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Dimension and Linear Layouts of Posets

Master Thesis of

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Statement of Authorship

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Abstract

We consider the Dushnik-Miller and the Boolean dimension of posets whose diagrams are structurally restricted with regard to their queue or stack number. We determine explicit upper bounds on the Dushnik-Miller dimension in terms of queue number and height, and similarly in terms of stack number and height based on the work of Joret et al.

Showing that every directed acyclic graph admits a 2-queue and a 3-stack-subdivision, we conclude that there is no polynomial function which bounds Dushnik-Miller dimension in terms of queue number and height and the same is true for stack number and height.

Finally, we prove that Boolean dimension is bounded by an exponential function in queue number and height and determine an upper bound on the Boolean dimension of subdivisions.

Deutsche Zusammenfassung

Wir setzen die Dushnik-Miller und die Boolsche Dimension von Posets in Bezug zur Queue Number und Stack Number ihrer Hasse Diagramme. Wir berechnen explizite obere Schranken für Dushnik-Miller Dimension durch Höhe, Queue Number und Stack Number mittels der Arbeiten von Joret et al.

Indem wir nachweisen, dass jeder gerichtete, azyklische Graph eine 2-Queue- und eine 3-Stack-Unterteilung besitzt, können wir beweisen, dass Dushnik-Miller Dimension nicht polynomiell in Queue Number und Höhe beschränkt ist und das Gleiche auch für Stack Number und Höhe gilt.

Des Weiteren berechnen wir eine exponentielle obere Schranke in Queue Number und Höhe für Boolsche Dimension und zeigen eine obere Schranke für die Boolsche Dimension von Unterteilungen von Posets.

Contents

1 Introduction

Partially ordered sets (or posets) are ubiquitous in combinatorics, logic, theoretical computer science and beyond. It is well-known that every poset has an associated Hasse diagram (or simply diagram), a certain directed acyclic graph that faithfully represents all relations of the poset.

This raises the natural question how the "complexity" of a poset and the "complexity" of its diagram are related. To make this question precise, we consider different complexity measures for posets and graphs and try to relate them, for example by bounding them in terms of each other.

The first complexity measure of posets we consider is Dushnik-Miller dimension, introduced in 1941 [24]. Well-structured posets are believed to have small dimension. Yet, a construction of Kelly shows that the Dushnik-Miller dimension of planar posets, posets whose diagrams can be drawn without any edge-crossings, is unbounded [64]. Realizing that the height of the planar posets which had been constructed by Kelly increases with the Dushnik-Miller dimension, Streib and Trotter were able to bound the Dushnik-Miller dimension of planar posets in terms of their height [85]. This result led to numerous publications, each of which bounds the Dushnik-Miller dimension of posets in terms of their height and some graph parameter of their diagrams [56, 58, 66, 71, 93]. Most notably, Joret, Micek, and Wiechert showed that the Dushnik-Miller dimension of any family of posets whose diagrams are sparse is bounded in terms of their height [59].

We are interested in two graph parameters which are defined via a minimization problem for vertex orderings. If we align the vertices of a graph along a horizontal line and draw the edges as half-circles above this line, we obtain a drawing of the graph where some edges cross and others nest. Considering edge partitions based on these two properties, we can define the stack number and the queue number of undirected graphs. The stack number is a graph parameter which has first been introduced by Bernhart and Kainen [8], the queue number later on by Heath and Rosenberg [48]. Since then, the concept has also been applied to directed acyclic graphs [45, 47, 60, 61] and posets in particular [33, 77, 86]. In the directed setting, we only consider topological vertex orderings. This is why results which hold in the undirected setting might not hold in the directed setting and vice versa.

Joret, Micek, and Wiechert showed that the Dushnik-Miller dimension of posets is in particular bounded in terms of their height and queue number, and the same is true if we replace queue number by stack number. We give explicit bounds which follow from their observations. Further, we are able to construct posets which show that the upper bound is exponential both in queue number and height, and the same holds for stack number.

As we are able to prove that every directed acyclic graph admits a 2-queue subdivision and a 3-stack-subdivision (facts which had already been known for undirected graphs [2, 23]), it follows easily that Dushnik-Miller dimension is not bounded in terms of queue or stack number, i.e. the bound established by Joret, Micek, and Wiechert does not hold independent of height.

For undirected graphs, upper bounds on the queue and stack number of subdivisions in terms of the queue and stack number of the initial graph have already been known [23]. We show similar bounds in the directed setting and determine a lower bound on the queue number of subdivisions where every edge has been subdivided at most h times in terms of the queue number of the initial directed acyclic graph and h. The latter result is similar to a bound established by Dujmović and Wood in the undirected setting [23].

Finally, we relate Boolean dimension, a generalization of Dushnik-Miller dimension which was introduced by Gambosi, Nešetřil, and Talamo [37], to queue and stack number. As the Boolean dimension of a poset does not exceed its Dushnik-Miller dimension, the bounds established by Joret, Micek, and Wiechert also apply to Boolean dimension. Yet, we are able to improve the upper bound on Boolean dimension in terms of queue number and height.

A construction given by Spinrad shows that the Dushnik-Miller dimension of subdivisions is not bounded in terms of the Dushnik-Miller dimension of the initial poset [83]. The same is true for Boolean dimension. Using a similar approach to the one given by Spinrad for Dushnik-Miller dimension [83], we determine an upper bound on the Boolean dimension of subdivisions in terms of the height and the Boolean dimension of the initial poset.

Outline

We first introduce the terminology of posets and Dushnik-Miller dimension in Section 2.1 and define queue and stack number in Section 2.2. In Section 2.3, we show that the stack number of posets is not bounded in terms of Dushnik-Miller dimension. Section 2.4 is dedicated to variants of the Erdős-Szekeres Theorem, which gives bounds on the minimum length of a monotone subsequence of an arbitrary sequence of real numbers. Using the Erdős-Szekeres Theorem, we show in Section 2.5 that the queue number of posets is not bounded by Dushnik-Miller dimension. In Section 2.6, we state results related to Dushnik-Miller dimension, queue and stack number, which apply to planar posets. Section 2.7 explores bounds on the Dushnik-Miller dimension for posets of queue or stack number at most 1. Section 2.8 includes linear upper bounds on the queue and stack number of subdivisions of directed acyclic graphs in terms of the queue and stack number of the initial graph and a lower bound on the queue number of subdivisions where every edge has only been subdivided a constant number of times. In Section 2.9, we restate results given by Spinrad regarding the Dushnik-Miller dimension of subdivisions. In Section 2.10, we show that each directed acyclic graph admits a 2-queue and a 3-stack subdivision and conclude that Dushnik-Miller dimension is neither bounded in terms of queue number nor in terms of stack number. Section 2.11 includes explicit bounds on the Dushnik-Miller dimension in terms of height and queue or stack number which follow from the work of Joret, Micek, and Wiechert. In Section 2.12, we construct posets which show that any bound on Dushnik-Miller dimension is exponential in height, queue and stack number.

In Chapter 3, we consider Boolean dimension instead of Dushnik-Miller dimension. Section 3.1 gives an introduction to Boolean dimension, with a focus on the relation between Dushnik-Miller and Boolean dimension. In Section 3.2, we relate Boolean dimension to queue and stack number. We compute the queue and stack number of a family of posets of unbounded Boolean dimension, give upper bounds on the Boolean dimension of height-2 posets in terms of queue number and stack number and establish an upper bound on Boolean dimension in terms of queue number and height. In Section 3.3, we introduce separated queue and stack layouts for directed acyclic graphs based on the work of Pemmaraju. We explain why a similar approach to the one given in the previous section does not yield a bound on Boolean dimension in terms of stack number and height. Section 3.4 focuses on upper bounds on the Boolean dimension of subdivisions in terms of the height and the Boolean dimension of the initial poset.

2 Dushnik-Miller Dimension

2.1 Introduction to Dimension

We only consider finite posets and graphs.

Definition 2.1.1. A *partially ordered set* P, abbreviated by *poset*, is a finite, non-empty set X together with a binary relation $\leq_P$ on X which is transitive, antisymmetric and reflexive. For two elements $x, y \in X$, we write $x \parallel_P y$ if neither $x \leq_P y$ nor $y \leq_P x$. In this case, we say that x and y are *incomparable*. Otherwise, x and y are called *comparable*.

A *subposet* of P is a non-empty subset of the elements of P with the inherited relation. A *chain* is a total order. An *antichain* is a poset with no comparability relations. The *size* of a chain and an antichain corresponds to the number of elements.

The *height* of a poset is the maximum size of a chain that is a subposet.

We can easily represent posets as directed graphs. Elements are represented by vertices while edges correspond to pairs of comparable elements. Orienting the edges, we are able to represent the respective order of the endpoints.

Definition 2.1.2. The *comparability digraph* of a poset P, denoted by $\mathrm{Comp}(P)$, is the directed graph on the elements of P where for distinct elements $x, y \in X$, we have that xy is an edge if and only if $x \leq_P y$. The *incomparability graph* of P, denoted by $\mathrm{Inc}(P)$, is defined as the undirected graph on the elements of P with edges corresponding to incomparable pairs.

As partial orders are antisymmetric and transitive, we see that comparability digraphs are acyclic. Thus, they are directed acyclic graphs, which we call *dags*.

A comparability digraph contains many edges representing relations which follow by transitivity. For example, the comparability digraph of a chain is an orientation of a complete graph, even though a chain has a simple structure.

We wish to consider a simpler representation of a poset which is obtained by deleting some of the edges of the comparability digraph.

Definition 2.1.3. We say that an edge ab of a dag G is *transitive* if there exists a vertex c such that $ac, cb \in E(G)$.

The graph G is called *transitive* if whenever ac, cb are edges of G, then G also contains the edge ab.

Note that the comparability digraph of a poset P is transitive. Relations which follow from transitivity correspond precisely to transitive edges of $\mathrm{Comp}(G)$. Deleting these edges, we obtain a smaller directed graph. This enables us for example to represent a chain as a directed path instead of an orientation of a complete graph.

Definition 2.1.4. The *diagram* of a poset P, also called *Hasse diagram*, is the directed graph on the elements of P that consists of all non-transitive edges of the comparability graph. The *cover graph* $\mathrm{Cov}(P)$ is the underlying undirected graph of the diagram.

The transitive closure of the relations represented by the diagram of a poset yields the poset itself.

Diagrams are subgraphs of comparability digraphs. Thus, they are also dags. In fact, dags with no transitive edges are in a one-to-one correspondence with posets defined on the same vertex set. The bijection is given by the function which maps every poset to its diagram.

As posets are partial orders, they can be extended to total orders which are also called linear orders. Such an extension is referred to as a linear extension. It can be constructed iteratively by choosing an arbitrary order of the minimal elements. Using the same approach on the poset we obtain after deleting all minimal elements, we get a linear extension of the initial poset by induction.

Definition 2.1.5. A poset is called a *linear order* or *chain* if all pairs of elements are comparable. We often represent a linear order L on elements $a_1, \ldots, a_n$ as

$$a_1 \leq_L a_2 \leq_L \cdots \leq_L a_n.$$

The *reversed linear order* of L, denoted by L^{rev}, corresponds to the poset defined by

$$a_n \leq_{L^{\mathrm{rev}}} a_{n-1} \leq_{L^{\mathrm{rev}}} \cdots \leq_{L^{\mathrm{rev}}} a_1.$$

If L_1 and L_2 are linear orders on disjoint sets X and Y, we denote by $L_1 \leq L_2$ the linear order on $X \cup Y$ obtained by preserving the comparability relations of L_1 and L_2 and defining $\ell_1 \leq \ell_2$ for all elements ℓ_1 of L_1 and all elements ℓ_2 of L_2.

We call a poset E a *linear extension* of a poset P, if E is a linear order on the elements of P and all comparability relations of P are preserved in E.

In fact for every ordered pair of incomparable elements (x, y) of a poset P, there exists a linear extension L of P such that $y \leq_L x$. Thus, a poset is the intersection of all its linear extensions [87, p. 9] where we use the definition of intersection for binary relations on the same set of elements.

Usually, a poset is the intersection of a relatively small number of linear extensions when compared to the total number of linear extensions; think for example of an antichain of size n. There are $n!$ linear extensions, but any linear order of the elements together with the reversed order yields the antichain.

Definition 2.1.6. A non-empty set $\mathcal{R}$ of linear extensions is a *Dushnik-Miller realizer* of a poset P if for every pair of incomparable elements (x, y), there exists a linear extension $L \in \mathcal{R}$ that reverses (x, y), i.e. $y \leq_L x$. When clear from context, we may refer to a Dushnik-Miller realizer as a realizer of P.

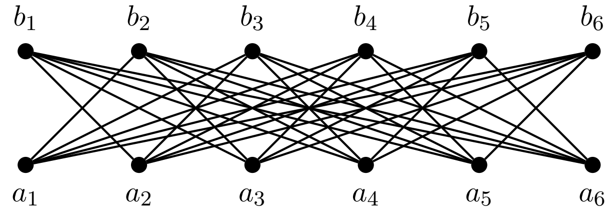


Figure 2.1: The diagram of a standard example of size 6

In other words, a realizer of a poset P is a set of linear extensions whose intersection yields P. As the intersection of all linear extensions is a realizer, we obtain the following.

Lemma 2.1.7 ([24, Theorem 2.32], [87, p. 9])**.** *Every poset admits a Dushnik-Miller realizer.*

We are interested in the minimum size of a realizer, which is called the dimension of a poset. The concept is due to Dushnik and Miller [24]. Their work founded a branch of combinatorics, referred to as Dimension theory, an introduction to which can be found in [87].

Definition 2.1.8 ([87, p. 9])**.** The Dushnik-Miller dimension of a poset P is the size of a smallest realizer of P.

Note that Dushnik-Miller dimension is well-defined by Lemma 2.1.7. The nomenclature might reflect the fact that every n-dimensional poset can be faithfully represented by a set of points in the n-dimensional real space [96, p. 1]. In Chapter 3, we introduce a different notion of dimension. Nevertheless, when there is no ambiguity, we refer to Dushnik-Miller dimension as dimension.

Clearly, the dimension of a linear order is 1. The dimension of an antichain of size at least 2 is 2; it suffices to consider any linear extension and the corresponding reversed linear order. In general, the dimension of a poset can be arbitrarily large. A well-known family of posets of large dimension is the family of standard examples which was introduced by Dushnik and Miller [24].

Definition 2.1.9 ([24, p. 604], [87, p. 12])**.** For $d \in \mathbb{N}$, let S_d be the poset on the $2d$-element set

$$\{a_1, \dots, a_d, b_1, \dots, b_d\}$$

where $a_i \leq b_j$ for all $i, j \in [d]$ with $i \neq j$, see Figure 2.1. The poset S_d is called the *standard example* of size d.

Indeed, the dimension of standard examples is unbounded.

Lemma 2.1.10 ([24, p. 604], [87, p. 12])**.** *The standard example of size d is a d-dimensional poset for any $d \geq 2$ and 2-dimensional for $d = 1$.*

Proof. The standard example of size 1 is an antichain, thus its dimension is 2.

Consider a standard example of size at least 2 and define for $i \in [d]$ the linear orders

$$L_i \coloneqq a_1\, a_2 \ldots\, a_{i-1}\, a_{i+1} \ldots a_d\, b_i\, a_i\, b_1 \ldots b_{i-1}\, b_{i+1} \ldots b_d.$$

Note that the linear orders $L_1, \ldots, L_d$ form a realizer of S_d. Thus, we obtain

$$\dim(S_d) \leq d.$$

We claim that every linear extension of S_d reverses at most one incomparable pair (a_i, b_i). If L is a linear extension of S_d which reverses a pair (a_i, b_i), we have $b_i \leq_L a_i$. As $a_j \leq b_i$ and $a_i \leq b_j$ for all $j \neq i$. we see that L reverses no other incomparable pair (a_j, b_j).

Suppose $\dim(S_d) < d$ and consider a realizer of minimum size. By pigeon-hole principle, at least one of the linear extensions of the realizer reverses at least two of the incomparable pairs (a_i, b_i). This yields a contradiction and $\dim(S_d) \geq d$ follows. $\square$

Hiraguchi showed that the dimension of any poset on n vertices is at most $\frac{n}{2}$ [50] [87, Theorem 10.8]. The standard example shows that this bound is tight.

As the restriction of a linear extension to a subposet is a linear extension of the latter, we see that dimension is a monotone property.

Lemma 2.1.11 ([87, p. 12])**.** *For every subposet Q of a poset P, we have*

$$\dim(Q) \leq \dim(P).$$

This property is often used to show that a poset has large dimension. For example, every poset which contains a standard example of size n has dimension at least n.

In general, it is much more difficult to determine the dimension of a poset. While there are efficient algorithms that recognize 1- and 2-dimensional posets, it is $\mathcal{NP}$-complete to determine whether a poset has dimension 3 [96].

Instead of giving an explicit construction of a realizer, a technique introduced by Rabinovitch and Rival is often used instead [80]. It suffices to identify specific incomparable pairs of the poset which are called critical pairs. Rabinovitch and Rival showed that a family $\mathcal{R}$ of linear extensions of a poset is a realizer, if and only if it contains for every critical pair a linear order which reverses its order. For a detailed explanation, the reader may consult [87, pp. 29–31].

Posets of dimension 2 are well-understood. They are characterized by a result of Dushnik and Miller, see Lemma 2.1.13, which relies on the following observation. Any transitive orientation of the incomparability graph defines a poset on the elements of P. Such orientations of $\mathrm{Inc}(P)$ and the comparability digraph of P are compatible in the following sense.

Lemma 2.1.12 ([24, Lemma 3.51])**.** *For any transitive orientation I of the incomparability graph of P, there exists a linear extension L of P such that for any two incomparable elements a and b of P, we have $a \leq_L b$ if and only if $ab \in E(I)$.*

Proof. Consider the complete graph K on the elements of P. For any two distinct elements a, b of P, we orient the edge $\{a, b\}$ of K from a to b if $a \leq_P b$. If $a \parallel b$ and the edge $\{a, b\}$ is oriented from a to b in the given transitive orientation I, we do the same. We obtain an orientation $\overrightarrow{K}$ of K and claim that $\overrightarrow{K}$ is transitive.

Suppose there are edges $ab, bc \in E(\overrightarrow{K})$. We need to show that $ac \in E(\overrightarrow{K})$. If $a \leq_P b$ and $b \leq_P c$, we see by transitivity that $a \leq_P c$, thus $ac \in E(\overrightarrow{K})$. Similarly, we obtain $ac \in E(\overrightarrow{K})$ if $a \parallel b$ and $b \parallel c$ as the orientation I is transitive. If one of the edges ab and bc is in $\mathrm{Inc}(P)$, while the other is an edge of the comparability digraph of P, we distinguish several cases.

Case 1. $a \leq_P b$, $b \parallel c$ and $a \parallel c$. Note that $bc \in E(I)$. If $ac \in E(I)$, we are done. Otherwise $ca \in E(I)$, and $ba \in E(I)$ follows by transitivity, which yields a contradiction.

Case 2. $a \leq_P b, b \parallel c$ and $a \nparallel c$. If $a \leq_P c$, we are done. Otherwise $c \leq_P a$, and by transitivity $c \leq_P b$, which yields a contradiction.

If $a \parallel b$, we proceed in a similar way. In each case, we obtain that $ac \in E(\overrightarrow{K})$. Thus, $\overrightarrow{K}$ is indeed a transitive orientation of K. We can interpret $\overrightarrow{K}$ as the comparability digraph of another poset L on the elements of P. As K is a complete graph, the poset L is a linear extension of P. □

Dushnik and Miller showed that if there exists a transitive orientation of the incomparability graph of a poset, it has dimension at most 2 [24, Theorem 3.61].

Lemma 2.1.13 ([24, Theorem 3.61])**.** *Let P be a poset. If its incomparability graph admits a transitive orientation, then* $\dim(P) \leq 2$.

Proof. Applying Lemma 2.1.12 to a transitive orientation I of the incomparability graph of P defines a linear extension L of P. Reversing the orientation of every edge in I, we obtain another transitive orientation I' of $\mathrm{Inc}(P)$. Lemma 2.1.12 applied to the orientation I' yields once again a linear extension L' of P. For incomparable elements a, b of P, we see that $a \leq_L b$ if and only if $b \leq_{L'} a$. Thus, $\{L, L'\}$ is a realizer of P and $\dim(P) \leq 2$ follows. □

Actually, the family of posets of dimension at most 2 can be characterized as posets whose incomparability graph admits a transitive orientation [24, Theorem 3.61].

If a poset contains a large standard example, its dimension is large. However, it is not necessary for a poset of large dimension to contain a large standard example. Universal interval orders provide an example of such a family of posets.

Definition 2.1.14 ([31, p. 675])**.** An interval $[a, b] \subseteq \mathbb{R}$ is called *non-degenerate* if $a < b$. We call a poset P an *interval order* if there exists an assignment $x \mapsto [\ell_x, r_x]$ that maps all elements x of P to non-degenerate intervals in $\mathbb{R}$ such that for any two elements u, v of P we have $u <_P v$ if and only if $r_u < \ell_v$. Such an assignment is called a *(closed) interval representation of* P. Sets of intervals in $\mathbb{R}$ define posets via the above correspondence.

For $n \in \mathbb{N}$, we define the *universal interval order* I_n as the poset corresponding to the set of all non-degenerate intervals with both endpoints in $[n]$.

There are several slightly different definitions of interval orders. While some authors require all intervals to be non-degenerate [35, p. 298], others give no further restriction [31, p. 675]. Instead of considering closed intervals, we can use open intervals. Spinrad defined interval orders using the following notion [83].

Definition 2.1.15. An *open interval representation* of a poset P is an assignment $x \mapsto (\ell_x, r_x)$ that maps all elements x of P to open (non-empty) intervals in $\mathbb{R}$ such that for any two elements u, v of P, we have $u <_P v$ if and only if $r_u \leq \ell_v$.

Note that the closed interval representation $x \mapsto [\ell_x, r_x]$ and the open interval representation $x \mapsto (\ell_x, r_x)$ might correspond to different posets. While the closed interval representation $\{[1, 2], [2, 3]\}$ represents an antichain, the open interval representation $\{(1, 2), (2, 3)\}$ yields a chain. Nevertheless, if a poset has a closed interval representation, it also admits an open one and vice versa.

Lemma 2.1.16. *A poset is an interval order if and only if it has an open interval representation.*

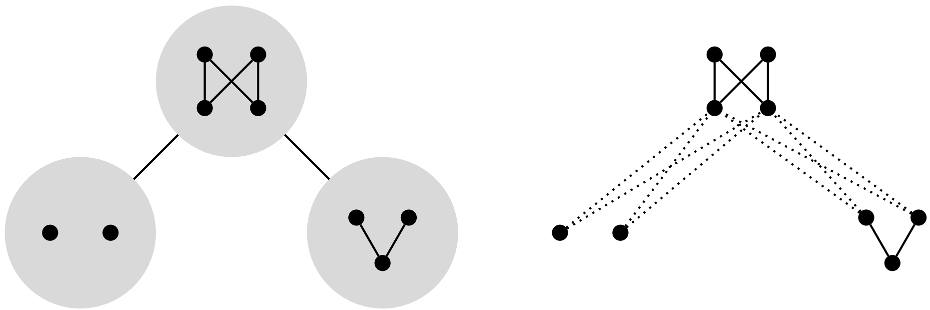
Proof. Let P be a poset. We say that an interval representation is *spaced* if there are no two elements u, v such that $r_u = \ell_v$. The assignment $f^\circ \colon x \mapsto (\ell_x, r_x)$ is the *interior* of $f \colon x \mapsto [\ell_x, r_x]$, and f is the *closure* of f°. Note that if P has a spaced, closed interval representation, its interior is an open interval representation of P. Similarly, the closure of a spaced, open interval representation of P is a closed interval representation of P. Thus, it suffices to construct spaced interval representations.

Suppose P is an interval order. Consider a closed interval representation f of P. If f is not spaced, there are incomparable elements u, v such that $r_u = \ell_v$. It suffices to slightly augment r_u by some real number $\varepsilon > 0$ such that $\ell_v < r_u + \varepsilon$. Choosing ε sufficiently small, the closed interval representation we obtain still yields P. Proceeding similarly for each such pair u, v, we obtain a spaced, closed interval representation of P.

Now suppose that P has an open interval representation. If there are two elements u, v such that $r_u = \ell_v$, we can slightly decrease r_u by some real number $\varepsilon > 0$ such that $r_u - \varepsilon < \ell_v$. For sufficiently small ε, the interval representation we obtain still represents P. Inductively, we obtain a spaced, open interval representation. As its closure is a closed interval representation of P, the claim follows. $\square$

Interval orders are characterized as posets which do not contain the standard example of size two as a subposet. We refer to the proof given in [87, p. 86]. The result was first explicitly shown by Fishburn [34, Theorem 4]. Former results implicitly yield the same characterization [40, 41], as was observed by Trotter [87, p. 86].

Even though interval orders do not contain large standard examples, their dimension can be arbitrarily large. Füredi et al. determined the exact value of the maximum dimension of the interval orders of bounded height up to a constant factor [35]. We are mostly interested in the lower bound for universal interval orders, see [35, Corollary 5.2] for a proof.



(a) The elements of P are represented by gray disks. The diagram of a poset Q_x corresponding to an element x of P is drawn within the disk of x.

(b) The diagram of the lexicographic sum is represented. Black edges belong to one of the posets Q_x. All other edges are represented as dotted lines.

Figure 2.2: Representations of a lexicographic sum $\sum_{x\in P} Q_x$.

Lemma 2.1.17 ([35, Corollary 5.2])**.** *For every $n \geq 4$, we have* $\dim(I_n) \geq \log\log(n)$.

The following definition provides an operation which enables us to combine several posets.

Definition 2.1.18 ([87, p. 24])**.** Let P be a poset and let $\mathcal{F} = \{Q_x \mid x \in P\}$ be a family of posets indexed by the elements of P. We denote by $\sum_{x\in P} Q_x$ the poset on $\{(x, y) \mid x \in P, y \in Q_x\}$ where $(x, y) \leq (x', y')$ if and only if either $x <_P x'$ or $x = x'$, $y \leq_{Q_x} y'$. The poset $\sum_{x\in P} Q_x$ is called the *lexicographic sum* of $\mathcal{F}$ over P.

An example of a lexicographic sum is given in Figure 2.2. Interestingly, if we know the dimension of each of the posets involved in the lexicographic sum, we can determine the dimension of the resulting poset as has been shown by Hiraguchi [50]. We refer to the proof given by Trotter in [87, p. 24].

Lemma 2.1.19 ([87, p. 24])**.** *For any poset P on a set X and any family of posets Q_x with $x \in X$, we have*

$$\dim(\sum_{x\in X} Q_x) = \max(\dim(P), \max_{x\in X} \dim(Q_x)).$$

It follows easily that the dimension of a poset is bounded by the maximum dimension of its components.

Lemma 2.1.20. *If P is a poset and we denote by $C_1, \dots, C_k$ the posets induced by the components of the diagram of P, then*

$$\dim(P) \leq \max(2, \max_{i\in[k]} \dim(C_i)).$$

Proof. We may assume that the diagram of P has at least two components. As P is the lexicographic sum of $\{C_1, \dots, C_k\}$ over an antichain with k elements labelled $1, \dots, k$, the claim follows immediately from Lemma 2.1.19. $\square$

Note that if P has at least two components, then the inequality in the previous lemma is an equality.

2.2 Stack and Queue Layouts

If we draw the vertices of a graph G on a horizontal line of the plane and draw the edges as half-circles above this line, we obtain a drawing of G. A layout is an ordering of the vertices together with an edge partition based on this drawing. The most common layouts are stack and queue layouts. Stack and queue number are two graph parameters which are based on minimization problems for stack respectively queue layouts.

Even and Itai, Ollmann as well as Bernhart and Kainen initiated the study of stack layouts of graphs, the latter using the term book embedding instead [8, 28, 62, 78]. Since then, many results connected to stack number followed [13, 26, 95]. Stack number is sometimes called book thickness or page number [22, p. 2]. Heath, Leighton, and Rosenberg developed the notion of queue layouts and explored the relationship between queue and stack number [44, 48]. Bounds on the stack number of a variety of graph classes have been established [20, 94]. The notions have also been applied to dags [45, 47, 60, 61] and posets [33, 77, 86]. An overview of further results connected to queue and stack number, as well as applications is given in [22].

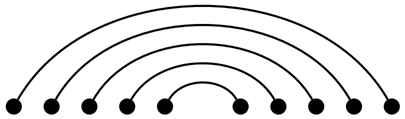
Definition 2.2.1. Consider a topological vertex ordering σ of a dag. An edge cd *lies inside* an edge ab if their endpoints are ordered $a <_\sigma c <_\sigma d <_\sigma b$ with respect to σ. If one of the edges ab and cd lies inside the other, we say that the edges ab and cd *nest*. A set of k pairwise nesting edges is called a *k-rainbow*; see Figure 2.3a. A topological vertex ordering of a dag together with a partition of its edges into k parts is called a *k-queue layout* if no two edges of the same part nest. If this is the case, the parts are referred to as *queues*. The *queue number* of a dag G, denoted by $\mathrm{qn}(G)$, is the minimum number k such that G admits a k-queue layout.

Given a topological vertex ordering σ of a dag, we say that edges ab and cd *cross* if their endpoints are ordered $a <_\sigma c <_\sigma b <_\sigma d$ or $c <_\sigma a <_\sigma d <_\sigma b$ with respect to σ. A *k-twist* consists of k pairwise crossing edges, see Figure 2.3b. A topological vertex ordering of a dag together with a partition of the edges into k parts such that no two edges of the same part cross, is a *k-stack layout*. The parts of a such a layout are called *stacks*. The *stack number* of a dag G, denoted by $\mathrm{sn}(G)$, is the minimum number k such that G admits a k-stack layout.

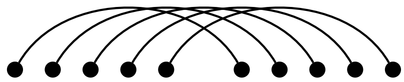
The queue and stack number of a poset is defined as the queue, respectively stack number of its diagram.

We define queue and stack number of undirected graphs in a similar way. Instead of considering topological vertex orderings, we consider all vertex orderings. The terms defined above are used analogously for undirected graphs as for dags.

There is a deep connection between (undirected) stack number and vertex colorings of circle graphs [28] [48, p. 932] [61, p. 8]. If we consider a circle C with chords, the



(a) A 5-rainbow.



(b) A 5-twist.

