Consistent Digital Rays

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Abstract

Given a fixed origin o in the d-dimensional grid, we give a novel definition of *digital rays* $\mathrm{dig}(op)$ from o to each grid point p. Each digital ray $\mathrm{dig}(op)$ approximates the Euclidean line segment $\overline{op}$ between o and p. The set of all digital rays satisfies a set of axioms analogous to the Euclidean axioms. We measure the approximation quality by the maximum Hausdorff distance between a digital ray and its Euclidean counterpart and establish an asymptotically tight $\Theta(\log n)$ bound in the $n \times n$ grid. Without a monotonicity property for digital rays the bound is improved to $O(1)$.

1 Introduction

The digital line segment $\mathrm{dig}(pq)$ between two grid points p and q is a fundamental digital geometric object, but its definition is not that obvious. Indeed, the digital representation of line segments has been an active subject of research for almost half a century now (see an excellent survey of Klette and Rosenfeld [2]). In digital geometry, a geometric object is represented by a set of d-dimensional grid points in a digital grid $\mathbf{G} = \mathbb{Z}^d$ and its topological properties are considered under a grid topology defined by a graph on the grid. In two dimensions, it is common to consider the orthogonal grid topology, where each point p is connected to the four grid points that are horizontally and vertically adjacent to p, and we focus on this topology; however, as a variant, we may consider the octagonal grid topology that connects each grid point to the eight neighboring grid points with a coordinate difference of at most 1 in each coordinate.

Since a digital line segment is the analogue of a line segment in Euclidean geometry, it is natural to set up the following axioms for a digital line segment:

- (S1) A digital line segment $\mathrm{dig}(pq)$ is a connected path between p and q.
- (S2) For any two grid points p and q there is a unique digital line segment $\mathrm{dig}(pq) = \mathrm{dig}(qp)$.
- (S3) If $s, t \in \mathrm{dig}(pq)$, then $\mathrm{dig}(st) \subseteq \mathrm{dig}(pq)$.
- (S4) For any p and q there is a grid point $r \notin \mathrm{dig}(pq)$ such that $\mathrm{dig}(pq) \subset \mathrm{dig}(pr)$.

Note that axiom (S3) implies that a non-empty intersection of two digital line segments is either a grid point or a digital line segment. Axiom (S4) implies that a digital line segment can be extended to a digital line. We often identify a path in a grid with its vertex set if the correspondence is clear. Accordingly, if we say a grid point p is in a path P, it means that p is a vertex of P.

Unfortunately, popular definitions of two-dimensional (2D) digital line segments in computer vision do not satisfy these axioms. For example, in the standard definition of a *digital straight segment (DSS)* [2], a digital line segment (in the octagonal topology) that corresponds to the line segment $y = mx + b$, $x_0 \le x \le x_1$ is defined as the set of grid points $\{(i, \lfloor mi + b \rfloor) \mid i \in \mathbb{Z}, x_0 \le i \le x_1\}$ if $|m| \le 1$. Using this definition the intersection of two DSSs is not always connected, and axiom (S3) is violated in some cases.

Another possibility to define digital line segments would be to use the system of L- and Γ-shaped shortest paths. An L- or Γ-shaped path between two points $p = (x_p, y_p)$ and $q = (x_q, y_q)$ such that $x_p \le x_q$, is the (at most) 2-link path that consists of the grid points on the vertical segment pp' and on the horizontal segment $p'q$ where $p' = (x_p, y_q)$. We can confirm that the system of these paths satisfies axioms (S1)–(S4) for digital line segments. A clear drawback is that an L-shaped path is visually very different from the Euclidean line segment, and the Hausdorff distance from $\overline{pq}$ to the L-shaped path becomes $n/\sqrt{2}$ for $p = (0, n)$ and $q = (n, 0)$. Therefore, it seems that there is a trade-off between the axiomatic requirements and the visual quality of digital line segments. It is a challenging problem to find a system of digital line segments that satisfies the axioms and is visually alike Euclidean line segments at the same time.

In this paper we study a less ambitious but important subproblem, motivated by geometric optimization applications like extracting digital star-shaped regions in pixel images [1]: we consider only digital line segments that have a fixed origin o as one of their endpoints. In other words, we consider digital halflines emanating from o, and $\mathrm{dig}(op)$ is defined as the unique portion of the halfline that has p as its second endpoint. We call such segments *digital ray segments* or simply *digital rays* emanating from o.

