over the trivial $\mathcal{O}(n)$ upper bound, where n denotes the number of vertices in G. Specifically, we first prove that the page number of every upward planar graph is bounded in terms of its width, as well as its height. We then combine both approaches to show that every n-vertex upward planar graph has page number $\mathcal{O}(n^{2/3}\log^{2/3}(n))$.

1 Introduction

In an *upward planar drawing* of a directed acyclic graph $G = (V, E)$, every vertex $v \in V$ is a point in the Euclidean plane, and every edge $(u, v) \in E$ is a strictly y-monotone curve[1] with lower endpoint u and upper endpoint v that is disjoint from other points and curves, except in its endpoints. A directed acyclic graph admitting such a drawing is called *upward planar*. In other words, a directed graph is upward planar if it allows a planar drawing with all edges "going strictly upwards". In Figure 1 we have an upward planar graph G on the left, while the planar directed acyclic graph G_k on the right is not upward planar.

In a *book embedding* of a directed acyclic graph $G = (V, E)$, the vertex set V is endowed with a topological ordering $<$, called the *spine ordering*, and the edge set E is partitioned into so-called *pages* with the property that no page contains two edges $(u_1, v_1), (u_2, v_2)$ that cross with respect to $<$, i.e., $u_1 < u_2 < v_1 < v_2$ or $u_2 < u_1 < v_2 < v_1$. Then the *page number* $\operatorname{pn}(G)$ of a directed acyclic graph G is the minimum k for which it admits a book embedding with k pages. In other words, $\operatorname{pn}(G) \leqslant k$ if the vertices can be ordered along the spine with all "edges going right" and there exists a k-edge coloring so that any two edges with alternating endpoints along the spine have distinct colors.

In Figure 2 we have book embeddings of the directed acyclic graphs in Figure 1 with three pages (left) and k pages (right), respectively. This shows that $\operatorname{pn}(G) \leqslant 3$ and $\operatorname{pn}(G_k) \leqslant k$. In fact, observe that G_k admits only one topological ordering $<$, as there is a directed Hamiltonian path $\ell_1, \ldots, \ell_k, r_1, \ldots, r_k$ in G_k. As the edges $(\ell_1, r_1), \ldots, (\ell_k, r_k)$ are pairwise crossing w.r.t. $<$, it follows that $\operatorname{pn}(G_k) = k$. It is easy to see (as observed for example in [9]) that for any directed graph G we have $\operatorname{pn}(G) \leqslant 2$ if and only if G is a spanning subgraph of an upward planar graph with a directed Hamiltonian path. (Recall that G_k from the right of Figure 1 is not upward planar.) It thus follows that $\operatorname{pn}(G) = 3$ for the graph G in the left of Figure 1.

The page number of *undirected* graphs (where the spine ordering may be any vertex ordering) was introduced by Bernhart and Kainen in 1979 [8], building upon the suggested notion of Ollmann [29]. Their conjecture that the page number of planar graphs is unbounded was quickly disproven [10, 20], with Yannakakis [31] giving the best upper bound of 4, which was just very recently shown to be best-possible [7, 32].

Book embeddings of *directed* graphs were first considered by Nowakowski and Parker [28] in 1989. They introduced the page number of a poset P by considering its cover graph $G(P)$ and restricting the spine ordering to be a topological ordering of $G(P)$, or equivalently, a linear extension of P. They then ask whether posets with a planar order diagram have bounded page number — equivalently, whether upward planar and transitively reduced graphs have bounded page number. Despite significant effort on posets [2–4, 21–24, 30] and general acyclic directed

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[1]Equivalently, straightline segments may be used [14].

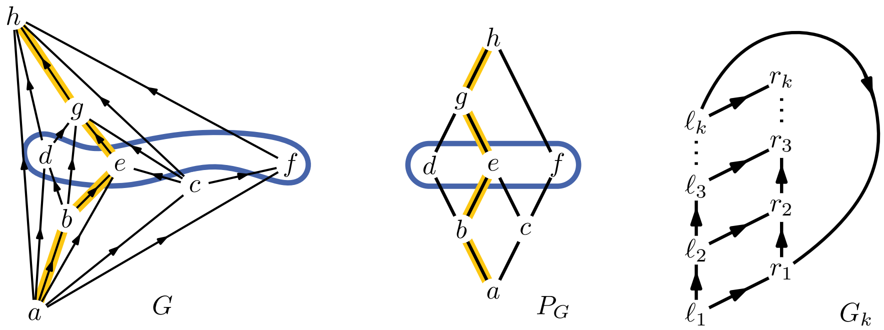

Figure 1: Left: An upward planar st-graph G of height 5 and width 3. Middle: The reachability poset P_G of G. Right: A planar directed acyclic graph G_k with $\mathrm{pn}(G_k) = k$.

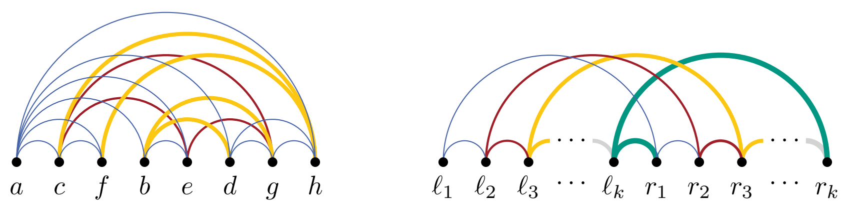

Figure 2: Book embeddings of the graphs in Figure 1.

graphs [1, 5, 9, 16–18, 26], this question is still open. In fact, the asymptotically best known upper bound is linear in n, the number of vertices, which can be obtained by simply putting each edge on a separate page.

In this paper, we provide the first upper bound for any upward planar graph that is sublinear in the number of vertices. Specifically, we prove that n-vertex upward planar graphs have page number $\mathcal{O}(n^{2/3} \log^{2/3}(n))$. We do so by bounding the page number of any upward planar graph first in terms of its width, then in terms of its height, and finally combining both approaches to achieve the desired bound in terms of its number of vertices.

Related Work. Nowakowski and Parker [28] (and independently Heath et al. [23]) show that directed forests have page number 1. Alzohairi and Rival [3] (see also [16]) show that series-parallel upward planar graphs have page number 2, which was later generalized to N-free upward planar graphs by Mchedlidze and Symvonis [26].

The best known upper bounds for the page number of upward planar graphs are due to Frati et al. [17], who prove that every n-vertex upward planar triangulation with $o(n/\log n)$ diameter has $o(n)$ page number, any n-vertex upward planar triangulation has page number at most $\min\{\mathcal{O}(k \log n), \mathcal{O}(2^k)\}$, where k is the maximum page number among its 4-connected subgraphs[2], and finally that every n-vertex upward planar triangulation has page number $o(n)$ if that is true for those with maximum degree $\mathcal{O}(\sqrt{n})$. According to the authors of [17] “Determining whether every n-vertex upward planar DAG has $o(n)$ page number [...] remains among the most important problems in the theory of linear graph layouts.”

For lower bounds, Nowakowski and Parker [28] present a planar poset with page number 3, while Hung [24] presents a planar poset with page number 4. For general upward planar graphs one can also easily derive the same lower bound of 4 from one of the undirected planar graphs of page number 4 [7, 32]. As nothing better is known here, we also present in this paper upward planar graphs with page number at least 5.

Preliminaries. We denote the directed reachability of a vertex v from another vertex u in a directed acyclic graph G by $u \prec_G v$ (omitting the index if it is clear from the context), and write $u \preccurlyeq v$ if $u \prec v$ or $u = v$. This way we obatin the *reachability poset* $P_G = (V, \prec)$ of G as the vertices of G partially ordered by their directed reachability. Transferring these notions from posets to directed acyclic graphs, we say that u and v are *comparable* if $u \prec v$ or $v \prec u$; otherwise u and v are *incomparable*. Consequently, the *height* $h(G)$ and *width* $w(G)$ of a

[2] A recent result by Davies and McCarty [13] automatically improves the result by Frati et al. [17] to $\min\{\mathcal{O}(k \log n), \mathcal{O}(k^2)\}$. An even more recent (and yet unpublished) result by Davies [12] further improves this to $\mathcal{O}(k \log k)$.

directed acyclic graph G is the largest number of pairwise comparable, respectively incomparable, vertices in G. Equivalently, $h(G)$ is the number of vertices in a longest directed path in G, while $w(G)$ is the largest number of vertices in G with no directed reachabilities among them. See the left and middle of Figure 1 for some example. Let us also define for a subset X of vertices of G its height $h(X)$ and width $w(X)$ as the maximum number of vertices in X that are pairwise comparable, respectively incomparable, in G.

An upward planar graph $G = (V, E)$ is an *st-graph* if there is a (unique) vertex s with $s \preccurlyeq v$ for all $v \in V$ and a (unique) vertex t with $v \preccurlyeq t$ for all $v \in V$. An st-path in G is a directed path from s to t in G. In particular, the height of an st-graph is the length of a longest st-path. It is known [25] that every upward planar graph G (on at least three vertices) is a spanning subgraph of some st-graph $\overline{G}$ whose faces are all bounded by triangles. Note that this augmentation is not unique. As $\mathrm{pn}(G) \leqslant \mathrm{pn}(\overline{G})$ whenever $G \subseteq \overline{G}$, we may restrict ourselves to st-graphs when proving upper bounds on the page number of upward planar graphs in terms of their number of vertices. (Note however that this is not true when working in terms of height.) Let us also remark that if G is an st-graph, its reachability poset P_G is called a *planar lattice* in order theory [6].

A notion closely related to the page number is the *twist number* $\mathrm{tn}(G)$, which is defined as the smallest k for which there exists a topological ordering $<$ of G with no $(k+1)$-twist, i.e., no $k+1$ edges that are pairwise crossing w.r.t. $<$. Clearly, $\mathrm{tn}(G) \leqslant \mathrm{pn}(G)$, as the k edges of a k-twist must be assigned to pairwise distinct pages. Indeed, having already decided on a spine ordering with no $(k+1)$-twist, assigning the edges to pages is equivalent to coloring the vertices of a corresponding circle graph H with no $(k+1)$-clique. As circle graphs are χ-bounded [19], one can actually bound the number of pages in terms of the largest twist size. The currently best result due to Davies and McCarty [13] states that $\chi(H) \leqslant 7\omega(H)^2$ for every circle graph H (where $\omega(H)$ is the clique number of H), which gives the following.[3]

Observation 1.1. *For every directed acyclic graph G we have* $\mathrm{pn}(G) \leqslant 7\,\mathrm{tn}(G)^2$.