Figure 2.3: On the left, a rainbow is represented, on the right a twist.

corresponding circle graph G is the undirected graph whose vertices correspond to the chords of C and where two vertices are adjacent if and only if the corresponding chords intersect. Considering the intersections of the chords with C in the order they appear on the circle, and interpreting these as vertices of an undirected graph H, we obtain a vertex ordering σ of H. The chords of C correspond to edges of the graph H. As any two crossing edges in σ correspond to two adjacent vertices in G, we see that a stack assignment of σ corresponds to a proper vertex coloring of G and vice versa.

Obviously, a layout which contains a k-rainbow requires at least k queues. Interestingly, this is the only obstruction as has been shown by Heath and Rosenberg [48, Theorem 2.3]. We follow the proof given by Dujmović and Wood [22, Lemma 1].

Lemma 2.2.2 ([22, Lemma 1])**.** *A topological vertex ordering of a dag admits a k-queue layout if and only if it contains no $(k+1)$-rainbow.*

Proof. Clearly, a k-queue layout contains no $(k+1)$-rainbow as all edges of the rainbow have to be assigned to different queues. Conversely, if the vertex ordering contains no $(k+1)$-rainbow, we define for an edge uv the value $q(uv)$ as the maximum size of a rainbow inside uv plus one. Note that if an edge $u'v'$ is nested inside an edge uv, we have $q(u'v') < q(uv)$. Thus, we obtain a valid queue assignment. As $q(e) \leq k$ for every edge e, we used at most k queues. □

To some extent k-twists behave in the context of stack number as k-rainbows for queue number. While both provide a lower bound for stack number and queue number respectively, there is no similar result to Lemma 2.2.2 for k-twists. There are families of layouts of graphs where the size of a largest twist is k, but which require at least $\Omega(k\log(k))$ stacks [14, Theorem 2]. Surprisingly, if a layout has large stack number, it also admits a large twist. More precisely, if the largest twist has size k, its stack number lies in $\mathcal{O}(k\log(k))$ [14, Theorem 1]. Although the largest size of a twist in a given vertex ordering can be determined in polynomial time [53], the problem of determining the minimum number of stacks for a vertex ordering is $\mathcal{NP}$-complete [48, Proposition 2.4]. This stands in clear contrast to queue layouts as given a vertex ordering with no $(k+1)$-rainbow, a k-queue layout can be determined in polynomial time [48, Theorem 2.3]. However, the problems of recognizing 4-queue dags and 6-stack dags are $\mathcal{NP}$-complete [45].

We now give bounds on the queue and stack number of some posets. As we wish to explore the connection between dimension, queue and stack number, we first consider the family of standard examples. These posets have high dimension and large queue and stack number.

Lemma 2.2.3. *For $d \geq 2$, there is a linear lower bound on the queue number of the standard example S_d, more precisely* $\mathrm{qn}(S_d) \geq \lfloor \frac{d}{2} \rfloor$. *The same is true if we replace queue by stack number.*

Proof. Note that the set $\{a_1, \ldots, a_{\lfloor \frac{d}{2} \rfloor}, b_{\lceil \frac{d}{2}+1 \rceil}, \ldots, b_d\}$ induces the complete bipartite graph $K \coloneqq K_{\lfloor \frac{d}{2} \rfloor, \lfloor \frac{d}{2} \rfloor}$ where all edges are directed from one part to the other. Therefore, it suffices to show that $\mathrm{qn}(K) \geq \lfloor \frac{d}{2} \rfloor$. Up to symmetry, there is only one topological ordering of K and this ordering contains a $\lfloor \frac{d}{2} \rfloor$-rainbow and a $\lfloor \frac{d}{2} \rfloor$-twist. □

Since all edges having the same left endpoint can be assigned to one queue, we see that the queue number is bounded from above by the number of vertices of the graph. The same is true for stack number. The lemma above establishes a linear lower bound on the queue and stack number of the standard example in the number of vertices. Note however that the given bound is only sharp up to a constant factor.

2.3 Stack Number is not bounded by Dimension

The cover graph of a standard example is an almost complete bipartite graph. Thus, the number of edges is quadratic in the number of vertices. We wish to construct posets with large stack number, but where the number of edges is linear.

Definition 2.3.1. Consider the poset on elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ whose diagram consists of paths $a_1 \ldots a_n$ and $b_1 \ldots b_n$ and all edges $a_i b_i$ for $i \in [n]$, see Figure 2.4a. We say that the poset is a *fin* of size n on $a_1, \ldots, a_n, b_1, \ldots, b_n$.

The *double fin* of size n is the poset consisting of a fin on $a_1, \ldots a_n, b_1 \ldots b_n$ and a fin on $c_1, \ldots, c_n, d_1, \ldots d_n$ where $a_n \leq d_1$ and $c_n \leq b_1$, see Figure 2.4b.

Using a similar idea as Jungeblut, Merker, and Ueckerdt [60, p. 1], we see that a fin of size n forces a large twist in certain topological orderings.

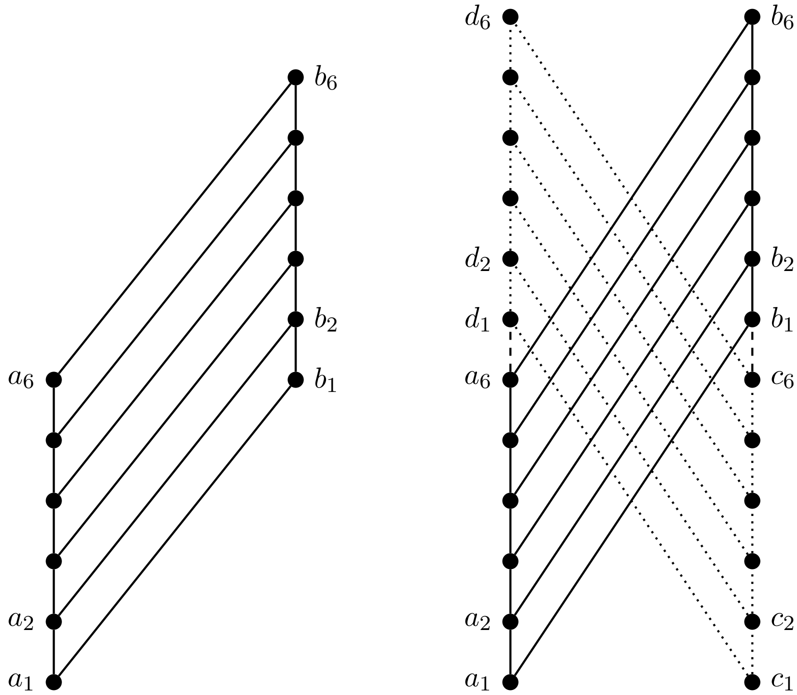
Lemma 2.3.2. *If a poset P contains a fin of size n on elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ and σ is a topological ordering of the vertices of P where $a_n \leq b_1$, then σ requires at least n stacks.*

Proof. As $a_n \leq_\sigma b_1$ and σ is a topological ordering, we obtain

$$a_1 \leq_\sigma a_2 \leq_\sigma \cdots \leq_\sigma a_n \leq_\sigma b_1 \leq_\sigma \cdots \leq_\sigma b_n.$$

We see that σ admits an n-twist as $a_i b_i$ is an edge for all i. □

The lemma above enables us to prove that double fins have indeed large stack number.



(a) A fin of size 6 on elements $a_1, \ldots, a_6, b_1, \ldots, b_6$.

(b) A double fin of size 6. One fin is represented by black lines, the other by dotted lines. Dashed lines represent edges which do not belong to any of the two fins.

Figure 2.4: Illustrations of a fin and a double fin.

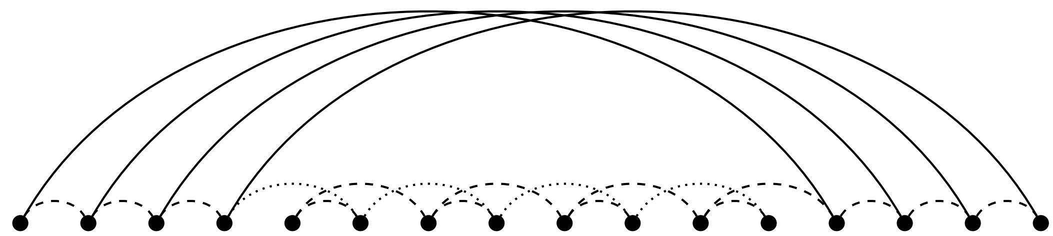


Figure 2.5: A 4-stack layout of the double fin of size 4. Dashed edges belong to one stack, dotted edges to another. The black edges form a 4-twist. Each of these needs to be assigned to a different stack.

Lemma 2.3.3. *A double fin of size $n \geq 2$ has stack number n, a double fin of size* 1 *has stack number* 2.

Proof. Up to symmetry, a double fin of size 1 has only one topological ordering. This ordering requires two stacks.

Let D_n be the double fin of size $n \geq 2$. Consider any topological ordering σ of the vertices of D_n. We may assume that $a_n \leq_\sigma c_n$. The case $c_n \leq_\sigma a_n$ is similar.

As $a_n \leq_\sigma c_n$, we obtain $a_n \leq_\sigma c_n \leq_\sigma b_1$. By Lemma 2.3.2, we see that σ contains an n-twist. As σ was an arbitrary topological ordering, $\mathrm{sn}(D_n) \geq n$ follows.

Conversely, the stack number is at most n, as there exists an n-stack layout of D_n, see Figure 2.5. We need n stacks for the edges of the n-twist formed by the edges $a_i b_i$ for $i \in [n]$. The remaining edges can be assigned to two different stacks. As none of these edges cross any of the edges of the n-twist, n stacks suffice. □

In particular, double fins provide an example of a family of posets with unbounded stack number, but whose cover graphs have bounded degree. For undirected graphs it is much harder to construct such examples. Only in 2023 has a family of undirected graphs with bounded degree and unbounded stack number been constructed [26], though it was known since 1987 that such families exist [13].

While the stack number of double fins is unbounded, their dimension is constant. This shows that stack number is not bounded by dimension.

Lemma 2.3.4. *A double fin has dimension* 2.

Proof. Let D_n be a double fin of size n. Consider the following two linear extensions

$$L_1 = a_1\, a_2\, \ldots\, a_n\, c_1\, d_1\, c_2\, d_2\, \ldots\, c_n\, d_n\, b_1\, \ldots\, b_n$$
$$L_2 = c_1\, c_2\, \ldots\, c_n\, a_1\, b_1\, a_2\, b_2\, \ldots\, a_n\, b_n\, d_1\, \ldots\, d_n.$$

Note that $a_i \leq_{L_1} c_j$ and $c_j \leq_{L_2} a_i$ for all i and j. Similarly, we see that all pairs (b_i, d_j) and (d_j, b_i) are reversed by L_1 and L_2 respectively. Further, we have $a_i \leq_{L_1} b_j$

and $b_j \leq_{L_2} a_i$ for $i > j$. By symmetry, we also obtain $c_i \leq_{L_2} d_j$ and $d_j \leq_{L_1} c_i$ for $i > j$. Therefore, $\{L_1, L_2\}$ is a Dushnik-Miller realizer of D_n and $\dim(D_n) \leq 2$ follows. As D_n is not a chain for any n, we see that its dimension is 2. □

Corollary 2.3.5. *There are posets with arbitrarily large stack number and dimension at most* 2*.*

Proof. By Lemma 2.3.4 and Lemma 2.3.3, the double fins form a family of 2-dimensional posets whose queue number is unbounded. □

Similarly, queue number is not bounded by dimension as we will see in Section 2.5.

2.4 Erdős-Szekeres

To some extent structure cannot be avoided in large configurations as has been shown by Ramsey [81], see [18, p. 284]. This result led to the development of a branch of mathematics, referred to as Ramsey Theory, with applications in the study of stack number [12, 25], dimension of posets [5, 56, 85, 87, 91] and graph theory [18]. Usually, the numbers involved are very large and often no explicit value is known. However, Erdős and Szekeres were able to show that long monotone subsequences cannot be avoided in any sequence of distinct numbers, thereby giving explicit bounds for a Ramsey-type argument [27]. There are several different proofs of the result [84], one of which is based on the pigeonhole principle and attributed to Seidenberg [82].

Theorem 2.4.1 ([82])**.** *Any sequence of at least $ab + 1$ distinct real numbers where a and b are any two positive, real numbers contains an increasing subsequence of length at least $\lfloor a + 1 \rfloor$ or a decreasing subsequence of length at least $\lfloor b + 1 \rfloor$.*

Proof. For every $i \in [n]$, we assign a label (a_i, b_i) to the i-th element s_i of the sequence where a_i denotes the length of a longest increasing and b_i the length of a longest decreasing subsequence ending with s_i. Note in particular that for any two elements s_i and s_j of the sequence where $i < j$, we have $a_i < a_j$ or $b_i < b_j$.

Suppose the sequence contains no increasing subsequence of length at least $\lfloor a + 1 \rfloor$ and no decreasing subsequence of length at least $\lfloor b + 1 \rfloor$. There are only $\lfloor a \rfloor \cdot \lfloor b \rfloor$ possible labels, each of which appears at most once. However, the sequence contains $ab + 1$ elements. This yields a contradiction by the pigeonhole principle. □

The same proof also yields another variant of the theorem, a formulation which is found in [13, Lemma 2.4].

Theorem 2.4.2 ([27])**.** *Let n be a natural number and let r be a positive, real number with $r \leq n$. Any sequence of n distinct real numbers contains an increasing subsequence of length at least $\lceil r \rceil$ or a decreasing subsequence of length at least $\lceil \frac{n}{r} \rceil$.*

These two results are in particular of great use in the study of layouts [13, 23, 33, 46, 60, 61, 79] and dimension theory [5, 24, 32, 55, 85, 92], most commonly in the construction of graphs or posets which meet given bounds.

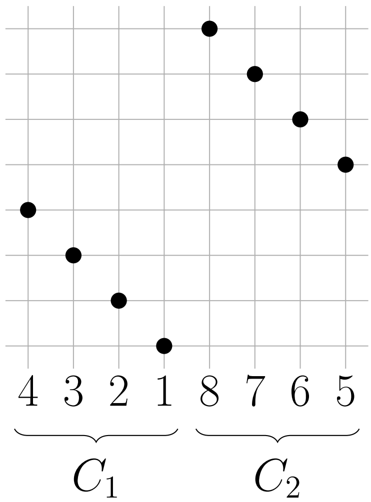


Figure 2.6: The ordering σ we obtain in Proposition 2.4.4 for $a = 2$ and $b = 4$. Every element v of $[ab]$ is represented by a point in $\mathbb{R}^2$ whose x-coordinate is equal to the position of v in σ and whose height corresponds to the value v.

In Theorem 2.4.2, we often set $r = \sqrt{n}$, see [46, Lemma 4.1] for instance. This special case is known as the symmetric variant of the Erdős-Szekeres Theorem.

Theorem 2.4.3. *Any sequence of n distinct real numbers contains a monotone subsequence of length at least $\lceil \sqrt{n}\, \rceil$.*

Well-known constructions show that the bounds given in Theorem 2.4.2 are tight.

Proposition 2.4.4. *For any two positive, real numbers a and b and any set S of ab distinct real numbers, there exists a sequence of the elements of S such that any increasing subsequence has length at most $\lceil a \rceil$ and every decreasing subsequence has length at most $\lceil b \rceil$.*

Proof. Let $n := ab$. Without loss of generality, we may assume that $S = [n]$. We partition S into sets $C_i := \{j \in [n] \mid (i-1)\lceil b \rceil < j \leq i \lceil b \rceil\}$ for $i \in [\lceil a \rceil]$. For every i, we define the ordering σ_i as the decreasing sequence on C_i. Define σ as the ordering of the elements of S such that

$$\sigma = \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{\lceil a \rceil}.$$

The ordering is exemplified in Figure 2.6. The ordering σ defines a sequence (i_k) of the elements of S. Note that any decreasing subsequence of (i_k) consists only of elements of one C_i. Thus, its length is at most $\lceil b \rceil$. Further, any increasing subsequence of (i_k) contains at most one element of each C_i. Therefore, its length is bounded by $\lceil a \rceil$. $\square$

The construction has for example been used by Heath and Pemmaraju [46, Theorem 4.2].

We will apply the next result in the following setting. Suppose we have n distinct real numbers and four numbers a, b, a', b' such that $n = ab = a'b'$. We arrange the n numbers according to the sequence (i_k) resulting from Proposition 2.4.4 with parameters a and b, followed by copies of the n numbers arranged according to the sequence (j_k) resulting from the same construction with parameters a' and b'. If we join all numbers of the same value, we obtain a layout of a matching. We are interested in the size ℓ of a largest rainbow of this layout. Note that such a rainbow corresponds to a longest subsequence of (i_k) which is reversed in (j_k). The following observation gives an upper bound on the size ℓ. It will be of use in the proof of Proposition 2.8.6 when we attempt to construct a $\preceq h$-subdivision of a poset which has relatively small queue number compared to the original poset.

Observation 2.4.5. *Let S be a set of n elements and let a, b, a', b' be positive, real numbers such that $ab = n$, $a'b' = n$ and $a \geq a'$. Let (i_k) and (j_k) be the sequences of the elements of S constructed in Proposition 2.4.4 for a, b and a', b' respectively. If (r_m) is a subsequence of (i_k) that is reversed in (j_k), then its length is at most $\frac{b'}{b} + 2$.*

Proof. Let $C_1, \dots, C_{\lceil a \rceil}$ and $C'_1, \dots, C'_{\lceil a' \rceil}$ be the partitions of S in the construction of (i_k) and (j_k) respectively. Note that the order of two elements in (i_k) is reversed in (j_k) if and only if they belong to different classes C_i, but the same class C'_j or vice versa. Let (r_m) be a subsequence of (i_k) of length ℓ that is reversed in (j_k). For $s \in [\ell]$, we denote by $C(r_s)$ the class C_i such that $r_s \in C_i$. Similarly, we use the notation $C'(r_s)$ for classes C'_i.

We may assume that $\ell \geq 3$, otherwise the claim follows immediately. As $a \geq a'$, the number of classes C'_i is at most equal to the number of classes C_i. Since (r_m) is reversed in (j_k), it follows that all classes $C(r_s)$ for $s \in [\ell]$ are distinct and all $C'(r_s)$ coincide as $\ell \geq 3$. In particular, all elements in $C(r_2), \dots, C(r_{\ell-1})$ lie in $C'(r_\ell)$. Thus,

$$b' \geq |C'(r_\ell)| \geq (\ell - 2)b$$

and we obtain $\ell \leq \frac{b'}{b} + 2$. □

2.5 Queue Number is not bounded by Dimension

In Section 2.3, we constructed a family of 2-dimensional posets with arbitrarily large stack number. The aim of this section is to provide a similar construction for queue number which is due to Heath and Pemmaraju [46].

Definition 2.5.1 ([46, p. 606])**.** The *wing* of size n is the poset on elements

$$a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$$

whose diagram consists of paths $a_1 \dots a_n$ and $b_n \dots b_1$ and all edges $a_i c_i, c_i b_i$ for $i \in [n]$, see Figure 2.7.

The family of wings does indeed constitute a family of low dimensional posets.

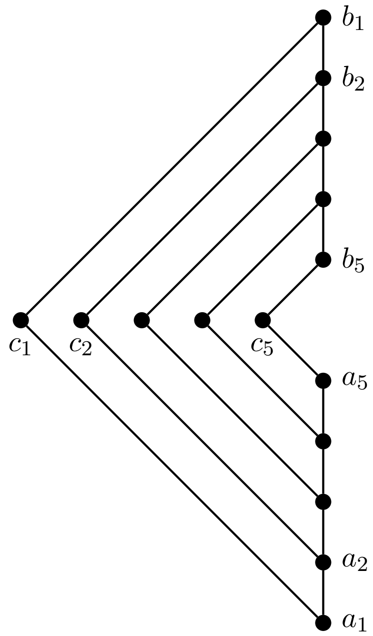


Figure 2.7: A wing of size 5.

Lemma 2.5.2. *All wings of size at least* 2 *have dimension* 2*. The wing of size* 1 *is a* 1*-dimensional poset.*

Proof. Note that the wing of size 1 is a chain. Thus, its dimension is 1.

Now consider a wing of size n at least 2. As it contains incomparable elements, its dimension is at least 2. The two linear extensions

$$L_1 = a_1\ c_1\ a_2\ c_2 \dots a_n\ c_n\ b_n \dots b_1$$
$$L_2 = a_1 \dots a_n\ c_n\ b_n\ c_{n-1}\ b_{n-1} \dots c_1\ b_1$$

form a realizer of the wing of size n. Thus, its dimension is at most 2. □

In fact, the result follows immediately from an observation of Trotter [87, p. 69], see Theorem 2.6.8.

Heath and Pemmaraju established bounds on the queue number of wings using the two different variants of the Erdős-Szekeres Theorem [46, Theorem 4.2] [79, Theorem 5.8], see Theorem 2.4.1 and Theorem 2.4.2. Combining the two results, we obtain the exact value up to one.

Lemma 2.5.3. *For the wing W_{n+1} of size $n+1$, we have*

$$\lfloor \sqrt{n} + 1 \rfloor \leq \mathrm{qn}(W_{n+1}) \leq \lceil \sqrt{n} + 1 \rceil.$$

Proof. We first show that W_{n+1} has queue number at least $\lfloor \sqrt{n} + 1 \rfloor$. Let σ be an arbitrary topological ordering of the vertices. Considering the ordering of the

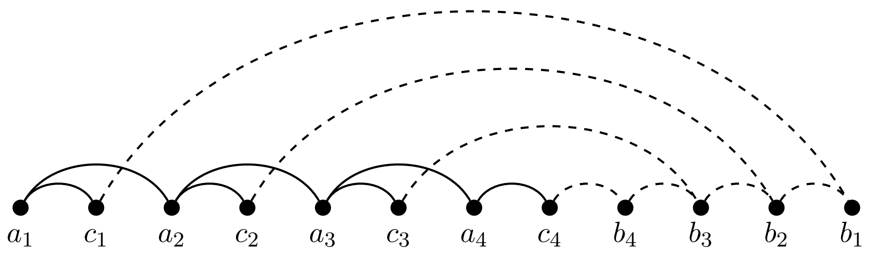


Figure 2.8: A 2-stack layout of a wing of size 4. Edges of the same color are assigned to the same stack.

vertices c_i induced by σ, we see by a well-known result of Erdős and Szekeres, see Theorem 2.4.1, that there is a monotone sequence (j_k) of length $\ell \geq \lfloor\sqrt{n}+1\rfloor$ such that $c_{j_1} \leq_\sigma \cdots \leq_\sigma c_{j_\ell}$. If (j_k) is increasing, the edges $c_{j_k}b_{j_k}$ form an ℓ-rainbow, if it is decreasing we obtain an ℓ-rainbow consisting of the edges $a_{j_k}c_{j_k}$ as σ is a topological ordering. Thus, $\operatorname{qn}(W_{n+1}) \geq \lfloor\sqrt{n}+1\rfloor$.

As the bound established by Erdős and Szekeres is tight, see Proposition 2.4.4, there exists a sequence (i_k) of the integers $[n]$ such that a longest monotone subsequence has length at most $\lceil\sqrt{n}\,\rceil$. Consider the topological ordering

$$\sigma = a_1 \leq \cdots \leq a_{n+1} \leq c_{i_1} \leq \cdots \leq c_{i_n} \leq c_{n+1} \leq b_{n+1} \leq \cdots \leq b_1$$

and a largest rainbow in σ. It suffices to show that its size is at most $\lceil\sqrt{n}+1\rceil$.

We call an edge of the rainbow with an endpoint c_i for $i \in [n]$ a *c-edge*. Define (j_m) as the sequence of integers such that $c_{j_1} < \cdots < c_{j_s}$ in σ and these vertices correspond to endpoints of the c-edges. As the c-edges form a rainbow, they are either all of the form a_ic_i or they are all of the form c_ib_i. In particular, we see that (j_m) is a monotone subsequence of (i_k). Its length is therefore at most $\lceil\sqrt{n}\,\rceil$.

We may assume that (j_m) is increasing; we can argue similarly if it is decreasing. As (j_m) is increasing, the rainbow contains the edges $c_{j_s}b_{j_s}$, but none of the edges a_ic_i. Further, either at most one of the edges b_ib_{i+1} or the edge $c_{n+1}b_{n+1}$ belongs to the rainbow. Thus, its size is at most $\lceil\sqrt{n}+1\rceil$. $\square$

While wings have large queue number, their stack number is at most 2, see Figure 2.8. It follows that the queue number of dags is not bounded by stack number. For undirected graphs, the problem is still open [19, p. 4], even though Dujmović et al. showed that the (undirected) stack number is not bounded by queue number. They constructed a family of graphs with queue number at most 4 and unbounded stack number [19]. For dags, the question is far easier to answer as the family of double fins has unbounded stack number, but queue number at most 2. Indeed, the layout of a double fin represented in Figure 2.5 requires only two queues.

Even if the queue number of a poset is large, its dimension can be small. The proof is similar to Corollary 2.3.5, which provides the analog result for stack number.

Corollary 2.5.4. *There are posets with arbitrarily large queue number and dimension at most* 2*.*

Proof. Wings form a family of posets whose dimension is at most 2 by Lemma 2.5.2. Further, their queue number is unbounded by Lemma 2.5.3. □

Similarly, there are posets with queue number at most 1 and arbitrarily large dimension, as we will see in Corollary 2.7.7. Thus, neither is dimension bounded by queue number, nor is queue number bounded by dimension. However, in Section 2.11 we will see that the dimension of a poset is bounded in terms of its queue number and height.

2.6 Planar Posets

Intuitively, a planar graph is a graph that can be drawn in the plane without any crossings. A rigorous definition of planar graphs and drawings is given in [18, Chapter 4].

Definition 2.6.1 ([60, p. 1])**.** A planar drawing of a dag is called *upward planar* if every edge ab is a strictly y-monotone curve with lower endpoint a and upper endpoint b. A *planar poset* is a poset whose diagram admits an upward planar drawing.

We may assume that the edges in an upward planar drawing are represented by straight line segments [17, Theorem 4.3]. While planar graphs can be recognized in linear time [52], the problem is $\mathcal{NP}$-complete for upward planar graphs [39]. For a fixed combinatorial embedding of a dag, it can be determined in polynomial time whether there exists an upward planar drawing which respects the embedding [9, Theorem 4].

As the standard example S_n is non-planar for $n \geq 5$, it was supposed for a short time that the dimension of planar posets might be bounded by a constant. However, in 1981, Kelly constructed planar posets which contain large standard examples as subposets [64, Section 2].

Definition 2.6.2 ([64, Section 2])**.** The poset Kelly_n is the poset on elements

$$a_1, \ldots, a_n, b_1, \ldots, b_n, \ell_1, \ldots, \ell_{n-1}, r_1, \ldots, r_{n-1}$$

where for all i, j we have

- $r_i \leq r_j$ if $i \leq j$
- $\ell_i \leq \ell_j$ if $j \leq i$.
- $a_i \leq r_j$ if $i \leq j$
- $a_i \leq \ell_j$ if $i < j$
- $\ell_i \leq b_j$ if $i \leq j$
- $r_i \leq b_j$ if $i < j$,

and all other relations follow by transitivity. It is represented in Figure 2.9.

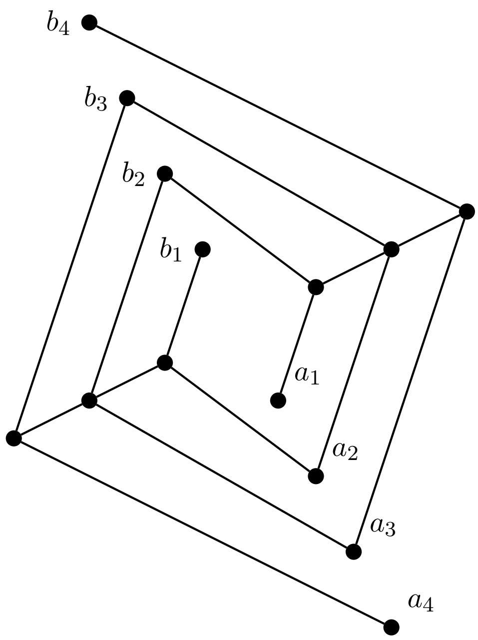


Figure 2.9: An illustration of the planar poset constructed by Kelly for $n = 4$.

The vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ induce a standard example of size n. In particular, we see that a subposet of a planar poset might not be planar. As any poset which contains an n-dimensional standard example has dimension at least n, it follows that $\dim(\text{Kelly}_n) \geq n$. Establishing a connection between the construction above and a poset called the $(n+1)$-dimensional Boolean lattice, Kelly was able to determine the dimension of Kelly_n, see [64, Section 2] for a proof.

Lemma 2.6.3 ([64, Section 2], [58, p. 2754])**.** *The poset* Kelly_n *is* $(n+1)$*-dimensional for every positive integer* n*.*

As the posets constructed by Kelly are planar and have arbitrarily large dimension, we see that the dimension of planar posets is unbounded.

Corollary 2.6.4 ([64, Section 2])**.** *The dimension of planar posets is unbounded.*

If a poset contains the standard example S_n for large n, its dimension is at least n. For posets with planar cover graphs, it has been conjectured that this is the only obstruction for small dimension [70, Conjecture B]; a proof for posets with planar cover graphs and a zero is known [70, Theorem 2]. Hodor et al. announced a proof for planar posets [51].

However, there are non-planar posets of large dimension which do not admit a large standard example as a subposet, for instance the incidence posets of complete graphs [92, p. 2].

While the queue number of undirected planar graphs is bounded by a constant [20], it is unbounded for planar posets.

Corollary 2.6.5 ([46, Theorem 4.2])**.** *The queue number of planar posets is unbounded.*

Proof. The family of wings is a family of planar posets with unbounded queue number by Lemma 2.5.3. □

As every stack of a stack layout induces a set of non-crossing edges, it is natural to wonder whether planar graphs have small stack number. Indeed, the undirected stack number of planar graphs is at most 4 [6, 95]. Actually, it is bounded for every proper minor-closed class of graphs [11, Theorem 1.2]. The directed case is somewhat different. It was first studied by Nowakowski and Parker who conjectured that the stack number of planar posets is bounded by a constant providing an example with stack number 3 [77, p. 217]. The lower bound was subsequently improved to 4 by Hung [54] and to 5 by Merker [68]. However, no planar poset with stack number 6 is known. In fact, whether the stack number of upward planar graphs is bounded by a constant is still open [61, Open Problem 3]. Interestingly, posets with planar cover graphs, but which are not necessarily planar posets, can have arbitrarily large stack number as has been shown by Heath and Pemmaraju [46, Theorem 5.1]. Since the question for planar posets has been asked by Nowakowski and Parker, significant progress has been made. Alzohairi and Rival investigated series-parallel planar posets and showed that their stack number is at most 2 [1, Theorem 8]. Heath, Pemmaraju, and Trenk proved upper bounds for oriented trees and unicyclic, directed graphs [47]. Algorithmic aspects have been investigated by Heath and Pemmaraju [45]. Most recently, the case of outerplanar, directed graphs has been settled. Jungeblut, Merker, and Ueckerdt showed that the stack number is indeed bounded for this graph class [61, Theorem 2].

