Towards a Topology-Shape-Metrics Framework for Ortho-Radial Drawings

Lukas Barth* Benjamin Niedermann* Ignaz Rutter† Matthias Wolf*

Abstract

Ortho-Radial drawings are a generalization of orthogonal drawings to grids that are formed by concentric circles and straight-line spokes from the center.

We show that bend-free planar ortho-radial drawings can be combinatorially described in terms of the distribution of the angles around the vertices. Previously, such a characterization was only known for paths, cycles, and theta graphs [5], and in the special case of rectangular drawings for cubic graphs [4], where the contour of each face is required to be a rectangle. This is an important ingredient in establishing an ortho-radial analogue of Tamassia's Topology-Shape-Metrics Framework for bend minimization in planar orthogonal drawings.

1 Introduction

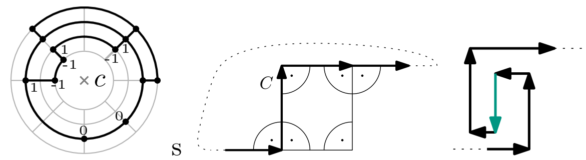
Grid drawings of graphs map vertices to grid points, and edges to internally disjoint curves on the grid lines connecting their endpoints. *Orthogonal grids*, where the grid lines are horizontal and vertical lines, are popular and widely used in graph drawing. Their strength lies in their simple structure, their high angular resolution, and the limited number of directions. Graphs admitting orthogonal grid drawings must be *4-planar*, i.e., they must be planar and have maximum degree 4.

It is well known that, a bend-free planar orthogonal drawing Γ of a 4-plane graph G, i.e., a 4-planar graph with a fixed combinatorial embedding, can be combinatorially described by the distribution of the angles around the vertices. For any incidence between a vertex v and a face f that lies to the right of the edges uv, vw, we measure the counterclockwise angle $a \in \{90°, 180°, 270°, 360°\}$ between vu and vw. In this way, we assign an angle to each vertex–face incidence. Consider two edges uv, vw not necessarily bounding a common face and let α be the sum of all the angles that lie locally to the right of uvw. We define the rotation of the path uvw as $\text{rot}(uvw) = 2 - \alpha/90°$, i.e., intuitively left and right turns correspond to rotations of -1 and 1, respectively, whereas going straight corresponds to a rotation of 0. We further generalize this to arbitrary paths $P = v_1, \dots, v_k$ as $\text{rot}(P) = \sum_{i=2}^{k-1} \text{rot}(v_{i-1}v_iv_{i+1})$ and to cycles $C = v_1, \dots, v_k, v_1$ as $\text{rot}(C) = \sum_{i=1}^{k} \text{rot}(v_{i-1}v_iv_{i+1})$, where $v_0 = v_k$ and $v_{k+1} = v_1$. For a face f, we define $\text{rot}(f) = \text{rot}(C_f)$, where C_f is the boundary of f directed such that f lies to its right. It is not hard to see that an angle assignment stemming from an orthogonal drawing satisfies the following conditions.

1. The sum of the angles around each vertex is 360°.
2. For each internal face f it is $\text{rot}(f) = 4$ and $\text{rot}(f) = -4$ for the outer face.

An angle assignment satisfying these conditions is called *orthogonal representation*. Tamassia [6] showed that, conversely, for any orthogonal representation there exists a corresponding planar orthogonal drawing with the given angles. It is this characterization, which decouples the shape of an orthogonal drawing (described in the form of an orthogonal representation) from its geometric realization, that has enabled a three-step framework for computing orthogonal planar drawings, the Topology-Shape-Metrics (TSM) framework, that is at the heart of various bend minimization algorithms for orthogonal drawings [6, 1, 2, 3]. Note that bends can be seen as subdivision vertices with a 90° and a 270° angle.

The goal of this work is to provide a similar result and thus to establish the existence of an analogous framework for *ortho-radial drawings*, which are based on *ortho-radial grids* formed by concentric circles and spokes emanating from the circles' center c; see Fig. 1a. In this case, our 4-plane input graph G comes with two designated faces, an *outer face*, which shall form the outer face of the drawing and a *central face* whose interior shall contain c. All other faces are *regular*. A simple cycle in G is *essential* if it contains c in its interior, otherwise it is *non-essential*. Throughout this paper we assume that G contains at least one



(a) (b) (c)

Figure 1: (a) An ortho-radial drawing on an ortho-radial grid, the small numbers give the rotations at the vertices in the central face. (b,c) Ortho-radial representations with locally correct angle sums that cannot be realized; the dotted curve is a single edge.

