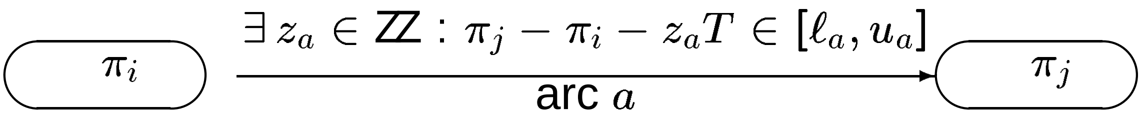
Periodic Event Scheduling Problem (PESP)

P. Serafini and W. Ukovich (1989)



Find a potential $\vec{\pi}$ and modulo parameter $\vec{z}$, such that for all arcs a there holds

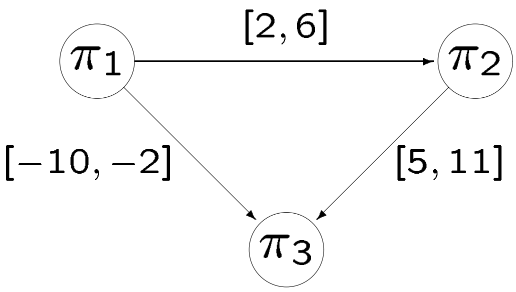
$$\pi_j - \pi_i - z_a T \in [\ell_a, u_a],$$

or prove infeasibility of this task.

Cylce inequalities (Odijk)

For each cycle $\vec{\gamma}$ there holds:

$$\left\lceil \frac{1}{T} \left(\vec{\ell}^t \vec{\gamma}^- - \vec{u}^t \vec{\gamma}^+ \right) \right\rceil \leq \vec{\gamma}^t \vec{z} \leq \left\lfloor \frac{1}{T} \left(\vec{u}^t \vec{\gamma}^- - \vec{l}^t \vec{\gamma}^+ \right) \right\rfloor$$



$$\begin{array}{rcccr}
2 & \leq & \pi_2 - \pi_1 - 10z_1 & \leq & 6 \\
5 & \leq & \pi_3 - \pi_2 - 10z_2 & \leq & 11 \\
2 & \leq & \pi_1 - \pi_3 + 10z_3 & \leq & 10 \\
\hline
9 & \leq & 10(z_3 - z_1 - z_2) & \leq & 27 \\
\hline
1 & \leq & z_3 - z_1 - z_2 & \leq & 2
\end{array}$$

Associated Polyhedra

Θ = incidence matrix of the event network
T = period

complete solution space

$$\mathcal{Q} := \left\{ \begin{pmatrix} \vec{\pi} \\ \vec{z} \end{pmatrix} \in \mathbb{Z}^n \times \mathbb{Z}^m \;\middle|\; \vec{\ell} \leq \Theta^t \vec{\pi} - T\vec{z} \leq \vec{u} \right\}$$

solution space of modulo parameter

$$\mathcal{Z} := \left\{ \vec{z} \in \mathbb{Z}^m \mid \exists \vec{\pi} : \; \vec{\ell} \leq \Theta^t \vec{\pi} - T\vec{z} \leq \vec{u} \right\}$$

$\mathcal{T}$ spanning tree of $\mathcal{N}$. For all $a' \in \mathcal{T}$ fix $z_{a'} = 0$:

$$\mathcal{Q}_{\mathcal{T}} := \left\{ \begin{pmatrix} \vec{\pi} \\ \vec{z_{co}} \end{pmatrix} \;\middle|\; \vec{\ell} \leq \Theta^t \vec{\pi} - T\vec{z} \leq \vec{u} \text{ for } \vec{z} = \begin{pmatrix} \vec{z_{co}} \\ \vec{0_{\mathcal{T}}} \end{pmatrix} \in \mathbb{Z}^m \right\}$$