Definition 2.6.6 ([87, p. 6])**.** Let P be a poset on elements X. An element $a \in X$ is a *lower bound* of a subset $S \subseteq X$ if $a \leq s$ for all $s \in S$. It is a *greatest lower bound* if for every lower bound a' of S, we have $a' \leq a$. *Upper bounds* and *least upper bounds* are defined similarly. If a is a (greatest) lower bound of X, we say that a is a *zero* of the poset. Similarly, we refer to a (least) upper bound of X as a *one*.

If every non-empty subset $S \subseteq X$ has a least upper and a greatest lower bound, the poset P is called a *lattice*; see Figure 2.10 for examples.

It is believed that Joseph A. Zilber, still a student at that time, communicated a result on planar lattices to Birkhoff who included it as Exercise 7 (c) in [10, Section II.4], see [4, 87]. Baker, Fishburn, and Roberts observed that the following theorem is a consequence [3, p. 18] [87, p. 69]. We refer to the proof given by Trotter [87, p. 69].

Theorem 2.6.7 ([87, p. 69])**.** *A lattice is a planar poset if and only if its dimension is at most* 2*.*

In fact, the proof given by Trotter even shows the following.

Theorem 2.6.8 ([87, p. 69])**.** *Every planar poset with a zero and a one has dimension at most* 2*.*

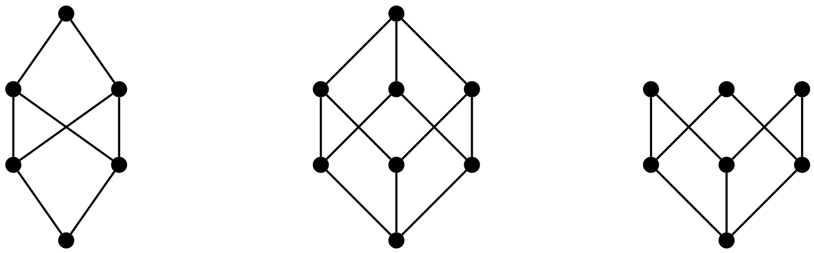


Figure 2.10: Three posets are represented. The poset on the left has a zero and a one, but is not a lattice. The one depicted in the center is a lattice, thus has in particular a zero and a one. The poset on the right has a zero, but does not have a one.

Actually, planar posets with a zero and a one correspond to the family of 2-dimensional lattices [87, p. 114]. Trotter and Moore showed in [89, Theorem 2] the following result for planar posets with a zero, but which do not necessarily have a one.

Theorem 2.6.9 ([89, Theorem 2], [87, p. 114])**.** *Every planar poset with a zero has dimension at most* 3.

An application of Theorem 2.6.9 shows that the dimension of any poset whose cover graph is a tree is at most 3 [89, Corollary 6] [87, p. 117]. It suffices to prove that the poset obtained by adding a minimal element is planar. Lemma 2.1.20 yields the following result.

Corollary 2.6.10 ([89, Corollary 6])**.** *Every poset whose cover graph is a forest has dimension at most* 3.

In 2010, Felsner, Li, and Trotter showed that the dimension of a poset of height 2 whose cover graph is planar is at most 4 [30, Corollary 5.1]. This result was extended by Streib and Trotter who proved that the dimension of any poset with a planar cover graph is bounded in terms of its height [85, Theorem 3.2]. The upper bound was later reduced to a polynomial in the height of the poset by Kozik, Micek, and Trotter [66, Theorem 1]. Further, it was shown that if the cover graph of a poset is outerplanar, its dimension is at most 4 [32, Theorem 1.8].

Clearly, planar posets have planar cover graphs. Therefore all previous results also apply to planar posets. The bounds for planar posets can be improved even further. Joret, Micek, and Wiechert showed that the dimension of planar posets is bounded by a linear function in the height [58, Theorem 1].

Theorem 2.6.11 ([58, Theorem 1])**.** *If P is a planar poset then*

$$\dim(P) \leq 192h(P) + 96.$$

2.7 1-queue and 1-stack Dags

Undirected 1-queue graphs have been characterized by Heath and Rosenberg as arched leveled-planar graphs [48]. Similarly, dags with queue number 1 correspond to the class of arched leveled-planar dags [47]. While recognizing undirected 1-queue graphs is $\mathcal{NP}$-complete, dags with queue number 1 can be recognized in linear time [45, Section 2]. As we are interested in the connection between queue and stack number and dimension, we consider 1-queue and 1-stack posets and ask whether their dimension is bounded.

Bernhart and Kainen showed that undirected graphs with stack number at most 1 correspond to the class of outerplanar graphs [8, Theorem 2.5]. Considering 1-stack posets restricts the class even further.

Lemma 2.7.1 ([77, p. 211], [86, p. 190])**.** *Every stack in the stack layout of a poset P induces a forest in the cover graph of P.*

Proof. Consider a stack layout σ of the diagram of P and its corresponding partition of the edges into stacks E_i. By definition of a stack, we see that each E_i is crossing-free. Suppose some E_i contains a cycle C on vertices $a_0, \ldots, a_n$ which appear in the given order in σ. We show that the edges of the cycle correspond to edges $a_j a_{j+1}$ where indices are taken modulo $n + 1$.

Suppose there is some $s < n$ such that both neighbors of a_s on C are to the left of a_s in σ. Consider the edge e which connects a_s to its left-most neighbor a_ℓ of C. As E_i is crossing-free, none of the vertices of C below the edge e in σ is connected to any of the vertices to the left of a_ℓ or to the right of a_s in σ. This is a contradiction as C is a cycle. Similarly, we see that there is no $s > 0$ such that both neighbors of a_s on C are right of a_s in σ. Thus, the edges of C correspond to edges $a_j a_{j+1}$.

As $a_0 a_1, \ldots, a_{n-1} a_n$ are edges of the diagram of P, the edge $a_0 a_n$ is transitive. This is a contradiction as the diagram of P contains no transitive edges. Therefore, no E_i contains an (undirected) cycle which finally shows that each E_i induces a forest in the cover graph of P. □

As the cover graphs of 1-stack posets are forests, their dimension is at most 3.

Corollary 2.7.2. *If a poset has stack number* 1*, its dimension is at most* 3*.*

Proof. The diagram of P is a forest by Lemma 2.7.1. Therefore, its dimension is at most 3 by Theorem 2.6.10. □

If a graph is dense, i.e. the number of edges is relatively large compared to the number of vertices, it has large stack number [8, Theorem 3.3]. For posets, we obtain the following.

Lemma 2.7.3. *If P is a poset on $n \geq 2$ elements whose cover graph contains m edges, then*

$$\operatorname{sn}(P) \geq \frac{m}{n-1}.$$

Proof. Consider a k-stack layout of P where k denotes the stack number of P. Each stack contains at most $n-1$ edges as it forms a forest by Lemma 2.7.1. Thus, the cover graph of P has at most $k \cdot (n-1)$ edges in total. □

Similarly, a dense graph has large queue number as has been shown by Heath and Rosenberg. They proved that a 1-queue graph on n vertices contains at most $2n-3$ edges, thus a graph on n vertices and m edges has queue number at least $\frac{m}{2n-3}$ [48, Theorem 3.6]. Pemmaraju showed that the queue number of a graph is bounded from above and from below in terms of the number of edges and vertices [79, Theorem 2.15], thereby improving the lower bound established by Heath and Rosenberg to $\frac{m}{2n-1}$ [79, Corollary 2.17].

If we wish to reduce the queue number of a graph, it is necessary to reduce its density. The following operation provides a means of such an undertaking.

Definition 2.7.4. Let G be a dag. We say that we *subdivided* an edge $ab \in E(G)$ k times if we replaced it by a directed path $a\, x_1^{(ab)} \dots x_k^{(ab)}\, b$. A directed graph obtained from G by subdividing edges is called a *subdivision* of G. Note that such a subdivision is also a dag. The vertices $x_1^{(ab)}, \dots, x_k^{(ab)}$ are called *division vertices*; vertices that correspond to the vertices of G are called *original vertices*.

We refer to the dag we obtain from G by subdividing each edge exactly k times for some $k \in \mathbb{N}$ as the *k-subdivision* of G. A subdivision of a dag where every edge has been subdivded at most k times is a *$\preceq k$-subdivision*. All terms defined for dags are used in a similar way for undirected graphs and posets. A subdivision of a poset corresponds to a subdivision of its diagram.

Heath and Rosenberg showed that 1-queue graphs are in particular planar [48]. Conversely, Dujmović and Wood were able to prove that a graph has a 1-queue subdivision if and only if it is planar [23, Theorem 20]. One direction does also hold for dags if we consider upward planarity instead of planarity.

Proposition 2.7.5. *Every upward planar dag admits a 1-queue subdivision.*

Proof. Consider an upward planar drawing of an upward planar dag G. Let $\ell_1, \dots, \ell_k$ be horizontal lines ordered by increasing y-coordinate such that at least one vertex lies on each of the horizontal lines and every vertex of G lies on some line. Subdividing the edges of G at the intersections with the horizontal lines, we obtain a subdivision S of G.

We denote by v_x the x- and by v_y the y-coordinate of any vertex v of S. For $i \in [k]$, let L_i denote the vertices of S which lie on the horizontal line ℓ_i. We define σ as the topological ordering where vertices of S are primarily ordered by increasing y-coordinate and secondly by increasing x-coordinate.

It suffices to show that σ is a 1-queue layout. Suppose two edges $ab, a'b'$ nest in σ. We may assume $a \leq_\sigma a' \leq_\sigma b' \leq_\sigma b$. By definition of S, there exists an integer $i \in [k-1]$ such that $a, a' \in L_i$ and $b, b' \in L_{i+1}$. Thus, $a_y = a'_y$ and $b_y = b'_y$ follows. Further, by the ordering of the vertices a, b, a', b' in σ, we see that $a_x < a'_x$

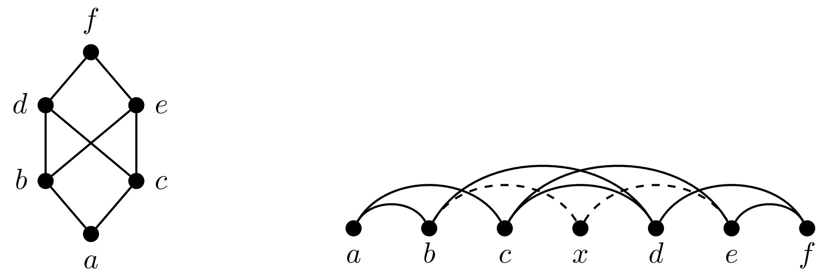


Figure 2.11: The poset represented on the left is non-planar. Subdividing the edge be, we obtain a poset which admits a 1-queue layout. The layout is represented on the right. The vertex x is a division vertex, dashed edges arose from subdivision.

and $b'_x < b_x$ which shows that the edges ab and $a'b'$ cross in the drawing of S. This yields a contradiction as the drawing of S is planar. □

Yet, a similar result does not hold for stack number. Indeed, there are planar posets which do not admit a 1-stack subdivision, think of the poset P whose diagram is a 4-cycle of height 3. Any subdivision of P has stack number 2.

Not only does every planar dag admit a 1-queue subdivision, in fact, every dag has a 2-queue subdivision as we will see in Corollary 2.10.6. However, there are also non-planar posets with 1-queue subdivisions.

Observation 2.7.6. *There exists a non-planar poset with a* 1*-queue subdivision.*

Proof. It is well-known that the poset represented on the left of Figure 2.11 is non-planar, see [16, Figure 9] for instance. Subdividing the edge be, we obtain a 1-queue subdivision witnessed by the queue layout represented on the right of Figure 2.11. □

Thus, the result of Dujmović and Wood [23, Theorem 20] does not hold in the directed setting.

While the dimension of 1-stack posets is at most 3, the dimension of 1-queue posets is unbounded.

Corollary 2.7.7. *There exists a family of posets with queue number* 1 *and unbounded dimension.*

Proof. By Proposition 2.7.5 there exists a 1-queue subdivision D_n of Kelly_n for every n. As Kelly_n is a subposet of D_n, we obtain $\dim(D_n) \geq n$ by Lemma 2.6.3. □

2.8 Queue and Stack Number of Subdivisions

The following definition is an adaptation of a definition given by Spinrad [83, p. 144]. It is of use when extending topological vertex orderings of a dag to a subdivision. If x is a division vertex of an edge ab, it has to be placed between the original vertices a and b in any topological ordering. Spinrad considered extensions of topological orderings where division vertices are placed directly after the left endpoint, or before the right endpoint of the edge they subdivided.

Definition 2.8.1. Let S be a subdivision of a dag G and let $x_1^{(ab)}, \ldots, x_k^{(ab)}$ denote the division vertices of some edge ab in the order they are encountered along ab. Consider a linear order σ' of the vertices of S. We say that the division vertices $x_1^{(ab)}, \ldots, x_k^{(ab)}$ are placed *low* with respect to the restriction of σ' to G if

$$a \leq_{\sigma'} x_1^{(ab)} \leq_{\sigma'} \cdots \leq_{\sigma'} x_k^{(ab)}$$

and no original vertex appears between a and $x_k^{(ab)}$. We say that they are placed *high* if

$$x_1^{(ab)} \leq_{\sigma'} \cdots \leq_{\sigma'} x_k^{(ab)} \leq_{\sigma'} b$$

and no original vertex appears between $x_1^{(ab)}$ and b.

Extending a linear extension of a poset to a subdivision by deciding whether to place the division vertices of every edge high or low, we obtain indeed a linear extension of the subdivision.

Lemma 2.8.2. *Let L be a linear extension of a poset P and let $E^\ell \cup E^h$ be a partition of the edges of the diagram of P. Any linear order L' of S that is an extension of L where all division vertices of edges that lie in E^ℓ are placed low and those of edges in E^h are placed high is a linear extension of S.*

Proof. The claim follows from the observation that division vertices of an edge ab are placed between the original vertices a and b in L'. $\square$

Note that the linear extension L' in Lemma 2.8.2 is not unique as we did not specify for instance the ordering of division vertices of distinct edges with the same right endpoint that have been placed high. Further, not every linear extension can be constructed in such a way.

Dujmović and Wood showed that the queue number of a 1-subdivision of a q-queue graph is at most $q + 1$ [23, Lemma 9] using a result of Dujmović, Pór, and Wood [21]. Further, they proved that the queue number of a 2-subdivision of a q-queue graph is at most $\lceil 2\sqrt{q}\,\rceil$ [23, Lemma 26]. In the directed setting, the queue number of any subdivision is bounded in terms of the queue number of the original graph. The proof is based on [23, Lemma 13].

Proposition 2.8.3. *Any subdivision S of a dag G satisfies*

$$\operatorname{qn}(S) \leq 2 \operatorname{qn}(G) + 2.$$

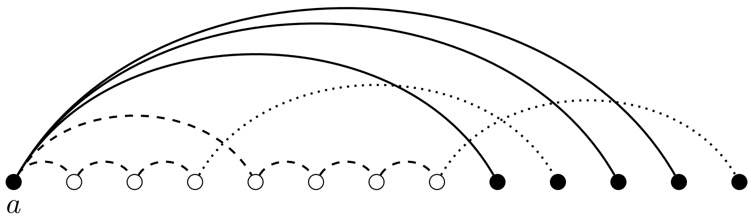


Figure 2.12: The vertex ordering σ' constructed in Proposition 2.8.3. Original vertices are represented in black, division vertices in white. The division vertices derived from edges whose left endpoint is a are placed immediately after a. The original vertices on the right are neighbors of a in the dag G. Dashed lines correspond to new, black lines to original, and dotted lines to old edges.

Proof. Let $q \coloneqq \mathrm{qn}(G)$ and consider a q-queue layout σ of G. We define a topological ordering σ' be the vertex ordering of S such that its restriction to the vertices of G is σ, division vertices of the same edge ab are placed consecutively according to their order on ab and all division vertices are placed low. Further, we require for two division vertices x and y of distinct edges ab and ab' having the same left endpoint that $x <_\sigma y$ if and only if $b <_\sigma b'$.

We call edges of S *original edges* if both endpoints correspond to original vertices. Edges where the right endpoint is a division vertex are *new edges*. The remaining edges are referred to as *old edges*. The layout σ is represented in Figure 2.12.

It is easy to see that the largest rainbow formed by new edges has size at most 2. Thus, we can assign all new edges to two queues. Further, as the restriction of σ' to G is σ, we see that q queues suffice for all original edges.

It remains to show that the old edges can be assigned to q more queues. Consider the queue assignment where every old edge inherits the queue of the edge it has been derived from. Suppose two old edges xa and yb are assigned to the same queue and nest. We may assume that $x \leq_{\sigma'} y \leq_{\sigma'} b \leq_{\sigma'} a$. Let $a'a$ and $b'b$ denote the edges from which xa and yb have been derived respectively.

Case 1. $a' = b'$. By definition of σ', we obtain $y \leq_{\sigma'} x \leq_{\sigma'} b \leq_{\sigma'} a$ as $b \leq_\sigma a$. This is a contradiction.

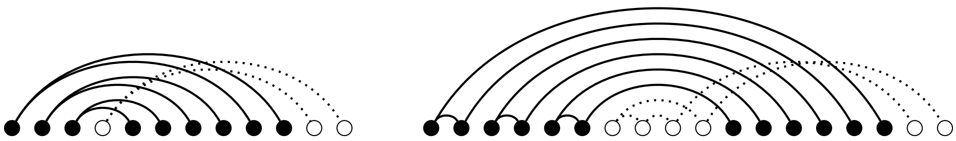
Case 2. $a' \neq b'$. As x and y have been placed low, we see that

$$a' \leq_{\sigma'} x \leq_{\sigma'} b' \leq_{\sigma'} y \leq_{\sigma'} b \leq_{\sigma'} a.$$

Thus, $a'a$ and $b'b$ nest in σ, which yields a contradiction.

Thus, q queues suffice for the old edges and we constructed a valid queue assignment using $2q + 2$ queues in total. $\square$

Observe that this result is best-possible if we extend a queue layout σ of a dag to a queue layout σ' of a subdivision as in the proof of Proposition 2.8.3, see Figure 2.13.



(a) A q-queue layout of a dag G. The black edges contain a q-rainbow.

(b) A $(2q+2)$-queue layout of the subdivision of G we obtain by subdividing every second black edge exactly once and subdividing the dashed edges once or twice respectively.

Figure 2.13: On the left, a q-queue layout σ of a dag G for $q = 3$ is represented. On the right we extended σ as in the proof of Proposition 2.8.3 to a $(2q+2)$-queue layout of a subdivision of G.

Indeed, the largest rainbow in the layout represented on the left of Figure 2.13 has size q, while the largest rainbow in the layout σ', represented on the right, has size $2q+2$.

Using Proposition 2.8.3, we may assume when considering a subdivision that every edge has been subdivided the same number of times at the cost of increasing the queue number slightly. This is of use in the proof of Lemma 2.8.5 for instance.

Similarly to queue number, the stack number of a subdivision of a dag G can be bounded by a function of the stack number of G. Our approach is based on a proof of Dujmović and Wood, who showed that the stack number of a 1-subdivision of an s-stack graph is at most $s+1$ [23, Lemma 13]. The proof is similar to the proof of Proposition 2.8.3.

Proposition 2.8.4. *Any subdivision S of a dag G satisfies* $\operatorname{sn}(S) \leq 2\operatorname{sn}(G)$.

Proof. Let s be the stack number of G and let σ be an s-stack layout. We define a topological ordering σ' of the vertices of S such that its restriction to the vertices of G is σ, division vertices of the same edge ab are placed consecutively according to their order on ab and all division vertices are placed low. Further, if x and y are division vertices of distinct edges ab and ac, we require that $x \leq_{\sigma'} y$ if and only if $c \leq_{\sigma} b$.

We call edges of S *original edges* if both endpoints correspond to original vertices. Edges where the right endpoint is a division vertex are *new edges*. The remaining edges are called *old edges*. The layout σ is represented in Figure 2.14.

Note that s stacks suffice for all original edges as the induced layout of the original vertices corresponds to σ. We may assign the new edges to any of these s stacks as no new edge crosses any other new edge and original edges and new edges do not intersect either.

It remains to show that s more stacks suffice for the old edges. Consider the stack assignment where every old edge inherits the stack of the edge it has been derived from. Suppose two old edges xa and yb which lie in the same stack cross. We may assume that $x \leq_{\sigma'} y \leq_{\sigma'} a \leq_{\sigma'} b$. Recall that the vertices x and y are division vertices

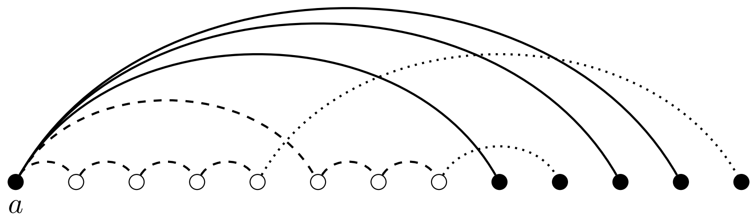


Figure 2.14: The vertex ordering σ' constructed in Proposition 2.8.4. Original vertices are represented in black, division vertices in white. The division vertices derived from edges whose left endpoint is a are placed immediately after a. The original vertices on the right are neighbors of a in the dag G. Dashed lines correspond to new, black lines to original, and dotted lines to old edges.

and the vertices a and b are original vertices. Let $a'a$ and $b'b$ be the edges xa and yb have been derived from respectively.

Case 1. $a' = b'$. By definition of σ', we obtain $y \leq_{\sigma'} x \leq_{\sigma'} a \leq_{\sigma'} b$ as $a \leq_\sigma b$. This is a contradiction.

Case 2. $a' \neq b'$. As x and y are placed low, we obtain

$$a' \leq_{\sigma'} x \leq_{\sigma'} b' \leq_{\sigma'} y \leq_{\sigma'} a \leq_{\sigma'} b.$$

Thus, the edges $a'a$ and $b'b$ cross in σ contradicting our assumption that xa and yb are assigned to the same stack.

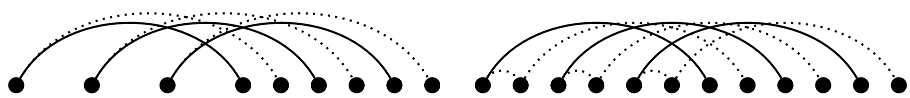
Thus, s more stacks suffice for the old edges. In total, we needed $2s$ stacks, thereby showing that σ' is indeed a $2s$-stack layout. □

If we extend the vertex ordering σ of an s-stack layout of a dag to a vertex ordering σ' of a subdivision as in the proof of Proposition 2.8.4, the smallest number of stacks required for σ' might indeed be $2s$; see Figure 2.15. The layout represented on the left requires only s stacks, while the layout of the subdivision contains a $2s$-twist, thus requires at least $2s$ stacks.

Dujmović and Wood bounded the (undirected) queue number of graphs in terms of the queue number of subdivisions were every edge has been subdivided at most h times. Explicitly, they showed that $\operatorname{qn}(G) \leq \frac{1}{2}(2\operatorname{qn}(S) + 2)^{2h} - 1$ in the undirected setting for every $\preceq h$-subdivision S of a graph G [23, Lemma 27].

We show a similar result which yields a slightly better bound for dags. Note however that the following result does not provide an improvement of the result of Dujmović and Wood as we only consider topological orderings of the vertices. Our approach is similar to the proof of the lower bound in Lemma 2.5.3.

Lemma 2.8.5. *If S is a $\preceq h$-subdivision of a dag G, then* $\operatorname{qn}(G) \leq (2\operatorname{qn}(S) + 2)^{h+1}$. *If S is an h-subdivision, then* $\operatorname{qn}(G) \leq \operatorname{qn}(S)^{h+1}$.



(a) An s-stack layout of a dag G. The layout contains an s-twist.

(b) A $2s$-stack layout of the subdivision of G we obtain by subdividing every dashed edge exactly once.

Figure 2.15: On the left, an s-stack layout of a dag G for $s = 3$ is represented. On the right we extended σ as in the proof of Proposition 2.8.4 to a $2s$-stack layout of a subdivision of G.

Proof. Let d be the queue number of G. We assume that every edge of S has been subdivided exactly h times. The general claim follows from Proposition 2.8.3 once we established the bound in the special case.

Consider any topological ordering σ of the vertices of S. We show that the ordering σ admits a rainbow of size at least $d^{1/(h+1)}$.

As G has queue number d, we see that the topological ordering of the original vertices induced by σ contains a d-rainbow, i.e. there are original vertices

$$a_1 \leq_\sigma \cdots \leq_\sigma a_d \leq_\sigma b_d \cdots \leq_\sigma b_1$$

and $a_i b_i$ is an edge in G for all $i \in [d]$.

For $i \in [d]$ and $s \in [h]$ let c_i^s be the s-th division vertex of the edge $a_i b_i$. Further, we define $b := d^{1/(h+1)}$ and $a_s := \frac{d}{b^s}$ for $s \in [h]$. It suffices to prove that σ contains a rainbow of size at least b.

We prove by induction on s that the ordering σ contains a $\lceil b \rceil$-rainbow or there exists an increasing sequence (i_k) of length $\ell \geq a_s$ such that $c_{i_1}^s \leq_\sigma \cdots \leq_\sigma c_{i_\ell}^s$ for all $s \in [h]$.

If $s = 1$, consider the ordering of the vertices $c_1^1, \ldots, c_d^1$ induced by the ordering σ. By Theorem 2.4.2, there exists a monotone sequence (i_k) of length ℓ such that

$$c_{i_1}^1 \leq_\sigma \cdots \leq_\sigma c_{i_\ell}^1$$

where $\ell \geq a_1$ if (i_k) is increasing, and $\ell \geq b$ otherwise. If (i_k) is increasing, the claim follows immediately. Otherwise, the sequence is decreasing and we obtain

$$a_{i_\ell} \leq_\sigma \cdots \leq_\sigma a_{i_1} \leq_\sigma c_{i_1}^1 \leq_\sigma \cdots \leq_\sigma c_{i_\ell}^1$$

as $a_{i_1} c_{i_1}^1$ is an edge in S and σ is a topological ordering. As $a_{i_k} c_{i_k}^1 \in E(S)$ for all $k \in [\ell]$, these edges form a ℓ-rainbow and the claim follows since $\ell \geq b$.

Suppose the claim holds for some $s \in [h-1]$. If σ contains a $\lceil b \rceil$-rainbow then we are already done. Otherwise, there exists an increasing sequence (i_k) of length $\ell \geq a_s$ such that $c_{i_1}^s \leq_\sigma \cdots \leq_\sigma c_{i_\ell}^s$. Now we proceed as in the base case. The role of the vertices a_i for $i \in [d]$ is now played by the vertices $c_{i_k}^s$ for $k \in [\ell]$.

We consider the ordering of the vertices $c_{i_1}^{s+1}, \dots, c_{i_\ell}^{s+1}$ induced by σ. By Theorem 2.4.2, there exists a monotone subsequence (j_k) of (i_k) of length $\hat{\ell}$ such that

$$c_{j_1}^{s+1} \leq_\sigma \cdots \leq_\sigma c_{j_{\hat{\ell}}}^{s+1}$$

where $\hat{\ell} \geq a_{s+1}$ if (j_k) is increasing and $\hat{\ell} \geq b$ otherwise. If (j_k) is increasing, the claim follows immediately. Otherwise (j_k) is decreasing and we see that the edges $c_{j_k}^s c_{j_k}^{s+1}$ form a rainbow of size at least b.

Suppose σ contains no $\lceil b \rceil$-rainbow. Setting $s = h$ it follows from the claim above that there is an increasing sequence (i_k) of length $\ell \geq a_h$ such that $c_{i_1}^h \leq_\sigma \cdots \leq_\sigma c_{i_\ell}^h$. As the division vertex $c_{i_k}^h$ is adjacent to the original vertex b_{i_k} for all $k \in [\ell]$, we obtain

$$c_{i_1}^h \leq_\sigma \cdots \leq_\sigma c_{i_\ell}^h \leq_\sigma b_{i_\ell} \leq_\sigma \cdots \leq_\sigma b_{i_1}.$$

The edges $c_{i_k}^h b_{i_k}$ form an ℓ-rainbow. This yields a contradiction to our assumption as $\ell \geq a_h$ and $a_h = b$.

Therefore, σ contains a $\lceil b \rceil$-rainbow. As σ was an arbitrary topological ordering, we obtain $\operatorname{qn}(S) \geq d^{1/(h+1)}$. This finally yields $\operatorname{qn}(G) \leq \operatorname{qn}(S)^{h+1}$. □

The result of Dujmović and Wood [23, Lemma 27] which provides a similar result to Lemma 2.8.5 in the undirected setting has been extended to shallow minors by Hickingbotham and Wood [49, Lemma 13].

The lemma above essentially shows that subdivisions of a dag with large queue number still have large queue number as long as every edge has been subdivided a small number of times. However, if we subdivide edges often, we might obtain a dag with small queue number as we will see in Corollary 2.10.6.

In Lemma 2.8.5, we showed that the queue number of $\preceq h$-subdivisions cannot be arbitrarily small for constant h. Yet, there are posets for which the queue number of $\preceq h$-subdivisions is relatively small when compared to the queue number of the initial poset. The family of wings provides an example.

Proposition 2.8.6. *For every $n \in \mathbb{N}$, there exists an $(n+1)$-queue poset P and a $\preceq h$-subdivision S of P such that $\operatorname{qn}(P) \geq (\operatorname{qn}(S) - 2)^{(h+2)/2}$.*

Proof. Let $n \in \mathbb{N}$ and consider the wing W of size $d := n^2 + 1$. By Lemma 2.5.3, we have $\operatorname{qn}(W) = n + 1$. Consider the subdivision S of W where each edge $c_i b_i$ has been subdivided h times. Let $q := d^{1/(h+2)}$.

