Drawing Trees with Perfect Angular Resolution and Polynomial Area

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Abstract. We study methods for drawing trees with perfect angular resolution, i.e., with angles at each vertex, v, equal to $2\pi/d(v)$. We show:

1. Any unordered tree has a crossing-free straight-line drawing with perfect angular resolution and polynomial area.
2. There are ordered trees that require exponential area for any crossing-free straight-line drawing having perfect angular resolution.
3. Any ordered tree has a crossing-free ***Lombardi-style*** drawing (where each edge is represented by a circular arc) with perfect angular resolution and polynomial area.

Thus, our results explore what is achievable with straight-line drawings and what more is achievable with Lombardi-style drawings, with respect to drawings of trees with perfect angular resolution.

1 Introduction

Most methods for visualizing trees aim to produce drawings that meet as many of the following aesthetic constraints as possible:

1. straight-line edges,
2. crossing-free edges,
3. polynomial area, and
4. perfect angular resolution around each vertex.

These constraints are all well-motivated, in that we desire edges that are easy to follow, do not confuse viewers with edge crossings, are drawable using limited real estate, and avoid congested incidences at vertices. Nevertheless, previous tree drawing algorithms have made various compromises with respect to this set of constraints; we are not aware of any previous tree-drawing algorithm that can achieve all these goals simultaneously. Our goal in this paper is to show what is actually possible with respect to this set of constraints and to expand it further with a richer notion of edges that are easy to follow. In particular, we desire tree-drawing algorithms that satisfy all of these constraints simultaneously. If this is provably not possible, we desire an augmentation that avoids compromise and instead meets the spirit of all of these goals in a new way, which, in the case of this paper, is inspired by the work of artist Mark Lombardi [18].

Problem Statement. The art of Mark Lombardi involves drawings of social networks, typically using circular arcs and good angular resolution. Figure 1 shows such a work of Lombardi that is crossing-free and almost a tree. Note that it makes use of both circular arcs and straight-line edges. Inspired by this work, let us define a set of problems that explore what is achievable for drawings of trees with respect to the constraints listed above but that, like Lombardi's drawings, also allow curved as well as straight edges.

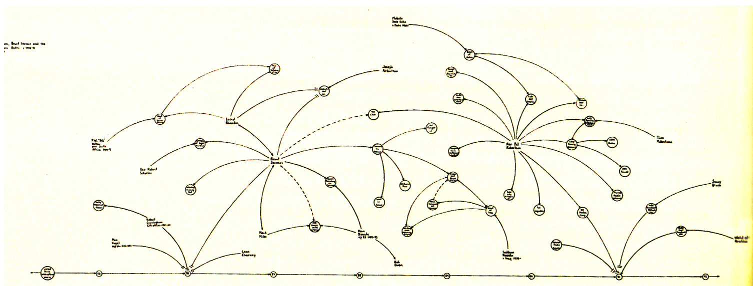
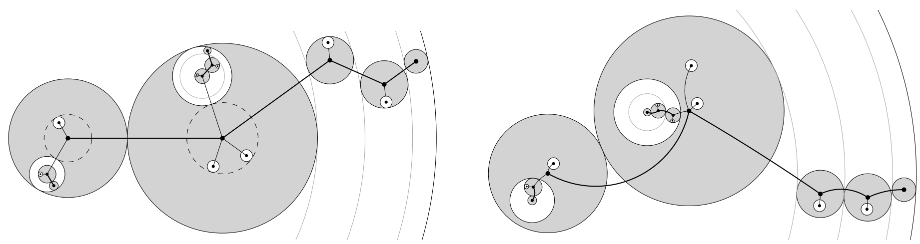


Fig. 1: Mark Lombardi, *Pat Robertson, Beurt Servaas, and the UPI Takeover Battle, ca. 1985-91*, 2000 [18].

