The Complexity of the Hausdorff Distance

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Abstract

We investigate the computational complexity of computing the Hausdorff distance. Specifically, we show that the decision problem of whether the Hausdorff distance of two semi-algebraic sets is bounded by a given threshold is complete for the complexity class $\forall\exists_<\mathbb{R}$. This implies that the problem is NP-, co-NP-, $\exists\mathbb{R}$- and $\forall\mathbb{R}$-hard.

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1 Introduction

The question of 'how similar are two given objects' occurs in numerous settings. One typical tool to quantify their similarity is the Hausdorff distance. Two sets have a small Hausdorff distance, if every point of one set is close to some point of the other set and vice versa. As a matter of fact, the Hausdorff distance appears in many branches of science. To illustrate the range of use cases, we consider two examples, for illustrations see Figure 1. In mathematics, the Hausdorff distance provides a metric on sets and henceforth also a topology. This topology can be used to discuss continuous transformations of one set to another [16]. In computer vision and geographical information science, the Hausdorff distance is used to measure the similarity between spacial objects [36, 44], for example the quality of quadrangulations of complex 3D models [51]. In this paper, we study the computational complexity of the Hausdorff distance from a theoretical perspective.

Definition. The *directed Hausdorff distance* between a non-empty set $A \subseteq \mathbb{R}^n$ and a non-empty set $B \subseteq \mathbb{R}^n$ is defined as

$$\vec{d}_{\mathrm{H}}(A, B) := \sup_{a\in A} \inf_{b\in B} \|a - b\|.$$

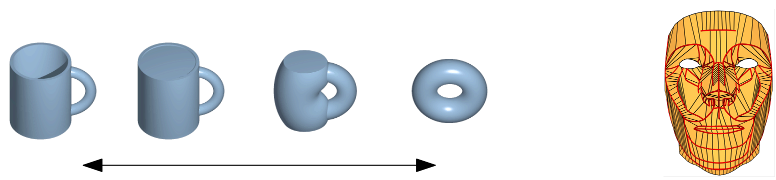

Figure 1 Left: Continuous deformation of a cup into a doughnut [21]. Right: Quadrangulation of a smooth surface used for rendering [51].

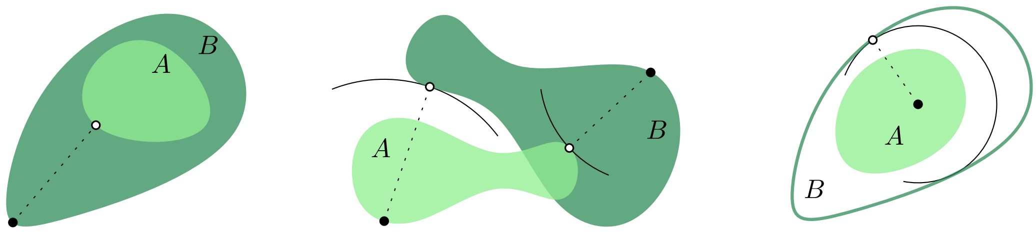

Figure 2 How similar are these sets?

The directed Hausdorff distance between A and B can be interpreted as the smallest value $\varepsilon \geq 0$ such that the (closed) ε-neighborhood of B contains A. Hence, it nicely captures the intuition of how much B has to be blown up to contain A. Note that $\vec{d}_{\mathrm{H}}(A, B)$ and $\vec{d}_{\mathrm{H}}(B, A)$ must not be equal. For an example, consider Figure 2; while $A \subset B$ and thus $\vec{d}_{\mathrm{H}}(A, B) = 0$, it holds that $\vec{d}_{\mathrm{H}}(B, A) > 0$. The (undirected) *Hausdorff distance* is symmetric and defined as

$$d_{\mathrm{H}}(A, B) := \max\{\vec{d}_{\mathrm{H}}(A, B), \vec{d}_{\mathrm{H}}(B, A)\}.$$

In this paper, we investigate the *computational complexity* of deciding whether the Hausdorff distance of two sets is at most a given threshold.

Semi-Algebraic Sets. The algorithmic complexity of the Hausdorff distance clearly depends on the type of their underlying sets. If we are given the sets in a way that we cannot even decide if they are empty, it seems near impossible to compute their Hausdorff distance. However, if the sets consists of finitely many points, their Hausdorff distance can be easily computed by checking all pairs of points. In practice, we are often somewhere between those two extreme situations. For instance, the sets could be a collection of disks in the plane or cubic splines, describing a surface in three dimensions, see also Figure 3.

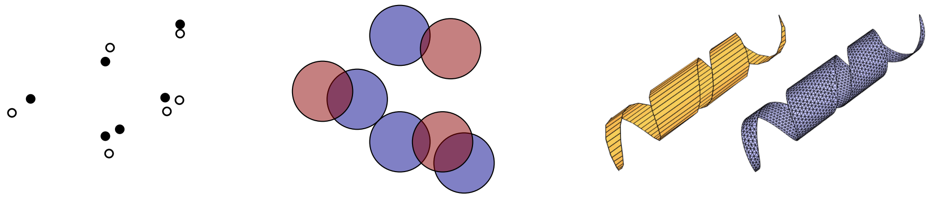

Figure 3 The Hausdorff distance can appear in simpler or more complicated settings. Left: Two finite point sets (black and white) in the plane. Middle: Two sets of blue and red disks in the plane. Right: Two surfaces in 3-space with different meshes, image taken from [51].

In this paper, we focus on semi-algebraic sets, i.e., sets that can be described by polynomial inequalities. Formally, a semi-algebraic set is the finite union of basic semi-algebraic sets. A basic semi-algebraic set S is specified by two families of polynomials $\mathcal{P}$ and $\mathcal{Q}$ such that

$$S = \{x \in \mathbb{R}^n \mid \bigwedge_{P \in \mathcal{P}} P(x) \leq 0 \wedge \bigwedge_{Q \in \mathcal{Q}} Q(x) < 0\}.$$

Semi-algebraic sets cover clearly the vast majority of practical cases and finding efficient algorithms for this problem would be a tremendous contribution. Simultaneously, when considering smooth sets, one is quickly in the situation that one needs to deal with polynomials anyway. So the step to general semi-algebraic sets is not a very big one.

Concrete Example. The following example was made up on the spot by Bernd Sturmfels at a workshop in Saarbrücken in 2019. The two polynomials

$$f(x,y) := x^4 + y^4 + 12x^3 + 2y^3 - 3xy + 11$$
$$g(x,y) := 7x^4 + 8y^4 - 1$$

define the sets $A = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = 0\}$ and $B = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 0\}$. For an illustration of A and B, consider the blue and green curve in Figure 4, respectively.

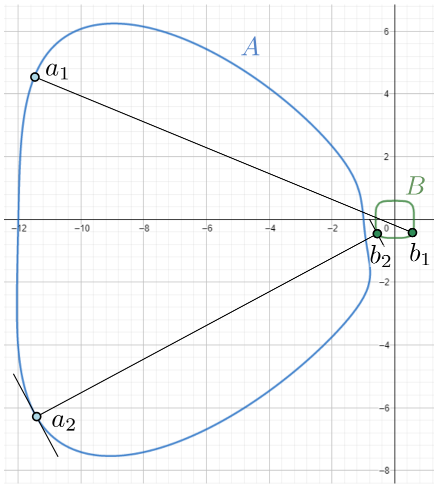

Figure 4 The Hausdorff distance between the compact semi-algebraic sets (in blue and green) is attained at points (a_2, b_2) such that the segment a_2b_2 is orthogonal to the tangents at a_2 and b_2. While the segment a_1b_1 is longer than a_2b_2, the pair (a_1, b_1) does not realize the Hausdorff distance because the segment a_1b_1 crosses both A and B.

It can be argued using convexity and continuity that the Hausdorff distance is attained at points $a \in A, b \in B$ such that the segment ab is orthogonal to the tangents at a and b. This yields a set of polynomial equations in four variables. The system has 240 complex solutions, eight of which are real. These 240 solutions can be computed using computer algebra systems based on Gröbner bases. For some of the real solutions (a, b), the segment ab crosses A and B, for example a_1b_1 as in Figure 4. Among the remaining solutions the points $a_2 \approx (-11.48362, -6.1760), b_2 \approx (-.56460, -.43583)$ realize the Hausdorff distance of approximately 12.33591. This approach does not easily generalize to general semi-algebraic sets. In the next paragraph, we present a slower, but more general method.

General Decision Algorithm. We consider a situation where we are given two semi-algebraic sets A and B as well as a threshold t; for simplicity, we assume here that A and B are closed. The statement $\vec{d}_{\mathrm{H}}(A, B) \leq t$ can be encoded into a logical sentence Φ of the form

$$\forall a \in A \,.\, \exists b \in B : \|a - b\|^2 \leq t^2,$$

where $\|x\|$ denotes the Euclidean norm of the vector x. We can decide the truth of this sentence by employing sophisticated algorithms from real algebraic geometry that can deal with *two blocks of quantifiers* [12, Chapter 14]. These algorithms are so slow that they would probably not work in the above example. Our main result roughly states that in general there is little hope for an improvement. To state this formally, we continue by defining suitable complexity classes.

Algorithmic Complexity. Let φ be a *quantifier-free formula in the first-order theory of the reals*, i.e., a formula formed over the alphabet $\Sigma = \{0, 1, +, \cdot, =, \neq, \leq, <, \vee, \wedge, \neg, \Rightarrow\}$. The Universal Existential Theory of the Reals (UETR) asks to decide if a sentence

$$\Phi := \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X, Y)$$

is true. An instance of UETR belongs to Strict-UETR if the corresponding formula φ is over the alphabet $\Sigma = \{0, 1, +, \cdot, <, \vee, \wedge\}$, i.e., if every atom is a strict inequality and negations and implications do not occur. The complexity classes $\forall\exists\mathbb{R}$ and $\forall\exists_<\mathbb{R}$ contain all decision problems for which there exists a polynomial-time many-one reduction to UETR and Strict-UETR, respectively. We propose to pronounce the complexity classe $\forall\exists\mathbb{R}$ as 'UER' or 'forall exists R' and $\forall\exists_<\mathbb{R}$ as 'Strict-UER' or 'strict forall exists R'. To the best of our knowledge, $\forall\exists\mathbb{R}$ was first introduced by Dobbins, Kleist, Miltzow, and Rzążewski [27]. The sister class co-$\forall\exists_<\mathbb{R} = \exists\forall_\leq\mathbb{R}$ was first studied by D'Costa, Lefaucheux, Neumann, Ouaknine and Worrel [24].

Concerning the relation of these complexity classes, it is easy to see that $\forall\exists_<\mathbb{R}$ is contained in $\forall\exists\mathbb{R}$. Yet, we are not aware of any algorithm that is faster for $\forall\exists_<\mathbb{R}$-complete problems than on $\forall\exists\mathbb{R}$ problems. Thus it is an intriguing open problem if those two classes coincide or are different. If the two classes are indeed different, it would imply that NP $\neq$ PSPACE. Because this is one of the biggest open problems in theoretical computer science, we do not expect such a proof any time soon. It is also conceivable that some extensions of known results in real algebraic geometry can be used to show $\forall\exists\mathbb{R} = \forall\exists_<\mathbb{R}$.

Problem and Results. We now have all ingredients to state our problem and main results. Let $\Phi_A(X)$ and $\Phi_B(X)$ be two quantifier-free formulas defining the semi-algebraic sets $A = \{x \in \mathbb{R}^n \mid \Phi_A(x)\}$ and $B = \{x \in \mathbb{R}^n \mid \Phi_B(x)\}$, and let $t \in \mathbb{Q}$ be a rational number. The Hausdorff problem asks whether $d_{\mathrm{H}}(A, B) \leq t$.

Our main result determines the algorithmic complexity of the Hausdorff problem.