*Karlsruhe Institute of Technology, Germany, firstname.lastname@kit.edu, Lukas Barth was partially supported by DFG Research Training Group 2153

†TU Eindhoven, The Netherlands, i.rutter@tue.nl

essential cycle. If it does not, then the central and the outer face are identical, and the ortho-radial drawing is equivalent to an orthogonal drawing [5].

It is not hard to see that also ortho-radial drawings induce angle assignments as above, which we call *ortho-radial representations*. It is clear that again the angle sums around each vertex must be 360°, and further similar to the orthogonal case, it is $\text{rot}(f) = 4$ for regular faces and $\text{rot}(f) = 0$ for the central and the outer face (recall that we assume them to be distinct). However, there are examples of such assignments that have no geometric realization; see Fig. 1b. Up to this point a characterization of the ortho-radial representations that have a corresponding drawing has been achieved only for paths, cycles, and theta-graphs [5] and for 3-regular *rectangular* graphs [4], whose ortho-radial representation is such that internal faces have exactly four 90° angles, while all other incident angles are 180°, and the central and outer face have only 180° angles. Our main result is a characterization of the ortho-radial representations of arbitrary 4-plane graphs that correspond to an ortho-radial drawing.

We introduce some notation and our definition of a *valid* ortho-radial representation in Section 2. Afterwards, we first show in Section 3 that valid ortho-radial representations characterize the ortho-radial drawings of rectangular graphs. Based on that special case of 4-planar graphs, we then present the characterization for general 4-planar graphs in Section 4.

2 Preliminaries and Ortho-Radial Representations

In this paper, all paths and cycles are directed. We implicitly direct cycles that do not cross themselves, e.g., facial cycles, such that their interior lies to the right, and we consider a cycle to be part of both its interior and its exterior, i.e., the interior and the exterior are closed. We further assume that paths are *simple* but cycles may be non-simple, though they may contain each edge at most once in each direction.

Making use of the view of ortho-radial drawings as orthogonal drawings of a cylinder, we classify the edges of a drawing as pointing *left*, *right*, *down* or *up*, respectively. Edges pointing left or right are *horizontal* edges and edges pointing up or down are *vertical* edges. Note further, that an ortho-radial representation determines the directions of all edges. Considering again the example from Fig. 1b it can be seen that the essential cycle C contains an up edge but no down edge, and thus there is no corresponding drawing. However, the existence of a down edge on each essential cycle containing an up edge is not sufficient for the existence of a drawing even in the case of cycles; see Fig. 1c. The reason is that, in some sense, the down edge in this case is too wound up to be of any help. Instead we need a somewhat more global measure than up and down, which we introduce next.

Ortho-Radial Representations For a 4-plane graph G with a given ortho-radial representation Γ, we fix an arbitrary reference edge $e^\star = rs$ on the outer face that points to the right, i.e., the outer face lies on its left. Let C be a simple essential cycle and let P be a path from s to a vertex v of C. We now define a labeling of the edges of C with respect to P and $e^\star$ as $\ell_C^P(e) = \text{rot}(e^\star + P + C[v, e])$, where $C[v, e]$ denotes the part of C from v to e. In the following we are mostly interested in labelings with respect to so-called *elementary paths* P, where v is the first vertex of C that lies on P. It can be shown that in this case the labeling does not depend on the choice of the elementary path. Thus, the labeling of C depends only on Γ, and we omit the superscript P. We are now ready to present our characterization.

Definition 1 *An ortho-radial representation is* valid *if the following conditions hold.*

1. *The sum of angles around each vertex is* 360°.
2. *For each face f, it is*

$$\text{rot}(f) = \begin{cases} 4, & \text{if } f \text{ is a regular face} \\ 0, & \text{if } f \text{ is the central/outer face.} \end{cases}$$

3. *For each simple essential cycle C in G, it is $\ell_C(e) = 0$ for all edges e of C, or there are edges e_+ and e_- on C with $\ell_C(e_+) > 0$ and $\ell_C(e_-) < 0$.*

We have already seen that the first two conditions are necessary. The last condition is new and guarantees that all cycles in the graph can be drawn consistently. For an essential cycle C that violates condition 3 either all labels of edges on C are non-negative or all are non-positive. Then C is called *decreasing* and *increasing*, respectively. Both increasing and decreasing cycles are called *monotone*. Note that an increasing (decreasing) cycle contains an edge with a negative (positive) label. Cycles with only the label 0 are not monotone. Our main result is as follows.

Theorem 1 *Let G be a 4-plane graph with an ortho-radial representation Γ. Then G has an ortho-radial drawing that corresponds to Γ if and only if Γ is valid.*

3 Rectangular Graphs

Let G be a 4-planar graph and let Γ be an ortho-radial representation of G where every face is rectangular. We use a flow method similar to Tamassia [6]. For each edge e, we find an arbitrary path P from $e^\star$ to e, and we determine e as pointing right, down, left or up, if $\text{rot}(P) \mod 4$ is $0, 1, 2$, or 3, respectively. We note that conditions 1 and 2 of valid ortho-radial representations guarantee that this is well-defined. We then reverse the downward and left edges so that all edges point either up or right. The rectangular property of the faces guarantees that each internal face is bounded by two vertical and by two horizontal paths.