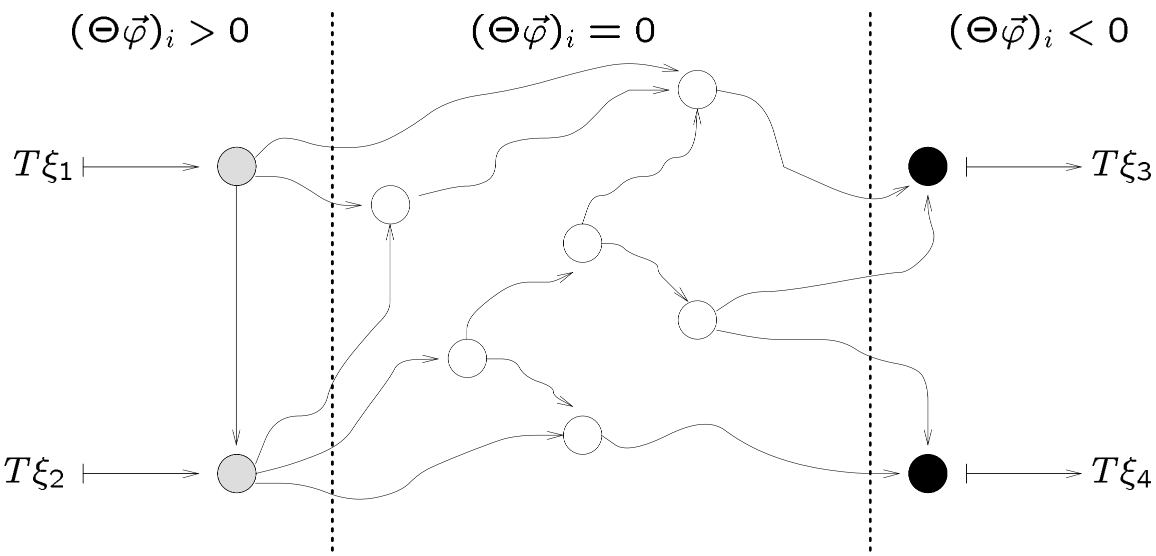
Polyhedral Structure of PESP

results by Thomas Lindner and Karl Nachtigall

Theorem
If $\vec{\xi}^t\vec{\pi} + \vec{\varphi}^t\vec{z} \geq \varphi_0$ is a valid inequality of $con\,(\mathcal{Q})$, then

$$\Theta\vec{\varphi} = -T\vec{\xi} \iff \forall i:\ (\Theta\vec{\varphi})_i = \sum_{a:i\to j}\varphi_a - \sum_{a:j\to i}\varphi_a = -T\xi_i$$

If $\vec{\varphi}^t\vec{z} \geq \varphi_0$ is valid for $con\,(\mathcal{Z})$, then $\Theta\vec{\varphi} = \vec{0}$.



flow conservation: $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0$

A Lifting Theorem

$\mathcal{T}$ spanning tree of $\mathcal{N}$. For each $\vec{\xi}, \vec{\varphi}_{co}$ with $\sum_i \xi_i = 0$ there exist unique flow values on the tree arcs $\vec{\varphi}_{\mathcal{T}} := f(\vec{\xi}, \vec{\varphi}_{co})$ such that $\vec{\varphi} := \begin{pmatrix} \vec{\varphi}_{\mathcal{T}} \\ \vec{\varphi}_{co} \end{pmatrix}$ is a flow fulfilling $\Theta\vec{\varphi} = -T\vec{\xi}$.

Theorem

$\vec{\xi}^t \vec{\pi} + \vec{\varphi}_{co}^t \vec{z}_{co} \geq \varphi_0$ is a valid (facet defining) inequality for $con\,(\mathcal{Q}_{\mathcal{T}})$, if and only if

$$\vec{\xi}^t \vec{\pi} + \vec{\varphi}_{co}^t \vec{z}_{co} + f(\vec{\xi}, \vec{\varphi}_{co})^t \vec{z}_{\mathcal{T}} \geq \varphi_0$$

is a valid (facet defining) inequality of $con\,(\mathcal{Q})$.