► **Theorem 1.** *The* Hausdorff *problem is* $\forall\exists_<\mathbb{R}$*-complete.*

Note that prior to our result, it was not even known if computing the Hausdorff distance was even NP-hard. As $\forall\exists_<\mathbb{R}$ contains, NP, co-NP, $\exists\mathbb{R}$ and $\forall\mathbb{R}$, we also get hardness for all of those complexity classes. Theorem 1 answers an open question posed by Dobbins, Kleist, Miltzow and Rzążewski [26].

One may wonder whether it is crucial for our results that the Hausdorff problem asks if the distance $\leq t$ rather than $< t$. We remark that all our proofs work with tiny modifications also for the case of a strict inequality. Furthermore, our results also hold for the directed Hausdorff distance. Note that one can compute the undirected Hausdorff distance trivially, by computing twice the directed Hausdorff distance. Thus intuitively, the directed Hausdorff distance is computationally at least as hard. Yet, this is not a many one reduction, as we need to compute the directed Hausdorff distance twice.

In the proof of $\forall\exists_<\mathbb{R}$-hardness for Theorem 1, we create instances with some additional properties. In particular, we can guarantee a gap, i.e., the Hausdorff distance is either below the threshold t or at least $t \cdot 2^{2^{\Omega(d)}}$, where d denotes the number of variables of Φ_A and Φ_B. Thus our result also rules out approximation algorithms.

► **Corollary 2.** *Let A and B be two semi-algebraic sets in $\mathbb{R}^d$ and $f(d) = 2^{2^{o(d)}}$. Then there is no $f(d)$-approximation algorithm to compute $d_{\mathrm{H}}(A, B)$, unless* P $= \forall\exists_<\mathbb{R}$.

We remark that our proof provides hard instances, where the threshold t is strictly larger than zero. By scaling of A and B, we can assume $t = 1$ without loss of generality. It is natural to wonder if $\forall\exists_<\mathbb{R}$-hardness also holds for the case of $t = 0$. This question is equivalent to checking whether the closure of two semi-algebraic sets is equal, i.e., $d_{\mathrm{H}}(A, B) = 0$ if and only if $\overline{A} = \overline{B}$. Computing the closure of a semi-algebraic set is non-trivial. In particular, it is not enough to replace all occurrences of $<$ by $\leq$. Yet testing, if two semi-algebraic sets are equal is likely slightly easier.

▶ **Theorem 3.** *Deciding if two semi-algebraic sets are equal is* $\forall\mathbb{R}$*-complete.*

Because the proof is rather simple, we present it at this point.

Proof. Given quantifier-free formulas $\Phi_A(X)$ and $\Phi_B(X)$, it holds that $A = B$ if and only if $\forall X \in \mathbb{R}^n : \Phi_A(X) \iff \Phi_B(X)$. This shows $\forall\mathbb{R}$-membership. To see $\forall\mathbb{R}$-hardness note that $\Psi := \forall X \in \mathbb{R}^n : \varphi(X)$ is equivalent to $\{x \in \mathbb{R}^n : \varphi(x)\} = \mathbb{R}^n$. ◀

1.1 Related Work

This subsection reviews previous work concerning two directions. First, we discuss the complexity of computing the Hausdorff distance for special sets. Afterwards, we investigate previous work on the complexity class $\forall\exists\mathbb{R}$.

Computing the Hausdorff Distance. The notion of the Hausdorff distance was introduced by Felix Hausdorff in 1914 [31]. Most of the early works focused on the Hausdorff distance for finite point sets. For a set of n points and a set of m points in any fixed dimension, the Hausdorff distance can be easily computed by checking all pairs, i.e., in time $O(mn)$. In the plane, this can be improved to $O((n+m)\log(m+n))$ by using Voronoi diagrams [7]. In fact, this method can be extended to sets consisting of pairwise non-crossing line segments in the plane, e.g., simple polygons and polygonal chains fulfill this property. If the polygons are additionally convex, their Hausdorff distance can even be computed in linear time [11].

More generally, the Hausdorff distance can be computed in polynomial time whenever the two sets can be described by a simplicial complex of fixed dimension. Based on the PhD thesis of Godau [29], Alt et al. [8, Theorem 3.3] show how to compute the directed Hausdorff distance between two sets in $\mathbb{R}^d$ consisting of n and m k-dimensional simplices in time $O(nm^{k+2})$. Using a Las Vegas algorithm for computing the vertices of the lower envelope, similar ideas yield an approach with randomized expected time in $O(nm^{k+\varepsilon})$ for $k > 1$ and every $\varepsilon > 0$ [8, Theorem 3.4]. They additionally present algorithms with better randomized expected running times for sets of triangles in $\mathbb{R}^3$ and point sets in $\mathbb{R}^d$.

Given two semi-algebraic sets $A, B \subseteq \mathbb{R}^n$, the Hausdorff problem can be encoded as a sentence of the form $\forall X \exists Y : \varphi(X, Y)$ with $\Theta(n)$ variables, where φ is quantifier-free. Such a sentence can be decided in time roughly equal to $(sd)^{O(n^2)}$ [12, Theorem 14.14] where d denotes the maximum degree of any polynomial of φ and s denotes the number of atoms.

In other contexts the two sets are allowed to undergo certain transformations (e.g. translations) such that the Hausdorff distance is minimized [17]. See Alt [9] for a survey.

Universal Existential Theory of the Reals. As mentioned above, the complexity class $\forall\exists\mathbb{R}$ was first studied by Dobbins, Kleist, Miltzow, and Rzążewski [27, 26] in the context of area-universality of graphs. A plane graph is *area-universal* if for every assignment of reals to the inner faces of a plane graph, there exists a straight-line drawing such that the area of each inner face equals the assigned number. Dobbins et al. conjecture that the decision problem whether a given plane graph is area-universal is complete for $\forall\exists\mathbb{R}$. They support this conjecture by proving hardness for several related notions [26]. Additionally, for future research directions, they present a number of candidates for potentially $\forall\exists\mathbb{R}$-hard problems. Among them, they stated a question motivating this paper as an open problem, namely whether the HAUSDORFF problem is $\forall\exists\mathbb{R}$-complete. The other candidates exhibit intrinsic connections to the notions of imprecision, robustness and extendability.

We point out that the computational complexity may also become easier when asking universal-type questions. For example, it is $\exists\mathbb{R}$-complete to decide whether a graph is a unit distance graph, i.e., whether it has a straight-line drawing in the plane in which all edges have the same length [46]. On the other hand, the decision problem whether for all reasonable assignments of weights to the edges, a graph has straight-line drawing in which the edge lengths correspond to the assigned weight lies in P [14]. Similarly, it is $\exists\mathbb{R}$-complete to decide for a given planar graph for which some vertices are fixed to the boundary of a polygon (with holes) whether there exists a planar straight-line drawing inside the polygon [32]. The case of simple polygons is open. In contrast, there is a polynomial time algorithm to test if a given graph G and a contained cycle C admit for *every* simple polygon P, representing C, a straight-line drawing of G inside P [38].

The sister class $\exists\forall\mathbb{R}$ was recently investigated by D'Costa et al. [24]. They show that it is $\exists\forall_{<}\mathbb{R}$-complete to decide for a given rational matrix A and a compact semialgebraic set $K \subseteq R^n$, whether there exists a starting point $x \in K$ such that $x_n := A^n x$ is contained in K for all $n \in N$. This and similar problems are generally referred to as *escape* problems.

We understand the complexity class $\forall\exists\mathbb{R}$ as a natural extension of the complexity class $\exists\mathbb{R}$ (pronounced as 'exists R', 'ER', or 'ETR'), which is defined similarly to $\forall\exists\mathbb{R}$, but without universally quantified variables. The complexity class $\exists\mathbb{R}$ has gained a lot of interest in recent years, specifically in the computational geometry community. It gains its significance because numerous well-studied problems from diverse areas of theoretical computer science and mathematics have been shown to be complete for this class. Famous examples from discrete geometry are the recognition of geometric structures, such as unit disk graphs [34], segment intersection graphs [33], visibility graphs [20], stretchability of pseudoline arrangements [37, 49], and order type realizability [33]. Other $\exists\mathbb{R}$-complete problems are related to graph drawing [32], Nash-Equilibria [15, 28], geometric packing [6], the art gallery problem [3], convex covers [2], non-negative matrix factorization [48], polytopes [25, 42], geometric embeddings of simplicial complexes [4], geometric linkage constructions [1], training neural networks [5], and continuous constraint satisfaction problems [35]. We refer the reader to the lecture notes by Matoušek [33] and surveys by Schaefer [45] and Cardinal [19] for more information on the complexity class $\exists\mathbb{R}$.

General Solution Strategies. We sometimes see that researchers make the *dichotomy* between tractable and intractable algorithmic problems. More precisely, when there exists a polynomial time algorithm the underlying problem is considered to be tractable. In contrast, in case of NP-hardness the underlying problem is considered intractable. Although most researchers are aware that this dichotomy does not match actual practical performance, it is often seen as a good enough yardstick.

In the last decades, a more *nuanced* perspective emerged. This new perspective acknowledges that there is a whole range of mathematical assumptions and models and that depending on the specific situation, different models can be more or less accurate [43]. One example is the so-called *smoothed analysis* of algorithms [50]. The underlying idea is that practical instances are subject to small noise. This small noise may tame a very difficult instance. In this context, we discuss four complexity classes: NP, $\exists\mathbb{R}$, Π_2^p, and $\forall\exists\mathbb{R}$.

NP Despite NP-hardness, huge practical instances can often be solved very fast. Prominent examples are ILPs that can be solved optimally using off the shelf solvers. Note that it is also possible to generate adversarial instances of moderate size for which no good tools exist.

$\exists\mathbb{R}$ Problems in $\exists\mathbb{R}$ are considerably harder. Still, we can often solve $\exists\mathbb{R}$-complete problems using suitable discretizations or using gradient descent. However, both methods usually have no guarantees to ever terminate. Furthermore, they may give solutions that are arbitrarily far from the optimum. Methods from real algebraic geometry are applicable if polynomials are explicitly given and contain only few variables, say around ten.

Π_2^p Describes problems on the second level of the polynomial time hierarchy [10]. We do not know many problems on this level, compared to the number of NP-complete problems. Due to the two blocks of quantifiers there are no effective general purpose tools like ILP-solvers. On the positive side, due to the combinatorial nature, it is possible to use exhaustive search.

$\forall\exists\mathbb{R}$ This class combines the difficulties of $\exists\mathbb{R}$ and Π_2^p. Note that we cannot even use gradient descent for problems in this class. Due to the continuous nature of the problem it is also not possible to use a simple brute-force algorithm. Furthermore, methods from real algebraic geometry cannot even solve small instances with up to say ten variables. The two different quantifiers limit those already impractical methods even further.

We want to point out that this classification of difficulty should not be taken dogmatically. For many algorithmic problems worst-case complexity is not an adequate model to explain practical performance. We rather take the perspective that this mathematical classification is a crude yardstick which measures algorithmic difficulty from the worst-case perspective. For each individual problem one has to judge, if the worst-case perspective is accurate.

1.2 Techniques and Proof Overview

In this subsection, we present the general idea behind the hardness reduction for the Hausdorff problem. The goal is to convey the intuition and to motivate the technical intermediate steps needed. The sketched reduction is oversimplified and thus neither in polynomial time nor fully correct. We point out both of these issues and give first ideas on how to solve them.