We construct a $(q+2)$-queue layout of S. For $i \in [d]$, we denote by $c_i^1, c_i^2, \dots, c_i^h$ the division vertices of the edge $c_i b_i$ in the order they appear along the former edge. Let $c_i^0 := c_i$ for $i \in [d]$. We partition the vertices of S into $h + 3$ classes $M_{-1} := \{a_1, \dots, a_d\}$, $M_s := \{c_1^s, \dots, c_d^s\}$ for $0 \leq s \leq h$ and $M_{h+1} := \{b_1, \dots, b_d\}$.

For $s \in \{0, \dots, h\}$, we define

$$b'_s := q^{s+1}, \quad a'_s := \frac{d}{q^{s+1}}.$$

By Proposition 2.4.4, there exists for every $s \in \{0, \dots, h\}$ a sequence $(i_k^{(s)})$ of the integers $[d]$ such that a longest increasing subsequence has length at most $\lceil a'_s \rceil$ and a longest decreasing subsequence has length at most $\lceil b'_s \rceil$. Let σ_s be the ordering

$$c^s_{i_1^{(s)}} \leq c^s_{i_2^{(s)}} \leq \dots c^s_{i_d^{(s)}}$$

on the vertices of M_s for $s \in \{0, \dots, h\}$. We define σ as the topological ordering

$$a_1 \leq a_2 \leq \dots \leq a_d \leq \sigma_0 \leq \sigma_1 \leq \dots \leq \sigma_h \leq b_d \leq \dots \leq b_1.$$

Note that a rainbow in σ of largest size is formed by edges between vertices of classes M_i and M_{i+1} for some i and at most one edge within one of the two classes. We show that such a rainbow has size at most $q + 2$.

Consider a largest rainbow between M_{-1} and M_0. Such a rainbow contains at most one of the edges $a_i a_{i+1}$. As the vertices a_i are in increasing order in σ, the endpoints c_i of the edges forming a largest rainbow have to be in decreasing order. A longest decreasing sequence in $(i_k^{(0)})$ has length $\lceil b'_0 \rceil = \lceil q \rceil$. Thus, the rainbow has size at most $\lceil q \rceil + 1$ and the claim follows. Similarly, we see that a rainbow between M_h and M_{h+1} has size at most $\lceil a'_h \rceil + 1 = \lceil q \rceil + 1$ as it requires an increasing sequence in $(i_k^{(h)})$ and contains at most one edge $b_{i+1} b_i$.

Now consider a largest rainbow between classes M_s and M_{s+1} for some s where $0 \leq s \leq h - 1$. If ℓ is its size, there exists a subsequence (r_m) of $(i_k^{(s)})$ such that

$$c^s_{r_1} \leq c^s_{r_2} \leq \dots c^s_{r_\ell} \leq c^{s+1}_{r_\ell} \leq \dots \leq c^{s+1}_{r_1}$$

in σ. The sequence (r_m) might not be a monotone. Yet, it is a subsequence of $(i_k^{(s)})$ that is reversed in $(i_k^{(s+1)})$. As $a'_s \geq a'_{s+1}$, we see by Observation 2.4.5 that

$$\ell \leq \frac{b'_{s+1}}{b'_s} + 2 = q + 2.$$

Therefore, a largest rainbow in σ has size at most

$$q + 2 = d^{1/(h+2)} + 2 \leq (n+1)^{2/(h+2)} + 2,$$

and we finally obtain $\mathrm{qn}(S) \leq \mathrm{qn}(W)^{2/(h+2)} + 2$. □

In fact, using the same argument, we can also show that Lemma 2.8.5 is relatively tight for dags. If we consider a d-rainbow and add all spine edges, i.e. edges between successive vertices in a topological order, the dag G we obtain has queue number d and admits only one topological ordering. If we subdivide each edge of the rainbow exactly $h + 1$ times, we are in the situation of the proof above. Using the same construction, we obtain the following.

Proposition 2.8.7. *For every $n \in \mathbb{N}$, there exists an n-queue dag G and a $\preceq h$-subdivision S of G such that* $\mathrm{qn}(G) \geq (\mathrm{qn}(S) - 2)^{h+1}$.

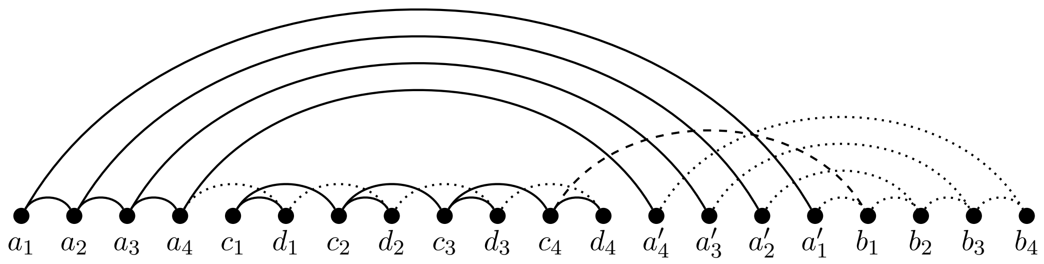


Figure 2.16: A 3-stack layout of a $\preceq$1-subdivision of the double fin of size 4. Edges of the same color are assigned to the same stack. The vertices a'_i are division vertices of the edges a_ib_i of the double fin.

Blankenship and Oporowski conjectured that a similar result to Lemma 2.8.5 holds in the undirected setting for stack number, providing a proof for complete and complete bipartite graphs [12, Conjecture 1.4]. However, in 2022, Dujmović et al. showed that the conjecture does not hold [19, p. 4].

In the directed setting, the conjecture does not hold either. We construct posets with arbitrarily large stack number that have $\preceq$1-subdivisions with stack number at most 3. Thus the (directed) stack number of a graph cannot be bounded from above in terms of the stack number of a subdivision and the number of times edges have been subdivided.

Examples of dags can be easily constructed. Adding all spine edges to an n-twist results in a dag which has only one topological ordering. As this ordering contains an n-twist, its stack number is at least n. Yet, if we subdivide each edge of the twist once, we obtain a dag with stack number at most 2. Note though that the given example is not the diagram of a poset as it contains transitive edges. An example for posets is the family of double fins.

Observation 2.8.8. *For every integer $n \geq 2$ there exists a poset with stack number n which admits a $\preceq 1$-subdivision that has stack number at most 3.*

Proof. Consider a double fin D of size $n \geq 2$. By Lemma 2.3.3, we have $\mathrm{sn}(D) = n$. Let S be the subdivision of D where all edges a_ib_i have been subdivided once. For $i \in [n]$, we denote by a'_i the division vertex of the edge a_ib_i. The subdivision S admits a 3-stack layout, see Figure 2.16. Thus, $\mathrm{sn}(S) \leq 3$ follows. □

2.9 Dimension of Subdivisions

Clearly, the Dushnik-Miller dimension of a subdivision S of a poset P is lower bounded by $\dim(P)$ as the restriction of any realizer of S to P yields a realizer of P. One might suspect that if we consider a realizer of P, we could get a realizer of S by defining for every linear order of the realizer two linear orders, one where all division vertices have been placed low and another one where all division vertices have been placed high, thus bounding $\dim(S)$ by $2\dim(P)$. However, in the situation represented in Figure 2.17 we see that the division vertex x appears before the division vertex y in all linear orders we constructed in such a way even though x and y are incomparable [83, p. 144]. In fact, the dimension of a subdivision is not bounded by the dimension of the initial poset as has been shown by Spinrad, see Proposition 2.9.3. In order to reverse a pair (x, y) as represented in Figure 2.17, it is useful to consider the virtual height of the two division vertices.

Definition 2.9.1. Let G be a dag. We define the *height* of a vertex v of G as the maximum number of vertices on a directed path ending in v and denote it by $h(v)$. The height of G, denoted by $h(G)$, is the maximum height of its elements.

For a subdivision S of G and a division vertex x of an edge ab, we define the *virtual height* of x with respect to G as the height of b in G. It is denoted by $h_{\mathrm{v}}(x)$. Note that the virtual height of division vertices of the same edge of G is identical while their heights in S differ. Further, the virtual height is bounded by $h(G)$.

If P is a poset, the height of an element x of P corresponds to its height in the diagram of P, i.e. the size of a longest chain ending in x.

Note that the height of a poset, as defined in Definition 2.1.1, corresponds to the height of its diagram.

Based on the bit representation of the virtual height of division vertices, Spinrad constructed linear extensions which reverse all pairs of incomparable division vertices, thereby showing that the dimension of a subdivision is bounded in terms of the dimension of the initial poset and its height.

Proposition 2.9.2 ([83, p. 145])**.** *For any subdivision S of a poset P, we have*

$$\dim(S) \leq \lfloor \log(h(P)) + 3 \rfloor \dim(P) + 1.$$

Proof. Let d be the dimension of P and $\mathcal{R}$ a minimum realizer. For every linear extension $L \in \mathcal{R}$, we define two linear extensions L^{low} and L^{high} of S where all division vertices have been placed low or high respectively. Note that the set

$$\mathcal{R}' \coloneqq \{L^{\mathrm{low}}, L^{\mathrm{high}} \mid L \in \mathcal{R}\}$$

realizes all non-relations between original vertices, and between original vertices and division vertices.

Suppose $\mathcal{R}'$ is not a realizer of S, then there are division vertices x, y in S with $x \parallel y$ such that $x \leq_{L'} y$ for all $L' \in \mathcal{R}'$. In particular x and y have to originate from distinct edges ab and cd. If $a \neq c$, we have $a \leq_L c$ for all $L \in \mathcal{R}$ as $a \leq_{L^{\mathrm{low}}} x \leq_{L^{\mathrm{low}}} c \leq_{L^{\mathrm{low}}} y$ showing that $a \leq c$ in P. If $a = c$, this clearly holds. Similarly, we obtain $b \leq d$ in P if we consider L^{high} for $L \in \mathcal{R}$. We say that the division vertices x and y form an *x-y-rhombus*; see Figure 2.17. Therefore, it suffices to find a set of linear extensions of S that contains for every x-y-rhombus a linear extension L' such that $y \leq_{L'} x$.

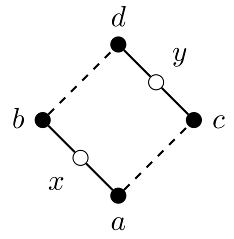


Figure 2.17: An x-y-rhombus. Original vertices are represented in black. The division vertices x and y of the edges ab and cd are represented in white. The directed paths corresponding to the relations $a \leq b$ and $c \leq d$ are represented by dashed lines.

Fix a linear extension $L_1 \in \mathcal{R}$ and let $\widetilde{L_1^{\text{high}}}$ denote the linear extension obtained by placing all division vertices high with respect to L_1 and reversing the order of division vertices with respect to L_1^{high} for which the order is not specified by L_1. Let

$$\mathcal{R}^h := \{L^k \mid L \in \mathcal{R}, k \in [\lfloor \log(h(P)) \rfloor + 1]\} \cup \{L_1^{\text{high}}, \widetilde{L_1^{\text{high}}}\}$$

where L^k denotes the linear extension of S where all division vertices x with a 0 in the k-th bit of the virtual height $h_{\text{v}}(x)$ are placed high and division vertices with a 1 in the k-th bit are placed low with respect to L. Further, we require that L^k has the following property. Whenever the order of division vertices x and y is not specified by the preceding property and x is placed high while y is placed low, we have $y \leq_{L^k} x$.

Consider an x-y-rhombus in S. We show that there exists a linear extension $L^k \in \mathcal{R}^h$ such that $y \leq_{L^k} x$. Let ab and cd be the edges that have been subdivided by x and y respectively. If $b \leq c$ in P, then $x \leq b \leq c \leq y$ in S which is a contradiction to $x \parallel y$. Therefore, there exists a linear extension $L \in \mathcal{R}$ such that $c \leq_L b$. If there is a $k \in [\lfloor \log(h(P)) + 1 \rfloor]$ such that the k-th bit of $h_{\text{v}}(x)$ is 0 and the k-th bit of $h_{\text{v}}(y)$ is 1, then we obtain

$$c \leq_{L^k} y \leq_{L^k} x \leq_{L^k} b$$

showing that the pair (x, y) is reversed by $\mathcal{R}^h$. Recall that $b \leq d$ in an x-y-rhombus. Thus, any chain in P ending in b can be extended to a chain ending in d. This observation shows that $h_{\text{v}}(x) \leq h_{\text{v}}(y)$; equality holds if b and d coincide. If $h_{\text{v}}(x) < h_{\text{v}}(y)$ the requested value of k exists. Otherwise, we have $b = d$ and the pair (x, y) is reversed by L_1^{high} or $\widetilde{L_1^{\text{high}}}$. Hence, $\mathcal{R}' \cup \mathcal{R}^h$ is a realizer of S and as

$$\left| \mathcal{R}' \cup \mathcal{R}^h \right| \leq 2d + (\lfloor \log(h(P)) + 1 \rfloor)d + 1$$

the claim follows. $\square$

To some extent the dimension of a subdivision does not exceed the dimension of the initial poset. In order to formulate the actual statement, we need the notion of the completion of a poset. Intuitively, it is the smallest lattice which contains the poset.

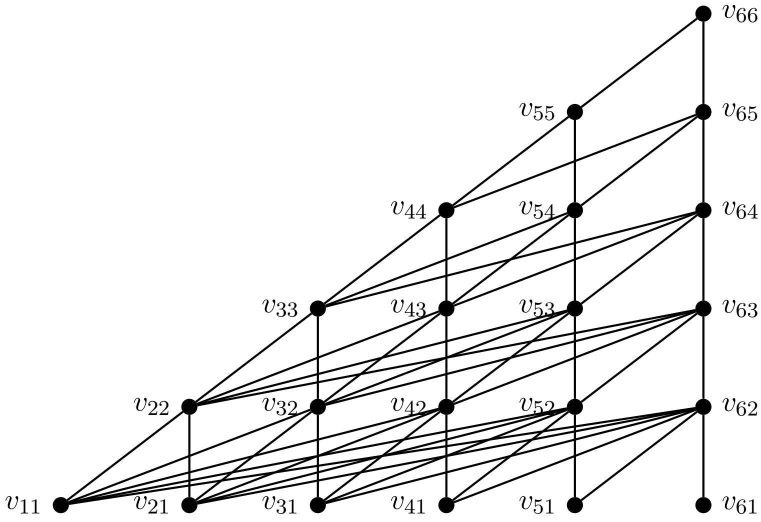


Figure 2.18: [83, Figure 4] A representation of the poset given in the proof of Proposition 2.9.3 for $n = 6$ based on the figure in [83, p. 146].

Lee et al. were able to prove that the dimension of any subdivision of the completion of a poset P equals the dimension of P [67, Theorem 1], thereby establishing another relationship between subdivisions and dimension.

Spinrad showed that every interval order is a subdivision of a 2-dimensional poset. As interval orders have arbitrarily large dimension, it follows that the dimension of a subdivision is not bounded in terms of the dimension of the initial poset.

Proposition 2.9.3 ([83, p. 146])**.** *For every $n \geq 2$, there exists a 2-dimensional poset P such that any interval order with an open interval representation with n distinct endpoints is a subdivision of P.*

In particular, for every $k \geq 4$, there exists a subdivision S of a 2-dimensional poset P such that $\dim(S) \geq \log\log(k)$.

Proof. Consider the poset P on elements $\{v_{i,j} \mid i \in [n], j \in [i]\}$ where for two distinct elements $v_{i,j}, v_{k,\ell}$ we have $v_{i,j} \leq_P v_{k,\ell}$ if and only if $i \leq k, j < \ell$; see Figure 2.18.

We first show that P is 2-dimensional. Clearly, the poset P is not a chain as it contains incomparable elements. Therefore, it suffices to show the upper bound on the dimension of P. We define two linear extensions

$$L_1 \coloneqq v_{1,1} \leq v_{2,1} \leq v_{2,2} \leq v_{3,1} \leq \cdots \leq v_{3,3} \leq \cdots \leq v_{n,n}$$
$$L_2 \coloneqq v_{n,1} \leq v_{n-1,1} \leq \cdots \leq v_{1,1} \leq v_{n,2} \leq \cdots \leq v_{2,2} \leq v_{n,3} \leq \cdots \leq v_{3,3} \leq \cdots \leq v_{n,n}.$$

Let $(v_{i,j}, v_{k,\ell})$ be a pair of incomparable elements of P. Note that we have $k < i$ or $\ell < j$, thus the incomparable pair is reversed by L_1 or L_2 respectively. Therefore, the set $\{L_1, L_2\}$ is indeed a realizer of P.

Let S be an interval order with an open interval representation with n distinct endpoints. We may assume that the n endpoints correspond to $[n]$. It remains to show that S is a subdivision of P. Consider a vertex u of S that is represented by an interval (i, j) in the interval representation of S. Note that $i+1 \leq j \leq n$. We assign u to the division vertex obtained by subdividing the edge $v_{i,i}v_{j,i+1}$ in the diagram of P.

We need to show that the given embedding of S in P respects the structure of S. Let u and w be elements of S corresponding to intervals (i, j) and (k, ℓ) respectively. We consider several cases.

Case 1. $j < k$. As we have $i + 1 \leq j < k$, we obtain $v_{j,i+1} \leq_P v_{k,k}$.

Case 2. $j = k$. If $i + 1 = j$, we obtain $v_{j,i+1} \leq_P v_{k,k}$ as the two elements coincide. Otherwise we have $i + 1 < j = k$ and $v_{j,i+1} \leq_P v_{k,k}$ follows by definition of P.

Case 3. $j > k$. We obtain $v_{j,i+1} \not\leq_P v_{k,k}$.

As $v_{j,i+1} \leq_P v_{k,k}$ if and only if $j \leq k$, we see that $u \leq w$ in the embedding of S in P if and only if the interval (i, j) precedes (k, ℓ). This observation shows that S is a subdivision of P.

As the universal interval order I_k with $k \geq 4$ has an open interval representation with n distinct endpoints for sufficiently large n and we have $\dim(I_k) \geq \log\log(k)$ by Lemma 2.1.17, the claim follows. □

If we subdivide each edge sufficiently often, the queue and stack number are constant, see Corollary 2.10.6. Yet, the dimension might increase. The following corollary shows how these two parameters interact for subdivisions of the standard example.

Corollary 2.9.4. *For $d \geq 2$, the dimension of a subdivision S of the standard example S_d is bounded by a function of the queue number and height of S, more precisely*

$$\dim(S) \leq 8 \cdot (2\,\mathrm{qn}(S) + 2)^{h(S)-1} + 5.$$

Proof. Let h be the maximum number of times an edge of S_d has been subdivided in S. As $\mathrm{qn}(S_d) \geq \lfloor \frac{d}{2} \rfloor$ by Lemma 2.2.3, we obtain by Lemma 2.8.5 $\lfloor \frac{d}{2} \rfloor \leq (2\,\mathrm{qn}(S) + 2)^{h+1}$ which yields the upper bound

$$d \leq 2(2\,\mathrm{qn}(S) + 2)^{h+1} + 1.$$

for the dimension of S_d. Further, as $h(S_d) = 2$, we obtain by Proposition 2.9.2

$$\dim(S) \leq \lfloor \log(2) + 3 \rfloor \cdot d + 1 \leq \lfloor \log(2) + 3 \rfloor \cdot (2(2\,\mathrm{qn}(S) + 2)^{h+1} + 1) + 1$$

which yields the claim as $h(S) = h + 2$ and $\lfloor \log(2) + 3 \rfloor = 4$. □

In fact, the dimension of any subdivision of S_n is equal to n [67, Theorem 2]. The approach above merely exemplifies how we can bound the dimension of a subdivision in terms of its queue number and height if we already know that there exists such a bound for the initial poset.

Actually, the dimension of any poset is bounded in terms of its queue number and height, as we will see in Section 2.11. However, the general bound is astronomical.

2.10 Subdivisions with small Queue or Stack Number

The aim of this section is to show that every dag admits a subdivision with small queue and stack number. In order to prove the result, we consider upward drawings of dags together with horizontal lines fulfilling certain properties.

Definition 2.10.1. We call crossings and vertices of an upward drawing of a dag G *events* and say that they occur at the corresponding y-coordinate of the drawing.

An upward drawing of G together with horizontal lines $\ell_1, \ell_1', \ell_2, \ell_2', \dots, \ell_t, \ell_t'$ of increasing y-coordinate is called a *horizontal k-division* for $k \geq 1$ if

(i) no event lies on one of the horizontal lines

(ii) no event occurs between ℓ_i' and ℓ_{i+1} for any i

(iii) for all i, exactly one of the two following properties holds:

a) only vertex-events occur between ℓ_i and ℓ_i' and they form an antichain of length at least one

b) only edge-crossing-events occur between ℓ_i and ℓ_i', and at most k edges drawn between ℓ_i and ℓ_i' pairwise intersect.

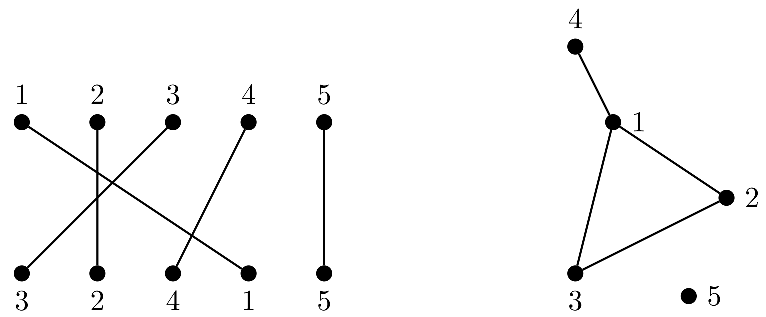
The subdivision we are interested in is obtained by subdividing each edge at its intersections with horizontal lines of a k-division. When constructing queue and stack layouts of such a subdivision, it will be useful to bound the number of colors needed for an edge-coloring of the edges drawn between horizontal lines ℓ_i and ℓ_i' in a horizontal k-division where no two edges of the same color intersect. If k such edges pairwise intersect, we clearly need at least k colors. Actually, k colors suffice as we will see in Lemma 2.10.4. The problem is closely related to the chromatic number.

In general, it is difficult to determine the chromatic number of an (undirected) graph [63, p. 94]. While the largest size of a clique provides a trivial lower bound on the chromatic number, these two numbers might differ largely as has been shown by Mycielski [72]. In 1961, Berge proposed the definition of perfect graphs, a class of graphs for which clique number and chromatic number coincide [7]. An example of such graphs are permutation graphs [42, Chapter 7].

Definition 2.10.2 ([42, Chapter 7])**.** Let π be a permutation of the numbers from 1 to n. Writing the numbers from 1 to n horizontally from left to right, and below the numbers $\pi(1), \dots, \pi(n)$ and connecting numbers of the same value with a straight line segment, we obtain a graph that is called the *matching graph* of π, see Figure 2.19a.

The straight line segments of the matching graph correspond to vertices of the *permutation graph* of π. The *permutation graph* is the graph on vertices $1, \dots, n$ where two vertices are adjacent if and only if their corresponding segments intersect in the matching graph, see Figure 2.19b.

Indeed, the maximum size of a clique provides an upper bound for the chromatic number of a permutation graph as has been shown by Even, Pnueli, and Lempel [29, p. 409].



(a) The matching graph of π. (b) The permutation graph of π.

Figure 2.19: The matching graph and the permutation graph of the permutation $\pi = 3\ 2\ 4\ 1\ 5$.

Lemma 2.10.3 ([29, p. 409]). *The chromatic number of a permutation graph is equal to the maximum size of a clique.*

As proper vertex-colorings of a permutation graph induce edge-colorings of the corresponding matching graph where no two edges of the same color cross, we obtain the following.

Lemma 2.10.4. *If H is a matching graph and at most k edges intersect pairwise, there exists a k-coloring of the edges such that no two edges of the same color intersect.*

Proof. Note that the largest clique in the corresponding permutation graph G has size at most k. By Lemma 2.10.3, there exists a proper k-vertex coloring of G. Coloring the edges of H with the color of the corresponding vertices in G, we obtain an edge-coloring of H where no two edges of the same color intersect. $\square$

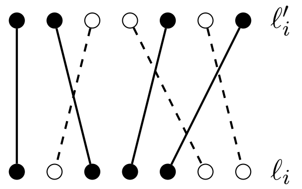
Observing that the segments drawn between horizontal lines ℓ_i and ℓ'_i of a horizontal division form a matching graph, we obtain the following.

Lemma 2.10.5. *If a dag G admits a horizontal k-division for $k \geq 2$, then the subdivision S of G we obtain by subdividing each edge at its intersection points with the horizontal lines admits a k-queue layout.*

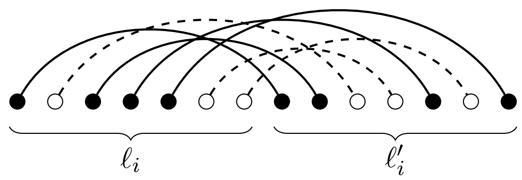
Proof. Consider a horizontal k-division of G and let $\ell_1, \ell'_1, \dots, \ell_t, \ell'_t$ be the corresponding horizontal lines of increasing y-coordinate. Define σ as the topological ordering where vertices of S are primarily ordered by increasing y- and secondly by increasing x-coordinate.

We show that σ admits a k-queue assignment. Let H_i be the graph induced by the vertices which lie between the lines ℓ_i and ℓ'_i.

Case 1. Only crossing events occur between ℓ_i and ℓ'_i. Possibly, no crossing occurs. Note that the graph H_i is a matching graph. By Lemma 2.10.4, we can color the edges of H_i with k colors such that no two crossing edges share the same color, see



(a) An edge-coloring of the matching graph H_i where no two edges of the same color intersect.



(b) A queue assignment of the edges between the horizontal lines ℓ_i and ℓ_i' induced by the edge-coloring of H_i. Edges of the same color are assigned to the same queue.

Figure 2.20: An illustration of the correspondence between edge-colorings of the matching graph H_i and a valid queue assignment illustrated for a horizontal 2-division and the case where only crossing events occur between the horizontal lines ℓ_i and ℓ_i'.

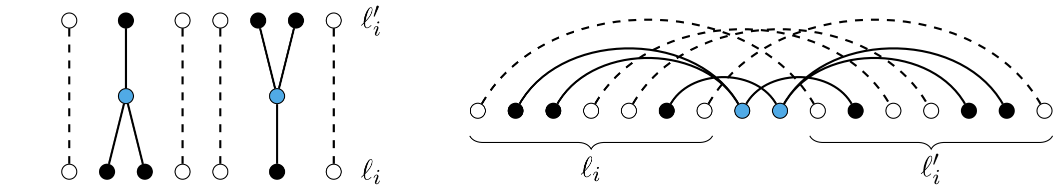
Figure 2.20. By definition of σ, no edges of the same color nest. Therefore, the edge-coloring induces a valid queue assignment.

Case 2. Only vertex events occur between ℓ_i and ℓ_i'. As the corresponding vertices form an antichain, we may assume that all these events occur at the same height. We color edges of H_i incident to an original vertex in green and all other edges in red, see Figure 2.21. Recall that no two edges between ℓ_i and ℓ_i' cross. Thus, the red edges form a twist in σ. Similarly, all edges which end in an original vertex intersect all edges ending in a different original vertex in σ as we assumed that all original vertices have the same height. In particular, they do not nest. The same is true for edges which start in an original vertex by definition of the vertex ordering. Note that any edge which ends in an original vertex cannot nest with any edge starting in another original vertex, as the original vertices are placed after all vertices lying on ℓ_i and before all vertices lying on ℓ_i'. Therefore, we may assign all edges of the same color to one queue.

Note that the edges of H_i lie completely to the left of the edges of H_j in σ for $i < j$. It follows that we only needed k queues so far as $k \geq 2$. As no event occurs between ℓ_i' and ℓ_{i+1} for any i, we see that the edges between these two horizontal lines form a twist in σ. They cannot nest with any edge that ends on ℓ_i' or starts on ℓ_{i+1}. Thus, we may assign these edges to any of the defined queues.

As we used k queues in total, σ is indeed a k-queue layout. □

The proof of the lemma above is somewhat similar to Proposition 2.7.5. In both proofs, we obtain a subdivision of the initial poset by subdividing all edges at their intersections with horizontal lines such that the subdivision has constant queue number. However, while we defined in Proposition 2.7.5 horizontal lines such that each vertex lies on such a line, we now require that no vertex does. This enables us to handle crossing events at the cost of needing possibly two queues instead of one for edges between two horizontal lines ℓ_i and ℓ_i' enclosing vertex events. If we also



(a) The edge-coloring of the matching graph H_i. Green edges are represented by black lines, red edges by dashed lines.

(b) A queue assignment of the edges between the horizontal lines ℓ_i and ℓ'_i induced by the edge-coloring of H_i. Edges of the same color are assigned to the same queue.

Figure 2.21: An illustration of the correspondence between edge-colorings of the matching graph H_i and a valid queue assignment illustrated for a horizontal k-division where only vertex events occur between the horizontal lines ℓ_i and ℓ'_i. Original vertices are represented in blue.

subdivide all edges between such lines ℓ_i and ℓ'_i which are not adjacent to an original vertex, we would only need one queue for all edges between ℓ_i and ℓ'_i. Thus, we could extend Lemma 2.10.5 to horizontal k-divisions for $k \geq 1$ if we consider different subdivisions.

Dujmović and Wood showed that every undirected graph admits a q-queue subdivision [23, Theorem 4 and 5] for any $q \geq 2$, in particular giving logarithmic lower and upper bounds in the queue number of the initial graph on the number of subdivisions per edge. We consider dags instead.

Corollary 2.10.6. *Every dag has a* 2*-queue subdivision.*

Proof. Consider any upward drawing of a dag G. Without loss of generality, we may assume that at most two edges intersect in a single point, and that no two events occur at the same height. We can easily construct a horizontal 2-division. By Lemma 2.10.5, there exists a subdivision of G that has queue number at most 2. □

As Heath and Rosenberg showed that all 1-queue graphs are planar, we see that no non-planar graph admits a 1-queue subdivision [48]. This shows that the result above is tight.

Every *undirected* graph has a 3-stack subdivision as has been shown by Atneosen [2, 12, 23].