Given a graph $G = (V, E)$, let $d(u)$ denote the ***degree*** of a vertex u, i.e., the number of edges incident to u in G. For any drawing of G, the ***angular resolution*** at a vertex u is the minimum angle between two edges incident to u. A vertex has ***perfect angular resolution*** if its minimum angle is $2\pi/d(u)$, and a drawing has perfect angular resolution if *every* vertex does. Drawings with perfect angular resolution cannot be placed on an integer grid unless the degrees of the vertices are constrained, so we do not require vertices to have integer coordinates. We define the ***area*** of a drawing to be the ratio of the area of a smallest enclosing circle around the drawing to the square of the distance between its two closest vertices.

Suppose that our input graph, G, is a rooted tree T. We say that T is ***ordered*** if an ordering of the edges incident upon each vertex in T is specified. Otherwise, T is ***unordered***. If all the edges of a drawing of T are straight-line segments, then the drawing of T is a ***straight-line*** drawing. We define a ***Lombardi drawing*** of a graph G as a drawing of G with perfect angular resolution such that each edge is drawn as a circular arc. When measuring the angle formed by two circular arcs incident to a vertex v, we use the angle formed by the tangents of the two arcs at v. Circular arcs are strictly more general than straight-line segments, since straight-line segments can be viewed as circular arcs with infinite radius. Figure 2 shows an example of a straight-line drawing and a Lombardi drawing for the same tree. Thus, we can define our problems as follows:

1. Is it always possible to produce a straight-line drawing of an unordered tree with perfect angular resolution and polynomial area?
2. Is it always possible to produce a straight-line drawing of an ordered tree with perfect angular resolution and polynomial area?
3. Is it always possible to produce a Lombardi drawing of an ordered tree with perfect angular resolution and polynomial area?



(a) Straight-line drawing for an unordered tree (b) Lombardi drawing for an ordered tree

Fig. 2: Two drawings of a tree T with perfect angular resolution and polynomial area as produced by our algorithms. Bold edges are heavy edges, gray disks are heavy nodes, and white disks are light children. The root of T is in the center of the leftmost disk.

Related Work. Tree drawings have interested researchers for many decades: e.g., hierarchical drawings of binary trees date to the 1970's [24]. Many improvements have been proposed since this early work, using space efficiently and generalizing to non-binary trees [2, 5, 13, 14, 15, 21, 23, 22]. These drawings do not achieve all the constraints mentioned above, however, especially the constraint on angular resolution.

Alternatively, several methods strive to optimize angular resolution of trees. Radial drawings of trees place nodes at the same distance from the root on a circle around the root node [11]. Circular tree drawings are made of recursive radial-type layouts [20]. Bubble drawings [16] draw trees recursively with each subtree contained within a circle disjoint from its siblings but within the circle of its parent. Balloon drawings [19] take a similar approach and heuristically attempt to optimize space utilization and the ratio between the longest and shortest edges in the tree. Convex drawings [4] partition the plane into unbounded convex polygons with their boundaries formed by tree edges. Although these methods provide several benefits, none of these methods guarantees that they satisfy all of the aforementioned constraints.

The notion of drawing graphs with edges that are circular arcs or other nonlinear curves is certainly not new to graph drawing. For instance, Cheng *et al.* [6] used circle arcs to draw planar graphs in an $O(n) \times O(n)$ grid while maintaining bounded (but not perfect) angular resolution. Similarly, Dickerson *et al.* [7] use circle-arc polylines to produce planar confluent drawings of non-planar graphs, Duncan *et al.* [8] draw graphs with fat edges that include circular arcs, and Cappos *et al.* [3] study simultaneous embeddings of planar graphs using circular arcs. Finkel and Tamassia [12] use a force-directed method for producing curvilinear drawings, and Brandes and Wagner [1] use energy minimization methods to place Bézier splines that represent express connections in a train network. In a separate paper [10] we study Lombardi drawings for classes of graphs other than trees.