For digital rays, the axioms for digital line segments should be modified as follows:

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(R1) A digital ray dig(op) is a connected path between o and p.
(R2) There is a unique digital ray dig(op) between o and any grid point p.
(R3) If $r \in$ dig(op), then dig$(or) \subseteq$ dig(op).
(R4) For any dig(op), there is a grid point $r \notin$ dig(op) such that dig$(op) \subset$ dig(or).

We also give an additional *monotonicity* axiom, which is not combinatorial, but a reasonable condition for a digital ray:

(R5) For any $r \in$ dig(op), $|\overline{or}| \leq |\overline{op}|$, where $|\overline{ab}|$ is the length of the Euclidean segment $\overline{ab}$.

A system of digital rays is called *consistent* if it satisfies axioms (R1)–(R5). From these axioms, it follows that the union of all digital rays forms an infinite spanning tree T of the grid graph on $\mathbf{G}$ rooted at o, such that dig(op) is the unique path between o and p in the tree. Because of axiom (R4), T cannot have leaves. Thus, the problem is basically to embed the infinite "star" consisting of the halflines emanating from o in the d-dimensional Euclidean space as a tree in the d-dimensional grid. Although embedding a tree in a grid is a popular topic in metric embedding and graph drawing, it is a novel and interesting problem to geometrically approximate ray segments by paths.

We give the asymptotically tight $\Theta(\log n)$ bound for the maximum Hausdorff distance between dig(op) and $\overline{op}$ among all p in an $n \times n$ grid. The lower bound argument is based on discrepancy theory, and the upper bound is attained by a simple and systematic construction of a tree T that can be extended to the d-dimensional case. Surprisingly, if we do not include the monotonicity axiom (R5), the bound can be reduced to $O(1)$.

2 The lower bound result

The Hausdorff distance $H(A, B)$ of two objects A and B in d-dimensional space is defined by $H(A, B) = \max\{h(A, B), h(B, A)\}$, where $h(A, B) = \max_{a \in A} \min_{b \in B} d(a, b)$ and $d(a, b)$ is some distance between the points a and b. We will use the L_∞-metric in the following for technical convenience, since the choice of the metric is irrelevant if we consider the bounds in big-O and big-Ω notations.

Let's consider the set $V = \{(i, j) \mid i, j \in \mathbb{Z}\}$ of grid points. We define a planar graph G on V that represents the adjacency relations of a pixel grid. In $G = (V, E)$ each vertex (i, j) is connected to its four neighbors $(i, j-1)$, $(i-1, j)$, $(i+1, j)$, and $(i, j+1)$. This also defines the orthogonal topology of the grid $\mathbf{G}$. A subset of V is *connected* in this topology if its induced subgraph in G is connected. We focus on the part $\mathbf{G}(n)$ of the planar orthogonal grid clipped

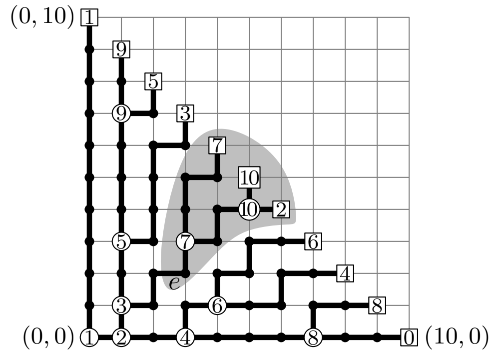

Figure 1: A spanning tree T of $\mathbf{G}(n)$ for $n = 10$. The labeled nodes are used in a low-discrepancy sequence.

to the region defined by $x + y \leq n$ in the first quadrant. From the monotonicity axiom it follows that dig$(op) \subset \mathbf{G}(n)$ for any $p \in \mathbf{G}(n)$, and that dig(op) is a shortest path in the grid. We show that there exists a point $p \in \mathbf{G}(n)$ such that the Hausdorff distance $H(\text{dig}(op), \overline{op})$ is $\Omega(\log n)$. Let T be the spanning tree of $\mathbf{G}(n)$ that is the union of dig(op) for all $p \in \mathbf{G}(n)$. An example spanning tree is shown in Fig. 1.