Let $\Phi := \forall X \in \mathbb{R}^n . \exists Y \in \mathbb{R}^m : \varphi(X,Y)$ be a Strict-UETR instance. We define two sets

$$A := \{x \in \mathbb{R}^n \mid \exists Y \in \mathbb{R}^m : \varphi(x,Y)\} \quad \text{and}$$
$$B := \mathbb{R}^n$$

and ask whether $d_{\mathrm{H}}(A,B) = 0$. If Φ is true, then $A = \mathbb{R}^n$ and we have $d_{\mathrm{H}}(A,B) = 0$ because both sets are equal. Otherwise, if Φ is false, then there exists some $x \in \mathbb{R}^n$ for which there is no $y \in \mathbb{R}^m$ satisfying $\varphi(x,y)$ and we conclude that $A \neq \mathbb{R}^n$. In general we call the set of all $x \in \mathbb{R}^n$ for which there is no $y \in \mathbb{R}^m$ satisfying $\varphi(x,y)$ the *counterexamples* $\perp(\Phi)$ of Φ. One might hope that $\perp(\Phi) \neq \emptyset$ is enough to obtain $d_{\mathrm{H}}(A,B) > 0$, but this is not the case. To this end, consider the formula $\Psi := \forall X \in \mathbb{R} . \exists Y \in \mathbb{R} : XY > 0$, which is false. The set $\perp(\Psi) = \{0\}$ contains only a single element, so we have $A = \mathbb{R} \setminus \{0\}$ and $B = \mathbb{R}$. However, their Hausdorff distance also evaluates to $d_{\mathrm{H}}(A,B) = 0$. We conclude that above reduction does not (yet completely) work, because it maps a yes- and a no-instances of Strict-UETR to a yes-instance of Hausdorff.

We solve this issue by blowing up the set of counter examples. Specifically, Theorem 12 establishes a polynomial time algorithm to transform a Strict-UETR instance Φ into an

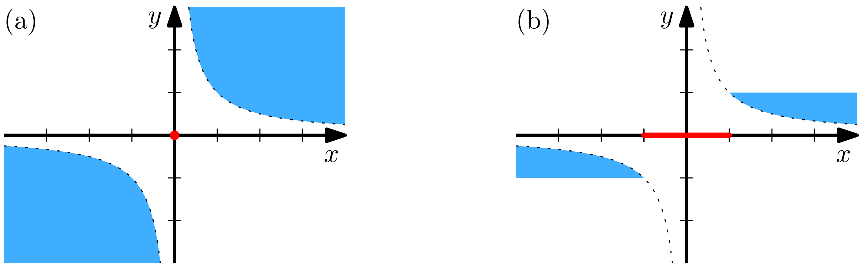

Figure 5 Consider the formula $\forall X \in \mathbb{R} \,.\, \exists Y \in \mathbb{R} : XY > 1$. (a) Each point $(x, y) \in \mathbb{R}^2$ in the blue open region satisfies $xy > 1$. Only for $x = 0$ (in red) no suitable $y \in \mathbb{R}$ exists. (b) Restricting the range of Y to $[-1, 1]$, then for all $x \in [-1, 1]$ (in red) no y with $xy > 1$ exists.

equivalent formula Φ' such that the set of counterexamples is either empty (if Φ' is true) or contains an open ball of positive radius (if Φ' is false). The radius of the ball serves as a lower bound on the Hausdorff distance $d_{\mathrm{H}}(A, B)$. Thus a reduction starting with Φ' is correct. As a key tool for this step, we restrict the variable ranges from $\mathbb{R}^n$ and $\mathbb{R}^m$ to small and compact intervals. Figure 5 presents an an example on how such a range restriction may enlarge the set of counterexamples from a single point to an interval.

We highlight that such a restriction of the variable ranges is not possible for general UETR formulas. However, we can exploit the fact that STRICT-UETR formulas are $\forall$-*strict*; a negation- and implication-free formula is $\forall$-strict if each atom involving universally quantified variables is a strict inequality. Being $\forall$-strict is a key property of many of the formulas considered throughout the paper, both for $\forall\exists_<\mathbb{R}$-hardness and -membership. We think that the special property of blown up counterexamples can prove useful in future reductions to show $\forall\exists_<\mathbb{R}$-hardness of other problems because it makes handling the no-instances easier.

A further challenge is given by the definition of the sets A and B. While the description complexity of B depends only on n, the definition of A contains an existential quantifier. This is troublesome because our definition of the HAUSDORFF problem requires quantifier-free formulas as its input, and in general there is no equivalent quantifier-free formula of polynomial length which describes the set A [23]. We overcome this issue by taking the existentially quantified variables as additional dimensions into account; it will be useful to scale them to a tiny range, so that their influence on the Hausdorff distance becomes negligible. Therefore instead of the above, in Section 5 we work with sets similar to

$$A := \{(x, y) \mid x \in [-1, 1]^n, y \in [-\varepsilon, \varepsilon]^m, \varphi_<(x, y)\} \quad \text{and}$$
$$B := [-1, 1]^n \times \{0\}^m$$

for some tiny value ε depending on the radius r (of the ball contained in the counterexamples) computed in Section 4. This definition of A and B introduces the new issue that even if Φ is true, the Hausdorff distance $d_{\mathrm{H}}(A, B)$ might be strictly positive. However, we manage to identify a threshold t, such that $d_{\mathrm{H}}(A, B) \leq t$ if and only if Φ is true. This completes the proof of $\forall\exists_<\mathbb{R}$-hardness.

Organization. The remainder of the paper is organized as follows. We introduce preliminaries concerning the first-order theory of the reals in Section 2 and essential tools form real algebraic geometry in Section 3. Section 4 presents the result for blowing up the set of counterexamples for $\forall$-strict-formulas and Section 5 the hardness proof. In Section 6, we show membership of HAUSDORFF in $\forall\exists_<\mathbb{R}$ (Theorem 17). For this, we translate the definition of the Hausdorff distance into a $\forall$-strict UETR instance and convert it via several intermediate steps into a STRICT-UETR instance. We conclude with a list of open problems in Section 7.

2 Preliminaries on First-Order Theory of the Reals and $\forall\exists\mathbb{R}$

Here, we give a short overview of notation and definitions used in the paper. We mostly introduce standard terminology following the book by Cox, Little, O'Shea [22].

An *atom* is an expression of the form $P \circ 0$ for some polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ and $\circ \in \{<, \leq, =, \neq, \geq, >\}$. We always assume that a polynomial is written as a sum of monomials, its *total degree* is the maximum number of occurrences of variables involved in any monomial. For example $P(X,Y,Z) = X^2Y^2 + XYZ$ has total degree four. A variable is called *free* if it is not bound by a quantifier. A *formula* is either

- an atom, or
- if φ_1, φ_2 are formulas, then $\varphi_1 \wedge \varphi_2$, $\varphi_1 \vee \varphi_2$, $\varphi_1 \implies \varphi_2$ and $\neg\varphi_1$ are formulas, or
- if X is a free variable of a formula $\varphi(X)$, then $\exists X : \varphi(X)$ and $\forall X : \varphi(X)$ are formulas in which X is bound.

In order to determine the *length* $|\varphi|$ of a formula φ, we count 1 for each fixed symbol, we encode integer coefficients in binary, exponents are encoded in unary, and we count $\log n$ for every occurrence of each variable, where n denotes the number of variables. We denote by QFF the family of quantifier free formulas that contain no negation or implication. Furthermore, $\text{QFF}_<$, $\text{QFF}_\leq$, and $\text{QFF}_=$ are the families in QFF that have only atoms involving $<$, $\leq$ and $=$ respectively.

A *sentence* is a formula without free variables and thus either equivalent to true or to false. The truth value is defined inductively, by interpreting the quantifiers over the real numbers $\mathbb{R}$. As a convention, we use capitalized Greek letters for sentences and use lower case Greek letter for formulas. We write $\Psi \equiv \Psi'$ if the two sentences have the same truth value. The *first order theory of the reals* (FOTR) is the family of all true sentences. If all quantifiers of a formula appear at its beginning, we say it is in *prenex normal form*. We usually write *blocks of variables*, i.e., $\forall X \in \mathbb{R}^n : \varphi(X)$. Here X is a shorthand notation for $X = (X_1, \ldots, X_n)$. We say n is the length of X in this case. All quantifiers quantify their bound variables over $\mathbb{R}$. The following are just shorthand notation:

$$\forall X \in [-1,1] : \varphi(X) \quad \equiv \quad \forall X \in \mathbb{R} : (X \geq -1 \wedge X \leq 1) \implies \varphi(X)$$
$$\exists X \in [-1,1] : \varphi(X) \quad \equiv \quad \exists X \in \mathbb{R} : (X \geq -1 \wedge X \leq 1) \wedge \varphi(X)$$

We use upper case letters for variables in formulas and lower case letters for specific values, i.e., symbol X denotes a vector of variables, while $x \in \mathbb{R}^n$ is a point. We sometimes write $\varphi(X,Y)$ to emphasize that X and Y are free variables of the formula φ. Often we do not mention the free variables of φ though.

Consider a formula $\Phi := \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X,Y)$, where $\varphi \in \text{QFF}$. Each atom of φ is of the form $P \circ 0$, where $\circ \in \{<, \leq, =, \neq, \geq, >\}$ and $P \in \mathbb{Z}[X,Y]$ is a multivariate polynomial in the variables X and Y. Without loss of generality we can restrict our attention to the case of $\circ \in \{<, \leq\}$, because the following transformations show that the other relations can be reformulated such that the length of the formula is at most doubled.

$$P > 0 \equiv -P < 0 \qquad\qquad P = 0 \equiv (P \leq 0) \wedge (-P \leq 0)$$
$$P \geq 0 \equiv -P \leq 0 \qquad\qquad P \neq 0 \equiv (P < 0) \vee (-P < 0)$$

Furthermore, we can assume that φ contains only the logical connectives $\wedge$ and $\vee$, because De Morgan's law allows to push all negations (and therefore also implications) down to the atoms transforming φ into *negation normal form.* With the following equivalences we obtain a formula without negations:

$$\neg(P < 0) \equiv -P \leq 0 \qquad\qquad \neg(P \leq 0) \equiv -P < 0$$

Given a formula φ, the set $S(\varphi) = \{x \in \mathbb{R}^n \mid \varphi(x)\}$ is semi-algebraic. The *complexity* of a semi-algebraic set S is the length of a shortest quantifier-free formula φ, such that $S = S(\varphi)$. We say $\varphi \equiv \varphi'$ if $S(\varphi) = S(\varphi')$.

Below we cite another helpful lemma, which is used again in Section 6. The lemma allows us to replace a formula $\varphi \in \mathrm{QFF}$ by a single polynomial (at the cost of adding a polynomial number of additional variables).

▶ **Lemma 4** ([47, Lemma 3.2]). *Let $\varphi(X) \in \mathrm{QFF}$ with n free variables X. We can construct in polynomial time $k = O(|\varphi|)$ and a polynomial $F : \mathbb{R}^{n+k} \to \mathbb{R}$ of degree at most 4 such that*

$$\{x \in \mathbb{R}^n \mid \varphi(x)\} = \{x \in \mathbb{R}^n \mid \exists U \in \mathbb{R}^k : F(x, U) = 0\}.$$

For any fixed $\circ \in \{<, \leq\}$, we denote by $\forall\exists_\circ\mathbb{R}$ the fragment of $\forall\exists\mathbb{R}$ containing all decision problems that polynomial-time many-one reduce to a UETR-instance where all formulas are contained in $\mathrm{QFF}_\circ$. Similarly, for $\circ \in \{<, \leq\}$, we denote the corresponding fragments of $\exists\mathbb{R}$ and $\forall\mathbb{R}$ by $\exists_\circ\mathbb{R}$ and $\forall_\circ\mathbb{R}$, respectively. The following lemma summarizes what we know about the relation between the complexity classes $\forall\exists_<\mathbb{R}$, $\forall\exists_\leq\mathbb{R}$ and $\forall\exists\mathbb{R}$ as well as their relation to the well-studied classes NP, co-NP, $\exists\mathbb{R}$, $\forall\mathbb{R}$, and PSPACE.