Observation 2.10.7 ([2] [12, Theorem 1.2] [23, Theorem 1])**.** *Every undirected graph admits a* 3*-stack subdivision.*

Proof. Consider an arbitrary vertex ordering σ of an undirected graph G. Let G'' be the subdivision of G where every edge is subdivided twice. Note that every division vertex is adjacent to exactly one original vertex which we call its original neighbor. We obtain a vertex ordering σ'' of G'' by extending σ such that every division vertex is placed directly after its original neighbor, i.e. these two vertices may only be separated in σ'' by other division vertices having the same original neighbor. As no edges between original vertices and division vertices cross in σ'', we can assign all these edges to the same stack. The remaining edges form a matching. Clearly, we can draw the remaining edges as non-intersecting curves in the plane. Whenever a curve crosses the x-axis, we subdivide the corresponding edge. Assigning the resulting edges above the x-axis to one stack and the edges below to another yields a 3-stack subdivision. □

The proof relies on the fact that there is no restriction on the placement of division vertices in the extension of a linear order of the original graph to a linear order of the subdivision. In particular, we are allowed to place division vertices of an edge ab before a or after b. Thus, it is not clear whether the approach of Atneosen can be adapted to directed graphs.

Nevertheless, a similar result to Observation 2.10.7 holds for directed graphs as we will see in Corollary 2.10.9. The result is an immediate consequence of the following technical lemma. The proof of Lemma 2.10.8 is similar to Lemma 2.10.5.

Lemma 2.10.8. *If a dag G admits a horizontal k-division for $k \geq 1$, then the subdivision S of G we obtain by subdividing each edge at its intersection points with the horizontal lines admits a $(k+1)$-stack layout.*

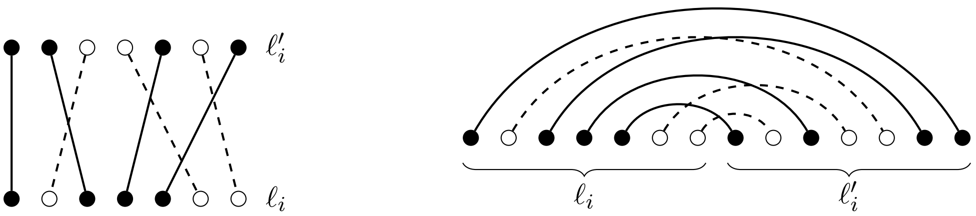
Proof. We proceed as in Lemma 2.10.5. Consider a horizontal k-division of G and let $\ell_1, \ell'_1, \ldots, \ell_t, \ell'_t$ be the corresponding horizontal lines of increasing y-coordinate. We may assume that every original vertex is adjacent to some division vertex on a horizontal line ℓ_i and to some division vertex on ℓ'_i. Otherwise, we add such edges with new endpoints on the horizontal lines in a planar way and observe that S is a subgraph of the obtained graph. We call these new vertices also division vertices.

Consider the topological ordering σ of the vertices of S where division vertices are primarily ordered by their y-coordinate in the drawing. Those lying on a horizontal line ℓ_i are placed in increasing order with respect to their x-coordinate, while those on a horizontal line ℓ'_i are placed in decreasing order. Original vertices are placed in σ just after their rightmost neighbor lying on a line ℓ_i with respect to σ. As the original vertices between horizontal lines ℓ_i and ℓ'_i form an antichain, σ is indeed a topological vertex ordering.

We show that σ admits a $(k+1)$-stack assignment. Let H_i be the graph induced by the vertices which lie between ℓ_i and ℓ'_i.

Case 1. Only crossing events occur between ℓ_i and ℓ'_i. Possibly, no crossing occurs. Note that H_i is a matching graph. Thus, we can color the edges of H_i with k colors such that no two edges of the same color intersect, see Figure 2.22. We see that no two edges of the same color cross in the layout σ. Therefore, we may assign all edges of the same color to one stack.

Case 2. Only vertex events occur between ℓ_i and ℓ'_i. We claim that one stack suffices for the edges of H_i. We color edges of H_i which are incident to an original vertex in green, and all other edges in red. Observe that if we were to replace all green stars by single edges, we would be in the situation of Case 1 with no edge crossings. We see



(a) An edge-coloring of the matching graph H_i where no two edges of the same color intersect.

(b) A stack assignment of the edges between the horizontal lines ℓ_i and ℓ_i' induced by the edge-coloring of H_i. Edges of the same color are assigned to the same stack.

Figure 2.22: An illustration of the correspondence between edge-colorings of the matching graph H_i and a valid stack assignment illustrated for a horizontal 2-division and the case where only crossing events occur between the horizontal lines ℓ_i and ℓ_i'.

that due to the definition of σ, none of the edges of H_i cross in σ, see Figure 2.23. Therefore, we may assign all these edges to one stack.

Note that all the edges of H_i lie completely to the left of the edges of H_j for $i \neq j$ in the topological ordering σ. Therefore, we needed k stacks so far.

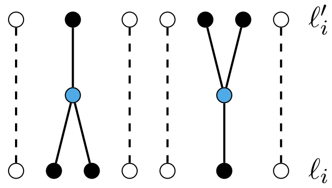
As the edges between horizontal lines ℓ_i' and ℓ_{i+1} form a rainbow in σ, a total of $k+1$ stacks suffices. $\square$

Dujmović and Wood showed that every undirected graph admits an s-stack subdivision for every $s \geq 3$. In particular, they determined logarithmic upper bounds in the queue respectively stack number of the initial graph on the number of subdivisions per edge [23, Theorem 7 and 9]. The lemma above enables us to prove the following for dags.

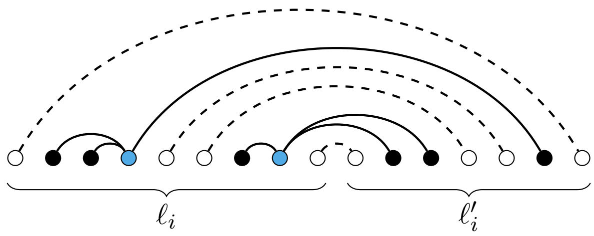
Corollary 2.10.9. *Every dag has a* 3*-stack subdivision.*

Proof. Consider any upward drawing of a dag G. We may assume that no two events occur at the same height. Thus, we can easily define a horizontal 2-division. As at most two edges cross between two successive horizontal lines, there exists a 3-stack subdivision S of G by Lemma 2.10.8. $\square$

Note that the subdivisions we consider in Corollary 2.10.9 and 2.10.6 are the same, i.e. every dag G has a subdivision S with queue number at most 2 and stack number at most 3. If we subdivide each edge of G often enough, the subdivision we obtain is in particular a subdivision of S. It follows from Proposition 2.8.3 and 2.8.4 that for each dag there exists a number h such that every subdivision where every edge has been subdivided at least h times has queue and stack number at most 6. Thus, large enough subdivisions have bounded queue and stack number. However, Lemma 2.8.5 shows that the constant h does indeed depend on the dag in question.



(a) The edge-coloring of the matching graph H_i.



(b) A stack layout of the edges between the horizontal lines ℓ_i and ℓ_i'. None of the edges cross.

Figure 2.23: An illustration of the correspondence between edge-colorings of the matching graph H_i and a valid stack assignment illustrated for a horizontal k-division where only vertex events occur between the horizontal lines ℓ_i and ℓ_i'. Original vertices are represented in blue. Green edges are represented by black lines, red edges by dashed lines.

The result of Corollary 2.10.9 is tight in the sense that there are dags which do not admit a 2-stack subdivision. Posets with non-planar cover graphs provide an example.

Observation 2.10.10. *No poset with a non-planar cover graph admits a* 2*-stack subdivision.*

Proof. Consider a poset P with a non-planar cover graph. Suppose it admits a 2-stack subdivision S. If we consider a 2-stack layout of S and draw all edges assigned to the first stack above the x-axis, and all edges assigned to the second stack below, we obtain a planar drawing of S. Thus P has a planar cover graph which yields a contradiction. □

A similar result holds for undirected graphs. In fact, the class of undirected graphs with stack number at most 2 corresponds to the class of planar graphs [23, Theorem 18].

In Corollary 2.7.7, we have seen that the dimension of a poset is not bounded in terms of its queue number. The following result provides the analog for stack number. Thus, dimension is neither bounded by queue, nor by stack number.

Corollary 2.10.11. *There exists a family of posets with stack number at most* 3 *and unbounded dimension.*

Proof. Consider the standard example S_d for $d \geq 2$. By Corollary 2.10.9, there exists a subdivision S_d' of S such that $\mathrm{sn}(S_d') \leq 3$. Further, as S_d is a subposet of S_d', we have $\dim(S_d') \geq \dim(S_d) = d$. Therefore, the family of subdivisions S_d' has unbounded dimension and queue number at most 3. □

2.11 Sparsity

Connections between the Dushnik-Miller dimension and graph parameters of the corresponding cover graphs have been studied since the 1980s. Füredi and Kahn bounded the dimension of posets whose cover graphs have bounded degree [36], improving a result which they attribute to Rödl and Trotter. In 2014, Streib and Trotter showed that the dimension of posets with planar cover graphs is bounded in terms of their height [85]. A polynomial bound was established by Kozik, Micek, and Trotter [66]. For planar posets, the upper bound is linear in the height [58], see Theorem 2.6.11. After Joret et al. established a bound on the Dushnik-Miller dimension in terms of height and the treewidth of the cover graph [56], a result by Walczak followed which bounded the dimension of posets whose cover graphs do not contain a fixed graph as a topological minor [93]. The proof relies on structure theorems; an elementary proof has been given by Micek and Wiechert [71].

Generalizing Walczak's result, Joret, Micek, and Wiechert showed that the dimension of posets with sparse diagrams is bounded in terms of their height. In particular, they proved that the dimension of every poset is bounded in terms of its queue number and height. The same is true if we replace queue by stack number [59, p. 1135]. They studied families of posets with bounded expansion which gives a restriction on minors of the cover graphs. It is one way to formalize sparsity of graphs. There are several equivalent definitions of bounded expansion [75]. The most common relies on a parameter referred to as the greatest reduced average density [73, pp. 766–767] [59, p. 1138]. We use the following definition which is equivalent to the definition given by Joret, Micek, and Wiechert in [59] by a result of Nešetřil, Ossona de Mendez, and Wood [75, Corollary 3.2].

Definition 2.11.1 ([75, pp. 354–355])**.** A graph H is a *shallow topological minor* of a graph G at depth d if a $\preceq 2d$-subdivision of H is a subgraph of G. We denote by $G \mathbin{\widetilde{\nabla}} d$ the class of graphs that are shallow topological minors of G at depth d.

Let G be an undirected graph. The *topological greatest reduced average density* of G of rank d, denoted by $\widetilde{\nabla}_d(G)$, is defined as

$$\sup_{H \in G \widetilde{\nabla} d} \frac{|E(H)|}{|V(H)|}.$$

A class $\mathcal{C}$ of undirected graphs has *bounded expansion* if there exists a function $f \colon \mathbb{N}_0 \to \mathbb{R}$ such that $\widetilde{\nabla}_d(G) \leq f(d)$ for every $G \in \mathcal{C}$ and every $d \in \mathbb{N}_0$.

In the remainder of this section, we give explicit bounds on the Dushnik-Miller dimension in terms of height, queue and stack number which follow from the arguments of Joret, Micek, and Wiechert in [59] and Nešetřil and Ossona de Mendez in [73].

Nešetřil, Ossona de Mendez, and Wood showed for several classes of graphs that they have bounded expansion, in particular for undirected graphs with bounded queue number [75, Theorem 7.4].

Theorem 2.11.2 ([75, Theorem 7.4])**.** *Undirected graphs of bounded queue number have bounded expansion. More precisely,*

$$\widetilde{\nabla}_d(G) \leq (2\,\mathrm{qn}(G) + 2)^{4d}$$

for every undirected graph G.

Similarly, they bounded the topological greatest reduced average density of undirected graphs with bounded stack number, showing that these graphs also have bounded expansion.

Theorem 2.11.3 ([75, Theorem 8.4])**.** *Undirected graphs of bounded stack number have bounded expansion. More precisely,*

$$\widetilde{\nabla}_d(G) \leq \frac{4\,\mathrm{sn}(G)(5\,\mathrm{sn}(G) - 5)^{2d+1}}{5\,\mathrm{sn}(G) - 6}$$

for every undirected graph G.

The proof of Joret, Micek, and Wiechert [59] relies on a characterization of graph classes with bounded expansion via specific vertex colorings which are called p-centered colorings.

Definition 2.11.4 ([73, p. 763])**.** A *p-centered coloring* of an undirected graph G is a vertex coloring of G with the property that the induced coloring of every connected subgraph H of G either uses some color exactly once or at least p distinct colors on H.

Nešetřil and Ossona de Mendez gave several characterizations of families of graphs of bounded expansion [73], one of which is based on p-centered colorings and appears in the proof of Joret, Micek, and Wiechert in [59]. We state it in Theorem 2.11.17. In order to a give an upper bound on the number of colors of p-centered colorings of a family of graphs, Nešetřil and Ossona de Mendez defined the following polynomials.

Definition 2.11.5 ([73, Notation 4.1 and 5.2])**.** Let $P_0(x, y) = x + y$ and define the polynomial $P_i(x, y)$ as

$$P_i(x, y) = P_{i-1}(x, y) + \big((2P_{i-1}(x, y) + 1)(x + y)\big)^{2i+1} y$$

for $i \geq 1$.

Further we define the polynomials $R_i(x, y)$ and $R'_i(x, y)$ recursively by setting

$$R_0(x, y) = x, \quad R'_0(x, y) = y$$

and defining

$$\begin{aligned} R_i(x, y) &= Q\big(R_{i-1}(x, y), R'_{i-1}(x, y)\big) \\ R'_i(x, y) &= P_{2i+1}\big(R_{i-1}(x, y) + 1, R'_{i-1}(x, y)\big) \end{aligned}$$

for $i \geq 1$ where $Q(x, y) \coloneqq x(x + 1) + P_1(x + 1, y)$.

Instead of solving the recurrence relations of the polynomials defined above, we establish bounds on the total degrees.

Definition 2.11.6. The total degree of a multivariate monomial is defined as the sum of the exponents of all variables which appear in it. The total degree of a multivariate polynomial P, denoted by $\deg(P)$, is the maximum degree of its monomials.

Lemma 2.11.7. *For every $i \in \mathbb{N}_0$, we have*

(i) $\deg(Q) = 7$

(ii) $2^i \cdot i! \leq \deg(P_i) \leq 8^i \cdot i!$

(iii) $\prod_{k=1}^{i} 2^{2k+1} \cdot (2k+1)! \leq \deg(R'_i) \leq \prod_{k=1}^{i} 8^{2k+1} \cdot (2k+1)!$

(iv) $7 \cdot \prod_{k=1}^{i-1} 2^{2k+1} \cdot (2k+1)! \leq \deg(R_i) \leq 7 \cdot \prod_{k=1}^{i-1} 8^{2k+1} \cdot (2k+1)!$.

Proof. Note that for $i \geq 1$, we have $\deg(P_i) = \big(\deg(P_{i-1}) + 1\big)(2i+1) + 1$ and $\deg(P_0) = 1$. In particular, we obtain $\deg(Q) = \deg(P_1) = 7$ and see that $(\deg(P_j)_j)$ is an increasing sequence.

We show Lemma 2.11.7(ii) by induction in i. Clearly, the claim holds for $i = 0$. Assuming it holds for some $i - 1 \geq 0$, we obtain

$$2^{i-1}(i-1)! \cdot 2i \leq \deg(P_{i-1}) \cdot 2i \leq \big(\deg(P_{i-1}) + 1\big)(2i+1) + 1 = \deg(P_i)$$

and

$$\deg(P_i) = \big(\deg(P_{i-1}) + 1\big)(2i+1) + 1 \leq \big(\deg(P_{i-1}) + 1\big)(2i+2) \leq 2\deg(P_{i-1}) \cdot 4i \leq 8^i \cdot i!$$

where we used in the second inequality that $i \geq 1$ and $\deg(P_{i-1}) \geq 1$. Therefore, the bounds on $\deg(P_i)$ hold for all i.

We now prove that P_i contains a monomial in the second variable y of degree $\deg(P_i)$. For $i = 0$, we see that y is a monomial of P_0 as $P_0(x, y) = x + y$. Assuming the claim holds for some $i - 1 \geq 0$, we see by definition of P_i that $(2\alpha y^{\deg(P_{i-1})} \cdot y)^{2i+1} \cdot y$ is a monomial of $P_i(x, y)$ for some $\alpha \neq 0$. Hence, P_i contains a monomial in the second variable of degree $\deg(P_i)$.

As $\deg(P_1) = 7$, the polynomial Q has degree 7 and contains a monomial αy^7 for some $\alpha > 0$. Inductively, we see that $\deg(R'_i) \geq \deg(R_i)$ as $(\deg(P_j))_j$ is an increasing sequence. This yields $\deg(R_i) = 7 \cdot \deg(R'_{i-1})$ and $\deg(R'_i) = \deg(P_{2i+1}) \cdot \deg(R'_{i-1})$ for $i \geq 1$.

We show by induction on i that

$$\prod_{k=1}^{i} 2^{2k+1} \cdot (2k+1)! \leq \deg(R'_i) \leq \prod_{k=1}^{i} 8^{2k+1} \cdot (2k+1)!.$$

As we take the products of an empty set on both sides for $i = 0$, the claim holds for R'_0. Suppose the claim holds for some $i - 1 \geq 0$. We obtain

$$\deg(R'_i) = \deg(P_{2i+1}) \deg(R'_{i-1}) \geq 2^{2i+1} \cdot (2i+1)! \deg(R'_{i-1})$$

and the lower bound follows from the induction hypothesis. The upper bound can be shown in a similar way.

Using $\deg(R_i) = 7 \cdot \deg(R'_{i-1})$, we obtain the bounds on $\deg(R_i)$. □

The following bounds on the factorial are well-known and derived from the Stirling formula.

Lemma 2.11.8 ([65, 1.2.5 Permutations and Factorials Exercise 24])**.** *For every $n \in \mathbb{N}$, we have*

$$\frac{n^n}{e^{n-1}} \leq n! \leq \frac{n^{n+1}}{e^{n-1}}.$$

Combining the bounds on the n-th factorial with our results of Lemma 2.11.7, we obtain more tangible bounds on the total degree of the polynomial R_i.

Lemma 2.11.9. *For every $i \geq 1$, we have*

$$(2i-1)^{\frac{1}{24}(2i-1)^2} \leq \deg(R_i) \leq (2i+1)^{3(i+1)^2}.$$

The upper bound holds for every $i \geq 0$.

Proof. We only show the upper bound. The lower bound can be obtained in a similar way. We first consider the case $i = 0$ and $i = 1$ separately. As $R_0(x, y) = x$, we have $\deg(R_0) = 1$ and by Lemma 2.11.7 we obtain $\deg(R_1) = 7$. Thus, the claim holds for $i \leq 1$.

We may now assume that $i \geq 2$. By Lemma 2.11.7, we have

$$\begin{aligned}
\deg(R_i) &\leq 7 \cdot \prod_{k=1}^{i-1} 8^{2k+1} \cdot (2k+1)! \leq 7 \cdot 8^{i-1+i(i-1)} \cdot \prod_{k=1}^{i-1} \frac{(2k+1)^{2k+2}}{e^{2k}} \\
&= 7 \cdot 2^{3i^2-3} \cdot \exp\left(-\sum_{k=1}^{i-1} 2k\right) \cdot \exp\left(\sum_{k=1}^{i-1} \log(2k+1)(2k+2)\right) \\
&\leq 2^{3i^2} \cdot e^{-i(i-1)} \cdot \exp\left(\sum_{k=1}^{i-1} \log(2k+1)(2k+2)\right)
\end{aligned}$$

where we used basic properties of arithmetic progressions and Lemma 2.11.8 in the second inequality. Define

$$\begin{aligned}
f \colon (0, \infty) &\to \mathbb{R}, \\
x &\mapsto \log(2x+1)(2x+2)
\end{aligned}$$

and

$$\begin{aligned}
F \colon (0, \infty) &\to \mathbb{R} \\
x &\mapsto \frac{1}{4}\Big((4x^2 + 8x + 3)\log(2x+1) - 2x^2 - 6x\Big).
\end{aligned}$$

Note that $\frac{d}{dx}F(x) = f(x)$. As f is increasing for $x \geq 1$, we obtain

$$\sum_{k=1}^{i-1} \log(2k+1)(2k+2) \leq \int_1^i f(x)\,dx = F(i) - F(1) \leq (i^2 + 2i + 1)\log(2i+1) + 2.$$

Further, observe that

$$3i^2 \log(2) - i(i-1) + 2 \leq 2(i+1)^2$$

as $(3\log(2) - 3)i^2 - 3i \leq 0$ for all $i \geq 0$. This shows for $i \geq 1$ that

$$\begin{aligned}
\deg(R_i) &\leq 2^{3i^2} \cdot e^{-i(i-1)} \cdot \exp\left(\sum_{k=1}^{i-1} \log(2k+1)(2k+2)\right) \\
&\leq 2^{3i^2} \cdot e^{-i(i-1)} \cdot \exp(F(i) - F(1)) \\
&\leq \exp\Big(3i^2 \log(2) - i(i-1) + (i^2 + 2i + 1)\log(2i+1) + 2\Big) \\
&\leq \exp\Big(2(i+1)^2 + (i+1)^2 \log(2i+1)\Big) \\
&\leq \exp\Big(3(i+1)^2 \log(2i+1)\Big)
\end{aligned}$$

where we used in the fifth step that $\log(2i+1) \geq 1$ for $i \geq 1$. □

In fact, the proof above shows in particular the following.

Lemma 2.11.10. *For every $i \geq 0$, we have*

$$7 \cdot \prod_{k=1}^{i-1} 8^{2k+1} \cdot (2k+1)! \leq (2i+1)^{3(i+1)^2}.$$

Our aim is to bound the polynomial R_i in terms of its total degree and the largest coefficient which appears in its expanded form in front of a monomial.

Definition 2.11.11. The largest among all coefficients in front of the monomials of a polynomial g is called the *largest coefficient* of g and denoted by $\mathrm{c}(g)$.

If we know the largest coefficient of a polynomial g, we can easily construct an upper bound on g.

Lemma 2.11.12. *If g is a polynomial in two variables of total degree d with $d \geq 1$, then*

$$g(x, y) \leq \mathrm{c}(g) \cdot 4d^2 \cdot x^d$$

for all x and y with $1 \leq y \leq x$. The same is true if the roles of x and y are reversed.

Proof. Let x and y be non-negative integers such that $1 \le y \le x$. Clearly, the value of $g(x, y)$ is bounded from above by the polynomial we obtain by replacing every monomial of g with $\mathrm{c}(g) \cdot x^d$. Therefore, it suffices to show that g has at most $4d^2$ monomials.

We use the twelvefold way to compute the number of monomials of g. Every monomial of g is of the form $x^i y^j$ for some integers i and j with $0 \le i + j \le d$. A monomial $x^i y^j$ can be represented by a string of length $d + 2$ which consists of d letters a and two letters b. The number of a's in front of the first b corresponds to i and the number of a's between the two b's corresponds to j. As there are $\binom{d+2}{2}$ such strings, there are at most $\binom{d+2}{2}$ monomials in g.

Note that

$$\binom{d+2}{2} = \frac{(d+2)(d+1)}{2} \le 4d^2$$

as $3t + 2 \le 7t^2$ for all $t \ge 1$. This observation yields the claim. $\square$

Thus, in order to bound the polynomial R_i, it suffices to compute its largest coefficient as we already determined upper bounds on the total degree.

Lemma 2.11.13. *For every $i \in \mathbb{N}_0$, we have*

(i) for the coefficient q of the polynomial Q

$$q = 1116$$

(ii) for the coefficient p_i of the polynomial P_i

$$p_i \le \left(2^{12i^2-3} \cdot 3^{2i+1} \cdot ((i-1)!)^{4i+2}\right)^{(2i+1)! \cdot i}$$

with the convention $(-1)! = 1$.

(iii) and for the coefficient r_i of the polynomial R_i

$$r_i \le 1116 \cdot \left(2^{48i^2} \cdot 3^{4i} \cdot ((2i)!)^{8i}\right)^{(4i)! \cdot i \cdot (2i+1)^{3(i+1)^2}}.$$

Proof. Considering the expanded form of

$$Q(x, y) = x(x+1) + x + 1 + y + ((2(x+1+y)+1)(x+1+y))^3 y,$$

we see that $q = 1116$.

We show the upper bound on the coefficients p_i inductively. As $P_0(x, y) = x + y$, we have $p_0 = 1$ and the claim holds. Suppose the claim holds for some $i - 1 \ge 0$.

We denote the degree of P_{i-1} by d and define $w = p_{i-1} \cdot 4d^2$. By definition of P_i, we obtain

$$\begin{aligned}
p_i &= \mathrm{c}(P_i(x,y)) \leq \mathrm{c}(P_i(x,x)) \\
&\leq p_{i-1} + \mathrm{c}\left(\left((2w \cdot x^d + 1) \cdot 2x\right)^{2i+1} x\right) \\
&\leq p_{i-1} + \mathrm{c}\left(\left(6w \cdot x^{d+1}\right)^{2i+1} x\right) \\
&\leq p_{i-1} + \mathrm{c}\left((6w)^{2i+1} \cdot x^{(2i+1)(d+1)+1}\right) \\
&= p_{i-1} + (6w)^{2i+1} \\
&\leq 2 \cdot p_{i-1}^{2i+1} \cdot \left(6 \cdot 4 \cdot d^2\right)^{2i+1}
\end{aligned}$$

where the first inequality follows from the fact that all coefficients of the polynomial P_i are non-negative, the second from Lemma 2.11.12 and the third inequality as $w \geq 1$. Applying Lemma 2.11.7, we obtain $d \leq 8^{i-1} \cdot (i-1)!$. Thus

$$\begin{aligned}
p_i &\leq 2 \cdot p_{i-1}^{2i+1} \cdot (6 \cdot 4 \cdot d^2)^{2i+1} \\
&\leq p_{i-1}^{2i+1} \cdot 2^{6i+4} \cdot 3^{2i+1} \cdot d^{4i+2} \\
&\leq p_{i-1}^{2i+1} \cdot 2^{6i+3} \cdot 3^{2i+1} \cdot 2^{3(i-1)(4i+2)} \cdot ((i-1)!)^{4i+2} \\
&\leq p_{i-1}^{2i+1} \cdot 2^{12i^2-3} \cdot 3^{2i+1} \cdot ((i-1)!)^{4i+2}.
\end{aligned}$$

By induction, this yields

$$\begin{aligned}
p_i \leq &\left(\left(2^{12i^2-3} \cdot 3^{2i+1} \cdot ((i-1)!)^{4i+2}\right)^{(2(i-1)+1)! \cdot (i-1)}\right)^{2i+1} \\
&\cdot 2^{12i^2-3} \cdot 3^{2i+1} \cdot ((i-1)!)^{4i+2} \\
\leq &\left(2^{12i^2-3} \cdot 3^{2i+1} \cdot ((i-1)!)^{4i+2}\right)^{(2i+1)! \cdot (i-1)} \\
&\cdot 2^{12i^2-3} \cdot 3^{2i+1} \cdot ((i-1)!)^{4i+2}
\end{aligned}$$

and the claim follows.

It remains to prove the upper bound on the coefficients r_i. Let r_i' denote the largest coefficient of R_i'. We proceed by induction on i. Clearly, the claim holds for r_0 and r_0' as $r_0 = r_0' = 1$. For $i \geq 1$, we have

$$\begin{aligned}
r_i &\leq q \cdot \max(r_{i-1}, r_{i-1}')^7 \\
r_i' &\leq p_{2i+1} \cdot \max(r_{i-1}+1, r_{i-1}')^{\deg(P_{2i+1})}.
\end{aligned}$$

as Q has degree 7 by Lemma 2.11.7

Define $d_k \coloneqq 8^k \cdot k!$ and let t_k be the upper bound on p_k we established for all $k \in \mathbb{N}_0$. Set $s_0 \coloneqq r_0$ and let $s_0' \coloneqq r_0'$. For $i \in \mathbb{N}$, we define

$$\begin{aligned}
s_i &\coloneqq q \cdot \max(s_{i-1}, s_{i-1}')^7 \\
s_i' &\coloneqq t_{2i+1} \cdot \max(s_{i-1}+1, s_{i-1}')^{d_{2i+1}}.
\end{aligned}$$

As d_{2i+1} is an upper bound on the degree of the polynomial P_{2i+1} by Lemma 2.11.7, we obtain $r_i \leq s_i$ and $r'_i \leq s'_i$. Thus, it suffices to determine bounds on s_i and s'_i for all i.

As $d_{2i+1} \geq 7$ and $t_{2i+1} > q = 1116$ for all $i \geq 1$, we obtain $s'_i > s_i$ for all $i \geq 1$. Since both s_i and s'_i are integers, this yields $s'_i \geq s_i + 1$. Thus, for all $i \geq 2$

$$\begin{aligned} s_i &= q \cdot (s'_{i-1})^7 \\ s'_i &= t_{2i+1} \cdot (s'_{i-1})^{d_{2i+1}}. \end{aligned}$$

We show by induction on i that

$$s'_i \leq \left(\prod_{k=1}^{2i+1} t_k \right)^{\prod_{j=1}^{i} 8^{2j+1} \cdot (2j+1)!}.$$

for all $i \in \mathbb{N}_0$. The claim clearly holds for $i \leq 1$ as $s'_0 = 1$ and $s'_1 = t_3$. Assume the claim holds for some $i - 1 \geq 1$. We obtain by induction and by definition of d_{2i+1}

$$\begin{aligned} s'_i &= t_{2i+1} \cdot (s'_{i-1})^{d_{2i+1}} \\ &\leq t_{2i+1} \cdot \left(\left(\prod_{k=1}^{2(i-1)+1} t_k \right)^{\prod_{j=1}^{i-1} 8^{2j+1} \cdot (2j+1)!} \right)^{d_{2i+1}}. \end{aligned}$$

which shows the upper bound on s'_i.

Thus, we obtain

$$s_i \leq q \cdot (s'_{i-1})^7 \leq q \cdot \left(\prod_{k=1}^{2i-1} t_k \right)^{7 \cdot \prod_{j=1}^{i-1} 8^{2j+1} \cdot (2j+1)!}$$

for all $i \geq 1$.