Already in 2007, Černý [11] proved that $\chi(H) \leqslant \mathcal{O}(\omega(H) \cdot \log(|V(H)|)$ for every circle graph H. As the vertices of H correspond to the edges of G in this application, this gives the following.

Observation 1.2. *For every n-vertex upward planar graph G we have* $\mathrm{pn}(G) \leqslant \mathcal{O}(\mathrm{tn}(G) \cdot \log(n))$.

In fact, we shall often times bound the twist number of the considered upward planar graph G and then conclude for its page number via Observation 1.1 or Observation 1.2.

All graphs considered in this paper are directed and in most figures we omit the arrows indicating an edge's direction. If not explicitly drawn otherwise, all edges are oriented upwards.

Our Results. First, we bound the page number of upward planar graphs G in terms of their width.

Theorem 1.1. *Every upward planar graph G of width w has* $\mathrm{pn}(G) \leqslant 14 \cdot w$.

Then, we bound the page number of st-graphs in terms of their height. In fact, we show that $\mathrm{tn}(G) \leqslant 4h(G)$, improving on the $\mathrm{tn}(G) \leqslant \mathcal{O}(h(G)\log(n))$ bound for every n-vertex st-graph G due to Frati et al. [17].[4] Together with Observation 1.1 this gives the following.

Theorem 1.2. *Every upward planar graph G of height h has* $\mathrm{pn}(G) \leqslant 112 \cdot h^2$.

We remark that with a very recent (and yet unpublished) improvement of Observation 1.1 by Davies [12], we obtain $\mathcal{O}(h \log(h))$ as an upper bound on the page number of upward planar graphs with height h.

Combining our approaches for bounded width and bounded height, we give the first sublinear upper bound on $\mathrm{pn}(G)$ in terms of the number of vertices in G.

Theorem 1.3. *Every upward planar graph G on n vertices has* $\mathrm{pn}(G) \leqslant \mathcal{O}(n^{2/3} \log^{2/3}(n))$.

Finally, we improve the best known lower bound on the maximum twist number and page number among upward planar graphs to 5.

Theorem 1.4. *There is an upward planar graph G with* $\mathrm{pn}(G) \geqslant \mathrm{tn}(G) \geqslant 5$.

[3] Davies [12] recently improved this bound to $\mathcal{O}(\omega(H) \log(\omega(H)))$.
[4] Frati et al. [17] refer to $h(G)$ as the diameter of G.

2 Bounded Width

Recall that the width $w(X)$ of a subset $X \subseteq V(G)$ of the vertex set of an *st*-graph G is the largest number of vertices in X that are pairwise incomparable in G. In this section we prove that the pagenumber is bounded by a linear function of the width. In fact, we show a more general statement: Given a subset $X \subseteq V(G)$, we embed all edges of $G[X]$ in $\mathcal{O}(w(X))$ pages, where $G[X]$ denotes the subgraph of G induced by X. This generalization will be used in Section 4, where we combine it with the results from Section 3. Theorem 1.1 will follow by setting $X = V(G)$.

The main lemma of this section (Lemma 2.1) takes as input an *st*-graph G and a subset $X \subseteq V(G)$ of the vertices. It describes how to assign all edges in $G[X]$ to few pages. Additionally, the lemma constructs a new *st*-graph G' that is used in Section 4 to handle the remaining edges, namely those with at most one endpoint in X. The vertex set of G' is a superset of the vertices of G. Further, for every two vertices $u, v \in V(G)$, whenever $u \prec_G v$, then also $u \prec_{G'} v$. Therefore every topological ordering of G' (restricted to the vertex set of G) yields a topological ordering of G. Note that for some u and v, we might have $u \prec_{G'} v$ but $u \not\prec_G v$. These additional comparabilities in G' make sure that the already assigned edges remain crossing-free on their respective pages, no matter which topological ordering of G' is chosen in later steps. All edges in $E(G) - E(G') =: E_\Delta$ that are removed while constructing G' are accounted for by Lemma 2.1 as well.

So consider the *st*-graph G and a set $X \subseteq V(G)$. All vertices in X can be covered by a set $\mathcal{P}$ of *st*-paths, where $|\mathcal{P}| = w(X)$. To see this, consider the directed acyclic graph H with vertex set X and an edge from $u \in X$ to $v \in X$ if and only if $u \prec_G v$. Its reachability poset P_H has width $w(P_H) = w(X)$. By Dilworth's Theorem, P_H can be decomposed into $w(X)$ chains, i.e. subsets of pairwise comparable elements. Each of these chains can be extended to an *st*-path in G. Given an upward planar embedding of G, we can define what it means for two of these paths to cross: Let $P, Q \in \mathcal{P}$ be two paths and v be the last vertex on the longest shared subpath beginning at s (the case $v = s$ is possible). Without loss of generality the next edge of P precedes the next edge of Q in the clockwise order of v's outgoing edges. We say that P and Q *cross* at another common vertex w if the next edge of P succeeds the next edge of Q in the clockwise order of w's outgoing edges. Note that this definition allows P and Q to have common vertices and edges, even if they do not cross. In the following we always assume $\mathcal{P}$ to be *non-crossing*, meaning there is a left-to-right ordering $P_1, \dots, P_{w(X)}$ of the *st*-paths in $\mathcal{P}$ such that no two consecutive paths cross. This assumption is justified, as a crossing between two paths P and Q at vertex w can be removed by swapping their subpaths starting at w. Thus, any set of crossing paths can be made non-crossing in every upward planar embedding (see for example the blue, yellow and red path in Figure 3, which cross in the left but not in the right subfigure).

For two consecutive paths $P_i, P_{i+1} \in \mathcal{P}$, a *lens L between P_i and P_{i+1}* is a subgraph of G enclosed by two subpaths $P'_i \subseteq P_i$ and $P'_{i+1} \subseteq P_{i+1}$ such that their endpoints coincide and they do not share any inner vertices. A lens L has a unique source s_L and a unique sink t_L with $s_L, t_L \in V(P'_i) \cap V(P'_{i+1})$ and $s_L \prec_G t_L$. As any two paths in $\mathcal{P}$ share the global source s and sink t, there is at least one lens between any two consecutive paths in $\mathcal{P}$. Given the *st*-paths in $\mathcal{P}$, we distinguish two kinds of edges of $G[\mathcal{P}]$, where $G[\mathcal{P}]$ is the subgraph of G induced by all vertices covered by $\mathcal{P}$: We call an edge e having both endpoints contained in the same path in $\mathcal{P}$ an *intra-path-edge* (e can be an edge of the path or a transitive edge). In contrast, an *inter-path-edge* e has its two endpoints in two different paths in $\mathcal{P}$. We note that if P_i and P_{i+1} share some vertices and edges, it is technically possible for an edge to be an intra-path-edge and an inter-path-edge at the same time. In this case, we consider it to be an intra-path-edge. See Figure 3 for a visualization of the terminology.

LEMMA 2.1. *Let G be an st-graph and let $X \subseteq V(G)$ be a subset of its vertices of width w. Then there is an st-graph G' with $V(G) \subseteq V(G')$ such that:*

- *For every two vertices $u, v \in V(G)$ with $u \prec_G v$ we have $u \prec_{G'} v$.*
- *Every topological ordering of G' admits an assignment of $E(G'[X])$ and E_Δ to $14w$ pages.*

Proof. We start by initially setting $G' = G$. As we go on, we add additional edges to G' and subdivide existing ones. Thus at the end G' is a supergraph of a subdivision of G and all reachabilities of G are maintained. All edges of G not in G' (exactly the ones that are subdivided) form the set E_Δ.

As X has width w, there is a set $\mathcal{P} = \{P_1, \dots, P_w\}$ of non-crossing *st*-paths in G' covering all vertices of X. Note that whenever we subdivide an edge on some path $P \in \mathcal{P}$, the new subdivision vertex and its two incident

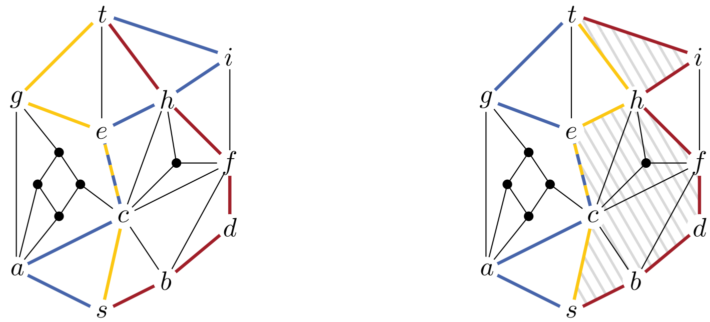

Figure 3: Left: An st-graph G with three paths covering the subset $X \subseteq V(G)$ of all labeled vertices. Right: The same graph with three non-crossing paths covering X. All colored edges as well as (a, g), (e, t), (c, h), (b, f) and (f, i) are intra-path-edges. On the other hand, (b, c) and (c, f) are inter-path-edges. The two shaded regions are the two lenses between the yellow and the red path.

edges are added to P (replacing the subdivided edge). This way, the subgraph $G'[\mathcal{P}]$ induced by the paths in $\mathcal{P}$ is well-defined at every step. Let the paths be numbered such that P_i is to the left of P_j whenever $i < j$. Let $P_i, P_{i+1} \in \mathcal{P}$ be two consecutive paths in the left-to-right ordering and let L_1, L_2 be two lenses between them. For $j = 1, 2$, let s_j and t_j denote the source, respectively the sink, of L_j. By definition, s_j and t_j are the only vertices bounding L_j common to both P_i and P_{i+1}. Thus we can assume without loss of generality that $s_1 \prec_{G'} t_1 \preccurlyeq_{G'} s_2 \prec_{G'} t_2$. We conclude that in every topological ordering of G' two edges from different lenses of P_i, P_{i+1} do not cross, allowing us to deal with each lens separately and to reuse the same set of pages for all lenses between P_i and P_{i+1}.