▶ **Lemma 5.** *It holds* NP $\subseteq \exists\mathbb{R} \subseteq \forall\exists_<\mathbb{R} \subseteq \forall\exists_\leq\mathbb{R} = \forall\exists\mathbb{R} \subseteq$ PSPACE*. Furthermore,* co-NP $\subseteq \forall\mathbb{R} \subseteq \forall\exists_<\mathbb{R}$*.*

Proof. The inclusions NP $\subseteq \exists\mathbb{R}$, co-NP $\subseteq \forall\mathbb{R}$ and $\forall\exists_\circ\mathbb{R} \subseteq \forall\exists\mathbb{R}$ for each $\circ \in \{<, \leq\}$ follow from the definitions of the involved complexity classes. The inclusion $\exists\mathbb{R} \subseteq \forall\exists_<\mathbb{R}$ follows from the fact that $\exists_<\mathbb{R} = \exists\mathbb{R}$ [47, Theorem 4.1]. The inclusion $\forall\exists\mathbb{R} \subseteq$ PSPACE was first established by Canny in his seminal paper [18].

For the first statement, it remains to show that $\forall\exists\mathbb{R} \subseteq \forall\exists_\leq\mathbb{R}$. To this end, consider a UETR instance

$$\Phi := \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X, Y).$$

We apply Lemma 4 to φ and obtain in polynomial time a single multivariate polynomial $F : \mathbb{R}^{n+m+k} \to \mathbb{R}$, such that

$$\Psi := \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^{m+k} : F(X, Y) \leq 0 \wedge -F(X, Y) \leq 0,$$

is equivalent to Φ and all atoms use $\leq$.

Let us now consider the inclusion $\forall\mathbb{R} \subseteq \forall\exists_<\mathbb{R}$. We show $\forall\mathbb{R} = \forall_<\mathbb{R} \subseteq \forall\exists_<\mathbb{R}$. Note that the second inclusion follows from the definition of the complexity classes. To show the first equation, it is sufficient to show that the two classes co-$\forall\mathbb{R} = \exists\mathbb{R}$ and co-$\forall_<\mathbb{R} = \exists_\leq\mathbb{R}$ are equal, because two complexity classes are equal whenever there sister classes are equal. Moreover, $\exists_\leq\mathbb{R} = \exists_=\mathbb{R} = \exists\mathbb{R}$ is a known result [47]. ◀

3 Mathematical Tools

In this section, we review already existing tools that are needed throughout the paper. In particular, we use two sophisticated results from algebraic geometry, namely singly exponential quantifier elimination and the so called Ball Theorem. While quantifier elimination provides equivalent quantifier free formulas of bounded length, the Ball Theorem guarantees that every non-empty semi-algebraic set contains an element not too far from the origin. We use the two results to establish useful properties of semi-algebraic sets.

We start with a result on quantifier-elimination which originates from a series of articles by Renegar [39, 40, 41]. We note that the time complexity of this algorithm is exponential and not double exponential for every fixed number of quantifier alternations.

▶ **Theorem 6** ([12, Theorem 14.16]). *Let $X_1, \dots, X_k, Y$ be vectors of real variables where X_i has length n_i, Y has length m, formula $\varphi(X_1, \dots, X_k, Y) \in \text{QFF}$ has s atoms and $Q_i \in \{\exists, \forall\}$ is a quantifier for all $i = 1, \dots, k$. Further, let d be the maximum total degree of any polynomial of $\varphi(X_1, \dots, X_k, Y)$. Then for any formula $\Phi(Y) := (Q_1 X_1) \dots (Q_k X_k) : \varphi(X_1, \dots, X_k, Y)$ there is an equivalent quantifier-free formula of size at most*

$$s^{(n_1+1)\cdots(n_k+1)(m+1)} d^{O(n_1)\cdots O(n_k)O(m)}.$$

We use the following corollary of Theorem 6 that is weaker but easier to work with.

▶ **Corollary 7.** *Given a formula $\Phi(Y)$ as in Theorem 6 with length $L = |\varphi(X_1, \dots, X_n, Y)|$. Then for a constant $\alpha \in \mathbb{R}$ independent of Φ, there exists an equivalent quantifier-free formula of size at most*

$$L^{\alpha^{k+1} n_1 \dots n_k m}.$$

Proof. Let Ψ be the quantifier-free formula equivalent to Φ obtained by Theorem 6. By observing that $d, s \le L$, we get

$$\begin{aligned}
|\Psi(Y)| &\le L^{(n_1+1)\cdots(n_k+1)(m+1)} \cdot L^{O(n_1)\cdots O(n_k)O(m)} \\
&\le L^{2n_1\cdots 2n_k \cdot 2m} \cdot L^{O(n_1)\cdots O(n_k)O(m)} \\
&\le L^{2n_1\cdots 2n_k \cdot 2m + O(n_1)\cdots O(n_k)O(m)} \\
&\le L^{\alpha_1 n_1 \cdots \alpha_k n_k \cdot \alpha_m m} && (\text{for } \alpha_1, \dots, \alpha_k, \alpha_m \in \mathbb{R}) \\
&\le L^{\alpha^{k+1} n_1 \cdots n_k m} && (\alpha := \max\{\alpha_1, \dots, \alpha_k, \alpha_m\})
\end{aligned}$$ ◀

Now, we turn our attention to the Ball Theorem. The Ball Theorem was first discovered by Vorob'ev [52] and Grigor'ev and Vorobjov [30]. Vorob'ev and Vorobjov are two different transcriptions of the same name from the Cyrillic to the Latin alphabet. Explicit bounds on the distance are given by Basu and Roy [13]. We use a formulation from Schaefer and Štefankovič [47].

▶ **Theorem 8** (Ball Theorem [47, Corollary 3.1]). *Every semi-algebraic set in $\mathbb{R}^n$ of complexity at most $L \ge 4$ contains a point of distance at most $2^{L^{8n}}$ from the origin.*

Recall that for any quantifier-free formula $\varphi(X)$ with free variables $X \in \mathbb{R}^n$, the set $S := \{x \in \mathbb{R}^n \mid \varphi(X)\}$ is semi-algebraic. Thus, a direct conclusion of Theorem 8 is that $\exists X \in \mathbb{R}^n : \varphi(X)$ is equivalent to $\exists X \in [-2^{L^{8n}}, 2^{L^{8n}}]^n : \varphi(X)$. This is how we are going to make use of Theorem 8 throughout this paper.

In the following, we deduce useful properties from Corollary 7 and Theorem 8, starting with a fact that was identified by D'Costa [24, Lemma 14] for two quantifiers. We are interested in a generalization to more quantifiers. Their proof also works with slight modifications in the more general case with k quantifiers.

▶ **Lemma 9.** *Let $X_1, \dots, X_k$ be vectors of variables where X_i has length $n_i \geq 1$ and let $\varphi(\varepsilon, X_1, \dots, X_k)$ be a quantifier-free formula of length L. Then the semi-algebraic set*

$$S = \{\varepsilon > 0 \mid (Q_1 X_1) \dots (Q_k X_k) : \varphi(\varepsilon, X_1, \dots, X_k)\},$$

where the Q_i are alternating existential and universal quantifiers, is either empty or it contains an element $\varepsilon^* \in S$ such that for some constant $\beta \in \mathbb{R}$ we have

$$\varepsilon^* \geq 2^{-L^{\beta^{k+2} n_1 \cdots n_k}}.$$

Proof. Let $\Phi(\varepsilon) := (Q_1 X_1) \dots (Q_k X_k) : \varphi(\varepsilon, X_1, \dots, X_k)$. By Corollary 7, $\Phi(\varepsilon)$ is equivalent to a quantifier-free formula $\psi(\varepsilon)$ of length $|\psi(\varepsilon)| \leq L^{2\alpha^{k+1} n_1 \cdots n_k}$ for some constant $\alpha \in \mathbb{R}$.

Replace each atom $P(\varepsilon) \circ 0$ of $\psi(\varepsilon)$ (where $\circ \in \{<, \leq\}$ and P is a univariate polynomial of degree d_P) by $\delta^{d_P} P(1/\delta) \circ 0$ and denote the new formula by $\psi^{-1}(\delta)$. By construction, for any $\varepsilon > 0$ we have that $\psi(\varepsilon)$ is satisfied if and only if for $\delta = 1/\varepsilon$ the sentence $\psi^{-1}(\delta)$ is satisfied. Further, the length of each atom increases by a factor of at most d_P, which is obviously at most $|\psi(\varepsilon)|$. We define $K := |\psi^{-1}(\delta)| \leq |\psi(\varepsilon)|^2$. Now consider the sentence $\exists \delta > 0 : \psi^{-1}(\delta)$, which is equivalent to $\exists \varepsilon > 0 : \psi(\varepsilon)$. By the Ball Theorem (Theorem 8), there is a δ^* satisfying $\psi^{-1}(\delta^*)$, such that $\delta^* \leq 2^{K^8}$. We get (for $\beta := \max\{\alpha, 16\}$) that

$$\varepsilon^* := \frac{1}{\delta^*} \geq 2^{-K^8} = 2^{-(|\psi(\varepsilon)|^2)^8} \geq 2^{-L^{16\alpha^{k+1} n_1 \cdots n_k}} \geq 2^{-L^{\beta^{k+2} n_1 \cdots n_k}}.$$ ◀

Given a semi-algebraic set $S \subseteq \mathbb{R}^n$ and any $\alpha \in \mathbb{Q}$, the scaled set $T = \{\alpha x \in \mathbb{R}^n \mid x \in S\}$ is semi-algebraic. The following lemmas proves that scaling any subset of the variables by a doubly exponentially large integer can be encoded by a formula of polynomial length.

We denote by the *type* of an atom whether it is a strict inequality, a non-strict inequality or an equation. We say that two formulas *have the same logical structure* if there is a bijection between their atoms such that identifying corresponding atoms leads to the same formula.

▶ **Lemma 10.** *For $N \in \mathbb{N}$ and $N+1$ variables $U = (U_0, \dots, U_N) \in [-1, 1]^{N+1}$ consider*

$$\chi(U) := (2 \cdot U_0 = 1) \wedge \bigwedge_{i=1}^{N} (U_i = U_{i-1}^2).$$

Then for each $u \in [-1, 1]^{N+1}$ and $i \in \{0, \dots, N\}$ we have $\chi(u)$ if and only if $u_i = 2^{-2^i}$.

Proof. The if-part is trivial. The only-if-part follows from a simple induction. ◀

▶ **Lemma 11** (Scaling Semi-Algebraic Sets)**.** *Let $\varphi(X, Y) \in \mathrm{QFF}$ with free variables $X \in \mathbb{R}^n$ and $Y \in \mathbb{R}^m$. Further, let N be an integer and $s \in \{-1, 1\}$. We can construct in time polynomial in $|\varphi|$ and N a formula $\psi(X, Y)$, such that for any $(x, y) \in \mathbb{R}^{n+m}$ we have $\varphi(x, y)$ if and only if $\psi(x \cdot 2^{s \cdot 2^N}, y)$. Further $\psi(X, Y)$ can be chosen to be of the form*

$$\psi(X, Y) \equiv \exists U \in [-1, 1]^{N+1} : \chi(U) \wedge \varphi'(X, Y, U) \quad \textit{or alternatively}$$
$$\psi(X, Y) \equiv \forall U \in [-1, 1]^{N+1} : \neg\chi(U) \vee \varphi'(X, Y, U).$$

In both cases, $\chi(U) \in \mathrm{QFF}_=$, formulas $\varphi'(X, Y, U)$ and $\varphi(X, Y)$ have the same logical structure and corresponding atoms have the same type.

Proof. For new variables $U = (U_0, \ldots, U_N) \in [-1,1]^{N+1}$ let $\chi(U)$ be the formula from Lemma 10. Recall that for every $u \in [-1,1]^{N+1}$ with $\chi(u)$ true, we have $u_i = 2^{-2^i}$. In the following, we distinguish whether $s = 1$ and we scale up or $s = -1$ and we scale down.