We use a classical result on pseudo-random number generation [3, 5]. Consider an infinite sequence $X = x_0, x_1, x_2, \ldots$ of real numbers in $[0, 1]$. For any given $a \in [0, 1]$ and any natural number m define $X_m(a) = |\{0 \leq i \leq m \mid x_i \in [0, a]\}|$. The *discrepancy* of the subsequence $x_0, x_1, \ldots, x_m$ is defined as $\sup_{a \in [0,1]} |am - X_m(a)|$. We use discrepancy theory in the form of the following theorem.

Theorem 1 (Schmidt [4]) *Given a sequence $X = x_0, x_1, x_2 \ldots$ of real numbers in $[0, 1]$ and a sufficiently large integer n, there exists an integer $m < n$ and a real number $a \in [0, 1]$ such that the subsequence $x_0, x_1, \ldots, x_m$ satisfies that $|am - X_m(a)| > c \log n$, where c is a positive constant independent of n.*

For $m = 1, 2, \ldots n+1$, let $L(m) = \{(i, m-1-i) \mid i = 0, \ldots, m-1\}$ be the subset of $\mathbf{G}(n)$ satisfying $x + y = m - 1$. Since there is no leaf of T in $L(m)$ for $m \leq n$ we must have exactly one branching node of degree 3 and $m - 1$ nodes of degree 2 in $L(m)$ in order to connect the m points of $L(m)$ to the $m + 1$ points in $L(m + 1)$.

We associate the leaf $(j, n-j) \in L(n+1)$ to the number j/n and define the set $N = \{j/n : j = 0, 1, \ldots n\} \subset [0, 1]$. For each edge e of T in $\mathbf{G}(n)$, the set of vertices of $L(n+1)$ in the subtree rooted at e forms an interval $I(e) \subset N$. Let $x(e)$ denote the largest element in $I(e)$. An example is given in Fig. 1, where $I(e) = \{0.4, 0.5, 0.6\}$ and $x(e) = 0.6$.

For a given spanning tree T we create a sequence $X(T)$ of values $x(e)$ for certain edges e. The lower bound on the discrepancy of $X(T)$ is used to show the following theorem.

Theorem 2 *For any spanning tree T there is a grid point $p \in L(n+1)$ and q in $\mathbf{G}(n)$ such that q is on the path $\mathrm{dig}(op)$ in T and the L_∞ distance from q to the line segment $\overline{op}$ exceeds $c \log n - 1$, where c is the constant considered in Theorem 1.*

Proof. For $m = 1, \ldots, n+1$ let $x_m = x(e_m)$, where e_m is the upper (i.e. vertical) branch of the unique branching node in $L(m)$. We artificially set $x_0 = 1$. Thus we obtain a sequence $X(T) = x_0, x_1, \ldots, x_n$, which is a permutation of N. Let $E(m)$ be the set of edges in T going from $L(m)$ towards $L(m+1)$. The following lemmas are obvious from the definitions:

Lemma 3 *The set $\{x(e) : e \in E(m)\}$ is equal to the set $\{x_0, x_1, \ldots, x_m\}$.*

Lemma 4 *Let e and f be edges in $E(m)$. If e is to the left of f then $x(e) < x(f)$.*

For example, the tree T in Fig. 1 creates the following sequence: $X(T) = 1, 0,\ 0.6,\ 0.3,\ 0.8,\ 0.2,\ 0.7,\ 0.4,\ 0.9,\ 0.1,\ 0.5$. The labels in Fig. 1 show the correspondence between the unique internal branching node in $L(i)$ and the leaf located at $(nx_i, n - nx_i)$ in $L(n+1)$ that is associated to the number x_i. For each $i = 1, \ldots, n$ the corresponding nodes are labeled by i. In other words, each branching node and the rightmost leaf in the subtree of the upper branch of that node have the same numbering.

We now consider the discrepancy of $X(T)$. From Theorem 1, we have $0 \le a \le 1$ and $m < n$ for n large enough such that $|am - X_m(a)| > c \log n$. The following two cases should be considered:

Case 1: $X_m(a) > am + c \log n$**.** Consider the node q located at $(X_m(a) - 1, m - (X_m(a) - 1)) \in L(m+1)$, and let e be the edge between q and its parent in T. By definition, q is on the path $\mathrm{dig}(op)$ from o to the node $p = (x(e)n, n - x(e)n) \in L(n+1)$. Because of the definition of $X_m(a)$ and Lemma 3, we have exactly $X_m(a)$ edges $f \in E(m)$ for which $x(f) \le a$. However, there are also exactly $X_m(a)$ edges of $E(m)$ to the left of e, including e itself, since q is the $X_m(a)$-th node in $L(m+1)$ counted from the left. Lemma 4 implies that no edge g to the right of e can attain $x(g) \le a$. Thus, e itself must satisfy $x(e) \le a$. Now, consider the L_∞ distance of the line segment $\overline{op}$ and q. The line op goes through $(x(e)m, m - x(e)m)$, which is the L_∞-nearest point from q on op. The L_∞ distance is $(X_m(a) - 1 - x(e)m) \ge (X_m(a) - 1 - am) > c \log n - 1$.