Note that

$$\begin{aligned} \prod_{k=1}^{2i-1} t_k &= \prod_{k=1}^{2i-1} \left(2^{12k^2-3} \cdot 3^{2k+1} \cdot ((k-1)!)^{4i+2} \right)^{(2k+1)! \cdot k} \\ &\leq \left(2^{12(2i-1)^2-3} \cdot 3^{2(2i-1)+1} \cdot ((2i-2)!)^{4(2i-1)+2} \right)^{(2(2i-1)+1)! \cdot (2i-1)^2} \\ &\leq \left(2^{48i^2} \cdot 3^{4i} \cdot ((2i)!)^{8i} \right)^{(4i)! \cdot i} \end{aligned}$$

for $i \geq 1$. Further, we have by Lemma 2.11.10

$$7 \cdot \prod_{j=1}^{i-1} 8^{2j+1} \cdot (2j+1)! \leq (2i+1)^{3(i+1)^2}.$$

This finally yields for all $i \geq 1$

$$r_i \leq s_i \leq q \cdot \left(2^{48i^2} \cdot 3^{4i} \cdot ((2i)!)^{8i} \right)^{(4i)! \cdot i \cdot (2i+1)^{3(i+1)^2}}.$$

As $r_0 = 1$ the inequality above holds for all $i \in \mathbb{N}_0$. The claim follows as $q = 1116$. $\square$

Definition 2.11.14. We define the function

$$\begin{aligned} r\colon \mathbb{N}_0 &\to \mathbb{R} \\ i &\mapsto 1116 \cdot \left(2^{48i^2} \cdot 3^{4i} \cdot ((2i)!)^{8i}\right)^{(4i)!\cdot i\cdot(2i+1)^{3(i+1)^2}} \cdot 4 \cdot (2i+1)^{6(i+1)^2}. \end{aligned}$$

An application of Lemma 2.11.12 yields an upper bound on the polynomial R_i.

Lemma 2.11.15. *For all $i \geq 0$ and all x and y with $1 \leq y \leq x$ we have*

$$R_i(x, y) \leq r(i) \cdot x^{\deg(R_i)}.$$

The same is true if the roles of x and y are reversed.

Proof. Let $i \in \mathbb{N}_0$ and let d_i denote the bound on the total degree of the polynomial R_i we determined in Lemma 2.11.9. It suffices to observe that $r(i)$ is the product of the upper bound on the largest coefficient of R_i we established in Lemma 2.11.13 and $4d_i^2$. The claim now follows from Lemma 2.11.12. □

This bound will be useful when we determine an upper bound on the number of colors of p-centered colorings of any class of graphs with bounded expansion.

Definition 2.11.16 ([73, Corollary 6.3])**.** We define the function

$$\begin{aligned} S\colon \mathbb{N} &\to \mathbb{R} \\ p &\mapsto 1 + (p-1)(2 + \lceil \log_2(p) \rceil). \end{aligned}$$

Nešetřil and Ossona de Mendez showed that if $\mathcal{C}$ is a class of graphs of bounded expansion, then there exists for every $p \in \mathbb{N}$ a number $X(p)$ such that every graph in $\mathcal{C}$ admits a p-centered coloring using at most $X(p)$ colors [73, Theorem 7.1]. In particular, their proof yields an upper bound on $X(p)$. Later on, Dębski et al. improved the bounds on $X(p)$ for certain classes of graphs, such as graphs avoiding a fixed graph as a topological minor [15].

Theorem 2.11.17 ([73, Theorem 7.1])**.** *A family $\mathcal{C}$ of undirected graphs has bounded expansion if and only if for every $p \in \mathbb{N}_0$, there exists an integer $X(p)$ such that every graph in $\mathcal{C}$ has a p-centered coloring using at most $X(p)$ distinct colors.*

In particular, if there exists an increasing function $f\colon \mathbb{N}_0 \to \mathbb{R}_{\geq 1}$ such that for every graph $G \in \mathcal{C}$ we have $\widetilde{\nabla}_d(G) \leq f(d)$, then there exists a p-centered coloring of every $G \in \mathcal{C}$ with at most

$$C(p) \cdot (p + p^p)^{\binom{C(p)}{p}}$$

colors where $C(p) \leq 2r(S(p)) \cdot \left(8f\left(2^{S(p)+1} - 1\right)\right)^{2^{2S(p)+2} \cdot \deg(R_{S(p)})} + 1.$

Proof. We only outline the argument given by Nešetřil and Ossona de Mendez [73, Theorem 7.1]. Let f be an increasing function such that $\widetilde{\nabla}_d(G) \leq f(d)$ and $f(d) \geq 1$ for all $G \in \mathcal{C}$ and $d \in \mathbb{N}_0$. Following the argument given by Nešetřil and Ossona de Mendez in combination with [75, Corollary 3.2] and [74, Lemma 4.6], we obtain that every graph in $\mathcal{C}$ admits a p-centered coloring using at most

$$C(p) \cdot N(p,p)^{\binom{C(p)}{p}} \tag{2.1}$$

colors where $N(p,p)$ is the constant defined in [74] and

$$C(p) \leq 2R_{S(p)}\left(2f(0), 4\left(4f\left(2^{S(p)+1}-1\right)\right)^{(2^{S(p)+1})^2}\right)+1.$$

The bound on the number of colors of a p-centered coloring given in (2.1) results from [73, Lemma 2.5].

As f is increasing and $f(d) \geq 1$ for all $d \in \mathbb{N}_0$, we obtain by Lemma 2.11.15

$$\begin{aligned} C(p) &\leq 2r(S(p)) \cdot \left(4\left(4f\left(2^{S(p)+1}-1\right)\right)^{(2^{S(p)+1})^2}\right)^{\deg(R_{S(p)})}+1 \\ &\leq 2r(S(p)) \cdot \left(8f\left(2^{S(p)+1}-1\right)\right)^{2^{2S(p)+2}\cdot\deg(R_{S(p)})}+1 \\ &\leq 2r(S(p)) \cdot \left(8f\left(2^{S(p)+1}-1\right)\right)^{2^{2S(p)+2}\cdot\deg(R_{S(p)})}+1 \end{aligned}$$

and as Nešetřil and Ossona de Mendez showed that $N(p,p) \leq p+p^p$ using a greedy coloring [74, Lemma 4.6] the claim follows. □

Intuitively, we might expect that posets with sparse cover graphs have small dimension. Planar posets are sparse; yet, Kelly's construction shows that they have arbitrarily large dimension, see Lemma 2.6.3. It was observed that these posets have large height and conjectured that the dimension of planar posets is bounded in terms of their height based on results of Felsner, Li, and Trotter [30, Corollary 5.1]. This was shown to be true by Streib and Trotter [85].

Similar results for different classes of posets with sparse cover graphs followed. An overview is given in [59, Figure 2]. Using the model of bounded expansion, Joret, Micek, and Wiechert were able to generalize previous results concerning such posets in the following theorem.

Theorem 2.11.18 ([59, Theorem 3])**.** *If P is a poset of height h whose cover graph admits a $2h$-centered coloring using c colors, then*

$$\dim(P) \leq 2^{2(c+1)^{h(P)}}.$$

The property of forming a nowhere dense class is a relaxation of bounded expansion. As Joret, Micek, and Wiechert were able to construct posets of height two with arbitrarily large dimension whose cover graphs form a class of nowhere dense graphs, the result above cannot be extended to nowhere dense cover graphs [59, p. 1139].

Based on the work of Nešetřil, Ossona de Mendez, and Wood who showed that graphs with bounded queue number have bounded expansion, see Theorem 2.11.2, Joret, Micek, and Wiechert conclude that the dimension of posets is bounded in terms of their height and queue number [59, p. 1135].

Theorem 2.11.19 ([59, p. 1135])**.** *The dimension of posets is bounded in terms of their height and (undirected) queue number. More precisely, we have for every poset P*

$$\dim(P) \leq 2^{2^{\left(\lambda\cdot\left(2h+(2h)^{2h}\right)^{\binom{\lambda}{2h}}+1\right)^{h}}}$$

where we write h for h(P) and where

$$\lambda \leq 2r(T)\cdot\left(8(2q+2)^{2^{T+3}}\right)^{2^{2T+2}\cdot(2T+1)^{3(T+1)^2}}+1$$

with $T = S(2h) \leq 8h\log_2(2h)$, $q = \mathrm{qn}(\mathrm{Cov}(P))$ and where r is the function defined in Definition 2.11.14.

Proof. Let $q \in \mathbb{N}$ and define $\mathcal{C}$ as the family of undirected graphs with (undirected) queue number at most q. The function f in Theorem 2.11.17 is given by

$$f(d) := (2q+2)^{4d}$$

for $d \in \mathbb{N}_0$. Indeed, Theorem 2.11.2 shows that $\widetilde{\nabla}_d(G) \leq f(d)$ for every $G \in \mathcal{C}$. Further, f is increasing and $f(d) \geq 1$ for all $d \in \mathbb{N}_0$. To improve readability, we may write h instead of $h(P)$ for the height of a poset P when P is clear from context. By Theorem 2.11.17, every poset P with *undirected* queue number at most q admits a $2h$-centered coloring using at most $\lambda\cdot(2h+(2h)^{2h})^{\binom{\lambda}{2h}}$ colors where

$$\begin{aligned}\lambda &\leq 2r(T)\cdot\left(8f\left(2^{T+1}-1\right)\right)^{2^{2T+2}\cdot\deg(R_T)}+1\\ &\leq 2r(T)\cdot\left(8(2q+2)^{4\cdot(2^{T+1}-1)}\right)^{2^{2T+2}\cdot\deg(R_T)}+1\\ &\leq 2r(T)\cdot\left(8(2q+2)^{2^{T+3}}\right)^{2^{2T+2}\cdot\deg(R_T)}+1\end{aligned}$$

and we write T for $S(2h)$. Further, we have $\deg(R_T) \leq (2T+1)^{3(T+1)^2}$ by Lemma 2.11.9. Thus,

$$\lambda \leq 2r(T)\cdot\left(8(2q+2)^{2^{T+3}}\right)^{2^{2T+2}\cdot(2T+1)^{3(T+1)^2}}+1$$

By Theorem 2.11.18, we obtain

$$\dim(P) \leq 2^{2^{\left(\lambda\cdot(2h+(2h)^{2h})^{\binom{\lambda}{2h}}+1\right)^{h}}}.$$

□

As the directed queue number of a poset is an upper bound on the undirected queue number of its cover graph, the result above also applies in our setting.

A similar result to Theorem 2.11.19 holds for stack number as graphs with bounded stack number have bounded expansion, see Theorem 2.11.3.

Theorem 2.11.20 ([59, p. 1135])**.** *The dimension of posets is bounded in terms of their height and (undirected) stack number. More precisely, we have for every poset P*

$$\dim(P) \leq 2^{2^{\left(\lambda\cdot(2h+(2h)^{2h})^{\binom{\lambda}{2h}}+1\right)^{h}}}$$

where we write h for $h(P)$ and where

$$\lambda \leq 2r(T) \cdot \left(8\left(\frac{4s(5s-5)^{2^{T+2}-1}}{5s-6}+1\right)\right)^{2^{2T+2}\cdot(2T+1)^{3(T+1)^2}} + 1$$

with $T = S(2h) \leq 8h\log_2(2h)$, $s = \mathrm{sn}(\mathrm{Cov}(P))$ and where r is the function defined in Definition 2.11.14.

Proof. The proof is similar to Theorem 2.11.19 and is therefore omitted. Using Theorem 2.11.3 instead of Theorem 2.11.2 in order to bound the number of colors of a $2h(P)$-centered coloring, we obtain the claim. □

Note that the result above remains true if we replace the stack number of the cover graph by the directed stack number of the poset P as the function dependent on $s = \mathrm{sn}(\mathrm{Cov}(P))$ is increasing in s and s is bounded from above by $\mathrm{sn}(P)$.

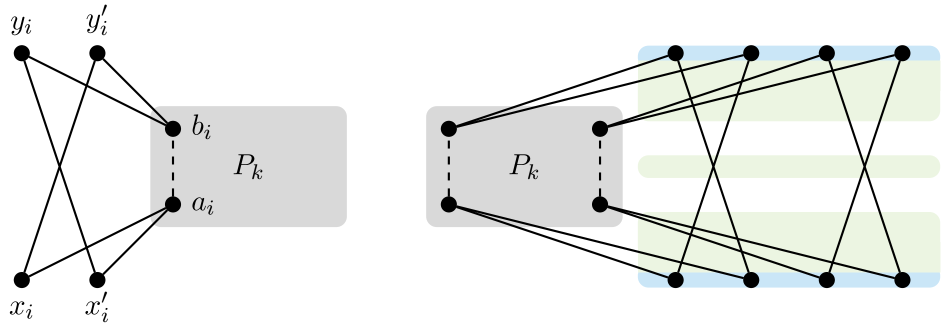
2.12 Lower Bounds

In Theorem 2.11.19 we determined an upper bound on the dimension in terms of queue number and height. The aim of this section is to show that any such upper bound has to be exponential in both parameters. The same is true for any upper bound on dimension in terms of stack number and height.

Similarly to queue and stack number, the dimension of a poset is bounded in terms of its height and the treewidth of its cover graph [56]. Joret et al. improved the previous upper bound [57, Corollary 13]. Constructing a family of posets whose dimension is exponential in height and treewidth, they conclude that the bound is essentially best-possible [57, Theorem 15]. Considering a restriction of this family of posets, we prove that there are posets whose dimension is exponential in height, queue or stack number. We first show the following technical lemma.

Lemma 2.12.1. *For every $k \geq 1$, there exists a poset P_k of height $2k$ that admits a horizontal 2-division consisting of $10(k-1)+4$ horizontal lines and has dimension at least 2^k.*

Proof. We show the proof by induction on k. We claim that for every $k \geq 1$, there exists a poset P_k such that the following holds.



(a) Construction of the poset P_{k+1}. The elements a_i and b_i are incomparable in the poset P_k which is represented by a gray rectangle.

(b) Extension of a 2-division of P_k to a 2-division of P_{k+1}. Areas where new vertex or crossing events occur are represented in blue and green respectively.

Figure 2.24: Illustrations of the proof of Lemma 2.12.1. Dashed lines join incomparable pairs a_i, b_i.

(i) $h(P_k) = 2k$

(ii) the minimal and maximal elements of P_k induce a standard example of size 2^k

(iii) there exists a horizontal 2-division of P_k consisting of $10(k-1)+4$ horizontal lines.

Clearly, the standard example S_2 fulfills (i) and (ii) for $k = 1$. As it can be represented by two non-intersecting edges, we see that S_2 has a horizontal 2-division consisting of four lines.

Suppose there exists such a poset P_k for some $k \geq 1$. Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ denote the elements of the standard example of size $n = 2^k$ formed by the minimal and maximal elements such that $a_i \parallel b_i$ for all i. For each i, we add four vertices x_i, x_i', y_i, y_i' such that $x_i, x_i' \leq a_i$, $b_i \leq y_i, y_i'$ and x_i, x_i', y_i, y_i' form a standard example of size 2, see Figure 2.24a.

Let P_{k+1} denote the poset we obtain. We see that the vertices $x_1, x_1', \ldots, x_n, x_n'$ are the minimal, and the vertices $y_1, y_1', \ldots, y_n, y_n'$ are the maximal elements of P_{k+1} which form a standard example of size $2 \cdot n = 2^{k+1}$. Further, the poset P_{k+1} has indeed height $2 + h(P_k) = 2(k+1)$.

We now prove that P_k admits a horizontal 2-division consisting of $10(k-1)+4$ horizontal lines. Suppose the claim holds for some $k \geq 1$. By definition of P_{k+1}, we can extend the horizontal 2-division of P_k to a 2-division of P_{k+1} using three more pairs of horizontal lines ℓ_i, ℓ_i' for crossing events and two more for vertex events, i.e. ten more horizontal lines in total, see Figure 2.24b. Thus, P_{k+1} admits a horizontal 2-division using $10k+4$ horizontal lines.

As P_k contains a standard example of size 2^k, we obtain $\dim(P_k) \geq 2^k$ by Lemma 2.1.10 and the claim follows. $\square$

Subdividing edges at their intersections with the horizontal lines of the 2-division, we obtain a poset with constant queue and stack number whose dimension is exponential in the height.

Corollary 2.12.2. *For every $k \geq 1$, there exists a poset of height at most $20k^2 - 12k$ and dimension at least 2^k that has stack number at most 3 and queue number at most 2.*

Proof. Consider the poset P_k given in Lemma 2.12.1. The poset P_k has height $2k$ and admits a 2-division consisting of at most $10(k-1)+4$ horizontal lines. Subdividing the edges of P_k at the intersections with the horizontal lines of the 2-division, we obtain a poset P'_k of height at most

$$h(P_k) \cdot (10(k-1)+4) = 20k^2 - 12k.$$

As P'_k contains P_k as a subposet, $\dim(P'_k) \geq 2^k$ follows. By Lemma 2.10.5 and Lemma 2.10.8, we see that P'_k has queue number at most 2 and and stack number at most 3. $\square$

Considering another construction of posets given in [57, Theorem 15], we find posets of constant height whose dimension is exponential in their queue and stack number.

Proposition 2.12.3. *For every $k \geq 1$, there exists a poset of height at most 4, dimension at least 2^k and stack and queue number at most $2k-1$.*

Proof. We proceed by induction on k. We claim that for each $k \geq 1$ there exists a poset T_k such that

(i) $h(T_k) \leq 4$

(ii) the minimal and maximal elements of T_k induce a standard example S of size 2^k

(iii) T_k admits a $(2k-1)$-stack layout and a $(2k-1)$-queue layout such that all minimal elements appear before the maximal elements.

Note that the standard example S_2 fulfills all properties for $k=1$.

Suppose the claim holds for some $k \geq 1$. Consider two copies $T_k^{(1)}$ and $T_k^{(2)}$ of T_k and a standard example S_2. Let x_1, x_2 be the minimal elements of S_2 and y_1, y_2 its maximal elements such that $x_i \parallel y_i$ for $i \in [2]$. Adding directed edges from all minimal elements of $T_k^{(i)}$ to x_i and from y_i to all maximal elements of $T_k^{(i)}$ for each $i \in [2]$, we obtain a poset T_{k+1}, see Figure 2.25.

As T_k has height at most 4, the same holds for T_{k+1}. Further, we see that the minimal and maximal elements of T_{k+1} form a standard example of size $2 \cdot 2^k$. By induction, there exists for each copy $T_k^{(i)}$ a $(2k-1)$-stack layout σ_i such that the minimal elements precede the maximal elements. We may split the ordering σ_i into two orderings $L^{(i)}$ and $H^{(i)}$ such that $L^{(i)}$ contains all minimal elements and $H^{(i)}$

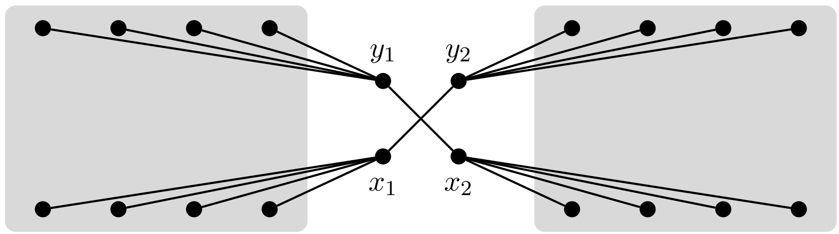


Figure 2.25: Construction of the poset T_{k+1} using two copies of T_k, represented in gray. Only edges which do not belong to any of the two copies are represented.

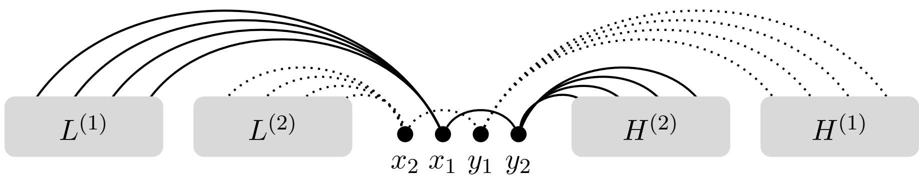


Figure 2.26: Given a stack layout of T_k where the minimal elements appear before the maximal elements, we obtain a stack layout of T_{k+1} with the same property using only two more stacks. Only edges which do not belong to any of the copies of T_k are represented. Edges of the same color are assigned to the same stack.

contains all maximal elements of $T_k^{(i)}$. We have $\sigma_i = L^{(i)} \leq H^{(i)}$. Consider the layout σ of T_{k+1} given by

$$L^{(1)} \leq L^{(2)} \leq x_2 \leq x_1 \leq y_1 \leq y_2 \leq H^{(2)} \leq H^{(1)},$$

see Figure 2.26. Note that none of the edges in $T_k^{(1)}$ intersects any of the edges of $T_k^{(2)}$ in σ. Thus, we only require $2k-1$ stacks for all these edges. Further, we can assign the edges between minimal elements of $T_k^{(1)}$ and x_1, the edge x_1y_2 and edges connecting y_2 to maximal elements of $T_k^{(2)}$ to one stack. Similarly, all edges between $T_k^{(2)}$ and x_2, the edge x_2y_1 and edges between y_1 and $T_k^{(1)}$ can be assigned to one stack. We used a total of $2k+1$ stacks, showing that σ is indeed a $(2k+1)$-stack layout where the minimal elements precede the maximal elements.

We can argue similarly to obtain a $(2k+1)$-queue layout of T_{k+1} if we consider the layout of T_{k+1} given by

$$L^{(1)} \leq L^{(2)} \leq x_1 \leq x_2 \leq y_1 \leq y_2 \leq H^{(1)} \leq H^{(2)}$$

where $L^{(i)} \leq H^{(i)}$ is a $(2(k-1)+1)$-queue layout of $T_k^{(i)}$ where $L^{(i)}$ contains the minimal and $H^{(i)}$ the maximal elements, see Figure 2.27.

As T_k contains a standard example of size 2^k, we obtain $\dim(T_k) \geq 2^k$ and the claim follows. $\square$

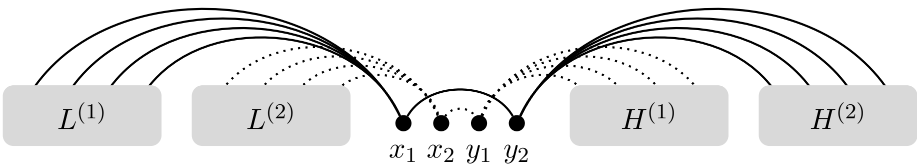


Figure 2.27: Given a queue layout of T_k where the minimal elements appear before the maximal elements, we obtain a queue layout of T_{k+1} with the same property using only two more queues. Only edges which do not belong to any of the copies of T_k are represented. Edges of the same color are assigned to the same queue.

Corollary 2.12.2 shows that there is no function that is polynomial in height and exponential in queue or stack number which bounds dimension. The same is true if we exchange the roles of height and queue or stack number by Proposition 2.12.3. However, the lower bounds we established are far from meeting the upper bounds on dimension determined in Theorem 2.11.19 and 2.11.20.

3 Boolean Dimension

In the remainder of this work, we study a variation of Dushnik-Miller dimension known as Boolean dimension. It was introduced by Gambosi, Nešetřil, and Talamo in 1987 [38], see [37] for the full version, and provides another means of studying a poset's complexity.

3.1 Introduction to Boolean Dimension

Boolean dimension is a parameter which is also based on linear orders; however, in contrast to Dushnik-Miller dimension, these do not need to be linear extensions of the poset in question.

Definition 3.1.1 (based on [5][p.2])**.** For a positive integer d, we denote by $\mathbf{2}^d$ the set of all binary vectors of length d. Consider a poset P and a family $\mathcal{B} = \{L_1, \ldots, L_d\}$ of d linear orders of the elements of P. We define for elements x, y of P the binary string $q(x, y, \mathcal{B})$ as the string of length d where the i-th element is 1 if $x \leq_{L_i} y$ and 0 otherwise. Let $\tau\colon \mathbf{2}^d \to \{0, 1\}$ be a function. The pair $(\mathcal{B}, \tau)$ is called a *Boolean realizer* if for all elements x, y of P, we have

$$x \leq_P y \iff \tau(q(x, y, \mathcal{B})) = 1.$$

The *Boolean dimension* of P, denoted by $\operatorname{bdim}(P)$, is the smallest positive integer d such that there exists a Boolean realizer $(\mathcal{B}, \tau)$ where $|\mathcal{B}| = d$.

Similar to Dushnik-Miller realizers, we can decide for two elements x and y of a poset P whether $x \leq_P y$ by considering their order in the linear orders of a Boolean realizer.

Note that the definition above requires that the function τ of a Boolean realizer $(\mathcal{B}, \tau)$ maps the binary string $1 \ldots 1$ to 1 by reflexivity of a partial order. If we only require for *distinct* elements x, y of a poset P that $x \leq_P y$ if and only if $\tau(q(x, y, \mathcal{B})) = 1$, we get a slightly different notion; see [5]. With our definition, the Boolean dimension of an antichain of size at least 2 is 2, with the definition of Barrera-Cruz et al. in [5] it is 1. For all other posets, the two definitions yield the same value [5, p. 2] due to the following observation. There is some pair of distinct elements x, y such that $x \leq y$, i.e. the pair (x, y) is associated with a binary string that is mapped to 1. Reversing some of the linear orders of the Boolean realizer and modifying the map accordingly, we can ensure that (x, y) is associated with the binary string $1 \ldots 1$. As every pair of equal elements is mapped to the same binary string as the pair (x, y), we constructed a

Boolean realizer which satisfies our definition and requires the same number of linear orders as the initial one.

Similarly to Dushnik-Miller dimension, Boolean dimension is monotone with respect to subposets. This follows from the fact that the restriction of a Boolean realizer to a subposet is still a Boolean realizer.

Lemma 3.1.2 ([5][p.2])**.** *For every subposet Q of a poset P, we have*

$$\mathrm{bdim}(Q) \leq \mathrm{bdim}(P).$$

As any Dushnik-Miller realizer together with the map which assigns 1 to the binary string consisting only of ones and 0 to all other strings is a Boolean realizer, we see that Dushnik-Miller dimension provides an upper bound on Boolean dimension.

Lemma 3.1.3 ([5][p.2])**.** *For every poset P, we have* $\mathrm{bdim}(P) \leq \dim(P)$.

Even though Boolean dimension provides a lower bound on Dushnik-Miller dimension and these two notions are not equivalent in general, they coincide for small dimension as has been shown by Gambosi, Nešetřil, and Talamo for a slightly different definition of Boolean dimension [37, Theorem 2.7]. The proof has been simplified and adapted to the definition of Boolean dimension we consider by Trotter and Walczak [90, p. 3]. While the proof of the 2-dimensional case is straight forward, it cannot be easily adapted to the 3-dimensional case. However, the observations of Trotter and Walczak do not only simplify the 2-dimensional case, but also yield an easy proof of the 3-dimensional case.

Lemma 3.1.4 ([90, p. 3])**.** *For a poset P and an integer $d \leq 3$, we have* $\mathrm{bdim}(P) = d$ *if and only if* $\dim(P) = d$.

Proof. By Lemma 3.1.3, it suffices to show that $\dim(P) \leq \mathrm{bdim}(P)$ for every poset of Boolean dimension at most 3.

If we consider a poset P with a Boolean realizer $(\mathcal{B}, \tau)$ such that $|\mathcal{B}| = 1$, we see that τ maps the binary string 1 to 1 by reflexivity of a partial order. As for any two elements x, y the corresponding binary string of (x, y) or (y, x) is 1, we see that any two elements are comparable, i.e. P is a chain. We obtain $\dim P = 1$.

Now let P be a poset of Boolean dimension $d \in \{2, 3\}$ and let $(\mathcal{B}, \tau)$ be a Boolean realizer. For any $\alpha \in \mathbf{2}^d$, we denote by $\overline{\alpha}$ the bitwise flipped binary vector.

If there exists at most one pair $\alpha, \overline{\alpha} \in \mathbf{2}^d$ such that $\tau(\alpha) = \tau(\overline{\alpha}) = 0$, we observe that for every incomparable pair (x, y) of P either the binary string associated with (x, y) or (y, x) is α. Thus, orienting edges xy of the incomparability graph from x to y if and only if $q(x, y, \mathcal{B}) = \alpha$ defines an orientation of $\mathrm{Inc}(P)$. Note that the orientation is transitive. Thus, $\dim(P) \leq 2$ follows by Lemma 2.1.13.

This concludes in particular the proof for $d = 2$. Indeed, as $|\mathbf{2}^2| = 4$ and τ maps the binary string 11 to 1, we see that there is at most one such α. We may therefore assume that $d = 3$.

If there exist at least two such α, at least four binary strings are mapped to 0 by τ. By antisymmetry of a partial order, at most two of the remaining four strings are mapped to 1. Let $\mathcal{B} = \{L_1, L_2, L_3\}$. We distinguish (up to symmetry) several cases.

Case 1. If $\tau(\alpha) = 1$ if and only if $\alpha \in \{111\}$, then $\mathcal{B}$ is a Dushnik-Miller realizer.

Case 2. If $\tau(\alpha) = 1$ if and only if $\alpha \in \{110, 111\}$, then $\{L_1, L_2\}$ is a Dushnik-Miller realizer. This yields a contradiction as $3 = \mathrm{bdim}(P) \leq \dim(P)$.

Case 3. If $\tau(\alpha) = 1$ if and only if $\alpha \in \{001, 111\}$, we see that for every two incomparable elements x, y either $q(x, y, \mathcal{B})$ or $q(y, x, \mathcal{B})$ lies in the set $\{101, 100\}$. Therefore, we obtain a transitive orientation of the incomparability graph of P by orienting edges from x to y if and only if $x <_{L_1} y$ and $x >_{L_2} y$. An application of Lemma 2.1.13 shows that the poset P has Dushnik-Miller dimension at most 2 which contradicts the assumption $\mathrm{bdim}(P) = 3$.

Thus, only the first case may occur and $\dim(P) \leq \mathrm{bdim}(P)$ follows. □

While Dushnik-Miller dimension and Boolean dimension coincide for posets of small dimension they may differ dramatically. The standard examples provide a family of posets of unbounded Dushnik-Miller dimension, but bounded Boolean dimension.

Lemma 3.1.5 ([5, p. 3][90, p. 3][37, Theorem 2.9]). *The Boolean dimension of any standard example of size at least* 4 *is* 4.

Proof. Consider the standard example S_n of size $n \geq 4$. Suppose its Boolean dimension is at most 3. By Lemma 3.1.4, we obtain $\dim(S_n) \leq 3$. This is a contradiction as the Dushnik-Miller dimension of S_n is n for $n \geq 2$, see Lemma 2.1.10. Thus, it suffices to show that the Boolean dimension of S_n is at most 4.