For a single lens L between P_i and P_{i+1} we partition the inter-path-edges in L into $\overrightarrow{E_L}$ (oriented from P_i to P_{i+1}) and $\overleftarrow{E_L}$ (oriented from P_{i+1} to P_i). From the planarity of G' we obtain a bottom-to-top ordering $e_1, \ldots, e_\ell$ of the inter-path-edges, i.e., we order them by their endpoints along P_i, using the endpoints on P_{i+1} as a tie-breaker.

Before we actually assign the edges to pages, let us give a short overview over the strategy: We will consider the inter-path-edges in $\overrightarrow{E_L}$ and $\overleftarrow{E_L}$ separately, distributing their edges (and all edges that are subdivided in the process) to six pages each. This results in a total of twelve pages for all lenses between two consecutive paths P_i and P_{i+1}. We will finish the proof by observing that each path itself (possibly with its subdivided edges) requires just two more pages. As there are w paths, this adds up to $2w + 12(w - 1) \leqslant 14w$ pages.

In the following we only consider the inter-path-edges in $\overrightarrow{E_L}$, the case for $\overleftarrow{E_L}$ works symmetrically. Some of these edges may be transitive in $G'[\mathcal{P}]$. We observe that for every transitive edge e there is a non-transitive edge $f = (v_j, w_j)$, such that e is either incident to v_j and above f, or incident to w_j and below f (above and below refer to the bottom-to-top ordering of the inter-path-edges)). See on the left of Figure 4 for some examples of transitive and non-transitive inter-path-edges.

Let $(v_1, w_1), \ldots, (v_k, w_k)$ be the subset of all inter-path-edges in $\overrightarrow{E_L}$ that are non-transitive in $G'[\mathcal{P}]$ ordered from bottom to top. We observe that these edges form a matching, as otherwise at least one of them would be transitive. Now subdivide each $e_j = (v_j, w_j)$ with $j \in \{1, \ldots, k\}$ in G' and call the subdivision vertex u_j. Further subdivide the edge of P_i outgoing from v_j and the edge of P_{i+1} incoming to w_j in G' calling the subdivision vertices v'_j and w'_j, respectively. By upward planarity, $v'_j \not\prec_{G'} w'_j$ and thus adding a directed path from w'_j to v'_j in G' (which we shall do next) maintains the acyclicity of G'. Additionally we have to ensure that G' remains a planar st-graph (and thus upward planar) with this new directed path (see the right of Figure 4): Call $E_{w,j}$ the set of edges incoming to w_j in clockwise order between (w'_j, w_j) and (u_j, w_j). Subdivide each edge in $E_{w,j}$ once and add a path from w'_j to u_j through the subdivision vertices in clockwise order. Analogously, $E_{v,j}$ contains the edges outgoing from v_j between (v_j, u_j) and (v_j, v'_j) in counterclockwise order. We subdivide all edges in $E_{v,j}$ and extend the new path from u_j to v'_j through all subdivision vertices in counterclockwise order. Now $w'_j \prec_{G'} v'_j$, as desired. Note that E_Δ consists of all edges that were subdivided in G. This includes all inter-path-edges and additionally some intra-path-edges and those edges incident to v_j or w_j with only one endpoint in $\mathcal{P}$.

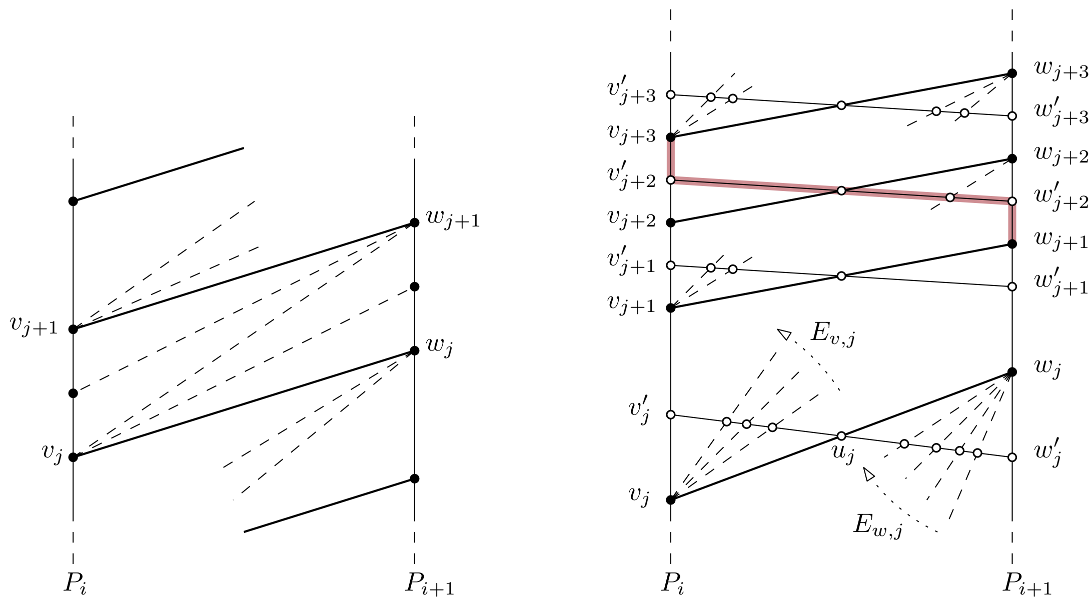

Figure 4: Left: Several inter-path-edges between P_i and P_{i+1}. Only the solid ones are non-transitive in $G'[\mathcal{P}]$. Right: The comparability between w'_j and v'_j was achieved by adding a w'_j-v'_j-path. To preserve that G' is a planar st-graph, all intersected edges are subdivided at the intersections. Further the comparability $w_{j+1} \prec_{G'} v_{j+3}$ is highlighted. Note that the dashed edges may have only one endpoint in P_i or P_{i+1}.

We now assign the edges in $G'[\mathcal{P}]$ and E_Δ to pages. Note that all edges of $G'[\mathcal{P}]$ are inter-path-edges or intra-path-edges as both their endpoints are in $\mathcal{P}$. The edges in $E_{v,j} \cup \{e_j\}$ form a star centered at v_j, so they can all be assigned to the same page in any topological ordering. Further, all of these edges have v_j as their lower endpoint. In an upward planar drawing of G', all edges in $E_{v,j}$ are inside the subregion of L enclosed by P_i and P_{i+1} to the sides and the subdivided e_j and e_{j+1} to the bottom and top. Thus in every topological ordering of G' they end at w_{j+1} or earlier and thus before v_{j+3} (because $w_{j+1} \prec_{G'} w'_{j+2} \prec_{G'} v'_{j+2} \prec_{G'} v_{j+3}$, see the red path in the right of Figure 4). Therefore the star centered at v_j can be embedded on the same page as the star centered at v_{j+3}. Generalizing this observation, we assign $E_{v,j} \cup \{e_j\}$ to a page $Q^r_{i,i+1}$ where r is the remainder of j divided by 3. With a symmetric argument all edges in $E_{w,j}$ can be assigned to three more pages $Q^r_{i+1,i}$.

The intra-path-edges are left to be embedded. Each path $P \in \mathcal{P}$ (including the added subdivision vertices) induces a planar directed Hamiltonian graph $H_P \subseteq G'$. The edges lost while subdividing can be added to H_P such that it remains planar and Hamiltonian. Therefore all intra-path-edges (of G and G') can be assigned to two further pages Q^1_i and Q^2_i in any topological ordering of G' [9].

Let us recap, that we use twelve pages for the inter-path-edges between any two consecutive paths in $\mathcal{P}$ and two pages per path for the intra-path-edges. In total we get that $2w + 12(w-1) \leqslant 14w$ pages suffice for every topological ordering of G'. ◻

As mentioned above, Theorem 1.1 now follows as a direct corollary from Lemma 2.1 by choosing $X = V(G)$. Let us remark that a more careful argumentation leads to a slightly better result. We are able to show that for every st-graph G we have $\mathrm{pn}(G) \leqslant 4w(G) - 2$ by using a different strategy to embed the inter-path-edges. However we were not able to show the more general statement of Lemma 2.1 (which we need in Section 4) with this approach and hence omit this improvement of Theorem 1.1 here.

3 Bounded Height

In this section, we prove Theorem 1.2, which bounds the page number of any st-graph in terms of its height. Recall that the height $h(X)$ of a subset $X \subseteq V(G)$ of the vertex set of an st-graph G is the largest number of vertices in X that are pairwise comparable in G. In combination with Lemma 2.1 from the previous section, the following lemma is central in the proof of our sublinear bound on the page number of upward planar graphs in terms of the

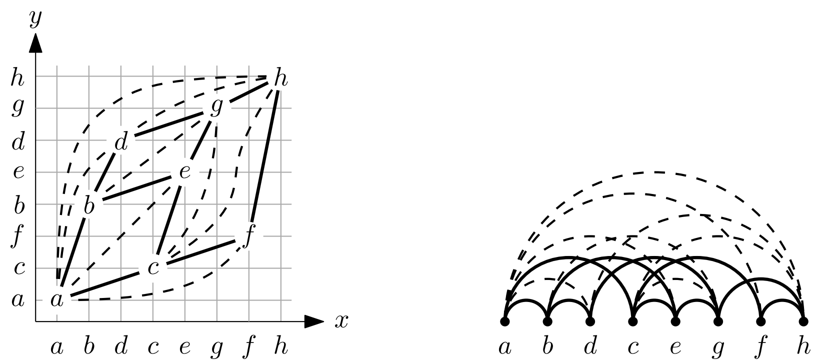

Figure 5: A dominance drawing (left) and the same graph with spine ordering $<_x$ (right). Transitive edges are drawn dashed for better readability.

number of vertices. As in Lemma 2.1, we prove a stronger statement than Theorem 1.2 by considering arbitrary subsets X of vertices of the graph.