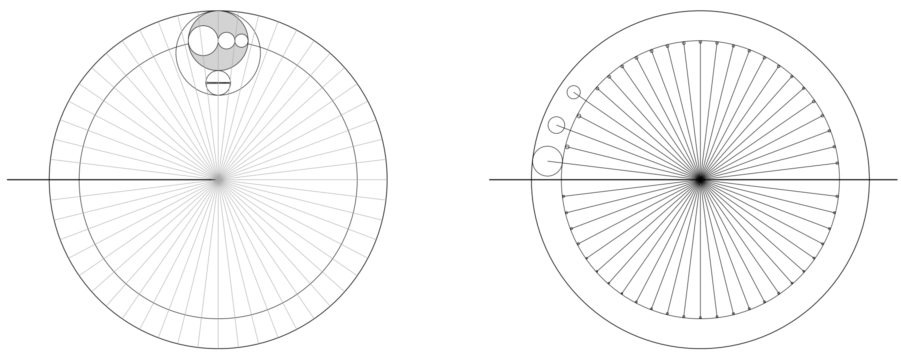
Our Contributions. In this paper we present the first algorithm for producing straight-line, crossing-free drawings of unordered trees that ensures perfect angular resolution and polynomial area. In addition we show, in Section 3, that if the tree is ordered (i.e., given with a fixed combinatorial embedding) then it is not always possible to maintain perfect angular resolution and polynomial drawing area when using straight lines for edges. Nevertheless, in Section 4, we show that crossing-free polynomial-area Lombardi drawings of ordered trees are possible. That is, we show that the answers to the questions posed above are "yes," "no," and "yes," respectively.

2 Straight-line drawings for unordered trees

Let T be an unordered tree with n nodes. We wish to construct a straight-line drawing of T with perfect angular resolution and polynomial area.

The main idea of our algorithm is, similarly to the common bubble and balloon tree constructions [16, 19], to draw the children of each node of the given tree in a disk centered at that node; however, our algorithm differs in several key respects:

- Before drawing the tree, we perform a ***heavy path decomposition*** [17]: for each node v, the ***heavy child*** of v is the child with the greatest number of descendants, and the other children are ***light children***, denoted $L(v)$. The paths that follow edges from nodes to their heavy children are ***heavy paths***, and they form a partition of the input tree with the property that the tree $H(T)$ formed by compressing each heavy path to a node has only logarithmic depth $h(T)$.
- In our drawing, each heavy path P is confined to a disk, whose radius is linear in the number of nodes descending from P and exponential in the level of P in the heavy path decomposition. In this way, at each step downwards in the heavy path decomposition, the total radius of the disks at that level shrinks by a constant factor, allowing room for disks at lower levels to be placed within the higher-level disks.
- For each heavy path P, and each node v on P, we form another disk, contained within the disk for P, that contains v at its center and also contains the disks for the lower-level heavy paths connected to v (the descendants of the light children of v). The disks for the nodes of the heavy path are placed within the disk for the heavy path, with the topmost node of the heavy path at the center of the disk for the heavy path and successive heavy path nodes placed on concentric circles within this disk.
- Because the radii of our disks are exponential in the level of the heavy path decomposition, the radii of the disks for the children of v add up to a constant fraction of the radii of the disk for v itself (Figure 3a). Within the disk centered at node v, we place the smaller disks containing the heavy paths descending from v in two concentric annuli (Figure 3b).
- The outer annulus surrounding v contains light children of v that are the ancestors of many nodes (relative to the total number of descendants of v and the degree of v); the disks for the heavy paths containing these light children are placed using a greedy algorithm so that the edges connecting them to v have angles that are multiples of the proper angular resolution.
- The inner annulus surrounding v contains the remaining light children, each of which is the ancestor of few enough nodes that the disk for its heavy path may be placed in the inner annulus at the perfect angular resolution for v, filling out all the positions incident to v that were not already filled by the two edges of the heavy path for v and the disks in the outer annulus.



(a) All light children fit into a disk of radius $r_v/4$ and are split into small and large disks.

(b) Large disks are placed in the outer annulus and small disks in the inner disk.

Fig. 3: Drawing a node v and its light children $L(v)$.