We construct a Boolean realizer of S_n. We define the following four linear orders of the vertices of S_n

$$\begin{aligned}
L_1 &= a_1 \dots a_n \; b_1 \dots b_n \\
L_2 &= a_n \dots a_1 \; b_n \dots b_1 \\
L_3 &= b_1 \; a_1 \; b_2 \; a_2 \dots b_n \; a_n \\
L_4 &= b_n \; a_n \; b_{n-1} \; a_{n-1} \dots b_1 \; a_1.
\end{aligned}$$

Further, we define a function

$$\begin{aligned}
\tau \colon V(S_n) \times V(S_n) &\to \{0, 1\} \\
(x, y) &\mapsto (x \leq_{L_1} y) \wedge (x \leq_{L_2} y) \wedge \big((x \leq_{L_3} y) \vee (x \leq_{L_4} y)\big)
\end{aligned}$$

where $x \leq_{L_i} y$ is true if and only if x precedes y in the linear order L_i. It is easy to check that $(\{L_1, \dots, L_4\}, \tau)$ is indeed a Boolean realizer of S_n. We obtain $\mathrm{bdim}(S_n) \leq 4$. □

Nešetřil and Pudlák showed in 1989 that the Boolean dimension of posets on n elements is in $\mathcal{O}(\log(n))$ [76, Proposition 1]. This is very different from Dushnik-Miller dimension as the Dushnik-Miller dimension of a poset on n elements can be linear in n, see Lemma 2.1.10. Further, Nešetřil and Pudlák proved the existence of a poset on n elements that meets the upper bound [76, Proposition 2]. However, their proof is non-constructive.

Later on, explicit constructions of posets with large Boolean dimension were found. For instance, universal interval orders do not only have large Dushnik-Miller dimension, as we have seen in Lemma 2.1.17, but also large Boolean dimension. The following lemma is a modification of an argument in [35] and is due to Felsner, Mészáros, and Micek, see [31, p. 676] for a proof.

Lemma 3.1.6 ([31, p. 676])**.** *For every $n \geq 2$, we have* $\mathrm{bdim}(I_n) \geq \log\log\log n$.

In Proposition 3.2.5 we will encounter yet another family of posets of unbounded Boolean dimension.

While the Dushnik-Miller dimension of planar posets is unbounded as we have seen in Corollary 2.6.4, it is still unknown whether the Boolean dimension of planar posets is bounded [70, Problem A]. The question was initially raised by Nešetřil and Pudlák who suggested a Ramsey-Type-argument in order to show that it is unbounded [76, Problem 3.1]. No progress has been made towards proving the negative result, but recent results suggest that the Boolean dimension of planar posets might be bounded. Felsner, Mészáros, and Micek showed that the Boolean dimension of a poset is bounded in terms of the treewidth of its cover graph [31]. While a construction of Trotter, called the wheel construction, shows that there are posets with planar cover graphs, a zero and a one with arbitrary large Dushnik-Miller dimension [88, Theorem 1] [70, p. 2], the same does not hold for Boolean dimension. Micek, Blake, and Trotter proved that the Boolean dimension of a poset with a planar cover graph and a zero is at most 13 [70, Theorem 1], thereby extending a result of Gambosi, Nešetřil, and Talamo who showed that the Boolean dimension of planar posets of height 2 is bounded [37, Theorem 3.4]. The result on posets with planar cover graphs and a zero seems somewhat similar to Theorem 2.6.9, which states that planar posets with a zero have Dushnik-Miller dimension at most 3.

Often, we relate the dimension of a poset to graph parameters of its cover graph, see [59, Figure 2] for an overview of such results. As graph partitions and coverings provide useful tools in graph theory, we might wonder whether it suffices to study the dimension of the components of a partition or covering of the diagram in order to determine the dimension of the poset.

Indeed, partitions are of help in some situations. For example, the Dushnik-Miller dimension of a poset is bounded by the maximum Dushnik-Miller dimension of its connected components plus one, see Lemma 2.1.20. No similar linear bound exists for Boolean dimension. In fact, there are posets whose Boolean dimension is exponential in the maximum Boolean dimension of its connected components [69, pp. 5–6]. Determining an upper bound in the maximum Boolean dimension of the connected components is much more complicated then for Dushnik-Miller dimension. An exponential upper bound has been established by Mészáros, Micek, and Trotter [69, Theorem 2.1].

However, neither graph partition nor coverings respect Dushnik-Miller dimension in general.

Observation 3.1.7. *The standard example S_n admits a partition of its edges into two posets, each of which has Dushnik-Miller dimension at most* 2*.*

Proof. Consider the partition of the standard example of size n into two posets P' and P'' on the elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ where $a_i <_{P'} b_j$ if and only if $i < j$ and $a_i <_{P''} b_j$ if and only if $i > j$ for all i and j.

As the two linear orders

$$L_1 = b_1\ a_1\ b_2\ a_2 \ldots b_n\ a_n$$
$$L_2 = a_n \ldots a_1\ b_n \ldots b_1$$

form a Dushnik-Miller realizer of P', we obtain $\dim(P) \leq 2$. The proof for P'' is similar. □

For Boolean dimension, the situation is somewhat different as is exemplified in the following lemma.

Lemma 3.1.8. *If P is a poset of height* 2 *and we partition the edges of its diagram into k sets $E_1, \ldots, E_k$, then we have $\operatorname{bdim}(P) \leq \sum_{i=1}^{k} \operatorname{bdim}(P_i)$ where P_i denotes the poset on the elements of P induced by the edges E_i.*

Proof. Let $d_i \coloneqq \operatorname{bdim} P_i$ and let $(\mathcal{B}_i, \tau_i)$ denote a Boolean realizer of P_i for each i. It suffices to show that for any two elements x, y of P we have $x \leq_P y$ if and only if

$$\bigvee_{i=1}^{k} \tau_i(q(x, y, \mathcal{B}_i)) = 1.$$

Let x, y be distinct elements of P. If $x \leq_P y$, then xy is an edge in the diagram and therefore there is an i such that $xy \in E_i$. In particular, $x \leq_{P_i} y$ and $\tau_i(q(x, y, \mathcal{B}_i)) = 1$ follows.

Suppose now that for distinct elements x, y of P, we have $\bigvee_{i=1}^{k} \tau_i(q(x, y, \mathcal{B}_i)) = 1$. Thus, there exists an index $i \in [k]$ such that $\tau_i(q(x, y, \mathcal{B}_i)) = 1$. We see that xy is an edge in E_i, therefore in particular an edge of the diagram of P. This observation yields $x \leq_P y$. □

In particular, the lemma above together with Observation 3.1.7 yields another proof of the fact that standard examples have Boolean dimension at most 4.

Any two elements which are incomparable in a subposet P' of a poset P are also incomparable in P as P' is the induced relation on a subset of elements of P. We wish to loosen this restriction.

Definition 3.1.9. A *weak subposet* P' of a poset P is a poset on a subset of the elements of P such that for all elements x and y of P' with $x \leq_{P'} y$ we have $x \leq_P y$.

Subposets are in particular weak subposets, yet the converse is not true. An antichain is for example a weak subposet of a chain of the same size, but not a subposet. In general, any poset whose diagram is a subgraph of the diagram of a poset P is a weak subposet of P. Yet, not every weak subposet has this property, think of the standard example which is a subposet of Kelly's construction.

Using the same idea as in the proof above, we can easily generalize Lemma 3.1.8 to posets of arbitrary height.

Lemma 3.1.10. *Let P be a poset. If $\mathcal{P}$ is a family of weak subposets of P such that*

(i) each $P' \in \mathcal{P}$ contains all elements of P

(ii) for every two distinct elements x and y of P with $x \leq_P y$ there exists a poset $P' \in \mathcal{P}$ whose diagram contains a directed x-y-path,

then

$$\mathrm{bdim}(P) \leq \sum_{P' \in \mathcal{P}} \mathrm{bdim}(P').$$

This observation will be useful in the proof of Proposition 3.2.15.

3.2 Boolean Dimension, Queue and Stack Number

Recall that Dushnik-Miller dimension is bounded from above in terms of queue number and height, see Theorem 2.11.19, and the same is true for stack number and height by Theorem 2.11.20. As the Boolean dimension of a poset does not exceed its Dushnik-Miller dimension, the same bounds also apply to Boolean dimension. The aim of this section is to study the relationship between queue and stack number and Boolean dimension.

We have already seen that the Dushnik-Miller dimension of a poset is neither bounded in terms of its queue number, nor in terms of its stack number, see Corollary 2.7.7 and 2.10.11. Yet, it is bounded in terms of stack number and height, and similarly in terms of queue number and height.

We may wonder whether if we substitute Dushnik-Miller dimension by Boolean dimension, a similar result holds independent of the height of the poset. Joret et al. showed that the Dushnik-Miller dimension of a poset can be bounded in terms of the treewidth of its cover graph and its height [56, Theorem 1.1]. While there are posets of arbitrarily large Dushnik-Miller dimension whose cover graphs have treewidth at most 3 [31, p. 656], the Boolean dimension of a poset is bounded in terms of the treewidth of its cover graph [31, Theorem 2]. Nevertheless, no similar result holds for stack or queue number of posets. Using the same approach as in Corollary 2.10.11, we show that Boolean dimension is not bounded in terms of stack or queue number.

Proposition 3.2.1. *There exists a family of poset with stack number at most* 3 *and unbounded Boolean dimension and a family of posets with queue number at most* 2 *and unbounded Boolean dimension.*

Proof. By Lemma 3.1.6, there exists for every $n \in \mathbb{N}$ a poset P such that

$$\operatorname{bdim}(P) \geq n.$$

Corollary 2.10.9 yields the existence of a subdivision S of P with stack number at most 3. As P is a subposet of P, we obtain $\operatorname{bdim}(S) \geq n$.

To obtain the analog result for queue number, we may apply Corollary 2.10.6 to the poset P. □

In Section 2.12, we showed that any upper bound on Dushnik-Miller dimension in terms of queue number and height has to be exponential in both parameters and the same is true for stack number and height. The posets we constructed in Proposition 2.12.3 and Proposition 2.12.2 have large Dushnik-Miller dimension as they contain large standard examples. However, standard examples have Boolean dimension at most 4. Thus, the lower bounds we established do not necessarily apply to Boolean dimension.

We are therefore interested in finding other constructions which determine lower bounds for Boolean dimension, i.e. we aim to construct posets with small height and queue number, but large Boolean dimension. In particular, we will study posets of height 2.

Barrera-Cruz et al. showed that the Boolean dimension of a poset P of height 2 is bounded by a function in the maximum degree of the maximal elements in the diagram of P [5, Theorem 2.5]. They attribute the proof to Gambosi, Nešetřil, and Talamo [37, Theorem 3.6].

Proposition 3.2.2 ([5, Theorem 2.5])**.** *If P is a height-2 poset and d is the maximum degree of the maximal elements of P in its cover graph, then* $\operatorname{bdim}(P) \leq 2d$.

Proof. Let A denote the minimal and let B denote the maximal elements of P. We color the edges of the diagram such that any two edges incident to the same vertex in B have distinct colors using at most d colors. Let $E_1, \ldots, E_d$ denote the color classes and let P_i denote the poset on the elements of P induced by the edges E_i.

By Lemma 3.1.8, it suffices to show that $\operatorname{bdim}(P_i) \leq 2$ for all i. As every vertex in B has degree at most 1 in the diagram G_i of P_i, we see that G_i is a union of stars and independent vertices. Thus, P_i can be easily extended to a planar poset with a zero and a one. By Theorem 2.6.8, we obtain $\operatorname{bdim}(P_i) \leq \dim(P_i) \leq 2$. □

There are posets of height 2 and arbitrarily large Dushnik-Miller dimension where every maximal element has degree at most 2 in the cover graph. The class of posets called incidence posets of complete graphs provides such an example as has often been noted, see [92, p. 2] for instance. As Proposition 3.2.2 shows that the Boolean dimension of these posets is at most 4, they provide yet another example of posets with constant Boolean dimension but unbounded Dushnik-Miller dimension.

The upper bound given in Proposition 3.2.2 is tight as has been shown by Barrera-Cruz et al. [5, Theorem 2.6] using the following observation.

Observation 3.2.3 ([5, p. 6])**.** *Replacing some of the linear orders of a Boolean realizer* $(\mathcal{B}, \tau)$ *of a poset* P *with the corresponding reversed linear orders, we obtain a Boolean realizer* $(\mathcal{B}', \tau')$ *of* P *where* τ' *arises from* τ *through the obvious modification.*

The bound given in Proposition 3.2.2 is established by the following family of posets.

Definition 3.2.4. For $d \leq n$, let $S(n, d)$ be the height-2 poset on all 1- and d-element subsets of $[n]$ ordered by inclusion.

Indeed, the Boolean dimension of $S(n, d)$ is $2d$ for large enough n as has been shown by Barrera-Cruz et al. [5, Theorem 2.6].

Proposition 3.2.5 ([5, Theorem 2.6])**.** *For every* $d \in \mathbb{N}$*, there exists a poset* P *of height* 2 *such that* $\operatorname{bdim}(P) = 2d$ *where* d *denotes the maximum degree of the maximal elements of* P *in the cover graph. In particular, for large enough* n*, we have* $\operatorname{bdim}(S(n, d)) = 2d$.

Proof. The claim obviously holds for $d = 1$. Now let $d \geq 2$. For every $n \geq d$, consider the poset $S(n, d)$. By Proposition 3.2.2, it suffices to show that $\operatorname{bdim}(S(n, d)) \geq 2d$ for sufficiently large n.

Suppose $\operatorname{bdim}(S(n, d)) < 2d$ for every $n \geq d$. Let $(\mathcal{B}, \tau)$ be a Boolean realizer of $S(n, d)$ and write $\mathcal{B} = \{L_1, \ldots, L_s\}$. Applying Theorem 2.4.3 s times, we find a set $A \subseteq [n]$ of size $2d + 1$ that appears in every linear order L_i in increasing or decreasing order if n is sufficiently large. By Observation 3.2.3, we may assume that

$$1 \leq_{L_i} 2 \leq_{L_i} \leq \cdots \leq_{L_i} 2d + 1$$

for every i after relabeling the elements of A.

There are $2d$ *gaps* between consecutive elements of A which we denote by $(j, j + 1)$ for $j \in [2d]$. Consider the element $S \coloneqq \{2, 4, \ldots, 2d\}$ of $S(n, d)$. As there are at most $2d - 1$ linear orders L_i and $2d$ gaps, there exists some gap $(j, j + 1)$ such that there is no linear order L_i with $j \leq_{L_i} S \leq_{L_i} j + 1$. Therefore, $q(j, S, \mathcal{B}) = q(j + 1, S, \mathcal{B})$. This shows that $j \in S$ if and only if $j + 1 \in S$. However, exactly one of the two numbers j and $j + 1$ lies in S yielding a contradiction. □

In particular, we see that there exists a family of posets of height 2 and unbounded Boolean dimension.

One might hope that the posets $S(n, d)$ have large Boolean dimension, small queue number and constant height. If this were true, they might provide a family of posets which shows that any upper bound on Boolean dimension in terms of queue number and height has to be exponential in queue number. However, no poset of height 2 provides such an exponential lower bound as we will see in Proposition 3.2.10. The same is true if we consider stack number instead of queue number, see Proposition 3.2.11. For the posets $S(n, d)$, we establish explicit upper bounds on the queue and stack number which directly yield polynomial upper bounds on the Boolean dimension.

Observation 3.2.6. *The queue number of $S(n,d)$ is at least $\sqrt{\frac{n-1}{2}}$ for $3 \leq d < n$ and $n \geq 15$. In particular, we have $\operatorname{bdim}(S(n,d)) \leq 4\operatorname{qn}(S(n,d))^2 + 2$. The same is true if we replace queue by stack number.*

Proof. Let $n' \coloneqq \lfloor \frac{n}{2} \rfloor$. First suppose that $d < \frac{n}{2}$. We construct distinct d-element subsets $A_1, \dots, A_{n'}$ of $[n]$ such that

$$i, n' - (i-1) \in A_i$$

for all $i \in [n']$ as follows. As $d \geq 3$ and $2n' \leq n$, we can define A_i as the union of any set

$$C_i \subseteq [n] \setminus (\{n'+1, \dots, 2n'\} \cup \{i, n'-(i-1)\})$$

of size $d-3$ and the set $\{i, n'-(i-1), n'+i\}$ for every i. Note that such sets C_i of size $d-3$ exist as $d < \frac{n}{2}$.

Let σ be any topological ordering of the elements of $S(n,d)$. Without loss of generality, we may assume that the elements $1, \dots, n'$ appear in increasing order in σ. Consider the sequence of the indices of the sets A_i induced by σ. By Theorem 2.4.3, there exists a monotone subsequence (i_k) of length $\ell \geq \sqrt{n'}$ such that

$$A_{i_1} \leq_\sigma A_{i_2} \leq_\sigma \cdots \leq_\sigma A_{i_\ell}.$$

As σ is a topological ordering, the elements i_1 and $n'-(i_1-1)$ precede A_{i_1} in σ. If (i_k) is increasing, we see that the edges between elements $n'-(i_k-1)$ and A_{i_k} form an ℓ-rainbow and the edges between elements i_k and A_{i_k} form an ℓ-twist. If (i_k) is decreasing we obtain an ℓ-rainbow between the elements i_k and A_{i_k}, and an ℓ-twist between the elements $n'-(i_k-1)$ and A_{i_k}. As this holds for any topological ordering, the queue and stack number of $S(n,d)$ is at least $\sqrt{n'} \geq \sqrt{\frac{n-1}{2}}$.

Now suppose that $d \geq \frac{n}{2}$. We construct n' distinct d-element subsets $A_1, \dots, A_{n'}$ of $[n]$ such that each A_i contains $[n'] \setminus \{i\}$ as a subset and does not contain the element i. Define A_i as the union of any set $C_i \subseteq [n] \setminus [n']$ of size $d-(n'-1)$ and the set $[n'] \setminus \{i\}$. Note that such sets C_i exist as $d < n$. As A_i does not contain i, but each A_j for $i \neq j$ does, we see that the induced poset on the elements $[n']$ and $A_1, \dots, A_{n'}$ forms the standard example. Thus, we obtain that the queue and stack number of $S(n,d)$ is at least $\lfloor \frac{n'}{2} \rfloor \geq \lfloor \frac{n-1}{4} \rfloor \geq \frac{n-4}{4}$ by Lemma 2.2.3. As $\frac{n-4}{4} \geq \sqrt{\frac{n-1}{2}}$ for $n \geq 15$, the claim follows.

Therefore, the queue and the stack number of $S(n,d)$ are at least $\sqrt{\frac{n-1}{2}}$. As $d \leq n$ by definition of $S(n,d)$, we obtain by Proposition 3.2.2

$$\operatorname{bdim}(S(n,d)) \leq 2d \leq 2n \leq 4\operatorname{qn}(S(n,d))^2 + 2$$

and a similar bound for stack number. □

Observation 3.2.6 does not give a lower bound on the queue or stack number of $S(n,2)$. Yet, using a similar argument, we see that these posets also have relatively large queue and stack number.

Observation 3.2.7. *The queue and stack number of $S(n,2)$ is at least $\sqrt{\frac{n-1}{2}}$ for all $n \geq 2$.*

Proof. We proceed as in the proof of Observation 3.2.6. Let $n' \coloneqq \lfloor \frac{n}{2} \rfloor$. Define

$$A_i \coloneqq \{i, n-(i-1)\}$$

for all $i \in [n']$. Note that all these sets are distinct and have size two. Consider any topological ordering σ of the elements of $S(n,2)$. We may assume that the elements $1,\dots,n$ appear in increasing order in σ. By Theorem 2.4.3, we obtain a monotone sequence (i_k) of length $\ell \geq \sqrt{n'}$ such that

$$A_{i_1} \leq_\sigma A_{i_2} \leq_\sigma \cdots \leq_\sigma A_{i_\ell}.$$

As σ is a topological ordering, the elements i_1 and $n-(i_1-1)$ precede A_{i_1} in σ for all $k \in [\ell]$. If (i_k) is increasing, the edges between $n-(i_k-1)$ and A_{i_k} for $k \in [\ell]$ form an ℓ-rainbow and the edges between i_k and A_{i_k} form an ℓ-twist. Similarly, we obtain a rainbow and a twist of size ℓ if (i_k) is decreasing.

As σ is an arbitrary ordering, we see that the stack and queue number of $S(n,2)$ is at least $\sqrt{n'} \geq \sqrt{\frac{n-1}{2}}$. □

The posets $S(n,2)$ correspond to the incidence posets of complete graphs, which have arbitrarily large Dushnik-Miller dimension [92, p. 2], but Boolean dimension at most 4.

In Corollary 2.3.5 and 2.5.4, we constructed posets with constant Dushnik-Miller dimension and arbitrarily large stack or queue number. As Boolean dimension is bounded from above by Dushnik-Miller dimension, these posets also have constant Boolean dimension. The posets $S(n,2)$ and the standard examples provide two more families of posets which show that neither queue nor stack number is bounded in terms of the Boolean dimension. However, they do not provide examples for the fact that the same is true for Dushnik-Miller dimension.

In the remainder of this section, we improve the upper bounds for Boolean dimension in terms of queue or stack number for posets of height 2. Further, we will be able to improve the upper bound for Boolean dimension in terms of queue number and height in general.

In the proof of Lemma 2.10.5, we partitioned an upward drawing of a dag S into smaller dags along horizontal lines. Some of the dags we obtained corresponded to matching graphs. Using a correspondence between the non-crossing edges in a matching graph and queues of a specific queue layout of its edges, we were able to bound the queue number of S.

Our aim is to construct planar weak subposets of a poset P which contain all directed paths of the diagram of P using the same correspondence. If we partition the diagram of P into matching graphs along horizontal lines and choose for each matching graph a set of non-crossing edges, we clearly obtain a planar weak subposet of P. Upward drawings for which such a partition into $k-1$ matching graphs exists are called k-leveled.

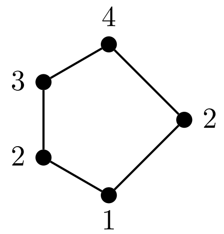


Figure 3.1: A poset that is not k-leveled for any k. The numbers associated to the elements correspond to their height in the poset.

Definition 3.2.8. A dag G is called *k-leveled* if its vertices can be partitioned into k sets $\mathcal{L}_1, \dots, \mathcal{L}_k$ such that for every edge $ab \in E(G)$ there exists an integer $i \in [k-1]$ such that $a \in \mathcal{L}_i$ and $b \in \mathcal{L}_{i+1}$. For $i \in [k]$ the set $\mathcal{L}_i$ is referred to as the *i-th level.* For a given partition of the vertex set of G into k levels, we call an embedding of G in $\mathbb{R}^2$ a *k-leveled drawing* if all vertices of the same level have the same y-coordinate and the y-coordinate of a vertex of the i-th level is smaller than the y-coordinate of a vertex of the j-th level for all i and j with $i < j$.

A poset is called *k-leveled* if its diagram is a k-leveled graph.

Note that the drawing induced by levels $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ is indeed a matching graph.

Clearly every directed, k-leveled graph admits a k-leveled drawing. Yet, the assignment of vertices to levels is not unique. An isolated vertex for instance can be part of any level.

It is tempting to suppose that every poset P is $h(P)$-leveled as we could try to define levels as vertices of the same height. However, such an assignment does not necessarily yield an $h(P)$-leveled drawing of the diagram; see Figure 3.1. In fact, the poset given in Figure 3.1 provides an example of a poset which is not k-leveled for any k.

For height-2 posets, 2-leveled drawings can be easily constructed.

Lemma 3.2.9. *For every k-queue poset P of height 2 there exists a 2-leveled drawing $\mathcal{E}$ of its diagram G and a decomposition of the edges of G into k sets $E_1, \dots, E_k$ such that each E_i induces a planar drawing in $\mathcal{E}$.*

Proof. Let k be the queue number of P. We consider a k-queue layout σ of the vertices of the diagram G of P. Let $E_1, \dots, E_k$ be the corresponding queue assignment. We define levels $\mathcal{L}_i := \{x \in P \mid h(x) = i\}$ for $i \in [2]$ where h is the function that assigns to each element x of P its height, i.e. the size of a longest chain ending in x. Indeed, the given assignment of vertices to levels shows that G is 2-leveled as there cannot be edges between vertices of the same height.

We obtain a 2-leveled drawing $\mathcal{E}$ of G in $\mathbb{R}^2$ by assigning a vertex of the j-th level the y-coordinate j and ordering vertices within a level according to σ. It remains to show that each E_i induces a planar drawing in $\mathcal{E}$. Suppose there is some $i \in [k]$ and edges $a'b', a''b'' \in E_i$ that cross in $\mathcal{E}$. Note that $a', a'' \in \mathcal{L}_1$ and $b', b'' \in \mathcal{L}_2$. Without loss of generality, we may assume $a' \leq_\sigma a''$. By construction, we obtain $b'' \leq_\sigma b'$. As σ is a topological ordering,

$$a' \leq_\sigma a'' \leq_\sigma b'' \leq_\sigma b'$$

follows. This is a contradiction, as no two edges in E_i nest. □

The lemma above shows that we can partition the edges of a height-2 poset P into $\mathrm{qn}(P)$ planar posets. By Lemma 3.1.8, we obtain the following upper bound on the Boolean dimension.

Proposition 3.2.10. *For every height-2 poset P, we have $\mathrm{bdim}(P) \leq 2\,\mathrm{qn}(P)$.*

Proof. Let $k := \mathrm{qn}(P)$. By Lemma 3.2.9, there exists a decomposition of the edges of the diagram of P into k sets $E_1, \ldots, E_k$ such that each E_i induces a planar poset. Let P_i be the planar poset on the elements of P induced by the edges E_i for $i \in [k]$. We see that for all i the poset P_i can be easily extended to a planar poset P_i' with a zero and a one such that $P_i \subseteq P_i'$. Thus for all i, we obtain

$$\mathrm{bdim}(P_i) \leq \mathrm{bdim}(P_i') \leq \dim(P_i') \leq 2$$

where the second inequality follows from Lemma 3.1.3 and the last from Theorem 2.6.8. An application of Lemma 3.1.8 yields the claim. □

Not only is the Boolean dimension of height-2 posets bounded from above by a linear function in the queue number, but also in terms of the stack number.

Proposition 3.2.11. *For every height-2 poset P, we have $\mathrm{bdim}(P) \leq 3\,\mathrm{sn}(P)$.*

Proof. Let $k := \mathrm{sn}(P)$. As P admits a k-stack layout, we can partition the edges of the diagram of P into k stacks $E_1, \ldots, E_k$. Each E_i induces a forest in $\mathrm{Cov}(P)$ by Lemma 2.7.1. Let P_i be the poset on the elements of P induced by the edges of E_i. By Corollary 2.6.10, we have $\dim(P_i) \leq 3$. As Boolean dimension is bounded by Dushnik-Miller dimension, we see that $\mathrm{bdim}(P_i) \leq 3$ for each i. An application of Lemma 3.1.8 yields the claim. □

Proposition 3.2.10 and Proposition 3.2.11 provide linear upper bounds on Boolean dimension in the queue respectively stack number of posets of height 2. This shows in particular that there is no family of height-2 posets whose Boolean dimension is exponential in their queue or stack number.

Lemma 3.2.9 enables us to cover a poset P of height 2 with relatively few planar posets depending on the queue number of P. We will now generalize this approach to posets of arbitrary height.

Definition 3.2.12. Let P be a poset that is not an antichain and let h be its height. We denote by E the edges of the diagram of P. For a q-queue layout σ of P, we define an edge-coloring $c\colon E \to [q]$ where we assign edges of the i-th queue the color i.

Let $k \in [h]$. Consider a strictly increasing sequence (ω_i) of heights in $[h]$ and a sequence (π_i) of colors in $[q]$ where (ω_i) and (π_i) are of length k and $k-1$ respectively.

We define a weak subposet P' of P whose diagram contains all elements of P and all edges in

$$\bigcup_{i \in [k-1]} \{e \in E \mid h(L(e)) = \omega_i, h(R(e)) = \omega_{i+1}, c(e) = \pi_i\}$$

where $L(e)$ denotes the left and $R(e)$ the right endpoint of an edge e. The poset P' is called a *weak k-subposet of P relative to σ* which is induced by the sequences (ω_i) and (π_i).

The queue number of antichains is 0 as their diagrams do not contain any edges. This is why we do not consider antichains in the definition above.

If we fix a queue layout σ of a poset P, we note that every directed path of the diagram of P is contained in a weak k-subposet relative σ for some k. Thus, it suffices to bound the Boolean dimension of all weak k-subposets in order to bound the Boolean dimension of P by Lemma 3.1.10.

In fact, weak k-subposets are planar posets as we will show in the following lemma. Thus, their Boolean dimension is bounded by Theorem 2.6.11 in terms of their height.

Lemma 3.2.13. *If P be a poset that is not an antichain, then the weak k-subposets relative to any queue layout of P are planar for every $k \in [h(P)]$.*

Proof. Let σ be a q-queue layout of the poset P and let $h = h(P)$. Consider a weak k-subposet P' of P relative σ which is induced by a strictly increasing sequence (ω_i) of heights in $[h]$ and a sequence (π_i) of colors in $[q]$.

Using a similar argument as in Lemma 3.2.9, we show that P' is a planar poset. We may assume that P' contains no independent vertices as these pose no obstruction to planarity.

Note that the poset P' is k-leveled. This can be seen as follows. Defining the i-th level $\mathcal{L}_i$ as the set of vertices of height ω_i in P, we can embed P' in $\mathbb{R}^2$ by assigning a vertex $v \in \mathcal{L}_i$ the coordinate (ℓ, i) where ℓ denotes the position of v in the ordering σ.

As the given drawing of P' is k-leveled, it suffices to prove that no two edges between two consecutive levels $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ cross. Observe that edges between levels $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ of P' belong to the same queue of σ. Suppose there are two crossing edges $a'b', a''b''$ between levels $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$. Without loss of generality, we may assume $a' \leq_\sigma a''$. By definition of the drawing, we see that $b'' \leq_\sigma b'$. Further, as σ is a topological ordering, we have $a'' \leq_\sigma b''$ and

$$a' \leq_\sigma a'' \leq_\sigma b'' \leq_\sigma b'$$

follows. This is a contradiction, as the edges $a'b'$ and $a''b''$ nest even though they lie in the same queue of σ. Thus, P' is a planar poset. □

As the Boolean dimension of all weak k-subposets of a poset P is bounded in terms of their height, it suffices to count the number of weak k-subposets in order to bound the Boolean dimension of P.

We first determine some identities for binomial sums which follow easily from the binomial theorem, see [43, p. 162] for further reference.