LEMMA 3.1. *Let G be an st-graph and let $X \subseteq V(G)$ be a subset of its vertices of height h. Then G admits a topological vertex ordering such that the size of every twist consisting of edges with at least one endpoint in X is at most $4h$.*

Proof. Di Battista, Tamassia, and Tollis [15] showed that for every st-graph, there is a *dominance drawing*: This is a planar drawing such that between any two vertices u and v, there is a path from u to v if and only if $x(u) \leqslant x(v)$ and $y(u) \leqslant y(v)$, where $x(w)$ and $y(w)$ denote the x-coordinate and y-coordinate of a vertex w, respectively (see Figure 5). Let $<_x$ denote the vertex ordering that is given by increasing x-coordinates, in case of ties, we define $u <_x v$ if $y(u) < y(v)$. Symmetrically, we define $u <_y v$ if and only if $y(u) < y(v)$ or if $y(u) = y(v)$ and $x(u) < x(v)$. We also write $v >_x u$ and $v >_y u$ instead of $u <_x v$ and $u <_y v$, respectively. Most importantly, we observe that

$$u \prec_G v \quad \Longleftrightarrow \quad u <_x v \text{ and } u <_y v. \tag{3.1}$$

Now we take $<_x$ as the linear vertex ordering for G and consider a largest twist $a_1 <_x \cdots <_x a_k <_x b_1 <_x \cdots <_x b_k$ consisting of edges in G with at least one endpoint in X. That is, $(a_i, b_i) \in E(G)$ and we have $a_i \in X$ or $b_i \in X$ for $i = 1, \ldots, k$. We assume for the sake of contradiction that $k > 4h$. By pigeonhole principle, more than $k/2$ of the a_i's are in X or more than $k/2$ of the b_i's are in X. Assume the first, the latter case works symmetrically. The symmetric case is shown in Figure 6 (right). Without loss of generality, we have $a_1, \ldots, a_{k'} \in X$, where $k' > 2h$. Consider the elements $a_1, \ldots, a_{k'}$ and their ordering with respect to $<_y$. As $k' > 2h$, by the Erdős-Szekeres theorem there exists at least one of the following:

- a sequence $i_1 < \cdots < i_{h+1}$ of indices with $a_{i_1} <_y \cdots <_y a_{i_{h+1}}$
- a sequence $i_1 < i_2 < i_3$ of indices with $a_{i_1} >_y a_{i_2} >_y a_{i_3}$

The first case would give together with (3.1) that $a_{i_1} \prec \cdots \prec a_{i_{h+1}}$, i.e., $h+1$ pairwise comparable vertices in X, a contradiction. Thus, we have the second case: Three vertices $a_{i_1}, a_{i_2}, a_{i_3}$ with opposing ordering with respect to $<_x$ and $<_y$, as illustrated in Figure 6. Together we have that $a_{i_2} <_x a_{i_3} <_x b_{i_1} <_x b_{i_2}$ and $a_{i_3} <_y a_{i_2} <_y a_{i_1} <_y b_{i_1}$. On one hand, this implies with (3.1) that $a_{i_3} \prec b_{i_1}$ and hence there is a path P in G from a_{i_3} to b_{i_1} that is monotone in x- and y-coordinates, i.e., P lies entirely inside the axis-aligned rectangle R spanned by the elements a_{i_3} and b_{i_1}, see Figure 6. On the other hand, the edge $e = (a_{i_2}, b_{i_2})$ crosses through the rectangle R from left to right. Note that edge e indeed lies below b_{i_1} as it does not cross the edge (a_{i_1}, b_{i_1}). We conclude that edge e crosses path P, which contradicts the planarity of the drawing. ◻

Choosing $X = V(G)$, Lemma 3.1 gives a topological ordering of any st-graph G whose maximum twist size is linear in its height. Together with Observation 1.1, this proves Theorem 1.2. We remark that for $X = V(G)$ we can strenghten the analysis above to $\mathrm{tn}(G) \leqslant 2h$: As all edges have both endpoints in X, we do not need to apply the pigeonhole principle to get that at least half the twisting edges have their lower (or equally their upper) endpoint in X, thus saving a factor of 2.

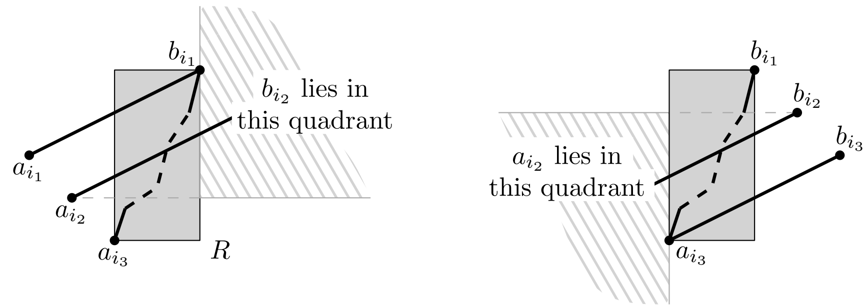

Figure 6: The situation for the final contradiction in the proof of Lemma 3.1, where $a_{i_1} >_y a_{i_2} >_y a_{i_3}$ (left), respectively $b_{i_1} >_y b_{i_2} >_y b_{i_3}$ (right, symmetric case with $b_1, \ldots, b_{k'} \in X$), by Erdős-Szekeres.

4 Bound in Terms of the Number of Vertices

In this section we combine our approaches of bounding the page number in terms of width and height and obtain the first sublinear upper bound on the page number of upward planar graphs and planar posets. We prove Theorem 1.3 which states that the page number of n-vertex upward planar graphs is $\mathcal{O}(n^{2/3} \log^{2/3}(n))$.

Proof. [Proof of Theorem 1.3] Let G be an n-vertex upward planar graph. Without loss of generality we may assume that G is an st-graph [25]. We first identify vertices that can be covered by few long directed paths and use Lemma 2.1 to embed the subgraph induced by these paths. We then apply Lemma 3.1 to the remaining vertices to find a topological ordering that admits an assignment of the remaining edges to few pages. However, as Lemma 2.1 introduces new directed reachabilities to the graph, we have to pick the first vertex set in a sequential way.

We construct a sequence $G_0, G_1, \ldots$ of graphs and a sequence $L_0 \subseteq L_1 \subseteq \ldots$ of sets containing the vertices of "long" directed paths in the respective graphs, starting with $G_0 = G$ and $L_0 = \emptyset$. We thereby ensure that $V(G_i) \subseteq V(G_{i+1})$ and that the width of L_i in G_i is at most i for each $i \geqslant 0$. Let $\ell = n^{2/3}/\log^{1/3}(n)$; we use this threshold to decide which paths are considered *long paths*. For ease of notation, let $E_\Delta(i, i+1) = E(G_i) - E(G_{i+1})$ denote the set of edges of G_i that is removed when defining the next graph G_{i+1}. We write $G[X]$ for the subgraph that is induced by $X \cap V(G)$, where X is a set of vertices of G_i which may include vertices that are not in G.

Assume that G_i and L_i are already defined and that there is an st-path P in G_i that contains at least ℓ vertices of G that are not contained in L_i. We include the vertices of P in the next set L_{i+1}. That is, we define $L_{i+1} = L_i \cup V(P)$. Note that adding the vertex set of a directed path to a set of vertices increases the width by at most 1. Hence, the width of L_{i+1} in G_i is at most $i+1$. Now apply Lemma 2.1 to G_i and L_{i+1} and obtain an st-graph G_{i+1} with $V(G_i) \subseteq V(G_{i+1})$. By Lemma 2.1, every topological ordering of G_{i+1} restricted to $V(G_i)$ admits an assignment of $E(G_{i+1}[L_{i+1}]) \cup E_\Delta(i, i+1)$ to $14(i+1)$ pages. As L_{i+1} can be covered by $i+1$ paths in G_i, the same holds in G_{i+1} as the reachabilities are preserved. Thus, the width of L_{i+1} in G_{i+1} is at most $i+1$.

Let t denote the largest i for which G_i and L_i are defined, i.e., there is no path in G_t that contains at least ℓ vertices of the initial graph G that are not covered by L_t. Note that $t \leqslant n/\ell = n^{1/3} \log^{1/3}(n)$, because we add at least ℓ vertices of G in each round. We claim that every topological ordering of G_t restricted to $V(G)$ admits an assignment of the edges of $G[L_t]$ to $\mathcal{O}(t^2)$ pages. To this end, fix an arbitrary topological ordering $<_t$ of G_t and consider the restriction $<$ of $<_t$ to the vertex set of G. Observe that $<$ is a topological ordering of G as directed reachabilities in G are maintained in G_t. For $i = 0, \ldots, t-1$, let $\mathcal{Q}_{i,i+1}$ denote the set of $14(i+1)$ pages used by Lemma 2.1 when applied to G_i. We restrict the pages to contain only edges of G. Observe that the edges in $E_\Delta(i, i+1) \cap E(G)$ are embedded in some page of $\mathcal{Q}_{i,i+1}$ for each $i = 0, \ldots, t-1$. Now, let E_t denote the remaining edges of $G[L_t]$. Note that these edges are contained in $G_t[L_t]$ and thus are embedded in some page of $\mathcal{Q}_{t-1,t}$ by Lemma 2.1. We conclude that the union of all $\mathcal{Q}_{i,i+1}$ covers all edges of $G[L_t]$ with $\sum_{i=0}^{t-1} |\mathcal{Q}_{i,i+1}| = \sum_{i=0}^{t-1} 14(i+1) = 7t(t+1)$ pages.

It is left to embed the set E_S of edges in G that are also contained in G_t and have at most one endpoint in L_t, i.e., at least one endpoint in $S = V(G) - L_t$. Recall that there is no path in G_t with at least ℓ vertices of G that are not contained in L_t, i.e., the height of S in G_t is less than ℓ. Applying Lemma 3.1 to S and G_t yields a topological ordering $<_t$ of G_t such that the edges in E_S form twists of size at most 4ℓ. By Observation 1.2, the same vertex ordering admits an assignment of the edges in E_S to $\mathcal{O}(\ell \log(n))$ pages. Restricting $<_t$ to G and

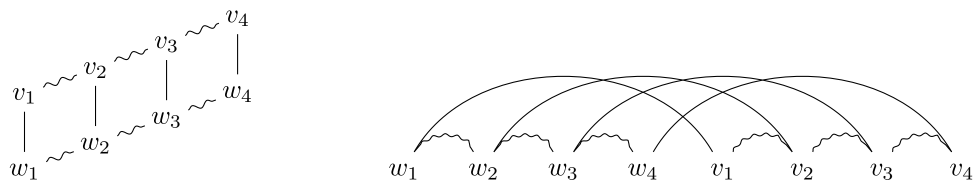

Figure 7: A 4-fence from v_1 to w_4 and a topological ordering with $w_4 < v_1$ yielding a 4-twist. Note that $(v_1, w_1), \ldots, (v_4, w_4)$ are edges, while all other shown reachabilites may be due to paths.

combining the page assignment of E_S with the page assignment of $G[L_t]$, we obtain a book embedding of G with $\mathcal{O}(\ell \log(n) + t^2) = \mathcal{O}(n^{2/3} \log^{2/3}(n))$ pages (recall that $\ell = n^{2/3}/\log^{1/3}(n)$ and $t \leqslant n/\ell = n^{1/3} \log^{1/3}(n)$). ◻

In view of Lemma 3.1 which bounds the twist number instead of the page number, the question arises whether our bound in terms of the number of vertices can be decreased by improving this step. We point out that (asymptotically) it does not make a difference whether we use Observation 1.2 giving a bound of $\mathcal{O}(\ell \log(n))$ pages for E_S or the result by Davies [12] which gives $\mathcal{O}(\ell \log(\ell))$ instead. We also remark that by choosing $\ell = n^{2/3}$, we obtain that every upward planar graph admits a topological vertex ordering whose maximum twist size is $\mathcal{O}(n^{2/3})$. That is, any improvement in bounding the page number of upward planar graphs in terms of their twist number also improves our result. Such an improvement, however, needs to make use of the structure of the graph and the constructed vertex ordering as Davies' result is asymptotically tight.