Case 2: $X_m(a) < am - c \log n$**.** Consider the node q located at $(X_m(a), m - X_m(a)) \in L(m+1)$ and the edge e between q and its parent. Since there are only $X_m(a)$ edges $f \in E(m)$ for which $x(f) \le a$ we have $x(e) > a$ (again by Lemma 4). Node q is on the path $\mathrm{dig}(op)$ to $p = (x(e)n, n - x(e)n)$. Similarly to Case 1, we can show that the L_∞ distance from q to $\overline{op}$ is greater than $c \log n$. This proves the theorem. □

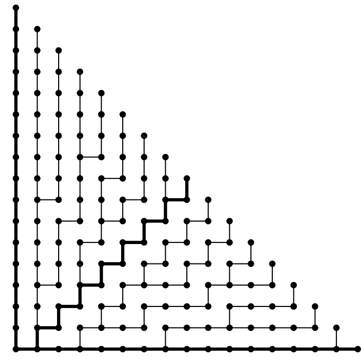

Figure 2: The spanning tree $DT(2)$ in $\mathbf{G}(n)$.

3 The upper bound results

As for the upper bound, we only give the flavor here and refer to the full version of this paper, which also gives a higher dimensional construction. We construct a spanning tree DT(2) of G, such that for every $p = (i, j) \in V$, the unique path from p to o in DT(2) defines the digital ray $\mathrm{dig}(op)$ simulating the line segment $\overline{op}$. This is illustrated in Fig. 2. The construction is recursive: We consider the diagonal (bold) center path. In the part below the center path, every edge in $E(2k-1)$ is horizontal for $k = 1, 2, \ldots$. As for the edges in $E(2k)$ below the center path, we copy the structure of $E(k)$. The part above the center path is constructed in a similar way.

The set of digital rays defined by DT(2) is consistent, and for any grid point $p \in \mathbf{G}(n)$, the L_∞-Hausdorff distance between $\mathrm{dig}(op)$ and $\overline{op}$ is less than $1 + \log n$.

The tree DT(2) is related to a famous low discrepancy sequence called van der Corput sequence [5]. Assume that n is a power of 2, and construct the sequence $X(DT(2))$ using the method of Section 2 (ignoring $x_0 = 1$). Then, we have $x_1 = 0$, $x_2 = 1/2$, $x_3 = 1/4$, $x_4 = 3/4$, and in general, if $b_1 b_2 b_3 \ldots b_s$ is the 2-adic expansion of $i - 1$, $x_i = 0.b_s b_{s-1} \ldots b_1$ in 2-adic decimal expansion for $1 \le i \le n$. This sequence is indeed the van der Corput sequence.

Surprisingly, if we omit the monotonicity axiom (R5), the lower bound does not hold, and we can give a constant upper bound on the Hausdorff distance. The digital ray that we construct is locally snake-like almost everywhere; however, considered from some distance it can approximate a line segment well.

The idea is as follows: We first consider a coarse grid of width 2, and construct a spanning forest T_1 of it allowing internal leaves. Then, we replace each node v of this tree by four nodes in the original unit-width grid such that v is located in the center of gravity of these four nodes. In the last step, we convert the forest T_1 into a tree T_2 in the original unit-width grid.

Let $c > 1$ be an irrational constant. The forest

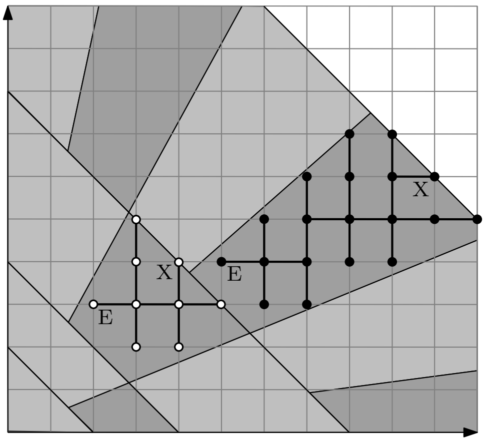

Figure 3: Trapezoid decomposition and two trees of the forest T_1.