In case $s = 1$, let $\varphi'(X, Y, U) := \varphi(U_N \cdot X, Y)$, where $U_N \cdot X$ denotes componentwise multiplication with U_N. For $X = (X_1, \ldots, X_n)$, this can be done by replacing each occurrence of X_i in $\varphi(X, Y)$ by $U_N X_i$.

The case $s = -1$ is similar. Instead of multiplying each occurrence of X_i by U_N, we now divide by U_N. As each atom of $\varphi(X, Y)$ is of the form $P \circ 0$ for $\circ \in \{<, \leq\}$, the division transforms each P into a rational function Q. To restore the form of the atoms, let d_P be the total degree of P. Multiply each atom with $U_N^{d_P}$ to obtain polynomial atoms again (this increases the length of each atom by a factor of d_P in the worst case). Let $\varphi'(X, Y, U)$ be the obtained formula.

In both cases $|\psi| \in O(|\varphi| + N)$. Correctness follows from the fact that all X_i have been scaled by the same factor and that both version of quantifying U force the variables to their desired values. ◀

4 Counterexamples of Strict-UETR

Let us recall the definition of *counterexamples* here that was already motivated in Section 1.2. Given a sentence $\Phi := \forall X \in \mathbb{R}^n . \exists Y \in \mathbb{R}^m : \varphi(X, Y)$ we call the set

$$\perp(\Phi) := \{x \in \mathbb{R}^n \mid \forall Y \in \mathbb{R}^m : \neg\varphi(x, Y)\}$$

its *counterexamples*. The counterexamples of Φ are exactly the values $x \in \mathbb{R}^n$ for which there is no $y \in \mathbb{R}^m$ such that $\varphi(x, y)$ is true.

The main result of this section, Theorem 12, is that we can transform a Strict-UETR instance Φ into an equivalent formula Ψ for which $\perp(\Psi)$ is either empty or contains an open ball. We achieve this by bounding the range over which the variables are quantified. The following theorem summarizes our findings. This open ball property is a key technical step and we believe is of independent interest.

▶ **Theorem 12.** *Given a* Strict-UETR *instance*

$$\Phi := \forall X \in \mathbb{R}^n . \exists Y \in \mathbb{R}^m : \varphi_<(X, Y),$$

with $\varphi_<(X, Y) \in \text{QFF}_<$, we can construct in polynomial time an equivalent UETR *instance*

$$\Psi := \forall X \in [-1,1]^n . \exists Y \in [-1,1]^\ell : \psi(X, Y),$$

where $\psi \in \text{QFF}$. Further, $\perp(\Psi)$ is either empty or contains an n-dimensional open ball.

In order to prove Theorem 12, we state three helpful lemmas. We start with the following lemma bounding the universally quantified variables, which (stated for the class $\exists\forall\mathbb{R}$) already appeared similarly in [24, Lemma 10].

▶ **Lemma 13.** *For each sentence $\Phi := \forall X \in \mathbb{R}^n . \exists Y \in \mathbb{R}^m : \varphi(X, Y)$ with $\varphi \in \text{QFF}$, there exists an integer N polynomial in $|\Phi|$, such that for $C := 2^{2^N}$ the sentence*

$$\Psi := \forall X \in [-C, C]^n . \exists Y \in \mathbb{R}^m : \varphi(X, Y)$$

is equivalent to Φ and the counterexamples are $\perp(\Psi) = \perp(\Phi) \cap [-C, C]^n$.

Proof. We rewrite Φ via a double negation to get

$$\Phi \equiv \neg\big(\exists X \in \mathbb{R}^n \,.\, \forall Y \in \mathbb{R}^m : \neg\varphi(X,Y)\big)$$

and let $L := |\neg\varphi|$ denote the length of the quantifier-free part. Applying quantifier elimination (Corollary 7) to the universally quantified variables Y yields an equivalent sentence $\neg\big(\exists X \in \mathbb{R}^n : \psi(X)\big)$, where ψ is quantifier-free and has size

$$|\psi| \leq L^{\alpha^2 nm} = 2^{\log(L)\alpha^2 nm}$$

for some constant $\alpha \in \mathbb{R}$. By the Ball Theorem (Theorem 8) there is some constant D with

$$D \leq 2^{|\psi|^{8n}} = 2^{(L^{\alpha^2 nm})^{8n}} = 2^{2^{8\log(L)\alpha^2 n^2 m}}$$

such that $\neg(\exists X \in I^n : \psi(X))$ is equivalent to Φ, for any set I containing the interval $[-D, D]$. Let N be the smallest integer, such that $C := 2^{2^N} \geq D$. Note that $N \in O(\log(L)n^2m)$, so it is polynomial in $|\Phi|$. Reversing the double negation we get that

$$\Psi := \forall X \in [-C, C]^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X,Y)$$

is equivalent to Φ. Because φ remained unchanged, the set of counter examples is given by $\bot(\Psi) = \bot(\Phi) \cap [-C, C]^n$. ◀

Next, we additionally bound the range of the existentially quantified variables. D'Costa et al. [24, page 9] present an example why this is impossible for general UETR formulas. They show, however, that such a range restriction is possible if all atoms are strict inequalities (and not negated). The statement and proof of the following lemma are a slight generalization of [24, Lemmas 12 and 13], proving that it is sufficient to be $\forall$-strict.

▶ **Lemma 14.** *Let N be an integer, $C = 2^{2^N}$ and Φ be a sentence of the form*

$$\Phi := \forall X \in [-C, C]^n \,.\, \exists Y \in \mathbb{R}^m : \varphi(X,Y)$$

where $\varphi \in \mathrm{QFF}$ is $\forall$-strict. Then there is an integer M polynomial in $|\Phi|$ and N such that for $D = 2^{2^M}$ the sentence

$$\Psi := \forall X \in [-C, C]^n \,.\, \exists Y \in [-D, D]^m : \varphi(X,Y)$$

is equivalent to Φ. Further $\bot(\Phi) \subseteq \bot(\Psi)$.

Proof. If Φ is false, then there exists $x \in [-C, C]^n$ such that no $y \in \mathbb{R}^m$ satisfies $\varphi(x, y)$. In particular, no $y \in [-D, D]^m \subseteq \mathbb{R}^m$ may satisfy $\varphi(x, y)$. Thus, Ψ is also false.

In the remainder of the proof, assume that Φ is true. The proof consists of two steps. First we show that Φ is equivalent to

$$\Phi_2 := \exists D > 0 \,.\, \forall X \in [-C, C]^n \,.\, \exists Y \in \mathbb{R}^m : \bigwedge_{i=1}^{m} |Y_i| \leq D \wedge \varphi(X,Y).$$

If Φ_2 is true, it follows directly that the we can bound the ranges of the Y variables. In a second step, we show that if Φ_2 is true, then we can assume $D \leq 2^{2^M}$ for an integer M polynomial in $|\Phi_2|$ and N.

Equivalence of Φ and Φ_2. Let $S = [-C, C]^n$. As Φ is true, for each $x \in S$ there is a $y(x) \in \mathbb{R}^m$ such that $\varphi(x, y(x))$ is true. Even stronger, as φ is $\forall$-strict, we even find an $\varepsilon(x) > 0$, such that for all $\widetilde{x} \in S$ with $\|x - \widetilde{x}\| < \varepsilon(x)$ we get that $\varphi(\widetilde{x}, y(x))$ is true. Recall that we denote by $B_n(x,r) = \{\widetilde{x} \in \mathbb{R}^n \mid \|\widetilde{x} - x\| < r\}$ the open ball with center x and radius r in $\mathbb{R}^n$. Then $\{B_n(x, \varepsilon(x)) \mid x \in S\}$ is an open cover of S. As S is compact, it has a finite subcover $B_n(x_1, \varepsilon(x_1)), \ldots, B_n(x_s, \varepsilon(x_s))$. Now, given some $x \in S$, there is an $i \in \{1, \ldots, s\}$, such that $\varphi(x, y(x_i))$ is true. We define $y_{\max} := \max\{\|y(x_1)\|_\infty, \ldots, \|y(x_s)\|_\infty\}$ and conclude that Φ and Φ_2 are equivalent as D can be chosen to be at least $y_{\max}$.

Bounding D. If we write down $C = 2^{2^N}$ explicitly, its binary encoding has exponential length. Using Lemma 11, we can rewrite Φ_2 equivalently as

$$\Phi_3 := \exists D > 0 \,.\, \forall X \in [-1,1]^n \,.\, \exists Y \in \mathbb{R}^m, U \in [-1,1]^{N+1} : \bigwedge_{i=1}^{m} |Y_i| \leq D \wedge \varphi'(X, Y, U),$$

where $\varphi' \in$ QFF. Let L be the length of the subformula of Φ_3 behind the existential quantification of D. Applying quantifier elimination (Corollary 7), there is an equivalent formula $\psi(D)$, such that $\Phi_3 \equiv \exists D > 0 : \psi(D)$ and the length of $\psi(D)$ is bounded by $L^{\alpha^3 n(m+N+1)}$ for some constant $\alpha \in \mathbb{R}$. Now the Ball Theorem (Theorem 8) states that the truth of Φ_3 remains invariant, if we bound D by

$$D \leq 2^{(L^{\alpha^3 n(m+N+1)})^{8n}} = 2^{L^{8\alpha^3 n^2 (m+N+1)}} = 2^{2^{8\log(L)\alpha^3 n^2 (m+N+1)}}.$$

Choose M to be the smallest integer such that $D \leq 2^{2^M}$. Note that $M \in O(n^2(m+N)\log L)$, so it is polynomial in $|\varphi|$ and N. ◀

We need one more lemma to prove the main theorem of this section, which is an observation from basic calculus.

▶ **Lemma 15.** *Given a continuous function $f : S \times T \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$ and $T \subseteq \mathbb{R}^m$ are compact. Then $g : S \to \mathbb{R}$, $x \mapsto \min_{y \in T}\{f(x,y)\}$ is continuous over S.*

Proof. We first observe that as S and T are compact, their Cartesian product $S \times T$ is compact as well. Thus, since f is continuous on $S \times T$ it is even uniformly continuous, meaning that for every $\varepsilon > 0$ there is a $\delta > 0$, such that $|f(x,y) - f(\widetilde{x}, \widetilde{y})| < \varepsilon$ whenever $\|(x,y) - (\widetilde{x}, \widetilde{y})\| < \delta$ for every two points $(x,y), (\widetilde{x}, \widetilde{y}) \in S \times T$.

Now consider $x, \widetilde{x} \in S$ with $\|x - \widetilde{x}\| < \delta$. We have

$$\begin{aligned} g(\widetilde{x}) - g(x) &= g(\widetilde{x}) - f(x,y) && \text{(for some } y \in T) \\ &< g(\widetilde{x}) - (f(\widetilde{x}, y) - \varepsilon) && \text{(by uniform continuity)} \\ &\leq g(\widetilde{x}) - (g(\widetilde{x}) - \varepsilon) && \text{(by definition of } g) \\ &= \varepsilon. \end{aligned}$$

By exchanging the role of x and $\widetilde{x}$, we get $g(x) - g(\widetilde{x}) < \varepsilon$. Combined, we get $|g(x) - g(\widetilde{x})| < \varepsilon$ for all $x, \widetilde{x} \in S$ with $\|x - \widetilde{x}\| < \delta$. This is the definition of g being continuous on S. ◀

Now, we are able to tackle the main result of this section, Theorem 12. Recall that it allows us to transform a Strict-UETR instance Φ into an equivalent UETR instance Ψ such that $\bot(\Psi)$ is either empty or contains an open ball.