5 Lower Bound

Recall that the twist number $\operatorname{tn}(G)$ of a directed acyclic graph G is the maximum k for which every topological ordering of G contains k pairwise crossing edges. In this section, we construct an upward planar graph whose twist number, and therefore in particular its page number, is at least 5. This improves on the previously best known bound of an upward planar graph that requires four pages (but has twist number 3) by Hung [24]. We remark that Merker [27] improved on our upward planar graph by transforming it into a planar poset whose twist number and page number are at least 5.

We identify a structure that can lead to large twists if the spine ordering is not chosen carefully. By adding additional edges, any topological ordering of the augmented graph avoids these twists. For $k \geqslant 2$, a *k-fence (from v_1 to w_k)* consists of $2 \cdot k$ distinct vertices $v_1 \prec \cdots \prec v_k$, $w_1 \prec \cdots \prec w_k$, and edges (w_i, v_i) for each $i = 1, \ldots, k$. The edges (w_i, v_i) are called *fence edges*. If k is not important, we simply say *fence*. Figure 7 (left) shows a 4-fence. Observe that v_1 and w_k are not necessarily comparable. However, we show that v_1 must preceed w_k in every spine ordering that has no k-twist. By transitivity, every $v \preccurlyeq v_1$ must therefore also preceed every $w \succcurlyeq w_k$.

Observation 5.1. *Every topological ordering of a k-fence from v_1 to w_k in which $w_k < v_1$ has a k-twist.*

Proof. Assuming $w_k < v_1$, we obtain $w_1 < \cdots < w_k < v_1 < \cdots < v_k$ as the unique topological ordering. Hence, the fence edges form a k-twist. See Figure 7 (right) for an illustration. ◻

Given an upward planar graph G, we augment it with additional edges that indicate how to avoid the k-twists that otherwise are present in k-fences. Let $k \geqslant 2$ and consider a k-fence from v_1 to w_k in G such that $v_1 \not\prec w_k$. We add a new edge (v_1, w_k) forcing $v_1 < w_k$, as every topological ordering with $w_k < v_1$ yields a k-twist (Observation 5.1). Since adding edges increases the set of reachabilities in G, new fences might emerge with each newly added edge. Here we consider only fences for which the fence edges (v_i, w_i) for $i = 1, \ldots, k$ are still edges of the original graph G but the reachabilities along the two paths $v_1 \prec \cdots \prec v_k$ and $w_1 \prec \cdots \prec w_k$ might consist (partly) of new edges. We continue adding new edges to each current and future k-fence. This process terminates, as there is only a finite number of possible comparabilities between the unchanged number of vertices The resulting graph is denoted by G_k^* and contains no k-fence from v_1 to w_k with $v_1 \not\prec w_k$.

Let us refer to Figure 8 for some illustrative examples. Note that if G is upward planar, then G_k^* is not necessarily upward planar; possibly not planar, nor even acyclic. We emphasize that the new edges in G_k^* are not part of $E(G)$, and as such, need not be assigned to any page in a book embedding. Their sole purpose is to restrict the set of possible topological orderings of G to those of G_k^*.

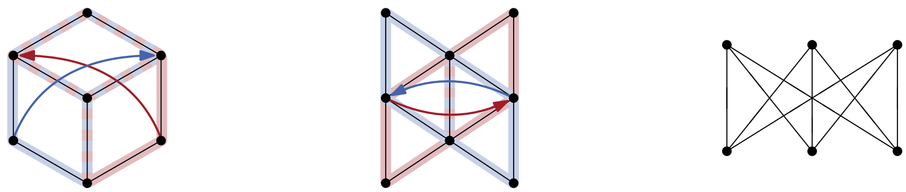

Figure 8: Examples of G_3^*. The edges in $E(G_3^*) - E(G)$ are drawn thick and the 3-fences are highlighted. Left: Taking a topological ordering of G_3^* shows that $\operatorname{tn}(G) \leqslant 2$. Middle: G_3^* is not acyclic and hence $\operatorname{tn}(G) > 2$. Right: $G_3^* = G$ as there is no 3-fence, but still $\operatorname{tn}(G) > 2$.

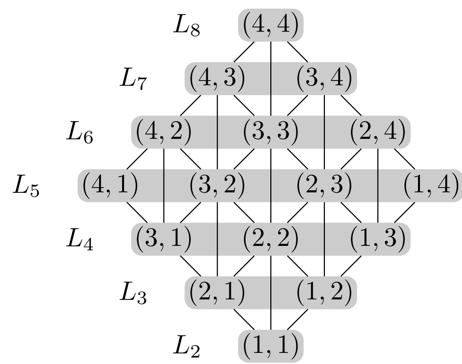

Figure 9: A 4×4 upward grid with levels $L_2, \ldots, L_8$. Consider the vertex $(2,2)$. The first left upper vertex is $(3,2)$, the second left upper vertex is $(4,1)$, the first right upper vertex is $(2,3)$, and the second right upper vertex is $(1,4)$.

Observation 5.1 shows that a topological ordering of G which is not a topological ordering of G_{k+1}^* yields a $(k+1)$-twist. In particular, G_{k+1}^* being acyclic is a necessary condition for G admitting a k-page book embedding.

Corollary 5.1. *Every book embedding of a directed acylic graph G without a $(k+1)$-twist (in particular every k-page book embedding) uses a topological ordering of G_{k+1}^* as spine ordering.*

However, using a topological ordering of G_k^* as a spine ordering is not sufficient to avoid k-twists; see e.g., the right of Figure 8. Quite the contrary, we find that for some small k, the augmented graph G_k^* might be cyclic and therefore not have any topological ordering at all. And even if G_k^* is acyclic, choosing any topological ordering of G_k^* can inescapably lead to arbitrarily large twists (which are not due to fences) even if the graph admits a book embedding with few (but more than k) pages. We shall force such a situation in our construction of an upward planar graph with twist number at least 5, which then proves Theorem 1.4.

For any integer $n > 0$, we define an $n \times n$ *upward grid* Grid_n as follows (see Figure 9). The vertices of Grid_n are the tuples (ℓ, r) of integers with $1 \leqslant \ell, r \leqslant n$. The vertices are partitioned into *levels*, where level L_h contains the vertices (ℓ, r) with $\ell + r = h$. The edge set of Grid_n consists of three subsets. There are *left edges* of the form $\big((\ell, r), (\ell+1, r)\big)$ for each $r = 1, \ldots, n$ and $\ell = 1, \ldots, n-1$. Symmetrically, the edges $\big((\ell, r), (\ell, r+1)\big)$ for $\ell = 1, \ldots, n$ and $r = 1, \ldots, n-1$ are called *right edges*. Finally, we have edges $\big((\ell, r), (\ell+1, r+1)\big)$ for $1 \leqslant \ell, r \leqslant n-1$ and call them *vertical edges*.

Consider a vertex $v = (\ell_v, r_v)$ in some level L_h of an upward grid. A vertex $w = (\ell_w, r_w)$ in level L_{h+1} is called an *i-th left (right) upper vertex of v* if $\ell_w = \ell_v + i$ ($r_w = r_v + i$). A vertex that is an i-th left upper vertex or an i-th right upper vertex of v is also called an *i-th upper vertex of v*. Note that every vertex in L_{h+1} is an i-th upper vertex of v for some $i > 0$.

Based on an $n \times n$ upward grid, we define an $n \times n$ N-grid, which we denote by N_n, for any integer $n > 0$. We shall show in this section that every $n \times n$ N-grid has a 5-twist in every topological ordering, provided n is large enough. The $n \times n$ N-grid N_n contains an $n \times n$ upward grid Grid_n as an induced subgraph and an additional vertex in each inner face of Grid_n. The additional vertices are called *N-vertices*, whereas the vertices that belong

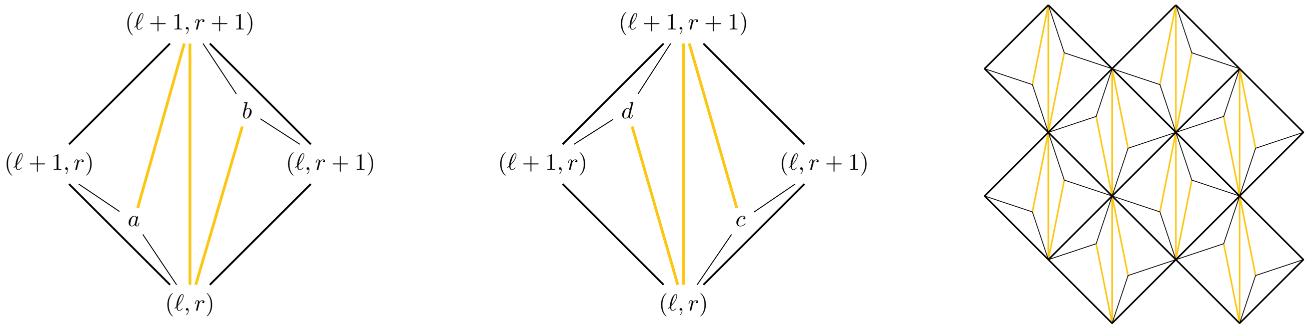

Figure 10: Parts of an N-grid with N-vertices $a = a_{\ell,r}$, $b = b_{\ell,r}$, $c = c_{\ell,r}$, and $d = d_{\ell,r}$, where $\ell - r$ is even (left), respectively odd (middle). The N-edges are highlighted yellow.