We create a radial flow network N_{rad} with a vertex for each face, and an edge from a face f to a face g if and only if there is a horizontal edge with f to its right and g to its left; see Fig. 2. Similarly, we define a vertical flow network N_{ver} that has a vertex for each internal face and an edge from f to g if and only if there is a vertical edge with f to its left and g to its right. We set the capacities of all edges to ∞ and require a minimum flow of 1 on each edge. It is then readily seen that drawings of Γ correspond bijectively to pairs $(F_{\text{rad}}, F_{\text{ver}})$ where F_{rad} is a flow from the central face to the outer face in N_{rad} and F_{ver} is a circulation in N_{ver}. The fact that such a flow exists in N_{rad} is analogous to the orthogonal case [6].

The key is to show that a circulation in N_{ver} exists if Γ is valid. The main idea is to determine for each arc a of N_{ver} a cycle C_a in N_{ver} that contains a. If F_a denotes the circulation that pushes one unit of flow along the arcs of C_a and is 0 elsewhere, then $F_{\text{ver}} = \sum_{a \in A} F_a$, where A denotes the arc set of N_{ver}, is the desired flow. The only reason why such a cycle might not exist is if there is a set S of vertices in N_{ver} such that there exists an arc entering S but no arc exiting S. Without loss of generality, we assume $N_{\text{ver}}[S]$ is weakly connected, which implies that S corresponds to a connected set $\mathcal{S}$ of faces in G. Note that S contains a directed cycle of N_{ver}, which is an essential cycle. Let C and C' denote the smallest and largest essential cycle of G, respectively, such that all faces in $\mathcal{S}$ lie in the interior of C and in the exterior of C'. We show that C is increasing or C' is decreasing.

Assume there is an incoming arc a that crosses C (an incoming arc crossing C' is analogous). Since all faces are rectangles, there is an elementary path P from $e^\star$ to a vertex v on G only using right and down edges of G. Thus, if w is the first vertex of C after v, it is $\ell_C(vw) = 0$ if vw is horizontal and $\ell_C(vw) = -1$ if vw points up. Since no edge on C is pointing downward, i.e., its label is congruent to $1 \mod 4$, and the labels between adjacent edges differ by $-1, 0$, or 1, it follows that $\ell_C(e') \in \{-2, -1, 0\}$ for all edges e' of C, i.e., $\ell_C(e') \leq 0$. However, the edge e corresponding to the incoming arc a of S is pointing upwards, and therefore $\ell_C(e) = -1$. Hence C is increasing.

Theorem 2 *Let (G, Γ) be a rectangular graph and its ortho-radial representation. There exists a bend-free ortho-radial drawing of G respecting Γ if and only if Γ is valid.*

4 General 4-planar Graphs

In this section we present the proof of Theorem 1. Following Tamassia's approach for orthogonal drawings [6], our approach is based on augmenting a graph G and its valid ortho-radial representation with additional edges so that it remains valid and becomes rectangular. Then the claim follows from Theorem 2. For the sake of simplicity, we assume that the outer face and the central face are already bounded by a horizontal cycle, if not, we can simply add these cycles and suitably attach them to our graph.

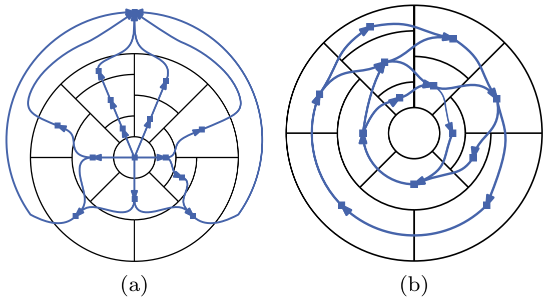


Figure 2: Networks N_{rad} (a) and N_{ver} (b) for assigning the lengths of radial and vertical edges, respectively.

A regular face f that is not a rectangle contains a left bend at a vertex on its boundary, and since it contains four more right bends than left bends by cond. 2, the boundary of f actually contains a vertex u that is followed by two right bends; see Fig. 3a. We call this a *U-shape*. Let z be a subdivision vertex on the edge e immediately after the second right bend of the U-shape and consider the graph G' and its representation Γ' obtained by adding the edge uz and setting the angles at u and z in the face left of uz to $90°$; see Fig. 3a. Tamassia shows that Γ' is a valid orthogonal representation of G'. Since G' has fewer left bends at internal faces than G, this procedure eventually terminates with a rectangular graph.