Proof of Theorem 12. The proof has three main steps: First, we construct a sentence Φ' equivalent to Φ. Second, we prove that if Φ' is false, there is an open ball in $\perp(\Phi')$. However, $|\Phi'|$ is exponential. Third, we construct Ψ, which has polynomial size and whose set of counterexamples $\perp(\Psi)$ is a scaled copy of $\perp(\Phi')$, thus also contains an open ball or it is empty.

We note that in our reduction, Ψ can be constructed directly from Φ, without ever writing down Φ' explicitly. This is important, as otherwise the reduction would not be in polynomial time. We take the conceptual detour via Φ' because it is much easier to argue that a non-empty $\perp(\Phi')$ contains an open ball.

Constructing Φ'. Let $P_i < 0$ be an atom of Φ, where $P_i \in \mathbb{Z}[X, Y]$. Replace $P_i < 0$ by the equivalent formula $\exists Z_i \in \mathbb{R} : Z_i^2 P_i + 1 < 0$. Here Z_i is a new variable exclusive to this atom. While this replacement may look innocent, it is very powerful. The key insight is the following. Once, we bound the range of Z_i to $[-D, D]$, we need $P_i < -1/D^2$ in order to satisfy the constraint. This stronger condition blows up the set of counterexamples as we show below. This replacement increases the length of the resulting sentence only by a constant amount per atom. Let k be the total number of atoms, which is clearly bounded by $|\Phi|$. Converting the new sentence into prenex normal form we obtain

$$\Phi_1 := \forall X \in \mathbb{R}^n \,.\, \exists Y \in \mathbb{R}^m, Z \in \mathbb{R}^k : \varphi'_<(X, Y, Z),$$

where $\varphi'_< \in \mathrm{QFF}_<$ and which has exactly the same logical structure as $\varphi_<$ (the only difference is in the transformed atoms). As each atom was replaced with an equivalent formula, we have Φ_1 equivalent to Φ. Moreover, $\perp(\Phi_1) = \perp(\Phi)$.

Because Φ_1 is $\forall$-strict, we can apply first Lemma 13 and then Lemma 14 to obtain integers N and M polynomial in $|\Phi_1|$ such that for $C = 2^{2^N}$ and $D = 2^{2^M}$ the following sentence is equivalent to Φ_1 and Φ:

$$\Phi' := \forall X \in [-C, C]^n \,.\, \exists Y \in [-D, D]^m, Z \in [-D, D]^k : \varphi'_<(X, Y, Z)$$

Moreover, it holds that $\perp(\Phi_1) \cap [-C, C] \subseteq \perp(\Phi')$.

Counterexamples of Φ'. Note that $\varphi_<$ and $\varphi'_<$ have the same logical structure (each atom has independently been replaced by an equivalent one). Therefore, a deterministic algorithm $\mathcal{A}$ that transforms a formula into a disjunctive normal form (DNF) applied to $\varphi_<$ and $\varphi'_<$ would yield formulas $\mathcal{A}(\varphi_<)$ and $\mathcal{A}(\varphi'_<)$ in DNF which again have the same logical structure. Further, for all $x \in [-C, C]^n$, $y \in [-D, D]^m$ and $z \in [-D, D]^k$ we have $\varphi_<(x, y) = \mathcal{A}(\varphi_<(x, y))$ and $\varphi'_<(x, y, z) = \mathcal{A}(\varphi'_<(x, y, z))$. This allows us to use only the formulas in DNF to argue about the counterexamples of Φ'. For the ease of notation, we do not write $\mathcal{A}(\varphi_<(x, y))$ and $\mathcal{A}(\varphi'_<(x, y, z))$, but instead assume without loss of generality that $\varphi_<$ and $\varphi'_<$ are in DNF.

Note that the conversion into a DNF leads to an exponential increase in the formula length in general. However, this is not a problem for us, because the DNF is only used in the proof, not in the reduction. The sentence Ψ (constructed from Φ' at the end of the proof) contains the subformula $\varphi'_<$ as it was constructed above.

By construction, the sentences Φ and Φ' are equivalent. Thus if Φ is true, we have $\perp(\Phi) = \emptyset$ and $\perp(\Phi') = \emptyset$. The case that Φ is false (and thus $\perp(\Phi) \neq \emptyset$) is more interesting: It is possible that $\perp(\Phi)$ consists of just a finite number of points. In contrast, we prove next that $\perp(\Phi')$ contains an n-dimensional open ball.

To this end, we consider the case that Φ is false. Let $x^* \in \bot(\Phi) \cap \bot(\Phi')$ be a counterexample, fixed until the end of the proof. In the following, we show that for some $r > 0$, all $x \in [-C, C]^n$ with $\|x^* - x\| < r$ are counterexamples as well. Then we have $B_n(x^*, r) \cap [-C, C]^n \subseteq \bot(\Phi')$ and x^* is the center of our desired open ball. It might happen, that $B_n(x^*, r)$ is not completely contained in $[-C, C]^n$. In this case, any $x' \in B_n(x^*, r) \cap (-C, C)^n$ can be used instead as the center of a smaller, but still open ball.

Let $\mathcal{C}(X, Y) := (\bigwedge_{i=1}^{s} P_i(X, Y) < 0)$ be one of the conjunctive clauses of (the DNF of) $\varphi_<(X, Y)$. For our fixed counterexample $x^* \in \bot(\Phi)$, every conjunctive clause of $\varphi_<(x^*, Y)$ evaluates to false independently of Y, and we get for all $y \in \mathbb{R}^m$ and thus in particular for all $y \in [-D, D]^m$ that

$$\neg\mathcal{C}(x^*, y) \equiv \bigvee_{i=1}^{s} P_i(x^*, y) \geq 0. \tag{1}$$

Let us point out, that for different choices of $y \in [-D, D]^m$, different subsets of the polynomials $P_i(x^*, y)$ may evaluate to non-negative values. We only know that for every y at least one of the polynomials is non-negative (here it is important that $\varphi_<(X, Y)$ is in DNF). To overcome this, the next step is to combine the polynomials into a single function.

Each of the $P_i \in \mathbb{Z}[X, Y]$, $i \in \{1, \ldots, s\}$, is a multivariate polynomial and thus continuous. The maximum over a finite number of continuous functions is continuous, so

$$P_{\max} : [-C, C]^n \times [-D, D]^m \to \mathbb{R} \quad \text{with} \quad (x, y) \mapsto \max_{i=1,\ldots,s} \{P_i(x, y)\}$$

is continuous. It follows from (1) that for our counterexample x^* and all $y \in [-D, D]^m$ we have

$$P_{\max}(x^*, y) \geq 0. \tag{2}$$

We want to argue about the value of $P_{\max}$ at points x in a small neighborhood around x^*. To this end, we consider the function

$$P^* : [-C, C]^n \to \mathbb{R} \quad \text{with} \quad x \mapsto \min_{y \in [-D, D]^m} P_{\max}(x, y),$$

which eliminates the dependency on y. The sets $[-C, C]^n$ and $[-D, D]^m$ are compact, so by Lemma 15 function P^* is again continuous. From (2), we get for our counterexample x^* that

$$P^*(x^*) \geq 0.$$

By the continuity of P^*, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in [-C, C]^n$ with $\|x^* - x\| < \delta$ we get $|P^*(x) - P^*(x^*)| < \varepsilon$. We choose $\varepsilon < 1/D^2$ and conclude that for a sufficiently small $\delta > 0$ and all $x \in [-C, C]^n$ with $\|x^* - x\| < \delta$ we have

$$P^*(x) > -\frac{1}{D^2}.$$

Fix one such x. Going backwards through our chain of defined functions, it follows for all $y \in [-D, D]^m$ that $P_{\max}(x, y) > -1/D^2$ and moreover that

$$\bigvee_{i=1}^{s} P_i(x, y) > -\frac{1}{D^2}. \tag{3}$$

Now also fix an arbitrary $y \in [-D, D]^m$ and choose $j \in \{1, \ldots, s\}$ such that $P_j(x, y) > -1/D^2$. Because $A := (P_j(X, Y) < 0)$ is an atom of $\varphi_<(X, Y)$, there is a corresponding atom $A' :=$

$(Z_j^2 P_j(X,Y) + 1 < 0)$ in $\varphi'_<(X,Y,Z)$. Recall that Z_j is an existentially quantified variable that only appears in A'. Note that A' can never be true for $Z_j = 0$. For $Z_j \neq 0$, the atom A' can be rewritten as $P_j(X,Y) < -1/Z_j^2$. From $Z_j \in [-D, D]$, we get that $Z_j^2 \leq D^2$ and therefore our considered atom A' can only ever be satisfied, if $P_j(X,Y) < -1/Z_j^2 \leq -1/D^2$. However, by the choice of j and (3), we know that $P_j(x,y) > -1/D^2$. Thus, because y was fixed arbitrarily, x must be a counterexample of Φ'. Additionally, because $x \in \mathbb{R}^n$ with $\|x^* - x\| < \delta$ was arbitrary, we conclude that all such x are counterexamples of Φ' forming an n-dimensional open ball.

Constructing Ψ from Φ'. Sentence Φ' already satisfies the condition, that its set of counterexamples is either empty or contains an open ball. The difference to Ψ as in the statement of the theorem is the range over which the variables are quantified. Also, the integers C and D in Φ' have size doubly exponentially in $|\Phi|$ and therefore in general require an exponential number of bits to be written down. Thus, a reduction from Φ to Φ' cannot be guaranteed to run in polynomial time.

We solve this problem by letting $K = \max\{N, M\}$ and applying Lemma 11 to Φ', which scales the values over which X, Y and Z are quantified down by 2^{2^K}. We obtain an equivalent formula of the form

$$\Psi := \forall X \in [-1,1]^n \, . \, \exists Y \in [-1,1]^m, Z \in [-1,1]^k, U \in [-1,1]^{K+1} : \psi(X,Y,Z,U),$$

which has the desired form (set $\ell = m + k + K + 1$ and group all existentially quantified variables into an ℓ-dimensional vector). Thus, Ψ is equivalent to Φ and $\perp(\Psi)$ is empty if Φ is true and contains an n-dimensional open ball otherwise. ◀

5 $\forall\exists_<\mathbb{R}$-Hardness

▶ **Theorem 16.** Hausdorff *and directed* Hausdorff *are* $\forall\exists_<\mathbb{R}$*-hard.*

Proof. Let $\Phi := \forall X \in \mathbb{R}^n \, . \, \exists Y \in \mathbb{R}^m : \varphi_<(X,Y)$ be an instance of Strict-UETR. We give a polynomial-time many-one reduction to an equivalent Hausdorff instance. The proof is split into three parts: First we transform Φ into an equivalent UETR instance Ψ' whose counterexamples contain an open ball (if there are any). Then we use Ψ' to define the semi-algebraic sets A and B as well as an integer t, such that (A, B, t) is a Hausdorff instance. Lastly we prove that Φ and (A, B, t) are indeed equivalent.