to Grid_n are called *grid vertices*. See Figure 10 for an illustration. Consider two triangles in Grid_n that share a vertical edge. That is, they consist of vertices (ℓ, r), $(\ell+1, r)$, $(\ell, r+1)$, and $(\ell+1, r+1)$ as shown in Figure 10. If $\ell - r$ is even, then we insert a vertex $a = a_{\ell,r}$ into the left triangle and add edges $\big((\ell, r), a\big)$, $\big(a, (\ell+1, r)\big)$, and $\big(a, (\ell+1, r+1)\big)$. In addition, we insert a vertex $b = b_{\ell,r}$ together with the edges $\big((\ell, r), b\big)$, $\big((\ell, r+1), b\big)$, and $\big(b, (\ell+1, r+1)\big)$ into the right triangle in this case. If $\ell - r$ is odd, then we insert vertices $c = c_{\ell,r}$ and $d = d_{\ell,r}$ into the right, respectively left, triangle and add edges $\big((\ell, r), c\big)$, $\big(c, (\ell, r+1)\big)$, $\big(c, (\ell+1, r+1)\big)$, $\big((\ell, r), d\big)$, $\big((\ell+1, r), d\big)$, and $\big(d, (\ell+1, r+1)\big)$. The definitions of levels and upper vertices remain as in Grid_n, the N-vertices do not belong to any level. Observe that every N-grid is upward planar. Whenever we refer to an embedding of an N-grid, we assume the upward grid induced by the grid vertices to be embedded in the canonical way shown in Figure 9 and the N-vertices to be placed in the respective triangular inner faces as shown in Figure 10.

The rest of this section is devoted to proving that every topological ordering of a sufficiently large N-grid yields a 5-twist. For this, we consider the graph $N^*_{n,5}$ that results from augmenting N_n via 5-fences as described above. By Corollary 5.1, every topological ordering of N_n that is not a topological ordering of $N^*_{n,5}$ yields a 5-twist. Hence, we only need to consider topological orderings of $N^*_{n,5}$. We say that two levels L_i, L_j $(2 \leqslant i < j \leqslant 2n)$ are *separated* by a topological ordering $<$, if for all grid vertices $(\ell_i, r_i) \in L_i$ and $(\ell_j, r_j) \in L_j$ we have $(\ell_i, r_i) < (\ell_j, r_j)$. We write $L_i < L_j$ in this case. We call a topological ordering $<$ of an N-grid *level-separating* if it separates every two consecutive levels, i.e., we have $L_2 < \cdots < L_{2n}$. We also say that $<$ *separates the levels of the N-grid* in this case. The next lemma shows that we can assume the levels of N_n to be separated if the vertex ordering is a topological ordering of $N^*_{n,5}$.

Lemma 5.1. *For every $n > 0$, there is an $n' \geqslant n$ such that every topological ordering $<$ of $N^*_{n',5}$ contains a copy of $N_n \subseteq N^*_{n',5}$ whose levels are separated by $<$.*

Proof. We choose $n' = n + 2(n-1)$ and use induction on $i = 1, \ldots, n$. For each i, we identify a set of vertices $V_i \subseteq V(N^*_{n',5})$ such that each grid vertex in V_i has an outgoing edge to all its j-th upper vertices that are contained in V_i for each $j = 1, \ldots, i$. We thereby ensure $V_i \subseteq V_{i-1}$ for $i > 1$ and that V_i induces a copy of $N_{n+2(n-i)}$ in $N_{n'}$. Finally we show that in $N^*_{n',5}[V_n]$ every grid vertex reaches every vertex of V_n in the subsequent level, and thus V_n induces the desired copy of N_n in $N_{n'}$.

For $i = 1$, define $V_1 = V(N^*_{n',5})$. Observe that in each N-grid, every grid vertex is adjacent to its first left upper vertex via a left edge and to its first right upper vertex via a right edge, which settles the base case. Now let $i > 1$ and assume that all grid vertices in V_{i-1} reach all j-th upper vertices also contained in V_{i-1} for each $j \leqslant i-1$. Consider the subgraph $N_{n'}$ of $N^*_{n',5}$ on the same vertex set but without the augmented edges. To obtain V_i from V_{i-1}, we drop all grid vertices incident to the outer face of $N_{n'}[V_{i-1}]$ and then remove all N-vertices that are now incident to the outer face. See Figure 11 to see how V_i lies in V_{i-1}. Note that every grid vertex in $N_{n'}[V_i]$ has an incoming vertical edge and and an outgoing vertical edge in $N_{n'}[V_{i-1}]$. Also observe that V_i induces an N-grid whose size is reduced by 2 in both directions compared to the N-grid $N_{n'}[V_{i-1}]$.

We next find a 5-fence from each grid vertex of V_i to its i-th upper vertices in V_i. Consider a grid vertex $v = (\ell_v, r_v) \in V_i$. Without loss of generality, we assume that $\ell_v - r_v$ is even. Swap left and right otherwise. Let $w = (\ell_w, r_w) \in V_i$ denote the i-th right upper vertex of v (if it exists). By definition of an i-th

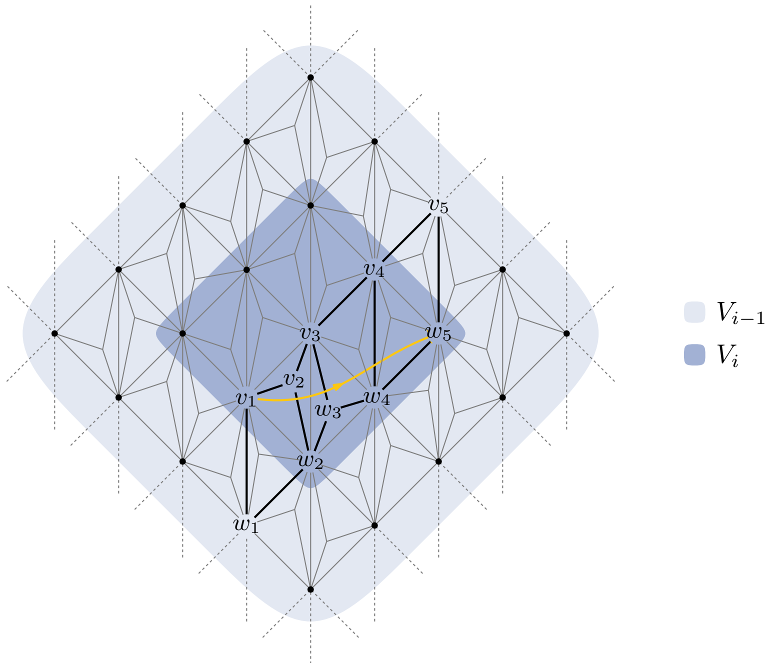

Figure 11: An inner N-grid $N_{n'}[V_i]$ (darkblue) inside an outer N-grid $N_{n'}[V_{i-1}]$ (lightblue). Observe that the shown 5-fence has vertices w_1 to v_5 outside $N_{n'}[V_i]$, but the yellow edge is inside.

right upper vertex, we have $r_w = r_v + i$. As the two vertices are in consecutive levels, we have $\ell_v + r_v = h$ and $\ell_w + r_w = h+1$, where L_h is the level of (ℓ_v, r_v). It follows that $\ell_w = \ell_v - i + 1$.

Now, consider the vertices

$$\begin{aligned} w_1 &= (\ell_v - 1, r_v - 1), \\ w_2 &= (\ell_v - 1, r_v), \\ w_3 &= c_{\ell_v - 1, r_v}, \\ w_4 &= (\ell_v - 1, r_v + 1), \text{ and} \\ w_5 &= (\ell_v - i + 1, r_v + i) = w. \end{aligned}$$

See Figure 12 for an illustration. These five vertices form the lower part of the desired 5-fence. Note that w_1 is not necessarily in V_i but is connected to v by a vertical edge in $N_{n'}[V_{i-1}]$ and thus is contained in V_{i-1} (see again Figure 11, where $v = v_1$). We next observe that $w_1, \ldots, w_5$ are pairwise comparable. The first four vertices induce a path in $N_{n'}$. The edge (w_4, w_5) exists in $N^*_{n',5}[V_{i-1}]$ by the induction hypothesis since w_5 is an $(i-1)$-th upper vertex of w_4. To see this, observe that w_4 and w_5 are in consecutive levels as $(\ell_v - i + 1 + r_v + i) - (\ell_v - 1 + r_v + 1) = 1$ and their r-coordinates differ by exactly $i-1$.

Now, consider the vertices

$$\begin{aligned} v_1 &= (\ell_v, r_v) = v, \\ v_2 &= d_{\ell_v - 1, r_v}, \\ v_3 &= (\ell_v, r_v + 1), \\ v_4 &= (\ell_v, r_v + 2), \text{ and} \\ v_5 &= (\ell_w + 1, r_w + 1) = (\ell_v - i + 2, r_v + i + 1). \end{aligned}$$

These five vertices serve as the upper part of the 5-fence from v to w. Again, we find that there is a path connecting the five vertices in $N^*_{n',5}[V_{i-1}]$. First, the edges (v_1, v_2) and (v_2, v_3) exist by construction of an N-grid. The edge (v_3, v_4) is a right edge in $N_{n'}$. We again remark that v_5 is not necessarily in V_i but is connected to w via

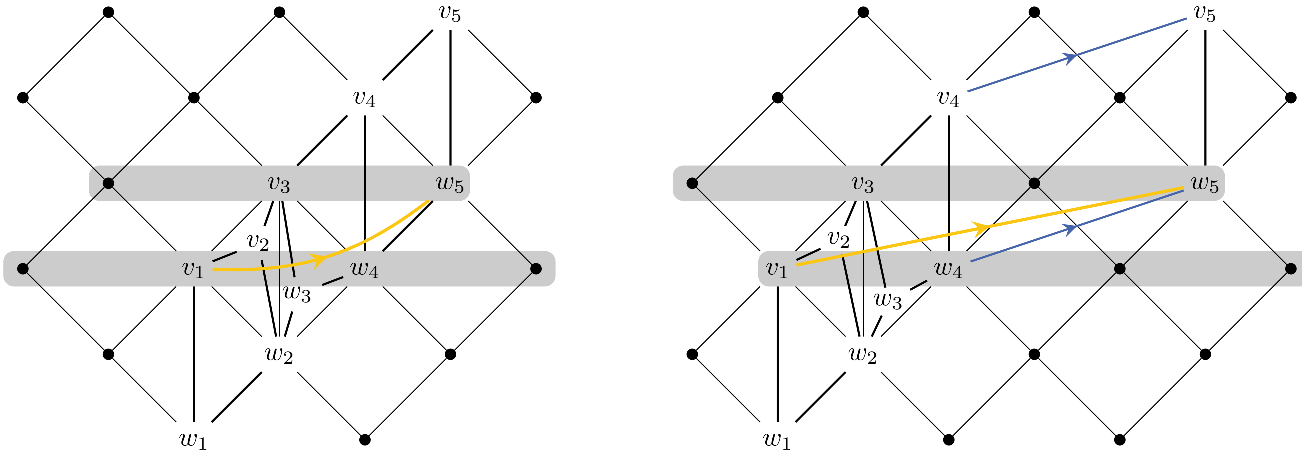

Figure 12: A 5-fence from $v = v_1$ to $w = w_5$, where w is the second/third right upper vertex of v. The blue edges (v_4, v_5) and (w_4, w_5) exist by induction.