As we show in the full paper [9], this method draws the given tree with perfect angular resolution and polynomial drawing area. However, our method may reorder the children of each node, so it does not respect a fixed embedding of the given tree. Figure 2a shows a drawing of an unordered tree according to our method.

3 Straight-line drawings for ordered trees

In many cases, the ordering of the children around each vertex of a tree is given; that is, the tree is ordered (or has a fixed combinatorial embedding). In the previous section we rely on the freedom to order subtrees as needed to achieve a polynomial area bound. Hence that algorithm cannot be applied to ordered trees with fixed embeddings. As we now show, there are ordered trees that have no straight-line crossing-free drawings with polynomial area and perfect angular resolution.

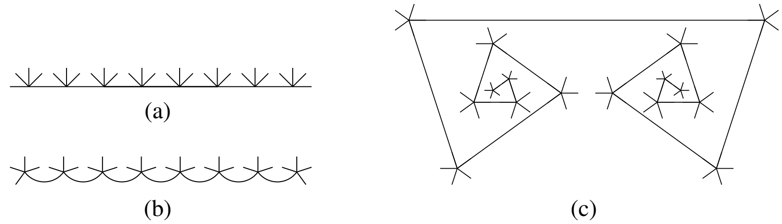


Fig. 4: (a) A Fibonacci caterpillar; (b) Lombardi drawing; (c) Straight-line drawing with perfect angular resolution and exponential area.

Specifically we present a class of ordered trees for which any straight-line crossing-free drawing of the tree with perfect angular resolution requires exponential area. Figure 4a shows a caterpillar tree, which we call the ***Fibonacci caterpillar*** because of its simple behavior when required to have perfect angular resolution. This tree has as its spine a k-vertex path, each vertex of which has 3 additional leaf nodes embedded on the same side of the spine. When drawn with straight-line edges, no crossings, and with perfect angular resolution, the caterpillar is forced to spiral (a single or a double spiral). The best drawing area, exponential in the number of vertices in the caterpillar, is achieved when the caterpillar forms a symmetric double spiral; see Figure 4c.

The Fibonacci caterpillar shows that we cannot maintain all constraints (straight-line edges, crossing-free, perfect angular resolution, polynomial area) for ordered trees. However, as we show next, using circular arcs instead of straight-line edges allows us to respect the remaining three constraints. See, for example, Figure 4b.

4 Lombardi drawings for ordered trees

In this section, let T be an ordered tree with n nodes. As we have seen in Section 3, we cannot find polynomial area drawings for all ordered trees using straight-line edges. An augmentation of the straight-line edge requirement is the use of circular arcs as edges. Circular arcs are curves that are not only still easy to follow visually but they also let us achieve all remaining three constraints, i.e., we can find crossing-free circular arc drawings with perfect angular resolution and polynomial area. We call a drawing with circular arcs and perfect angular resolution a Lombardi drawing, so in other words we aim for crossing-free Lombardi drawings with polynomial area.

The flavor of the algorithm for Lombardi tree drawings is similar to our straight-line tree drawing algorithm of Section 2: We first compute a heavy-path decomposition $H(T)$ for T. Then we recursively draw all heavy paths within disks of polynomial area. Unlike before, we need to construct the drawing in a top-down fashion since the placement of the light children of a node v now depends on the curvature of the two heavy edges incident to v.

Our construction in this section uses the invariant that a heavy path P at level j is drawn inside a disk D of radius $2 \cdot 4^{h(T)-j} n(P)$, where $n(P) = |T_v|$ for the root v of P.

4.1 Drawing heavy paths

Let $P = (v_1, \dots, v_k)$ be a heavy path at level j of the heavy-path decomposition that is rooted at the last node v_k. We denote each edge $v_i v_{i+1}$ by e_i. Recall that the angle in an intersection point of two circular arcs is measured as the angle between the tangents to the arcs at that point. We define the angle $\alpha(v_i)$ for $2 \le i \le k-1$ to be the angle between e_{i-1} and e_i in node v_i (measured counter-clockwise). The angle $\alpha(v_k)$ is defined as the angle in v_k between e_{k-1} and the light edge $e = v_k u$ connecting the root v_k of P to its parent u. Due to the perfect angular resolution requirement for each node v_i, the angle $\alpha(v_i)$ is obtained directly from the number of edges between e_{i-1} and e_i and the degree $d(v_i)$.