Transforming Φ into Ψ'. We apply Theorem 12 to Φ and obtain an equivalent sentence

$$\Psi := \forall X \in [-1,1]^n \, . \, \exists Y \in [-1,1]^\ell : \psi(X,Y)$$

in polynomial time, where $\psi(X,Y) \in$ QFF. Additionally, we get that $\perp(\Psi) = \emptyset$ if Ψ is true and that it contains an n-dimensional open ball $B_n(x,r)$ centered at some $x \in \perp(\Psi) \subseteq [-1,1]^n$ of radius $r > 0$ otherwise. We remark that Ψ is an instance of UETR and not necessarily of Strict-UETR. Using the tools from Section 3, we shall prove next, that we can give a lower bound on the radius r of the open ball of counterexamples centered at x. For this, assume that Ψ is false, so $\perp(\Psi) \neq \emptyset$ and therefore

$$\neg\Psi = \exists X \in [-1,1]^n \, . \, \forall Y \in [-1,1]^\ell : \neg\psi(X,Y)$$

is true. Utilizing our knowledge about the open ball of counterexamples around x, we can strengthen this to

$$\exists r > 0 \, . \, \exists X \in [-1,1]^n \, . \, \forall \widetilde{X} \in [-1,1]^n, Y \in [-1,1]^\ell : \|X - \widetilde{X}\|^2 < r^2 \implies \neg\psi(\widetilde{X}, Y),$$

which is still equivalent to $\neg\Psi$. Let L denote the length of the quantifier-free part of this formula. We see that L is clearly polynomial in $|\Psi|$, which by Theorem 12 is polynomial in $|\Phi|$. The above sentence has the form required to apply Lemma 9, and we get that there is an r satisfying above sentence with

$$r \geq 2^{-L^{\beta^4 n(n+\ell)}} \tag{4}$$

for some constant $\beta \in \mathbb{R}$. Let N be the smallest integer, such that

$$r \cdot 2^{2^N} > \ell. \tag{5}$$

By Equation (4), it holds that $N \in O(n(n+\ell)\log(L))$. Using Lemma 11 on Ψ and N, we can again in polynomial time create a sentence in which we scale up the range of the universally quantified variables and get

$$\Psi' := \forall X \in [-2^{2^N}, 2^{2^N}]^n . \exists Y \in [-1,1]^\ell, U \in [-1,1]^{N+1} : \psi'(X,Y,U),$$

where $\psi'(X,Y,U) \in \text{QFF}$, we have $\bot(\Psi')$ equals $\bot(\Psi)$ scaled up by 2^{2^N} in all dimensions and for all $(x,y,u) \in \mathbb{R}^{n+\ell+N+1}$ with $\psi'(x,y,u)$ we have $u_i = 2^{-2^i}$. In particular, the radius of the open ball of counterexamples around $2^{2^N} \cdot x \in \bot(\Psi')$ is now

$$r' := r \cdot 2^{2^N} > \ell$$

by the choice of N.

Defining HAUSDORFF **instance** (A, B, t). We first define three sets A', B' and C' as follows:

$$\begin{aligned}
A' &:= \left\{(x,y,u) \in [-2^{2^N}, 2^{2^N}]^n \times [-1,1]^\ell \times [-1,1]^{N+1} \;\middle|\; \psi'(x,y,u)\right\} \\
B' &:= [-2^{2^N}, 2^{2^N}]^n \times \{0\}^\ell \times \{2^{-2^0}\} \times \ldots \times \{2^{-2^N}\} \\
C' &:= \{2^{2^{N+1}}\}^{n+\ell} \times \{2^{-2^0}\} \times \ldots \times \{2^{-2^N}\}
\end{aligned}$$

Note that $A', B', C' \subseteq \mathbb{R}^{n+\ell+N+1}$ and all three sets have polynomial description complexity. We further define

$$\begin{aligned}
A &:= A' \cup C', \\
B &:= B' \cup C' \quad \text{and} \\
t &:= \ell.
\end{aligned}$$

The reason to include C' into both A and B is to guarantee that both semi-algebraic sets are non-empty. Otherwise, if $\bot(\Psi') = [-2^{2^N}, 2^{2^N}]^n$, the set A is the empty set and the Hausdorff distance between A and B would not be well-defined. The triple (A, B, t) is the desired HAUSDORFF instance.

Equivalence of Φ **and** (A, B, t). We first note that we can ignore C' in our argumentation about $d_{\mathrm{H}}(A, B)$: In fact, assuming that both A' and B' are non-empty, we have $d_{\mathrm{H}}(A, B) = d_{\mathrm{H}}(A', B')$. To prove this, observe first that adding the same set of points to A' and B' can only decrease their Hausdorff distance. Second, C' was chosen to have $d_{\mathrm{H}}(A', C') \geq d_{\mathrm{H}}(A', B')$, so for no $a \in A$, the distance to the closest $b \in B$ has decreased (and vice versa).

To see that Φ and (A, B, t) are equivalent, assume first that Φ is true. Let $u \in [-1,1]^{N+1}$ such that $u_i = 2^{-2^i}$. As seen above, this is necessary in every satisfying assignment of the variable vector U in Ψ'. Then for every $x \in [-2^{2^N}, 2^{2^N}]^n$ there is at least one $y \in [-1,1]^\ell$ such that $a = (x, y, u) \in A$. At the same time, $b = (x, \{0\}^\ell, u) \in B$. We get

$$\|a - b\| = \|(x, y, u) - (x, \{0\}^\ell, u)\| = \|y - \vec{0}\| \leq \sqrt{\sum_{i=1}^{\ell} 1} = \sqrt{\ell} \leq \ell = t.$$

As x was chosen arbitrarily, we get an upper bound for the directed Hausdorff distance $\vec{d}_{\mathrm{H}}(A, B) \leq \ell$. On the other hand, for every $b = (x, \{0\}^\ell, u) \in B$ there is an $y \in [-1,1]^\ell$ such that $a = (x, y, u) \in A$, as we assume that Φ is true. As above, we get $\vec{d}_{\mathrm{H}}(B, A) \leq \ell$ and thus

$$d_{\mathrm{H}}(A, B) \leq \ell = t. \tag{6}$$

Now assume that Φ is false. By construction Ψ' is also false and contains a counterexample $x \in \bot(\Psi')$ such that $B_n(x, r') \subseteq \bot(\Psi')$. Consider $b = (x, \{0\}^\ell, u) \in B$. Since Ψ' is false, for no $\widetilde{x}$ with $\|x - \widetilde{x}\| < r'$ and no $y \in [-1,1]^\ell$ there is a point $a = (\widetilde{x}, y, u) \in A$. We conclude

$$d_{\mathrm{H}}(A, B) \geq \vec{d}_{\mathrm{H}}(B, A) \geq r' > \ell = t. \tag{7}$$

Equations (6) and (7) prove that $d_{\mathrm{H}}(A, B) \leq t$ (and $\vec{d}_{\mathrm{H}}(B, A) \leq t$) if and only if Φ is true. ◀

In the proof of Theorem 1, we could choose $N' := N + 1$ instead of N in Equation (5). Then in the case that Φ is false, the Hausdorff distance is at least

$$r' > 2^{2^{N+1}} r > 2^{2^{N+1} - 2^N} \ell = 2^{2^N} \ell = 2^{2^N} t.$$

Note that the dimension d of the resulting sets A, B equals $d = n + \ell + N' + 1 = \Theta(N)$. Thus, we created a gap of size $2^{2^{\Theta(d)}}$. This implies the following inapproximability result.

▶ **Corollary 2.** *Let A and B be two semi-algebraic sets in $\mathbb{R}^d$ and $f(d) = 2^{2^{o(d)}}$. Then there is no $f(d)$-approximation algorithm to compute $d_{\mathrm{H}}(A, B)$, unless* $\mathsf{P} = \forall\exists_{<}\mathbb{R}$.

6 $\forall\exists_{<}\mathbb{R}$-Membership

This section is devoted to show the following theorem.

▶ **Theorem 17.** *Deciding whether two semi-algebraic sets A and B have (directed) Hausdorff distance at most $t \in \mathbb{Q}$ is in $\forall\exists_{<}\mathbb{R}$.*

Note that $\forall\exists\mathbb{R}$-membership is already shown by Dobbins et al. [26]. Proving Theorem 17 requires some preliminary work. Recently, D'Costa et al. [24] showed the following.

▶ **Proposition 18** ([24, Theorem 3])**.** *It is $\exists\forall_{\leq}\mathbb{R}$-complete to decide sentences of the form*

$$\exists X \in [-1,1]^n \,.\, \forall Y \in [-1,1]^m : \varphi_{\leq}(X, Y),$$

where $\varphi_{\leq}(X, Y) \in \mathrm{QFF}_{\leq}$.

Observing that co-$\exists\forall_{\leq}\mathbb{R} = \forall\exists_{<}\mathbb{R}$ and that $\neg\varphi_{\leq}(X, Y) \in \mathrm{QFF}_{<}$, we can state the following corollary. We denote the corresponding decision problem as Bounded-Strict-UETR.

▶ **Corollary 19.** *It is $\forall\exists_{<}\mathbb{R}$-complete to decide the truth of sentences of the form*

$$\forall X \in [-1,1]^n \,.\, \exists Y \in [-1,1]^m : \varphi_{<}(X, Y),$$

where $\varphi_{<}(X, Y) \in \mathrm{QFF}_{<}$.

The remainder of this section deals with reformulating a Hausdorff instance into a Bounded-Strict-UETR instance, so into the form of Corollary 19. Let (A, B, t) be a Hausdorff instance, where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ are described by quantifier-free formulas $\Phi_A(X)$ and $\Phi_B(X)$ with a vector X of n free variables each. For simplicity, we only consider the directed Hausdorff distance here, namely the question whether

$$\vec{d}_{\mathrm{H}}(A, B) := \sup_{a\in A} \inf_{b\in B} \|a - b\| \overset{?}{\leq} t.$$

It is obvious, that $d_{\mathrm{H}}(A, B) \leq t$ if and only if $\vec{d}_{\mathrm{H}}(A, B) \leq t$ and $\vec{d}_{\mathrm{H}}(B, A) \leq t$. So if we can formulate the decision problem for the directed Hausdorff distance as a Bounded-Strict-UETR instance, their conjunction is a formula for the general Hausdorff problem. Assuming, that no variable name appears in both operands of this conjunction, this formula can be converted into prenex normal form by just moving the quantifiers to the front.

From the definition of the directed Hausdorff distance we can formulate

$$\vec{d}_{\mathrm{H}}(A, B) \leq t \quad \equiv \quad \forall \varepsilon > 0, a \in A \,.\, \exists b \in B : \|a - b\|^2 < (\varepsilon + t)^2. \tag{8}$$

Let us remark that introducing the real variable ε is necessary to also consider the points in the closures of A and B. Moreover, we work with the squared distance between a and b, because the Euclidean norm is a square root of a sum of squares; and the syntax of first-order formulas does not allow square roots.

Below we will transform formula (8) in multiple technical steps into a form such that we can apply Corollary 19. Before we do this, we need a few helpful lemmas. The first lemma establishes a bound on the image of a polynomial with compact domain.

▶ **Lemma 20** ([24, Lemma 18]). *Given a multivariate polynomial $P : [-1, 1]^n \to \mathbb{R}$, we can compute in polynomial time an integer B such that for all $x \in [-1, 1]^n$ we have $|P(x)| \leq B$.*

The second lemma helps to compute a Bounded-Strict-UETR instance from a UETR instance of special structure.

▶ **Lemma 21.** *Given a multivariate polynomial $H : [-1, 1]^{n+m} \to \mathbb{R}$ and a* UETR *instance*

$$\Phi := \forall X \in [-1, 1]^n \,.\, \exists Y \in [-1, 1]^m : \varphi_<(X, Y) \vee H(X, Y) = 0,$$

with $\varphi_<(X, Y) \in \mathrm{QFF}_<$. We can compute an equivalent Bounded-Strict-UETR *instance in polynomial time.*

Proof. We start by proving that there is an integer N polynomial in $|\Phi|$, such that for every fixed $\varepsilon < 2^{-2N}$ the Strict-UETR instance

$$\Psi := \forall X \in [-1, 1]^n \,.\, \exists Y \in [-1, 1]^m : \varphi_<(X, Y) \vee H(X, Y)^2 < \varepsilon$$

is equivalent to Φ. In a second step, we encode ε into the formula.