Box Constraints

$\mathcal{T}$ spanning tree of $\mathcal{N}$.

$$\mathcal{Z}_{\mathcal{T}} := \left\{ \vec{z}_{co} \,|\, \exists \vec{\pi} : \; \vec{\ell} \leq \Theta^t \vec{\pi} - T\vec{z} \leq \vec{u} \text{ for } \vec{z} = \begin{pmatrix} \vec{z}_{co} \\ \vec{0}_{\mathcal{T}} \end{pmatrix} \right\}$$

For each co-tree arc a the subgraph $\mathcal{T} + \{a\}$ contains a unique cylce $\vec{\gamma}_a$. Then

$$\underline{z_a} := \left\lceil \frac{1}{T} \left(\vec{\ell}^t \vec{\gamma_a}^- - \vec{u}^t \vec{\gamma_a}^+ \right) \right\rceil \leq z_a$$

and

$$z_a \leq \left\lfloor \frac{1}{T} \left(\vec{u}^t \vec{\gamma_a}^- - \vec{l}^t \vec{\gamma_a}^+ \right) \right\rfloor =: \overline{z_a}$$

The Single-bound Kernel

Let $\vec{\varphi}^t\vec{z} \geq \varphi_0$ be a valid inequality of $\mathcal{Z}$. If $\varphi_a > 0$, then

$$z_a \geq \left\lceil \frac{1}{\varphi_a}\left(\varphi_0 - \sum_{a\neq a',\ \varphi_{a'}>0} \varphi_{a'}\overline{z}_{a'} - \sum_{a\neq a',\ \varphi_{a'}<0} \varphi_{a'}\underline{z}_{a'}\right)\right\rceil$$

If $\varphi_a < 0$, then

$$z_a \leq \left\lfloor \frac{1}{\varphi_a}\left(\varphi_0 - \sum_{a\neq a',\ \varphi_{a'}<0} \varphi_{a'}\overline{z}_{a'} - \sum_{a\neq a',\ \varphi_{a'}>0} \varphi_{a'}\underline{z}_{a'}\right)\right\rfloor$$

Denote those bounds by $\underline{\alpha}(\vec{\varphi}, \varphi_0) \leq z_a \leq \overline{\alpha}(\vec{\varphi}, \varphi_0)$

For a class $\mathcal{C}$ of valid inequalities define the single bound kernel $\tilde{\mathcal{Z}}^*(\vec{\underline{z}}, \vec{\overline{z}})$ to be the maximal subset $\tilde{\mathcal{Z}}(\vec{\underline{z}}', \vec{\overline{z}}') \subseteq \tilde{\mathcal{Z}}(\vec{\underline{z}}, \vec{\overline{z}})$ with

$$\forall(\vec{\varphi}, \varphi_0) \in \mathcal{C}:\ \underline{z}'_a \geq \underline{\alpha}(\vec{\varphi}, \varphi_0) \text{ and } \overline{z}'_a \leq \overline{\alpha}(\vec{\varphi}, \varphi_0)$$

The Single-bound Kernel

Theorem

If the single bound separation problem for a class $\mathcal{C}$ of valid inequalities is polynomial solvable, then the single bound kernel $\tilde{\mathcal{Z}}^*(\underline{\vec{z}}, \overline{\vec{z}})$ for each box $\underline{\vec{z}} \le \vec{z} \le \overline{\vec{z}}$ obtainted from a spanning tree can be calculated within polynomial time.

Proof: For each box $\underline{\vec{z}} \le \vec{z} \le \overline{\vec{z}}$, generated by a spanning tree, there holds

$$\sum_a (\overline{z}_a - \underline{z}_a) \le \sum_{a \text{ co-tree arc}} (n-1) \le (m-n-1)(n-1)$$