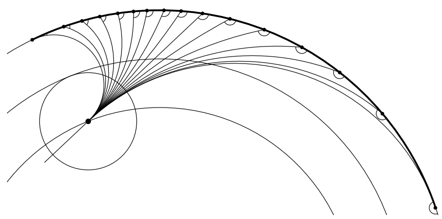


Fig. 5: Any angle $\alpha \in [0, \pi]$ can be realized.

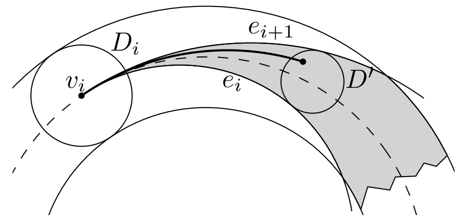


Fig. 6: Placing a single disk D' in the extended small zone of D_i (shaded gray).

Lemma 4.1. *Given a heavy path $P = (v_1, \ldots, v_k)$ and a disk D_i of radius r_i for the drawing of each v_i and its light subtrees, we can draw P with each v_i in the center of its disk D_i inside a large disk D such that the following properties hold:*

1. *each heavy edge e_i is a circular arc that does not intersect any disk other than D_i and D_{i+1};*
2. *there is a stub edge incident to v_k that is reserved for the light edge connecting v_k and its parent;*
3. *any two disks D_i and D_j for $i \neq j$ are disjoint;*
4. *the angle between any two consecutive heavy edges e_{i-1} and e_i is $\alpha(v_i)$;*
5. *the radius r of D is $r = 2\sum_{i=1}^{k} r_i$.*

Proof. We draw P incrementally starting from the leaf v_1 by placing D_1 in the center M of the disk D of radius $r = 2\sum_{i=1}^{k} r_i$. We may assume that D_1 is rotated such that the edge e_1 is tangent to a horizontal line at v_1 and that it leaves v_1 to the right. All disks $D_2, \ldots, D_k$ will be placed with their centers $v_2, \ldots, v_k$ on concentric circles $C_2, \ldots, C_k$ around M. The radius of C_i is $r_1 + 2\sum_{j=2}^{i-1} r_j + r_i$ so that D_{i-1} and D_i are placed in disjoint annuli and hence by construction no two disks intersect (property 3). Each disk D_i will be rotated around its center such that the tangent to C_i at v_i is the bisector of the angle $\alpha(v_i)$.

We now describe one step in the iterative drawing procedure that draws edge e_i and disk D_{i+1} given a drawing of $D_1, \ldots, D_i$. Disk D_i is placed such that C_i bisects the angle $\alpha(v_i)$ and hence we know the tangent of e_i at v_i. This defines a family $\mathscr{F}_i$ of circular arcs emitted from v_i that intersect the circle C_{i+1}, see Figure 5. We consider all arcs from v_i until their first intersection point with C_{i+1}. Observe that the intersection angles of $\mathscr{F}_i$ and C_{i+1} bijectively cover the full interval $[0, \pi]$, i.e., for any angle $\alpha \in [0, \pi]$ there is a unique arc in $\mathscr{F}_i$ that has intersection angle α with C_{i+1}. Hence we choose for e_i the unique circular arc that realizes the angle $\alpha(v_{i+1})/2$ and place the center v_{i+1} of D_{i+1} at the endpoint of e_i. We continue this process until the last disk D_k is placed. This drawing of P realizes the angle $\alpha(v_i)$ between any two heavy edges e_{i-1} and e_i (property 4). Note that for the edge from v_k to its parent we can only reserve a stub whose tangent at v_k has a fixed slope (property 2). Figure 7 shows an example.