The direction $\Phi \implies \Psi$ is trivial. To prove $\Psi \implies \Phi$ we show its contraposition $\neg\Phi \implies \neg\Psi$. We start with considering $\neg\Phi$ and have that

$$\neg\Phi \equiv \exists X \in [-1, 1]^n \,.\, \forall Y \in [-1, 1]^m : \neg\varphi_<(X, Y) \wedge H(X, Y)^2 > 0$$

is true. Thus for at least one $x \in [-1, 1]^n$ we have for all $y \in [-1, 1]^m$ that $H(x, y)^2 > 0$ and that therefore $H(x, Y)^2$ (a polynomial in Y with real coefficients) is positive everywhere on its domain $[-1,1]^m$. Because $[-1,1]^m$ is compact $H(x,Y)^2$ attains its minimum and it follows that

$$\exists \varepsilon > 0 \,.\, \exists X \in [-1,1]^n \,.\, \forall Y \in [-1,1]^m : \neg\varphi_<(X,Y) \wedge H(X,Y)^2 \geq \varepsilon \tag{9}$$

is true. Let L be the length of the subformula in (9) behind the quantification of ε. Using Lemma 9, we get a lower bound on ε of

$$\varepsilon \geq 2^{-L^{\beta^4 nm}} \tag{10}$$

for some constant $\beta \in \mathbb{R}$. Let N be the smallest integer, such that $2^{-2^N} \leq \varepsilon$. Note that $N \in O(nm \log L)$, so it is polynomial in the input size. Formula (9) and Equation (10) imply $\neg\Psi$ and we conclude that Φ and Ψ are indeed equivalent.

To construct a Bounded-Strict-UETR instance, we need to express ε inside the formula. For $N+1$ new variables $U = (U_0, \dots, U_N) \in [-1,1]^{N+1}$, let $\chi(U)$ be the formula from Lemma 10. Recall that for every $u \in [-1,1]^{N+1}$ where $\chi(u)$ is true we have $u_i = 2^{-2^i}$. Including $\chi(U)$ into our formula, we can conclude that

$$\forall X \in [-1,1]^n, U \in [-1,1]^{N+1} \,.\, \exists Y \in [-1,1]^m : \neg\chi(U) \vee \neg\varphi_<(X,Y) \vee H(X,Y)^2 < U_N$$

is equivalent to Φ. Note that this formula has polynomial length and $\neg\chi(U) \in \mathrm{QFF}_<$, so it is a Strict-UETR instance. ◀

Now we have all the tools needed to prove Theorem 17 which states that the Hausdorff problem is in $\forall\exists_<\mathbb{R}$. We do this by transforming formula (8) in several steps into the form required by Lemma 21. This in turn yields an equivalent Bounded-Strict-UETR instance. By Corollary 19 this shows $\forall\exists_<\mathbb{R}$-membership.

Proof of Theorem 17. In a first step, we resolve the shorthand notations $a \in A$ and $b \in B$ in formula (8) and we obtain

$$\Psi_1 := \forall \varepsilon \in \mathbb{R}, a \in \mathbb{R}^n : (\varepsilon > 0 \wedge \Phi_A(a)) \implies \big(\exists b \in \mathbb{R}^n : \Phi_B(b) \wedge \|a-b\|^2 < (\varepsilon + t)^2\big).$$

Next, we consider the (quantifier-free) subformulas $\Phi_A(a)$ and $\Phi_B(b)$. Using Lemma 4 we obtain in polynomial time two polynomials $F_A : \mathbb{R}^{n+k} \to \mathbb{R}$ and $F_B : \mathbb{R}^{n+\ell} \to \mathbb{R}$, such that $\Phi_A(a)$ is equivalent to $\exists U_a \in \mathbb{R}^k : F_A(a, U_a) = 0$ and similarly $\Phi_B(b)$ is equivalent to $\exists U_b \in \mathbb{R}^\ell : F_B(b, U_b) = 0$. This yields the equivalent formula

$$\begin{aligned}\Psi_2 := \forall \varepsilon \in \mathbb{R}, a \in \mathbb{R}^n : (\varepsilon > 0 \wedge \exists U_a \in \mathbb{R}^k : F_A(a, U_a) = 0) \implies \\ \big(\exists b \in \mathbb{R}^n, U_b \in \mathbb{R}^\ell : F_B(b, U_b) = 0 \wedge \|a-b\|^2 < (\varepsilon + t)^2\big).\end{aligned}$$

We rewrite the condition that ε is positive by introducing a new variable and observing that $\varepsilon > 0$ is equivalent to $\exists S \in \mathbb{R} : S^2\varepsilon - 1 = 0$:

$$\begin{aligned}\Psi_3 := \forall \varepsilon \in \mathbb{R}, a \in \mathbb{R}^n : (\exists S \in \mathbb{R} : S^2\varepsilon - 1 = 0 \wedge \exists U_a \in \mathbb{R}^k : F_A(a, U_a) = 0) \implies \\ \big(\exists b \in \mathbb{R}^n, U_b \in \mathbb{R}^\ell : F_B(b, U_b) = 0 \wedge \|a-b\|^2 < (\varepsilon + t)^2\big).\end{aligned}$$

Rewriting the implication $X \implies Y$ as $\neg X \vee Y$ changes the existential quantifiers for S and U_a into universal quantifiers, which we can move to the front. Also, the two equations get negated. Substituting $\neg(X = 0)$ by the equivalent $X^2 > 0$ we get

$$\begin{aligned}\Psi_4 := \forall \varepsilon \in \mathbb{R}, a \in \mathbb{R}^n, S \in \mathbb{R}, U_a \in \mathbb{R}^k : \ (S^2\varepsilon - 1)^2 > 0 \vee F_A(a, U_a)^2 > 0 \vee \\ \big(\exists b \in \mathbb{R}^n, U_b \in \mathbb{R}^\ell : F_B(b, U_b) = 0 \wedge \|a-b\|^2 < (\varepsilon + t)^2\big).\end{aligned}$$

Moving the existential quantifiers behind the universal ones yields a sentence in prenex normal form:

$$\Psi_5 := \forall \varepsilon \in \mathbb{R}, a \in \mathbb{R}^n, S \in \mathbb{R}, U_a \in \mathbb{R}^k . \exists b \in \mathbb{R}^n, U_b \in \mathbb{R}^\ell : \\ (S^2\varepsilon - 1)^2 > 0 \vee F_A(a, U_a)^2 > 0 \vee \big(F_B(b, U_b) = 0 \wedge \|a - b\|^2 < (\varepsilon + t)^2\big)$$

At this point we abstract from the details and introduce several abbreviations to highlight the overall structure of this formula. We define $n_6 := 1 + n + 1 + k$, $m_6 := n + \ell$ and

$$\begin{aligned} X &:= (\varepsilon, a, S, U_a) \in \mathbb{R}^{n_6}, \\ Y &:= (b, U_b) \in \mathbb{R}^{m_6}, \\ \varphi_<(X) &:= \big((S^2\varepsilon - 1)^2 > 0 \vee F_A(a, U_a)^2 > 0\big) \in \mathrm{QFF}_<, \\ F_B(Y) &:= F_B(b, U_b), \\ G(X, Y) &:= \|a - b\|^2 - (\varepsilon + t)^2 \end{aligned}$$

and can now write Ψ_5 as

$$\Psi_6 := \forall X \in \mathbb{R}^{n_6} . \exists Y \in \mathbb{R}^{m_6} : \varphi_<(X) \vee (F_B(Y) = 0 \wedge G(X, Y) < 0).$$

We intend to use Lemma 21. To this end, we reformulate $G < 0$ into an equation $G' = 0$ using familiar tools. Then we can merge $F_B = 0$ and $G' = 0$ into $H = F_B^2 + G'^2 = 0$.

We observe that Ψ_6 is $\forall$-strict. This allows us to apply first Lemma 13 bounding the range of the universally quantified variables and then Lemma 14 dealing with the existentially quantified variables. The lemmas give us two integers N and M polynomial in $|\Psi_6|$, such that for $C := 2^{2^N}$ and $D := 2^{2^M}$ we can restrict the ranges of the universally and existentially quantified variables to $[-C, C]$ and $[-D, D]$, respectively. For $n_7 := n_6$ and $m_7 := m_6$ we get

$$\Psi_7 := \forall X \in [-C, C]^{n_7} . \exists Y \in [-D, D]^{m_7} : \varphi_<(X) \vee (F_B(Y) = 0 \wedge G(X, Y) < 0).$$

Now recall that $G(X, Y) < 0$ is a shorthand for $\|a - b\|^2 < (\varepsilon + t)^2$. Using that ε is universally quantified, we observe that instead of $<$ we could equivalently use $\leq$ in this atom. Substituting $<$ by $\leq$ and defining $n_8 := n_7$ and $m_8 := m_7$ yields

$$\Psi_8 := \forall X \in [-C, C]^{n_8} . \exists Y \in [-D, D]^{m_8} : \varphi_<(X) \vee (F_B(Y) = 0 \wedge G(X, Y) \leq 0).$$

Using Lemma 11 we can further restrict the ranges of the variables to $[-1, 1]$ at the cost of adding a few additional (universally quantified) variables. Thus for $n_9 \in O(n_8 + |N| + |M|)$ and $m_9 = m_8$, sentence Ψ_8 is equivalent to

$$\Psi_9 := \forall X \in [-1, 1]^{n_9} . \exists Y \in [-1, 1]^{m_9} : \varphi'_<(X) \vee (F'_B(Y) = 0 \wedge G'(X, Y) \leq 0),$$

where $\varphi'_<$, F'_B and G' are obtained from $\varphi_<$, F_B and G as described in Lemma 11 (note that $\varphi'_<$ also includes the subformula $\neg\chi$ which constructs the scaling factor).

The next step is to replace $G'(X, Y) \leq 0$ in Ψ_9 by $\exists Z \in \mathbb{R} : G'(X, Y) + Z^2 = 0$. From Lemma 20 we know that we can compute in polynomial time an integer B such that for all $(x, y) \in [-1, 1]^{n_9 + m_9}$ we have $|G(x, y)| < B$. We conclude that Z can actually be restricted to be chosen from $[-B, B]$. Again, with Lemma 11, we can even restrict Z to be from $[-1, 1]$ by adding a few more universally quantified variables, which are exclusively used for scaling. For $n_{10} \in O(n_9 + |B|)$, $m_{10} = m_9 + 1$ we obtain

$$\Psi_{10} := \forall X \in [-1, 1]^{n_{10}} . \exists Y \in [-1, 1]^{m_{10}} : \varphi''_<(X) \vee (F''_B(Y) = 0 \wedge G''(X, Y) = 0),$$

where again $\varphi''_<$, F''_B and G'' are obtained from $\varphi'_<$, F'_B and G' as described in Lemma 11 (and $\varphi''_<$ contains the subformula $\neg\chi$ which constructs the scaling factor).

Lastly, we define a polynomial $H(X, Y) := F''_B(Y)^2 + G''(X, Y)^2$ and get for $n_{11} = n_{10}$ and $m_{11} = m_{10}$ a formula

$$\Psi_{11} := \forall X \in [-1, 1]^{n_{11}} . \exists Y \in [-1, 1]^{m_{11}} : \varphi''_<(X) \vee H(X, Y) = 0,$$

which has exactly the form required by Lemma 21. We apply it to get an equivalent Bounded-Strict-UETR instance. Since Bounded-Strict-UETR is $\forall\exists_<\mathbb{R}$-complete, we conclude that Hausdorff is in $\forall\exists_<\mathbb{R}$. ◀

7 Open Problems

We showed that the Hausdorff problem is $\forall\exists_<\mathbb{R}$ complete. One important open question asks if the two complexity classes $\forall\exists\mathbb{R}$ and $\forall\exists_<\mathbb{R}$ are actually the same. An answer to this question is interesting in its own right. Furthermore, it is interesting to see if our hardness result can be extended to simpler settings. For instance, we wonder about the complexity of the Hausdorff problem for *algebraic* sets or *compact semi-algebraic* sets. Additionally, we wonder if we can also prove hardness for the case that two semi-algebraic sets have Hausdorff distance 0, i.e., if they have the same closure. Consequently, this question is closely related to the problem of deciding whether a given point is contained in the closure of a semi-algebraic set. Similarly interesting is the question if a semi-algebraic set is equal to its closure.

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