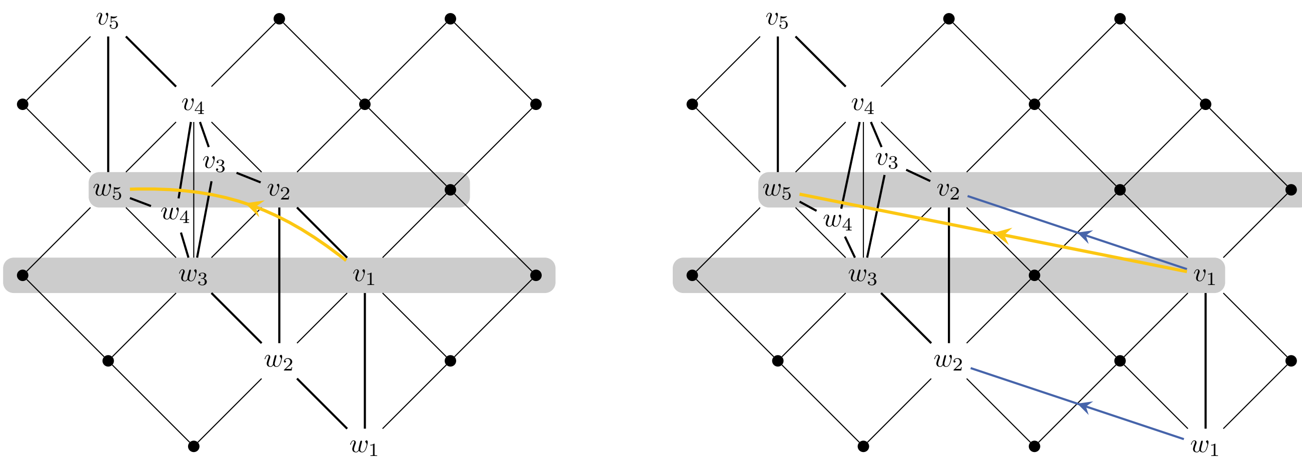

Figure 13: A 5-fence from $v = v_1$ to $w = w_5$, where w is the second/third left upper vertex of v. The blue edges (v_1, v_2) and (w_1, w_2) exist by induction.

a vertical edge and thus is contained in V_{i-1}. We obtain the remaining edge (v_4, v_5) by induction as v_5 is an $(i-1)$-th right upper vertex of v_4. Thus, we find a 5-fence from v to w using $(v_1, w_1), \ldots, (v_5, w_5)$ as fence edges.

The proof for the i-th left upper vertex works nearly symmetrically. In contrast to the right upper vertex, we first use the edges obtained by the induction hypothesis and then the edges of $N_{n'}$ to find the two paths of the 5-fence. Let $w = (\ell_w, r_w) = (\ell_v + i, r_v - i + 1) \in V_i$ denote the i-th left upper vertex of v (if it exists). We find a 5-fence from v to w using the vertices

$$
\begin{aligned}
w_1 &= (\ell_w - i - 1, r_w + i - 2) = (\ell_v - 1, r_v - 1), \\
w_2 &= (\ell_w - 2, r_w), \\
w_3 &= (\ell_w - 1, r_w), \\
w_4 &= a_{\ell_w - 1, r_w}, \text{ and} \\
w_5 &= (\ell_w, r_w) = w
\end{aligned}
$$

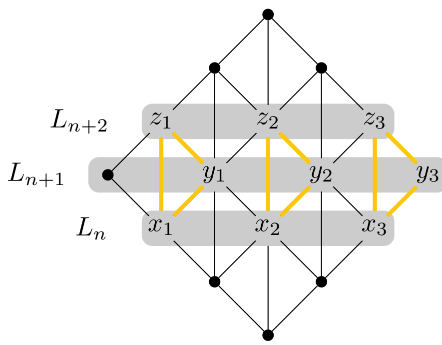

Figure 14: Three triangles in $\mathrm{Grid}_4 \subseteq N_4$ with vertices in levels L_4, L_5, and L_6.

for the lower part, while the upper part is formed by the vertices

$$\begin{aligned} v_1 &= (\ell_w - i, r_w + i - 1) = v, \\ v_2 &= (\ell_w - 1, r_w + 1), \\ v_3 &= b_{\ell_w - 1, r_w}, \\ v_4 &= (\ell_w, r_w + 1), \text{ and} \\ v_5 &= (\ell_w + 1, r_w + 1). \end{aligned}$$

We refer to Figure 13 for an illustration. Note that the coordinates of w_3 have an even difference as $(\ell_w - 1) - r_w = (\ell_v + i - 1) - (r_v - i + 1) = \ell_v - r_v + 2i - 2$, which means that the claimed a- and b-vertices indeed exist. The edges (w_1, w_2) and (v_1, v_2) exist by induction as their upper endpoints are $(i-1)$-th left upper vertices of the lower endpoints. The other vertices are connected by two paths using only edges of $N_{n'}$. We again obtain a 5-fence using the edges $(v_1, w_1), \ldots, (v_5, w_5)$ as fence edges.

To conclude the proof, recall that V_n induces a copy of N_n in $N_{n'}$. Observe that in N_n, no vertex has an i-th upper vertex for $i > n$. Thus by the induction above, we have that in $N^*_{n',5}[V_n]$ every grid vertex reaches all its upper vertices that are contained in V_n, i.e., all vertices of the subsequent level of N_n. Therefore, the levels of $N^*_{n',5}[V_n]$ are separated by every topological ordering of $N^*_{n',5}$. ◻

Having Lemma 5.1, we know that we may assume the levels of an N-grid N_n to be separated when we try to avoid 5-twists, i.e., when we consider topological orderings of $N^*_{n,5}$. The next lemma, however, shows that separated levels imply not only 5-twists but arbitrarily large twists, finishing the proof of Theorem 1.4.

Lemma 5.2. *For every $p \geqslant 0$, there is an n such that every level-separating topological ordering $<$ of N_n yields a $(p+1)$-twist. In particular, $<$ does not admit a p-page book embedding.*

Proof. Let $r = p^3 + 1$ and $n = r + 1$. We identify r triangles in N_n, each of which has exactly one vertex in each of the three levels L_n, L_{n+1} and L_{n+2}. Observe that each of these levels has at least r vertices. For $i = 1, \ldots, r$, we define the triangle T_i consisting of the vertices $x_i = (n - i, i) \in L_n$, $y_i = (n - i, i + 1) \in L_{n+1}$, and $z_i = (n - i + 1, i + 1) \in L_{n+2}$. See Figure 14 for an example. By our assumption we have $L_n < L_{n+1} < L_{n+2}$.

We now define an ordering $<_T$ on the triangles and use it to find a $(p+1)$-twist. We define $T_i <_T T_j$ if and only if $x_i < x_j$. A subsequence $y_{i_1}, \ldots, y_{i_s}$ of $y_1, \ldots, y_r$ is *increasing* if its ordering corresponds to $<_T$, that is if $T_{i_1} <_T \cdots <_T T_{i_s}$. Similarly, a subsequence $y_{i_1}, \ldots, y_{i_s}$ of $y_1, \ldots, y_r$ is called *decreasing* if their reverse ordering corresponds to $<_T$, that is if $T_{i_1} >_T \cdots >_T T_{i_s}$. Increasing and decreasing subsequences of $z_1, \ldots, z_r$ in level L_{n+2} are defined analogously.

We now only consider the subgraph of N_n that is given by the triangles $T_1, \ldots, T_r$. That is, a neighbor of a vertex v refers to a vertex in the same triangle as v. If there is an increasing subsequence of $y_1, \ldots, y_r$ or of $z_1, \ldots, z_r$ of length $p + 1$, then we have a $(p+1)$-twist between these vertices and their neighbors in L_n. Hence, the longest increasing subsequences of $y_1, \ldots, y_r$ and $z_1, \ldots, z_r$ have length at most p. By the Erdős-Szekeres theorem, there exists a decreasing subsequence $y_{i_1}, \ldots, y_{i_s}$ of $y_1, \ldots, y_r$ of length $s = p^2 + 1$. Again by the

Erdős-Szekeres theorem, there exists a decreasing subsequence $z_{i'_1}, \ldots, z_{i'_t}$ of $z_{i_1}, \ldots, z_{i_s}$ of length $t = p+1$. But then $y_{i'_t} < \cdots < y_{i'_1} < z_{i'_t} < \cdots < z_{i'_1}$ form a $(p+1)$-twist as $L_{n+1} < L_{n+2}$. ◻

We conclude by Lemma 5.2 that for $p = 4$ and $n = p^3 + 2 = 66$ every level-separating topological ordering of N_n contains a 5-twist. Further, by Lemma 5.1, there is an $n' \geqslant n$ such that every topological ordering $<$ of $N^*_{n',5}$ contains a copy of N_n whose levels are separated by $<$ (i.e., $n' = n + 2(n-1) = 192$ as in the proof). Together this yields $\mathrm{pn}(N_{n'}) \geqslant \mathrm{tn}(N_{n'}) \geqslant 5$, proving Theorem 1.4.