Note that each edge e_i is contained in the annulus between C_i and C_{i+1} and thus does not intersect any other edge of the heavy path or any disk other than D_i and D_{i+1}

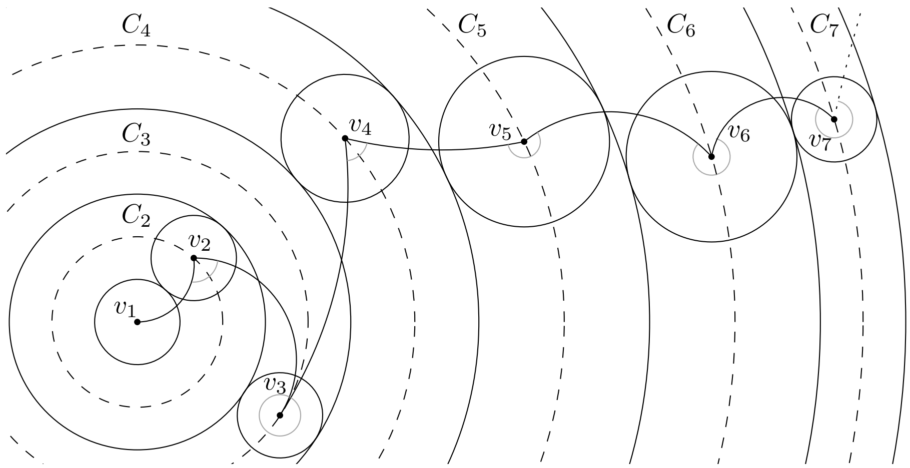


Fig. 7: Drawing a heavy path P on concentric circles with circular-arc edges. The angles $\alpha(v_i)$ are marked in gray; the edge stub to connect v_7 to its parent is dotted.

(property 1). Furthermore, the disk D with radius $r = 2\sum_{i=1}^{k} r_i$ indeed contains all the disks $D_1, \ldots, D_k$ (property 5). □

Lemma 4.1 shows how to draw a heavy path P with prescribed angles between the heavy edges and an edge stub to connect it to its parent. Since each heavy path P (except the path at the root of $H(T)$) is the light child of a node on the previous level of $H(T)$ that light edge is actually drawn when placing the light children of a node, which we describe next.

4.2 Drawing light children

Once the heavy path P is drawn as described above, it remains to place the light children of each node v_i of P. For each node v_i the two heavy edges incident to it partition the disk D_i into two regions. We call the region that contains the larger conjugate angle the ***large zone*** of v_i and the region that contains the smaller conjugate angle the ***small zone***. If both angles equal π, then we can consider both regions small zones.

For a node v_i at level j of $H(T)$ we define the radius r_i of D_i as $r_i = 4^{h(T)-j}(1 + \sum_{u \in L(v_i)} |T_u|) = 4^{h(T)-j} l(v_i)$. All light children of v_i are at level $j+1$ of $H(T)$ and thus by our invariant every light child u of v_i is drawn in a disk of radius $r_u = 2 \cdot 4^{h(T)-j-1}|T_u|$. Thus we know that $r_u \leq r_i/2$; in fact, we even have $\sum_{u \in L(v_i)} r_u \leq r_i/2$.

Light children in the small zone. Depending on the angle $\alpha(v_i)$, the small zone of a disk D_i might actually be too narrow to directly place the light children in it. Fortunately, we can always place another disk D' of radius at most $r_i/2$ in an extension of the small zone along the annulus of D_i in the drawing of P such that D' touches e_{i-1} and e_i and does not intersect any other previously placed disk, see Figure 6. If there is a single child u

in the small zone then $D' = D_u$ and we are done. The next lemma shows how to place more than one child; the proof can be found in the full paper [9].