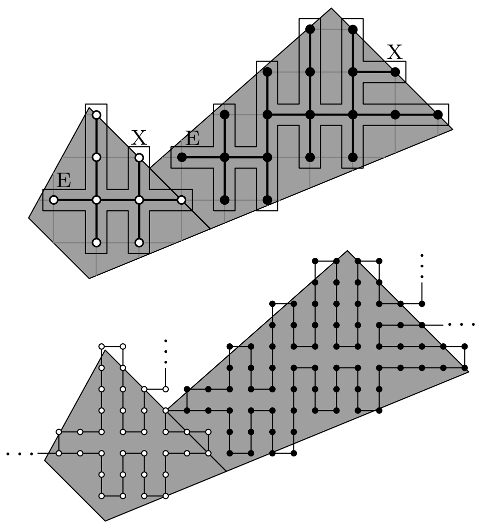

Figure 4: Walks around the two trees (top) and the corresponding part of the tree T_2 (bottom).

T_1 is constructed as follows: We consider the belt $R(k) \supset \mathbf{G}(2^{k+1}) \setminus \mathbf{G}(2^k)$ defined by $2^k < x+y \leq 2^{k+1}$ in the first quadrant, and subdivide it into trapezoids by lines $\ell_t : y = \frac{2^k - tc}{tc} x$ passing through the non-grid points $(tc, 2^k - tc)$ on the line $x + y = 2^k$ for $t = 1, 2, \ldots, \lfloor 2^k/c \rfloor$. The widths of the two parallel edges of each trapezoid are (at most) $\sqrt{2}c$ and $2\sqrt{2}c$, respectively. Further, each trapezoid F is adjacent to one trapezoid $p(F)$ in $R(k-1)$ called the parent of F, and two trapezoids $l(F)$ and $r(F)$ in the belt $R(k+1)$ that are called the left and right child, respectively. Let q be the intersection of $x + y = 2^{k+1}$ and the dividing line of $l(F)$ and $r(F)$. The nearest grid point to q in F is called the exit node of F, and the nearest grid points to q in $l(F)$ and $r(F)$ are called their entry nodes. Each trapezoid has exactly one entry and one exit node. In Fig. 3, the entry and exit nodes are marked by "E" and "X", respectively.

By gathering these trapezoids for all $k \geq \lceil \log c \rceil$, we have a decomposition of the first quadrant of the plane. Since $c > 1$, each trapezoid is wide enough so that the induced subgraph of the grid points in a trapezoid is connected. It is easy to find a spanning tree of the vertices in each trapezoid consisting of a *stem* that is shortest path from its entry node to its exit node, together with branches such that the length of each branch (i.e., the path length from the stem to the furthest leaf) is at most $2c$ as seen in Fig. 3. This gives a forest T_1 consisting of small trees, one in each trapezoid. Now, let's convert T_1 to T_2 as shown in Fig. 4. Each node of T_1 is replaced by four nodes at the corners of the surrounding unit square. Thus, we can realize the walk around each subtree of T_1 in a trapezoid F as a Hamiltonian cycle in the finer grid. We cut the cycle at the exit node, and connect it to the entry nodes of the trees in the two child trapezoids $l(F)$ and $r(F)$ as in Fig. 4. We obtain a tree T_2 that has no internal leaf.

Theorem 5 *If the monotonicity axiom (R5) is not considered, the tree T_2 defined above gives a system of digital rays in the plane grid such that the Hausdorff distance between each digital ray and its corresponding Euclidean line segment is $O(1)$.*

Proof. For any grid point p in a trapezoid F, the line segment $\overline{op}$ is contained in the union of the ancestor trapezoids of F, and also all ancestors of p in the tree T_2 are in the same union of trapezoids. Since the width of each trapezoid is at most $2\sqrt{2}c$, the distance from any point q in the path $\mathrm{dig}(op)$ in T_2 to the line op is at most $2\sqrt{2}c$. It might happen that the nearest point from q to the line op is not in the segment $\overline{op}$ since we do not assume the monotonicity axiom. However, since the length of each branch of a subtree in T_1 is at most $2c$, the Hausdorff distance between the segment $\overline{op}$ and the path from o to p in the tree is at most $(2\sqrt{2} + 2)c$. $\square$

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