Lemma 3.2.14. *For positive integers n and a, we have*

(i) $\sum_{k=1}^{n} \binom{n}{k} a^{k-1} \cdot k = n(a+1)^{n-1}$

(ii) $\sum_{k=1}^{n} \binom{n}{k} a^{k-1} = \frac{1}{a}\left((a+1)^n - 1\right)$

Proof. Let n and a be positive integers. We have

$$\sum_{k=1}^{n} \binom{n}{k} a^{k-1} \cdot k = \sum_{k=1}^{n} n \cdot \binom{n-1}{k-1} a^{k-1} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k} a^{k} = n(a+1)^{n-1}$$

where we used the well-known identity $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$ [43, p. 157] in the first and the binomial theorem [43, p. 162] in the third equality. This shows the first part of the lemma.

For the second part, observe that

$$\sum_{k=1}^{n} \binom{n}{k} a^{k-1} = \frac{1}{a} \sum_{k=1}^{n} \binom{n}{k} a^{k} = \frac{1}{a} \left(\sum_{k=0}^{n} \binom{n}{k} a^{k} - 1 \right) = \frac{1}{a}\left((a+1)^n - 1\right).$$

□

As Boolean dimension is bounded by Dushnik-Miller dimension, the Boolean dimension of a poset is in particular bounded in terms of its height and queue or stack number. The bounds on Dushnik-Miller dimension we established in Theorem 2.11.19 and Theorem 2.11.20 are astronomical. For queue number, we are able to improve the bound for Boolean dimension. However, the bound is still exponential in the height.

Proposition 3.2.15. *The Boolean dimension of every poset is bounded in terms of its queue number and height. More precisely, for a poset P of queue number q and height h we have*

$$\operatorname{bdim}(P) \leq 192(h+1)(q+1)^{h-1}.$$

Proof. Let P be a poset, let q denote its queue number and let h be its height. First assume that P is an antichain. We obtain

$$\operatorname{bdim}(P) \leq \dim(P) \leq 2 \leq 384$$

as its height is 1 and its queue number is 0. Thus, the claim holds for antichains.

We may now assume that P is not an antichain. Therefore, its diagram contains at least one edge and $q \geq 1$ and $h \geq 2$ follow.

For every height $k \in [h]$, we can construct $\binom{h}{k}$ strictly increasing sequences (ω_i) of length k in $[h]$ and q^{k-1} sequences (π_i) of length $k-1$ in $[q]$. Let $\mathcal{P}_k$ denote the set of weak k-subposets of P relative σ. Note that all posets in $\mathcal{P}_k$ are of height k. By Theorem 2.6.11, we have for all $P' \in \mathcal{P}_k$

$$\dim(P') \leq 192k + 96.$$

As every directed path in the diagram of P lies in the diagram of some weak k-subposet relative σ we obtain by Lemma 3.1.10

$$\mathrm{bdim}(P) \le \sum_{k=1}^{h} \sum_{P' \in \mathcal{P}_k} \mathrm{bdim}(P') \le \sum_{k=1}^{h} \binom{h}{k} q^{k-1} \cdot (192k + 96)$$

where the last inequality follows from Lemma 3.1.3. Recall that $q \ge 1$ as we assumed that P is not an antichain. By Lemma 3.2.14, we get

$$\mathrm{bdim}(P) \le 192 \sum_{k=1}^{h} \binom{h}{k} q^{k-1} k + 96 \sum_{k=1}^{h} \binom{h}{k} q^{k-1} = 192h(q+1)^{h-1} + \frac{96}{q}((q+1)^h - 1).$$

As $q + 1 \le 2q$ for $q \ge 1$, this finally yields

$$\mathrm{bdim}(P) \le 192h(q+1)^{h-1} + \frac{96}{q}(q+1)^h \le 192h(q+1)^{h-1} + 192(q+1)^{h-1}.$$

□

While we were able to establish exponential lower bounds on functions which bound Dushnik-Miller dimension, the same approach does not work for Boolean dimension as the argument in Corollary 2.12.2 and Proposition 2.12.3 resides on the fact that the standard example has high Dushnik-Miller dimension. Whether there are exponential lower bounds on functions which bound Boolean dimension in terms of height and queue, or height and stack number is not known.

3.3 Separated Layouts

In the previous section, we improved the upper bound on Boolean dimension in terms of queue number and height. The aim of this section is to show that a similar approach does not work for stack number.

In the proof of Lemma 3.2.13, we showed that every weak k-subposet P' relative to a given queue layout is planar. Defining weak k-subposets relative to a given stack layout analogously, we could attempt to prove that these posets are also planar using a similar approach. However, this might not be the case. In order to succeed, we need separated layouts which have been introduced by Pemmaraju for undirected graphs [79, Chapter 3]. While he was interested in general partitions of the vertices, we consider a restricted version for directed graphs where vertices are partitioned according to their height.

Definition 3.3.1. Let G be a dag. We partition the vertex set of G into classes V_i corresponding to vertices of height i. Note that the classes V_i are independent sets in G as if ab is an edge any chain ending in a can be extended to a chain ending in b.

A queue layout σ of G is called *separated* if $V_1 \le_\sigma \cdots \le_\sigma V_h$. The minimum number of queues necessary among all separated queue layouts is called the *separated queue* *number* of G and is denoted by $\operatorname{sqn}(G)$. We define the *separated stack number* $\operatorname{ssn}(G)$ analogously.

If P is a poset, its separated queue and stack number, denoted by $\operatorname{sqn}(P)$ and $\operatorname{ssn}(P)$ respectively, correspond to the separated queue and stack number of its diagram.

The separated queue number of a dag is equal to the separated queue number of the underlying undirected graph if we consider the same partition, i.e. vertices are partitioned according to their height in the dag. Therefore, results that hold in the undirected setting also hold in the directed setting.

The weak k-subposets relative a separated stack layout are indeed planar.

Observation 3.3.2. *If P is a poset that is not an antichain, then the weak k-subposets relative to any separated stack layout of P are planar for every $k \in h(P)$.*

Proof. Let σ be an s-stack layout of P. Let P' be a weak k-subposet relative σ induced by an increasing sequence (w_i) of heights in $[h(P)]$ and a sequence (π_i) of colors in $[s]$. We may assume that P' contains no independent vertices as these pose no obstruction to planarity.

We define the i-th level $\mathcal{L}_i$ of P' as the elements of height w_i in P. Consider the k-leveled drawing of P' where all elements of a level $\mathcal{L}_i$ have y-coordinate i, elements within a level $\mathcal{L}_{2i+1}$ are ordered with respect to σ, and elements within a level $\mathcal{L}_{2i+2}$ are ordered with respect to σ^{rev}.

As the drawing is k-leveled, it suffices to prove that no two edges between levels $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ cross. Note that edges between levels $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$ belong to the same stack. Suppose there are two crossing edges $a'b', a''b''$ between levels $\mathcal{L}_i$ and $\mathcal{L}_{i+1}$. Without loss of generality, we may assume $a' \leq_\sigma a''$. By definition of the drawing, we obtain $b' \leq_\sigma b''$. Observe that a' and a'' have smaller height than b' and b'' in P. Thus, as σ is a separated layout of P, we obtain in particular $a'' \leq_\sigma b'$ and

$$a' \leq_\sigma a'' \leq_\sigma b' \leq_\sigma b''$$

follows. This is a contradiction, as the edges $a'b'$ and $a''b''$ cross in σ even though they belong to the same stack. Thus, P' is a planar poset. $\square$

As the weak k-subposets of a poset relative a separated stack layout are planar, it would suffice to bound the separated stack number of a poset in terms of its stack number and height in order to show a similar result to Proposition 3.2.15 for stack number. However, we will see in Observation 3.3.7 that such a bound does not exist.

The separated queue number of a dag cannot be bounded in terms of its queue number.

Observation 3.3.3. *For all $n \in \mathbb{N}$, there exists a poset of queue number at most 2 whose separated queue number is at least n.*

Proof. Consider the poset P on elements $A \cup B \cup C$ where

$$A = \{a_1, \dots, a_n\}$$
$$B = \{b_1, \dots, b_n\}$$
$$C = \{b_1^{(i)}, \dots, b_{2n-i}^{(i)} \mid i \in [n]\}$$

whose diagram consists of the directed path $a_1\, a_2 \dots a_n$, the directed paths $b_1^{(i)} \dots b_{2n-i}^{(i)}\, b_i$ and the edges $a_i b_i$ for all $i \in [n]$.

Define the ordering $\sigma_i := b_1^{(i)} \leq \dots \leq b_{2n-i}^{(i)} \leq b_i$ for all $i \in [n]$ and note that the topological ordering

$$\sigma = a_1 \leq \dots \leq a_n \leq \sigma_1 \leq \dots \leq \sigma_n$$

requires at most two queues. Thus, we obtain $\mathrm{qn}(P) \leq 2$.

It remains to show that the separated queue number of P is at least n. Consider an arbitrary separated queue layout σ' of P. By definition of P, we have $h(a_i) = i$ for all $i \in [n]$ and $h(b_i) = \max(i+1, 2n-i+1) = 2n-i+1$ for all $i \in [n]$. In particular, we obtain $h(a_1) < h(a_2) < \dots < h(a_n) < h(b_n) < \dots < h(b_1)$ and

$$a_1 <_{\sigma'} \dots <_{\sigma'} a_n <_{\sigma'} b_n <_{\sigma'} \dots <_{\sigma'} b_1$$

follows as σ' is a separated queue layout. Thus, the edges $a_i b_i$ form an n-rainbow and the separated queue number of P is at least n as σ' was an arbitrary separated queue layout.

□

Note that the separated queue number increases with the height of the posets in the construction above. In fact, the separated queue number of every dag can be bounded in terms of its queue number and height. This observation is due to Pemmaraju who showed that the queue number of undirected graphs is bounded in the queue number and the number of partition classes, which corresponds in our setting to the height of the dag in question. The following lemma is an easy adaptation of the proof given by Pemmaraju [79, Theorem 3.19].

Lemma 3.3.4. *For every dag G, we have* $\mathrm{sqn}(G) \leq \mathrm{qn}(G) \cdot (h(G) - 1)$.

Proof. Let q be the queue number of G and h its height. Consider a q-queue layout σ of G and let $E_1, \dots, E_q$ be the corresponding queues. Let σ' be the ordering of the vertices of G where for distinct vertices x, y we have $x \leq_{\sigma'} y$ if and only if $h(x) < h(y)$ or x and y are of same height and $x \leq_\sigma y$. The ordering σ' is by definition a separated queue layout of G.

We partition each queue E_i into $h-1$ sets $E_i^\ell := \{ab \in E_i \mid h(b) - h(a) = \ell\}$ where $\ell \in [h-1]$. Note that no two edges $ab, cd \in E_i$ where $h(b) - h(a) = h(d) - h(c)$ nest in σ'. Thus, we constructed a $q(h-1)$-queue assignment of σ'. □

In Observation 3.3.3, we have constructed posets of height $2n$, queue number at most 2 and separated queue number at least n. Yet, these posets do not meet the upper bound established in the lemma above. Pemmaraju showed that the bound of [79, Theorem 3.19], from which the lemma is derived, is tight for bipartite graphs [79, Theorem 3.9]. Whether the theorem is tight in general in the undirected setting is unknown [79, p. 106].

In fact, the proof of the lemma above shows that the queue number of a poset is bounded in terms of its height and the undirected queue number of its cover graph. Using the same argument as in Proposition 3.2.15, we obtain a bound on the Boolean dimension of posets in terms of their height and the undirected queue number of their cover graphs.

Proposition 3.3.5. *The Boolean dimension of every poset P is bounded in terms of its height h and the undirected queue number q of its cover graph. More precisely,*

$$\operatorname{bdim}(P) \leq 288 \cdot q^{h-1} \cdot h^h$$

with the convention $0^0 = 1$.

While the separated queue number of dags is bounded in terms of height and queue number, this does not hold for the separated stack number.

Observation 3.3.6 ([13, Theorem 2.3])**.** *For $n \in \mathbb{N}$, let $T(n)$ be the dag on vertices $\{a_i, b_i, c_i \mid i \in [n]\}$ with edges $\{a_i b_i, b_i c_i, a_i c_i \mid i \in [n]\}$. The dag $T(n)$ has height 3 and stack number 1, while its separated stack number is at least $n^{1/3}$.*

Proof. As $T(n)$ is the union of n triangles, its stack number is 1.

It remains to show that its separated stack number is at least $n^{1/3}$.

Consider a separated stack layout σ of $T(n)$. Without loss of generality, we may assume that

$$a_1 \leq_\sigma \cdots \leq_\sigma a_n.$$

By a well-known result of Erdős and Szekeres, see Theorem 2.4.2, there exists a monotone sequence (i_k) of length ℓ such that

$$b_{i_1} \leq_\sigma \cdots \leq_\sigma b_{i_\ell}$$

and $\ell \geq n^{1/3}$ if (i_k) is increasing and $\ell \geq n^{2/3}$ otherwise.

If (i_k) is increasing, the edges $a_{i_k} b_{i_k}$ form an ℓ-twist as σ is a separated stack layout. Otherwise (i_k) is a decreasing sequence of length $\ell \geq n^{2/3}$. Applying Theorem 2.4.2 to the sequence of vertices c_i with indices in (i_k), we obtain a monotone subsequence (j_k) of length $\ell' \geq n^{1/3}$ such that

$$c_{j_1} \leq_\sigma \cdots \leq_\sigma c_{j_{\ell'}}.$$

If (j_k) is decreasing, the edges $b_{j_k} c_{j_k}$ form an ℓ'-twist, otherwise (j_k) is increasing and the edges $a_{j_k} c_{j_k}$ form an ℓ'-twist.

Therefore, the separated stack layout σ contains a twist of size at least $n^{1/3}$ and the claim follows. □

As the graphs $T(n)$ contain transitive edges, they do not provide an example of posets with stack number 1 and unbounded separated stack number. Note that if we delete all transitive edges in $T(n)$, we obtain the diagram of a poset with separated stack number at most 2. Yet, using a similar construction as Chung, Leighton, and Rosenberg in the observation above, we obtain families of posets with unbounded separated stack number, but bounded stack number.

Observation 3.3.7. *There exists a family of* 2*-stack posets* $\{R(n) \mid n \in \mathbb{N}\}$ *of height* 4 *and unbounded separated stack number. More precisely, we have* $\mathrm{ssn}(R(n)) \geq (n-1)^{1/5}$ *for all* $n \in \mathbb{N}$.

Proof. Consider the height 4 poset R whose cover graph is a 5-cycle. For $n \in \mathbb{N}$, we define $R(n)$ as the union of n posets which are isomorphic to R, i.e. $R(n)$ is the poset on vertices $\{a_i, b_i, c_i, d_i, e_i \mid i \in [n]\}$ with relations

$$a_i \leq b_i \leq c_i \leq e_i$$
$$a_i \leq d_i \leq e_i$$

for $i \in [n]$. As R has stack number 2, we see that $\mathrm{sn}(R(n)) = 2$.

We define *partition classes* A, B, C, D and E consisting of all a_i, b_i, c_i, d_i and e_i respectively.

Consider a separated stack layout σ of $R(n)$. Without loss of generality, we may assume that

$$a_1 \leq_\sigma \cdots \leq_\sigma a_n.$$

By Theorem 2.4.2, there exists a monotone sequence (i_k) of length ℓ_B such that

$$b_{i_1} \leq_\sigma \cdots \leq_\sigma b_{i_{\ell_B}}$$

and $\ell_B \geq n^{1/5}$ if (i_k) is increasing and $\ell_B \geq n^{4/5}$ otherwise. If (i_k) is increasing, the edges $a_{i_k}b_{i_k}$ form an ℓ_B-twist. Otherwise, (i_k) is a decreasing sequence of length at least $n^{4/5}$. By Theorem 2.4.2, there exists a monotone subsequence (j_k) of (i_k) of length ℓ_C such that

$$c_{j_1} \leq_\sigma \cdots \leq_\sigma c_{j_{\ell_C}}.$$

and $\ell_C \geq n^{1/5}$ if (j_k) is decreasing and $\ell_C \geq n^{3/5}$ otherwise. If (j_k) is decreasing, the edges $b_{j_k}c_{j_k}$ form an ℓ_C-twist. Otherwise (j_k) is an increasing sequence of length $\ell_C \geq n^{3/5}$.

Using the same argument as before, we obtain a twist of size at least $n^{1/5}$ between the partition classes of C and E or a decreasing sequence (z_k) of length $\ell_E \geq n^{2/5}$ such that

$$e_{z_1} \leq_\sigma \cdots \leq_\sigma e_{z_{\ell_E}}.$$

In the latter case, we consider the vertices of D with indices in (z_k). An application of Theorem 2.4.2 yields a twist of size at least $n^{1/5}$ between the partition classes A and D, or between D and E.

Therefore, σ contains a twist of size at least $n^{1/5}$ and $\mathrm{ssn}(R(n)) \geq n^{1/5}$ follows as σ is an arbitrary separated stack layout. □

Even though weak k-subposets relative to a given separated stack layout are planar, a similar approach to the proof of Proposition 3.2.15 does not work for stack number as the separated stack number of a poset is not bounded in terms of height and stack number.

3.4 Boolean Dimension of Subdivisions

In Section 2.9, we investigated the relationship between the Dushnik-Miller dimension of posets and their subdivisions. The aim of this section is to obtain similar results for Boolean dimension.

If S is a subdivision of a poset P, it is easy to see that P is a subposet of S. Thus, the Boolean dimension of the initial poset provides a lower bound on the Boolean dimension of any subdivision. Following closely the proof of Spinrad of the analog result for Dushnik-Miller dimension, see Proposition 2.9.2, we determine an upper bound.

Proposition 3.4.1. *For any subdivision S of a poset P, we have*

$$\operatorname{bdim}(S) \leq 4 + \lfloor \log(h(P)) + 1 \rfloor \cdot (\operatorname{bdim}(P) + 2) + 2 \operatorname{bdim}(P).$$

Proof. Let G be the diagram of a poset P and let S be a subdivision of P. We write $e_1, \ldots, e_m$ for the edges of G. Using the same notation as in Proposition 2.9.2, we denote by $x_1^{(ab)}, \ldots, x_k^{(ab)}$ the division vertices of an edge ab of G where

$$x_1^{(ab)} \leq_S \cdots \leq_S x_k^{(ab)}.$$

We fix an ordering π_{orig} of the original vertices of P and define an ordering π_{div} of the division vertices as follows. For every $i \in [m]$, let $\tilde{\pi}_i$ be the topological vertex ordering of the division vertices of the edge e_i. We define π_{div} as the linear order of the division vertices given by

$$\pi_{\text{div}} \coloneqq \tilde{\pi}_1 \leq \tilde{\pi}_2 \leq \cdots \leq \tilde{\pi}_m.$$

Consider the linear orders

$$\pi_1 \coloneqq \pi_{\text{orig}} \leq \pi_{\text{div}}, \quad \pi_2 \coloneqq \pi_{\text{orig}} \leq \pi_{\text{div}}^{\text{rev}}, \quad \pi_3 \coloneqq \pi_{\text{orig}}^{\text{rev}} \leq \pi_{\text{div}}.$$

Note that using $\{\pi_1, \pi_2, \pi_3\}$ we are able to decide for distinct elements x and y of S which of them (if any) is a division vertex and which (if any) is an original vertex solely based on the binary vector $q(x, y, \{\pi_1, \pi_2, \pi_3\})$.

We still need yet another linear order which enables us to determine for two division vertices x, y whether they subdivided the same edge of the diagram G, and if so, whether $x \leq_S y$. Let π'_{div} be the linear order of the division vertices of S defined via

$$\pi'_{\text{div}} \coloneqq \tilde{\pi}_1^{\text{rev}} \leq \tilde{\pi}_2^{\text{rev}} \leq \cdots \leq \tilde{\pi}_m^{\text{rev}}$$

and let $\pi_4 \coloneqq \pi_{\text{orig}} \leq \pi'_{\text{div}}$. We observe that π_4 is very similar to π_1, but that the ordering of division vertices of the same edge is reversed. By construction, we obtain for two distinct division vertices x and y that $q(x, y, \{\pi_1, \pi_4\}) = (1, 0)$ if and only if x and y are division vertices of the same edge and $x \leq_S y$. We set $\mathcal{B}' \coloneqq \{\pi_1, \dots, \pi_4\}$.

Consider a linear order L of the vertices of S. We use the terms *high* and *low* for the placement of division vertices of an edge ab in L with respect to its restriction to P as in Definition 2.8.1. The linear order we obtain by placing all division vertices of S high with respect to a given linear order L of P is denoted by L^{high}. Similarly, we define L^{low} as the linear order obtained by placing all division vertices low.

Let $d \coloneqq \operatorname{bdim}(P)$ be the Boolean dimension of P and let $(\mathcal{B}, \tau)$ be a minimum Boolean realizer. Let $\mathcal{B}^{\text{low}} = \{L^{\text{low}} \mid L \in \mathcal{B}\}$ and $\mathcal{B}^{\text{high}} = \{L^{\text{high}} \mid L \in \mathcal{B}\}$.

Consider distinct elements x, y of S such that at least one of these two elements is an original vertex. We show that we are able to decide whether $x \leq_S y$ based on the binary vector $q(x, y, \mathcal{B}' \cup \mathcal{B}^{\text{low}} \cup \mathcal{B}^{\text{high}})$.

Case 1. x and y are original vertices. Note that $q(x, y, \mathcal{B}^{\text{low}}) = q(x, y, \mathcal{B})$. Thus

$$\begin{aligned} x \leq_S y &\iff x \leq_P y \\ &\iff \tau(q(x, y, \mathcal{B})) = 1 \\ &\iff \tau(q(x, y, \mathcal{B}^{\text{low}})) = 1. \end{aligned}$$

Case 2. x is an original vertex and y is a division vertex of an edge ab. Note that $x \leq_S y$ if and only if $x \leq_P a$. Further, as y is placed low in every linear order $L^{\text{low}} \in \mathcal{B}^{\text{low}}$, we see that $q(x, y, \mathcal{B}^{\text{low}}) = q(x, a, \mathcal{B})$. Thus

$$\begin{aligned} x \leq_S y &\iff x \leq_P a \\ &\iff \tau(q(x, a, \mathcal{B})) = 1 \\ &\iff \tau(q(x, y, \mathcal{B}^{\text{low}})) = 1. \end{aligned}$$

Case 3. x is a division vertex of an edge ab and y is an original vertex. A similar argument as in Case 2 yields

$$x \leq_S y \iff \tau(q(x, y, \mathcal{B}^{\text{high}})).$$

We are able to distinguish these cases based on the binary vector $q(x, y, \mathcal{B}')$ for elements x, y of S. Therefore, it suffices to show how to determine for two distinct division vertices x, y whether $x \leq_S y$.

Recall that for a division vertex x of an edge ab, the virtual height $h_{\text{v}}(x)$ is the height of b in P, see Definition 2.9.1. Let $\ell \coloneqq \lfloor \log(h(P)) + 1 \rfloor$. For $i \in [\ell]$ define $\mathcal{B}^i \coloneqq \{L^i \mid L \in \mathcal{B}\}$ where L^i is the linear order obtained from L where all division vertices x with a 0 in the i-th bit of $h_{\text{v}}(x)$ are placed high, and low otherwise. In addition, we require for division vertices x and y for which the order in L^i has not been specified by the preceding property that $y \leq_{L^i} x$ if y is placed low and x is placed high.

For $s \in \{0,1\}$, we denote by σ_i^s a fixed linear order of all vertices of S with a s in the i-th bit of their (possibly virtual) height in P. We define for $i \in [\ell]$ two linear orders

$$\sigma_i \coloneqq \sigma_i^0 \leq \sigma_i^1, \quad \sigma_i' \coloneqq (\sigma_i^0)^{\mathrm{rev}} \leq (\sigma_i^1)^{\mathrm{rev}}.$$

Let $\mathcal{B}'' \coloneqq \{\sigma_i, \sigma_i' \mid i \in [\ell]\}$. Note that using $q(x, y, \mathcal{B}'')$ we are able to determine a value i for distinct division vertices x, y of S such that the i-th bit of $h_{\mathrm{v}}(x)$ is 0 while the i-th bit of $h_{\mathrm{v}}(y)$ is 1, or conclude that no such i exists.

Let x, y be distinct division vertices. Recall that using the linear orders π_1 and π_4, we can determine whether x and y subdivide the same edge of the diagram of P and if so whether $x \leq_S y$. Therefore, we only need to consider the case where x and y are division vertices of distinct edges ab and cd.

Note that $x \leq_S y$ if and only if $b \leq_P c$. Suppose there exists an integer $i \in [\ell]$ such that x is placed high, and y is placed low in all linear orders of $\mathcal{B}^i$. We claim that $q(x, y, \mathcal{B}^i) = q(b, c, \mathcal{B})$. Consider a linear order $L \in \mathcal{B}$. If $b \leq_L c$, we obtain $x \leq_{L^i} b \leq_{L^i} c \leq_{L^i} y$. If $c \leq_L b$, we have $c \leq_{L^i} y \leq_{L^i} x \leq_{L^i} b$ by definition of L^i. Thus, we see that $q(x, y, \mathcal{B}^i) = q(b, c, \mathcal{B})$ which finally yields

$$x \leq_S y \iff \tau(q(x, y, \mathcal{B}^i)) = 1.$$

As we are able to determine whether such a value i exists using $\mathcal{B}''$, it suffices to show that if $x \leq_S y$ there exists an i such that the i-th bit of $h_{\mathrm{v}}(x)$ is 0 while the i-th bit of $h_{\mathrm{v}}(y)$ is 1. Note that if $x \leq_S y$ then every chain ending at b can be extended to a chain ending at d and as $b \neq d$, we obtain $h_{\mathrm{v}}(x) < h_{\mathrm{v}}(y)$. Therefore, there exists an integer $i \in [\ell]$ with the required property.

Let $\mathcal{B}_S \coloneqq \mathcal{B}' \cup \mathcal{B}^{\mathrm{low}} \cup \mathcal{B}^{\mathrm{high}} \cup \mathcal{B}'' \cup \bigcup_{i \in [\ell]} \mathcal{B}^i$ and let

$$d_S \coloneqq |\mathcal{B}_S| = 4 + d + d + 2 \cdot \lfloor \log(h(P)) + 1 \rfloor + \lfloor \log(h(P)) + 1 \rfloor \cdot d.$$

Using the argumentation above, we can construct a function $\tau_S \colon \mathbf{2}^{d_S} \to \{0,1\}$ such that $(\mathcal{B}_S, \tau_S)$ is a Boolean realizer of S. □

One might hope that the Boolean dimension of a subdivision of a poset P is bounded from above in terms of the Boolean dimension of P, independent of the height. However, this is not the case. Using the construction of Spinrad given in [83] and a result of Felsner, Mészáros, and Micek we obtain the following:

Observation 3.4.2. *For every $n \geq 2$, there exists a poset P with* $\mathrm{bdim}(P) = 2$ *and a poset S that is a subdivision of P such that* $\mathrm{bdim}(S) \geq \log\log\log(n)$.

Proof. Let $n \geq 2$. Consider the 2-dimensional poset P given in Proposition 2.9.3. As every interval order with an open interval representation with at most k distinct endpoints is a subdivision of P, this holds in particular for the universal interval order I_n if k is sufficiently large. By Lemma 3.1.6, we have $\mathrm{bdim}(I_n) \geq \log\log\log n$. The claim follows as Boolean dimension and Dushnik-Miller dimension coincide for low dimensional posets by Lemma 3.1.4. □

4 Conclusion

In this thesis, we related different notions of the dimension of posets to the queue and stack number of their diagrams, with a separate consideration of subdivisions. In particular, we investigated how subdivision affects dimension, queue number and stack number. We determined upper bounds on the queue and stack number of subdivisions. The bounds we established are best-possible for the specific layouts we constructed. Yet, it remains open whether these bounds are tight in general.

Question 4.1. *Is there for every $q \in \mathbb{N}$ a dag G of queue number q and a subdivision S of G such that $\operatorname{qn}(S) = 2q + 2$?*

Question 4.2. *Is there for every $s \in \mathbb{N}$ a dag G of stack number s and a subdivision S of G such that $\operatorname{sn}(S) = 2s$?*

If every edge of a dag is only subdivided a constant number of times, the queue number remains relatively large. More precisely, we showed that every $\preceq h$-subdivision S of a dag G has queue number at least $\frac{1}{2}(\operatorname{qn}(G)^{1/(h+1)} - 2)$. For dags, this bound is tight up to a constant factor as there are dags G which admit $\preceq h$-subdivisions of queue number at most $\operatorname{qn}(G)^{1/(h+1)} + 2$. Yet, for posets, we were only able to construct $\preceq h$-subdivisions of posets P whose queue number is at most $\operatorname{qn}(P)^{2/(h+2)} + 2$.

Question 4.3. *Is there for every $h \in \mathbb{N}$ and every $q \in \mathbb{N}$ a poset P of queue number q and a $\preceq h$-subdivisions of P whose queue number is at most $q^{1/(h+1)} + 2$?*

In Section 2.11, we determined a function which bounds the Dushnik-Miller dimension in terms of height and queue number, based on the work of Joret, Micek, and Wiechert [59]. Further, we showed in Section 2.12 that any such function is exponential both in height and queue number. However, there is still a large gap between these two bounds. The same holds for the bounds we developed in terms of stack number and height.

Question 4.4. *Is Dushnik-Miller dimension bounded by an exponential function in queue number and height?*

Question 4.5. *Is Dushnik-Miller dimension bounded by an exponential function in stack number and height?*

While Dushnik-Miller dimension has been studied intensively over the last decades, interest in Boolean dimension has only recently rekindled. In Section 3.2, we determined an exponential upper bound on the Boolean dimension in terms of queue number and height. Yet, we are unaware of posets which meet this bound.

Question 4.6. *Are there posets whose Boolean dimension is exponential in their queue number and height?*

While we were able to establish an exponential upper bound on the Boolean dimension in terms of queue number and height, a similar attempt for stack number failed.

Question 4.7. *Is Boolean dimension bounded by an exponential function in stack number and height?*

We considered the Boolean dimension of structurally restricted classes of posets, posets with bounded queue or stack number. Other parameters such as treewidth have also been considered in the literature [31]. Yet, the most intriguing question related to Boolean dimension remains a problem originally posed by Nešetřil and Pudlák [76, Problem 3.1]. They asked whether the Boolean dimension of planar posets is unbounded. While some progress has been made [69, 70], the question remains open to this date.

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