In the case of ortho-radial drawings the situation is not so simple, since we additionally have to ensure that the insertion does not create any monotone cycles. However, this case cannot occur if the edge uz is vertical. Namely, if inserting the vertical edge uz created a monotone cycle C' in G', then, instead of the edge uz, one can use the cycle $C' - uz$ and (a part of) the U-shape to find a monotone cycle C in G.

Lemma 3 *Vertical augmentation does not create monotone cycles.*

There are, however, faces that do not have a U-shape whose last segment is horizontal; see Fig. 3b. Fix again a vertex u with a U-shape in face f. Indeed, it can be the case that subdividing the last edge of the U-shape by a vertex z and inserting uz as above creates a monotone cycle. In this case, instead of just considering subdividing the last edge of the U-shape as above, we consider all the *candidate edges* e_i incident to f that are opposite of u in the sense that $\text{rot}(C_f[u, e]) = 2$, where C_f denotes the facial cycle of f. Let $e_1, \dots, e_k$ denote the candidate edges as they occur clockwise starting from u. We call the

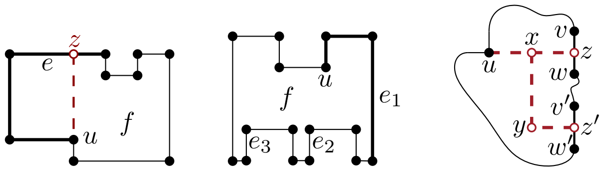


Figure 3: (a) U-shape (thick black) leading to a vertical augmentation (dashed edge). (b) Face f whose U-shapes all require horizontal augmentations and the candidates e_i for u. (c) Structure for simulating the simultaneous augmentation with edges vw and $v'w'$.

process of subdividing e_i with a new vertex z and adding the edge uz *augmenting with* e_i. Intuitively, augmenting with e_i makes a jump further downwards as i increases. Indeed, using similar arguments as the ones for the vertical case, one can show that the first candidate never creates an increasing cycle, and the last candidate never creates a decreasing cycle.

Lemma 4 *Augmenting with e_1 (resp. with e_k) never creates an increasing (resp. decreasing) cycle.*

Moreover, it can be shown that an increasing and a decreasing cycle cannot intersect strictly, and therefore, it is not possible that augmenting with a candidate creates both an increasing and a decreasing cycle. Thus, if each augmentation with respect to the edges e_i yields an increasing or a decreasing cycle, then by Lemma 4, there exists a pair of candidates such that e_i creates a decreasing cycle and e_{i+1} creates an increasing cycle. Let $e_i = vw$ and $e_{i+1} = v'w'$ be directed so that f lies to their right and note that possibly $v' = w$. Let z and z' denote subdivision vertices of e_i and e_{i+1}, respectively. We simulate augmentation with both candidates simultaneously by inserting two vertices x and y as shown in Fig. 3c. By construction, we then find both a decreasing cycle C using the edges ux, xz and a cycle C' using the edges ux, xy, yz' that is increasing except for possibly the edge xy. Since these cycles both contain u but one is decreasing and the other one is increasing, and such cycles cannot strictly intersect, we can infer that, outside of the face f both cycles actually coincide, i.e., their edges all have label 0 outside of f. We thus find a path P in G consisting of only horizontal edges that starts at v' or at w, ends at u and contains all these vertices. Thus, augmenting the graph G by adding the edge from u to the starting point of P creates a cycle consisting of only horizontal edges. It can be argued that such an augmentation is always safe. In fact one can show that no monotone cycle can share a vertex with a horizontal cycle. The following lemma summarizes this discussion.

Lemma 5 *Let f be a face and let u be a vertex on the boundary of f that forms a left bend in f. Let further $e = vw$ and $e' = v'w'$ be two consecutive candidates. If augmentation with e creates an increasing cycle and augmenation with e' creates a decreasing cycle, then augmenting with one of uw or uv' does not create a monotone cycle.*

Altogether, this proves that for each left bend in a regular face f there exists an augmentation such that the resulting graph and ortho-radial representation are still valid. Eventually, we thus arrive at a rectangular graph, and Theorem 2 applies.

5 Conclusion

In this work we considered orthogonal drawings of graphs on cylinders. Our main result is a characterization of the 4-plane graphs that can be drawn bend-free on a cylinder in terms of a ortho-radial representation of such drawings.

While our proof for both the rectangular case and the general case are algorithmic, only the former currently has an efficient implementation, e.g., in terms of a flow algorithm. In contrast, the rectangulation procedure from Section 4 requires checking whether augmentations create monotone cycles. Our most important open problem is whether such a check can be performed in polynomial time.

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