Finally, we remark that N-grids have bounded page number but it is not obvious whether five pages suffice for all N-grids. However, separating the levels of N-grids works only with 5-fences, which is why new ideas are needed for any significant improvement.

6 Conclusions

In this paper, we improve both the lower and the upper bound on the maximum page number among upward planar graphs. Concerning the lower bound, we remark that Lemma 5.2 does not depend on the size of the twist to be enforced but yields arbitrarily large twists. That is, for pushing the lower bound further it suffices to find a large enough upward planar graph whose vertices can be partitioned into levels, i.e., into sets of vertices that are separated by any topological ordering. However, it is crucial that there are edges connecting non-consecutive levels. We also expect the concept of fences to prove useful for improving the lower bound further as we only need to consider topological orderings that respect the augmented edges.

The main contribution of this paper is the first sublinear upper bound on the page number of upward planar graphs in terms of their number of vertices. We remark that when applying Lemma 2.1 repeatedly, many edges are embedded multiple times. In fact, we only need the edges of $G'[X]$ in the last application of the lemma, whereas we use the embedding of the edges in E_Δ in all rounds. In light of this observation, we see potential improvements in reducing the number of pages needed for E_Δ (at the expense of the number of pages needed for $G'[X]$) or in reducing the number of applications of Lemma 2.1 (e.g., by covering the edges of E_Δ with Lemma 3.1). Both would lead to an upper bound of $\mathcal{O}(\sqrt{n \log(n)})$. To improve the bound beyond that, we think that new approaches are necessary.

In addition to the sublinear upper bound, we attack the problem of bounding the page number of upward planar graphs by showing that families of upward planar graphs with bounded width or bounded height have bounded page number. However, the initial question by Nowakowski and Parker [28] whether planar posets, and more generally upward planar graphs, have bounded page number, still remains open.

References

[1] Hugo A. Akitaya, Erik D. Demaine, Adam Hesterberg, and Quanquan C. Liu. Upward Partitioned Book Embeddings. In Fabrizio Frati and Kwan-Liu Ma, editors, *Graph Drawing and Network Visualization*, volume 10692 of *Lecture Notes in Computer Science*, pages 210–223, Cham, 2018. Springer. doi:10.1007/978-3-319-73915-1_18.

[2] Mustafa Alhashem, Guy-Vincent Jourdan, and Nejib Zaguia. On The Book Embedding Of Ordered Sets. *Ars Combinatoria*, 119:47–64, 2015.

[3] Mohammad Alzohairi and Ivan Rival. Series-Parallel Planar Ordered Sets Have Pagenumber Two. In Stephen North, editor, *Graph Drawing*, pages 11–24, Berlin, Heidelberg, 1997. Springer. doi:10.1007/3-540-62495-3_34.

[4] Mohammad Alzohairi, Ivan Rival, and Alexandr Kostochka. The Pagenumber of Spherical Lattices is Unbounded. *Arab Journal of Mathematical Sciences*, 7(1):79–82, 2001.

[5] Patrizio Angelini, Marco Di Bartolomeo, and Giuseppe Di Battista. Implementing a Partitioned 2-Page Book Embedding Testing Algorithm. In Walter Didimo and Maurizio Patrignani, editors, *Graph Drawing*, volume 7704 of *Lecture Notes in Computer Science*, pages 79–89, Berlin, Heidelberg, 2013. Springer. doi:10.1007/978-3-642-36763-2_8.

[6] Kirby A. Baker, Peter C. Fishburn, and Fred S. Roberts. Partial Orders of Dimension 2. *Networks*, 2(1):11–28, 1972. doi:10.1002/net.3230020103.

[7] Michael A. Bekos, Michael Kaufmann, Fabian Klute, Sergey Pupyrev, Chrysanthi Raftopoulou, and Torsten Ueckerdt. Four Pages are Indeed Necessary for Planar Graphs. *Journal of Computational Geometry*, 11(1):332–353, 2020. doi:10.20382/jocg.v11i1a12.

[8] Frank Bernhart and Paul C. Kainen. The Book Thickness of a Graph. *Journal of Combinatorial Theory, Series B*, 27(3):320–331, December 1979. doi:10.1016/0095-8956(79)90021-2.

[9] Carla Binucci, Giordano Da Lozzo, Emilio Di Giacomo, Walter Didimo, Tamara Mchedlidze, and Maurizio Patrignani. Upward Book Embeddings of st-Graphs. In Gill Barequet and Yusu Wang, editors, *35th International Symposium on Computational Geometry (SoCG 2019)*, volume 129 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 13:1–13:22, Dagstuhl, Germany, 2019. Schloss Dagstuhl–Leibniz-Zentrum für Informatik. doi:10.4230/LIPIcs.SoCG.2019.13.

[10] Jonathan F. Buss and Peter W. Shor. On the Pagenumber of Planar Graphs. In *Proceedings of the sixteenth annual ACM symposium on Theory of computing*, STOC '84, pages 98–100, New York, NY, USA, December 1984. Association for Computing Machinery. doi:10.1145/800057.808670.

[11] Jakub Černý. Coloring Circle Graphs. *Electronic Notes in Discrete Mathematics*, 29:457–461, 2007. doi:10.1016/j.endm.2007.07.072.

[12] James Davies. Improved bounds for colouring circle graphs, 2021. arXiv:2107.03585.

[13] James Davies and Rose McCarty. Circle Graphs are Quadratically χ-Bounded. *Bulletin of the London Mathematical Society*, 53(3):673–679, June 2021. doi:10.1112/blms.12447.

[14] Giuseppe Di Battista and Roberto Tamassia. Algorithms for Plane Representations of Acyclic Digraphs. *Theoretical Computer Science*, 61(2–3):175–198, November 1988. doi:10.1016/0304-3975(88)90123-5.

[15] Giuseppe Di Battista, Roberto Tamassia, and Ioannis G. Tollis. Area Requirement and Symmetry Display of Planar Upward Drawings. *Discrete & Computational Geometry*, 7(4):381–401, 1992. doi:10.1007/BF02187850.

[16] Emilio Di Giacomo, Walter Didimo, Giuseppe Liotta, and Stephen K. Wismath. Book Embeddability of Series-Parallel Digraphs. *Algorithmica*, 45:531–547, February 2006. doi:10.1007/s00453-005-1185-7.

[17] Fabrizio Frati, Radoslav Fulek, and Andres J. Ruiz-Vargas. On the Page Number of Upward Planar Directed Acyclic Graphs. *Journal of Graph Algorithms and Applications*, 17(3):221–244, 2013. doi:10.7155/jgaa.00292.

[18] Francesco Giordano, Giuseppe Liotta, Tamara Mchedlidze, Antonios Symvonis, and Sue H. Whitesides. Computing Upward Topological Book Embeddings of Upward Planar Digraphs. *Journal of Discrete Algorithms*, 30:45–69, January 2015. doi:10.1016/j.jda.2014.11.006.

[19] András Gyárfás. On the Chromatic Number of Multiple Interval Graphs and Overlap Graphs. *Discrete Mathematics*, 55(2):161–166, July 1985. doi:10.1016/0012-365X(85)90044-5.

[20] Lenwood S. Heath. Embedding Planar Graphs In Seven Pages. In *25th Annual Symposium on Foundations of Computer Science, 1984*, pages 74–83, 1984. doi:10.1109/SFCS.1984.715903.

[21] Lenwood S. Heath and Sriram V. Pemmaraju. Stack and Queue Layouts of Posets. *SIAM Journal on Discrete Mathematics*, 10(4):599–625, 1997. doi:10.1137/S0895480193252380.

[22] Lenwood S. Heath and Sriram V. Pemmaraju. Stack and Queue Layouts of Directed Acyclic Graphs: Part II. *SIAM Journal on Computing*, 28(5):1588–1626, 1999. doi:10.1137/S0097539795291550.

[23] Lenwood S. Heath, Sriram V. Pemmaraju, and Ann N. Trenk. Stack and Queue Layouts of Directed Acyclic Graphs: Part I. *SIAM Journal on Computing*, 28(4):1510–1539, 1999. doi:10.1137/S0097539795280287.

[24] Le Tu Quoc Hung. A Planar Poset which Requires 4 Pages. *Ars Combinatoria*, 35:291–302, 1993.

[25] David Kelly. Fundamentals of Planar Ordered Sets. *Discrete Mathematics*, 63(2–3):197–216, 1987. doi:10.1016/0012-365X(87)90008-2.

[26] Tamara Mchedlidze and Antonios Symvonis. Crossing-Free Acyclic Hamiltonian Path Completion for Planar st-Digraphs. In Yingfei Dong, Ding-Zhu Du, and Oscar Ibarra, editors, *Algorithms and Computation*, volume 5878 of *Lecture Notes in Computer Science*, pages 882–891, Berlin, Heidelberg, 2009. Springer. doi:10.1007/978-3-642-10631-6_89.

[27] Laura Merker. Ordered Covering Numbers. Master's thesis, Karlsruhe Institute of Technology, 2020. URL: https://i11www.iti.kit.edu/_media/teaching/theses/ma-merker-20.pdf.

[28] Richard Nowakowski and Andrew Parker. Ordered Sets, Pagenumbers and Planarity. *Order*, 6:209–218, 1989. doi:10.1007/BF00563521.

[29] L. Taylor Ollmann. On the Book Thicknesses of Various Graphs. In *Proc. 4th Southeastern Conference on Combinatorics, Graph Theory and Computing*, volume 8, 1973.

[30] Maciej M. Sysło. Bounds to the Page Number of Partially Ordered Sets. In Manfred Nagl, editor, *Graph-Theoretic Concepts in Computer Science*, volume 411 of *Lecture Notes in Computer Science*, pages 181–195, Berlin, Heidelberg, 1990. Springer. doi:10.1007/3-540-52292-1_13.

[31] Mihalis Yannakakis. Embedding Planar Graphs in Four Pages. *Journal of Computer and System Sciences*, 38(1):36–67, February 1989. doi:10.1016/0022-0000(89)90032-9.

[32] Mihalis Yannakakis. Planar Graphs that Need Four Pages. *Journal of Combinatorial Theory, Series B*, 145:241–263, November 2020. doi:10.1016/j.jctb.2020.05.008.