Lemma 4.2. *If a single disk D' of radius r' can be placed in the possibly extended small zone of the disk D_i, then we can correctly place any sequence of l disks $D'_1, \dots, D'_l$ with radii $r'_1, \dots, r'_l$ and $\sum_{i=1}^{l} r'_i = r'$ in the (extended) small zone of D_i.*

Light children in the large zone. Placing the light children of a vertex v_i in the large zone of D_i must be done slightly different from the algorithm for the small zone since Lemma 4.2 holds only for opening angles of at most π. On the other hand, the large zone does not become too narrow and there is no need to extend it beyond D_i. Our approach splits the large zone into two parts that again have an opening angle of at most π so that we can apply Lemma 4.2 and draw all children accordingly.

Let l be the number of light children in the large zone of D_i. We first place a disk D' of radius at most $r_i/2$ such that it touches v_i and such that its center lies on the line bisecting the opening angle of the large zone. The disk D' is large enough to contain the disjoint disks $D'_1, \dots, D'_l$ for the light children of v_i along its diameter. We need to distinguish whether l is even or odd. For even l we create a container disk D''_1 for disks $D'_1, \dots, D'_{l/2}$ and a container disk D''_2 for $D'_{l/2+1}, \dots, D'_l$. Now D''_1 and D''_2 can be tightly packed on the diameter of D'. Using a similar argument as in Lemma 4.2 we separate the two disks by a circular arc through v_i that is tangent to the bisector of $\alpha(v_i)$ in v_i. Since D' is centered on the bisector this is possible even though the actual opening angle of the large zone is larger than π. If l is odd, we create a container disk D''_1 for disks $D'_1, \dots, D'_{\lfloor l/2 \rfloor}$ and a container disk D''_2 for $D'_{\lceil l/2 \rceil+1}, \dots, D'_l$. The median disk $D'_{\lceil l/2 \rceil}$ is not included in any container. Then we apply Lemma 4.2 to D' and the three disks $D''_1, D'_{\lceil l/2 \rceil}, D''_2$ along the diameter of D', see Figure 8a. The separating circular arcs in v_i are again tangent to the bisector of $\alpha(v_i)$, which is, since l is odd, also the correct slope for the circular arc connecting v_i to the median disk $D'_{\lceil l/2 \rceil}$.

In both cases we split the large zone and the sequence of light children to be placed into two parts that each have an opening angle at v_i of at most π between a separating circular arc and the edge e_{i-1} or e_i, respectively. Next, we move D''_1 and D''_2 along the separating circular arcs keeping their tangencies until they also touch the edge e_{i-1} or e_i, respectively. Then we can apply Lemma 4.2 to both container disks and thus place all light children in the large zone, see Figure 8b.

Drawing light edges The final missing step is how to actually connect a heavy node v_i to its light children given a position of v_i and positions of all disks containing its light subtrees. Let u be a light child of v_i and let D_u be the disk containing the drawing of T_u. When placing the disk D_u in the small or large zone of v_i we made sure that a circular arc from v_i with the tangent required for perfect angular resolution at v_i can reach any point inside D_u without intersecting any other edge or disk.

On the other side, we know by Lemma 4.1 that u is placed in the outermost annulus of D_u and that it has a stub for the edge $e = uv_i$. This stub is the required tangent for e in order to obtain perfect angular resolution in u. Let C_u be the circle that is the locus of u if we rotate D_u and the drawing of T_u around the center of D_u.

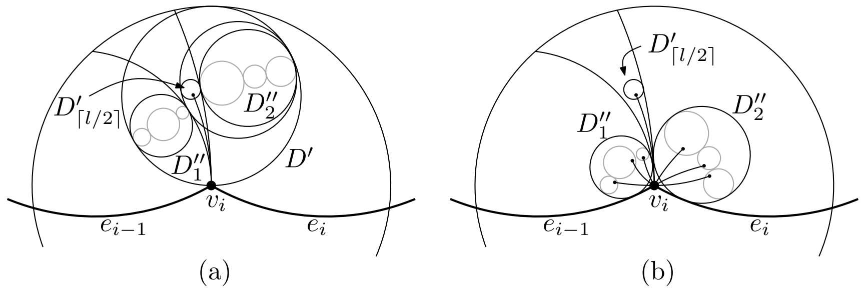


Fig. 8: Placing light children in the large zone by first splitting it into two parts (a) and then applying the algorithm for small zones to each part (b).