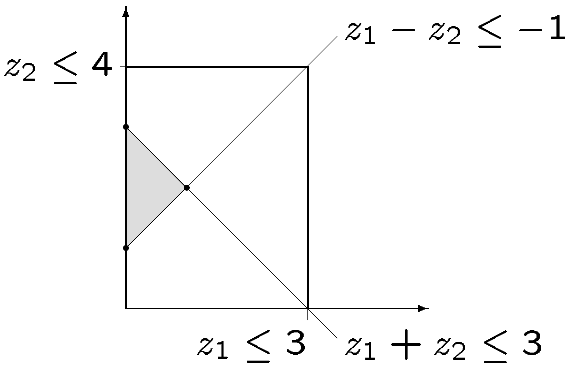
The Single-bound Kernel

Theorem
The single bound separation problem for the class of cycle inequalities can be solved by calculating for each arc $a : i \to j$ a minimum cost path $\vec{p}$ from i to j:

$$z_a \leq \left\lfloor \frac{1}{T} \left(\ell_a + (\vec{p}^{+})^t(\vec{u} + T\vec{\overline{z}}) - (\vec{p}^{-})^t(\vec{\ell} + T\vec{\underline{z}}) \right) \right\rfloor$$

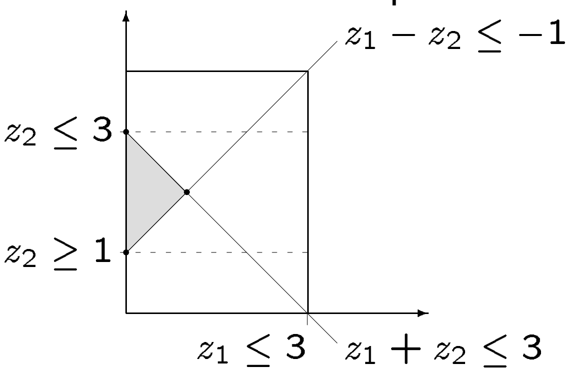
Example

Consider the system



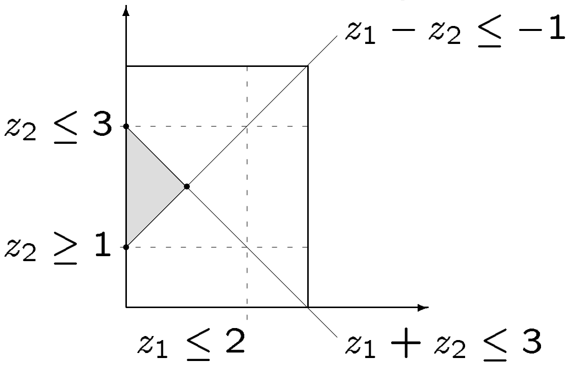
$$
\begin{array}{rcrcrcr}
 & & z_1 & + & z_2 & \le & 3 \\
 & & z_1 & - & z_2 & \le & -1 \\
0 & \le & z_1 & & & \le & 3 \\
0 & \le & & & z_2 & \le & 4
\end{array}
$$

The first iteration step leads to



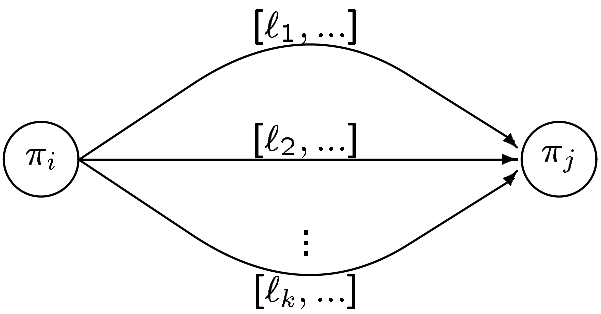
$$
\begin{array}{rcrcrcr}
 & & z_1 & + & z_2 & \le & 3 \\
 & & z_1 & - & z_2 & \le & -1 \\
0 & \le & z_1 & & & \le & 3 \\
1 & \le & & & z_2 & \le & 3
\end{array}
$$

The final iteration step leads to the kernel



$$
\begin{array}{rcrcrcr}
 & & z_1 & + & z_2 & \le & 3 \\
 & & z_1 & - & z_2 & \le & -1 \\
0 & \le & z_1 & & & \le & 3 \\
0 & \le & & & z_2 & \le & 4
\end{array}
$$