There is again a family $\mathscr{F}$ of circular arcs with the correct tangent in u that lead towards D_u and intersect the circle C_u. As observed in Lemma 4.1 the intersection angles formed between $\mathscr{F}$ and C_u bijectively cover the full interval $[0,\pi]$, i.e., for any angle $\alpha \in [0,\pi]$ there is a unique circular arc in $\mathscr{F}$ that has an intersection angle of α with C_u. In order to correctly attach u to v_i we first choose the arc a in $\mathscr{F}$ that realizes an intersection angle of $\alpha(u)/2$ with C_u, where $\alpha(u)$ is the angle between e and the heavy edge from u to its heavy child that is required for perfect angular resolution in u. Let p be the intersection point of that arc with C_u. Then we rotate D_u and the drawing of T_u around the center of D_u until u is placed at p, see node v_7 in Figure 7. Since the stub of u for e also has an angle of $\alpha(u)/2$ with C_u, the arc a indeed realizes the edge e with the angles in both u and v_i required for perfect angular resolution. Furthermore, a does not enter the disk bounded by C_u and hence it does not intersect any part of the drawing of T_u other than u.

We can summarize our results for drawing the light children of a node as follows:

Lemma 4.3. *Let v be a node of T at level j of $H(T)$ with two incident heavy edges. For every light child $u \in L(v)$ assume there is a disk D_u of radius $r_u = 2 \cdot 4^{h(T)-j-1}|T_u|$ that contains a fixed drawing of T_u with perfect angular resolution and such that u is exposed in the outer annulus of D_u. Then we can construct a drawing of v and its light subtrees inside a disk D, potentially with an extended small zone, such that the following properties hold:*

1. *the edge between v and any light child $u \in L(v)$ is a circular arc that does not intersect any disk other than D_u;*
2. *the heavy edges do not intersect any disk D_u;*
3. *any two disks D_u and $D_{u'}$ for $u \neq u'$ are disjoint;*
4. *the angular resolution of v is $2\pi/d(v)$;*
5. *the disk D has radius $r_v = 4^{h(T)-j} l(v)$.*

By combining Lemmas 4.1 and 4.3 we obtain the following theorem:

Theorem 4.4. *Given an ordered tree T with n nodes we can find a crossing-free Lombardi drawing of T that preserves the embedding of T and fits inside a disk D of radius $2 \cdot 4^{h(T)} n$, where $h(T)$ is the height of the heavy-path decomposition of T. Since $h(T) \leq \log_2 n$ the radius of D is no more than $2n^3$.*

Figure 2b shows a drawing of an ordered tree according to our method. We note that instead of asking for perfect angular resolution, the same algorithm can be used to construct a circular-arc drawing of an ordered tree with any assignment of angles between consecutive edges around each node that add up to 2π. The drawing remains crossing-free and fits inside a disk of radius $O(n^3)$.

5 Conclusion and Closing Remarks

We have shown that straight-line drawings of trees can be performed with perfect angular resolution and polynomial area, by carefully ordering the children of each vertex and by using a style similar to balloon drawings in which the children of any vertex are placed on two concentric circles rather than on a single circle. However, using our Fibonacci caterpillar example we showed that this combination of straight lines, perfect angular resolution, and polynomial area could no longer be achieved if the children of each vertex may not be reordered. For trees with a fixed embedding, Lombardi drawings in which edges are drawn as circular arcs allow us to retain the other desirable qualities of polynomial area and perfect angular resolution. In [9] we report on a basic implementation and some practical improvements of the straight-line drawing algorithm.

Our work opens up new problems in the study of Lombardi drawings of trees, but much remains to be done in this direction. In particular, our polynomial area bounds seem unlikely to be tight, and our method is impractically complex. It would be of interest to find simpler Lombardi drawing algorithms that achieve perfect angular resolution for more limited classes of trees, such as binary trees, with better area bounds.

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