Bundle Cutting Planes

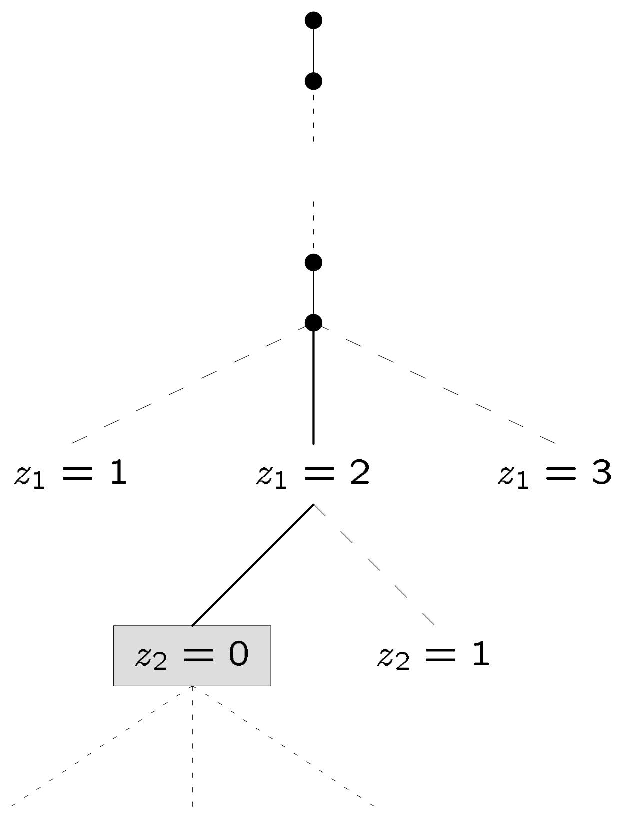


$$
\begin{aligned}
\ell_1 &\leq \pi_j - \pi_i - z_1 T \\
\ell_2 &\leq \pi_j - \pi_i - z_2 T \\
&\vdots \\
\ell_k &\leq \pi_j - \pi_i - z_k T
\end{aligned}
$$

If $0 \leq \ell_1 \leq .... \leq \ell_k < T$, then

$$\pi_j - \pi_i - (T - \ell_k) z_1 - (\ell_2 - \ell_1) z_2 - ... - (\ell_k - \ell_{k-1}) z_k \geq \ell_k$$

A Backtracking Algorithm (Serafini, Ukovich)



tree arcs a with $z_a = 0$

active arcs with fixed z_a

actual search level

non-active arcs (ignored!)

Some disadvantages:

- the arcs are processed in a pre-defined **fixed** order
- non-active arcs are not taken into account in any way

A Branch-and-Cut Algorithm

1. Compute a box $\underline{\vec{z}} \leq \vec{z} \leq \vec{\overline{z}}$ by using a spanning tree. Calculate the single bound kernel $\tilde{\mathcal{Z}}^*(\underline{\vec{z}}, \vec{\overline{z}})$.

2. Find $\vec{\tilde{z}} \in \tilde{\mathcal{Z}}^*(\underline{\vec{z}}, \vec{\overline{z}})$. Minimize the number of fractionals by applying a heuristic.

3. If $\vec{\tilde{z}}$ is integral, STOP, a solution is found. Otherwise

4. (a) Choose a fractional arc $\tilde{z}_a \notin \mathbb{Z}$ and decide to investigate either $z_a \leq \lfloor \tilde{z}_a \rfloor =: \overline{z}'_a$ or $z_a \geq \lceil \tilde{z}_a \rceil =: \underline{z}'_a$

   (b) Calculate the single bound kernel. If $\tilde{\mathcal{Z}}^*(\underline{\vec{z}}', \vec{\overline{z}}') \neq \emptyset$, then return to 2). Otherwise bracktrack. If the backtracking procedure terminates at the root node, STOP, the problem is infeasible.