Two Approaches to Model Periodic Timetabling With Different Event Frequencies

Christian Liebchen, TU Berlin

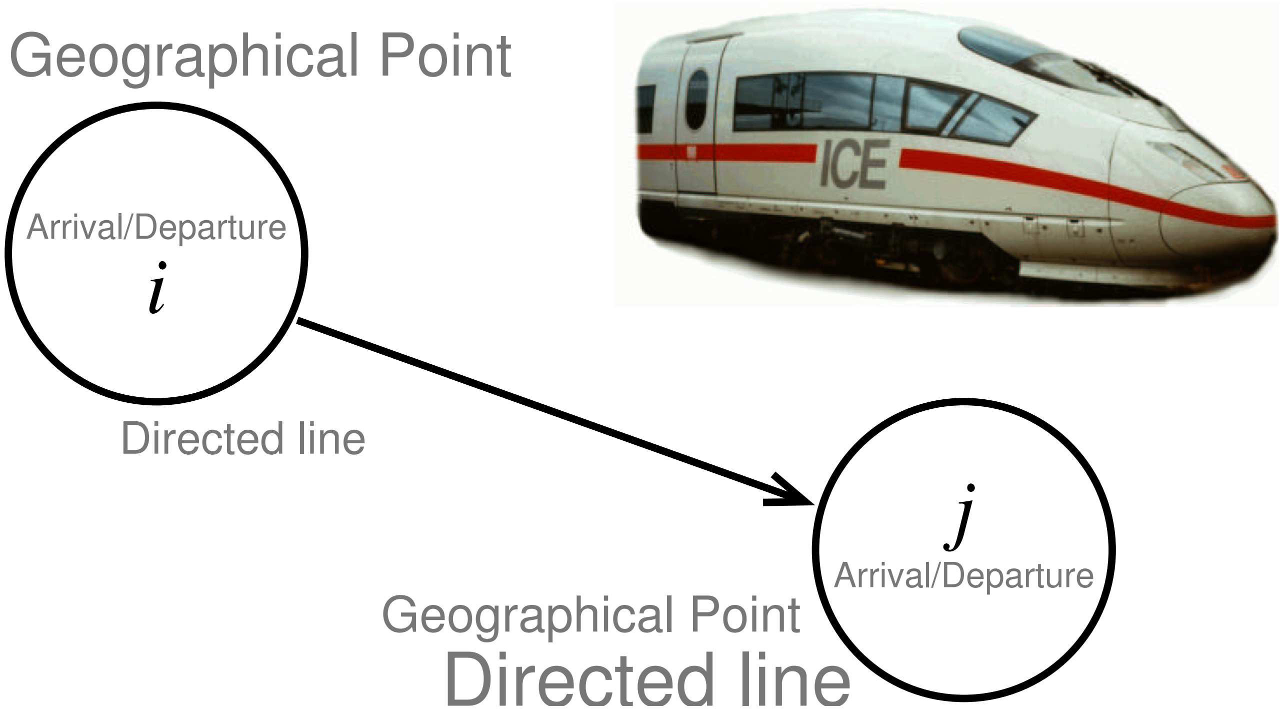
- The Standard Single Period Length Timetabling Model
- The Two Models Incorporating Different Frequencies
- Properties of the Two Models

The (Omni-) Presence of Periodic Timetables

Zug	RE 16317	ICE 1557 MAX BECKMANN	IR 2455 HEINZ RÜHMANN 15	RE 16319	RE 3659	IR 2102 STRELASUND	RE 16321	ICE 1559 AUGUST BEBEL	IR 2457 WARTBURG
von		Saarbrücken/ Frankfurt	Düsseldorf/ Dortmund		Göttingen	Frankfurt am Main		Frankfurt am Main	Düssel-dorf
Eisenach	10 09	**11 01**	**11 04**	11 09		**12 05**	12 09	**13 01**	**13 04**
Wutha	10 14			11 14			12 14		
Schönau (Hörsel)	10 17			11 17			12 17		
Sättelstädt	10 20			11 20			12 20		
Mechterstädt	10 23			11 23			12 23		
Fröttstädt 3 ○	10 26			11 26			12 26		
Fröttstädt 606	10 27			11 27			12 27		
Gotha 4 ○	10 33		**11 20**	11 33	11 48	**12 20**	12 33		**13 20**
Gotha 572,602	10 34		**11 21**	11 34	11 54	**12 21**	12 34		**13 21**
Seebergen	10 39			11 39			12 39		
Wandersleben	10 43			11 43			12 43		
Neudietendorf 3 ○	10 46			11 46	12 03		12 46		
Neudietendorf	10 47			11 47	12 04		12 47		
Erfurt-Bischleben	10 52			11 52			12 52		
Erfurt Hbf 7 ○	10 58	**11 29**	**11 37**	11 58	12 13	**12 39**	12 58	**13 29**	**13 37**
nach	Halle (Saale)	Dresden	Weimar	Halle (Saale)	Zwickau/ Glauchau	Stralsund	Halle (Saale)	Dresden	Weimar

The *Standard* Single Period Length Timetabling Model

The Core of the Model



$$\ell_{ij} \leq \pi_j - \pi_i \qquad \leq u_{ij}$$

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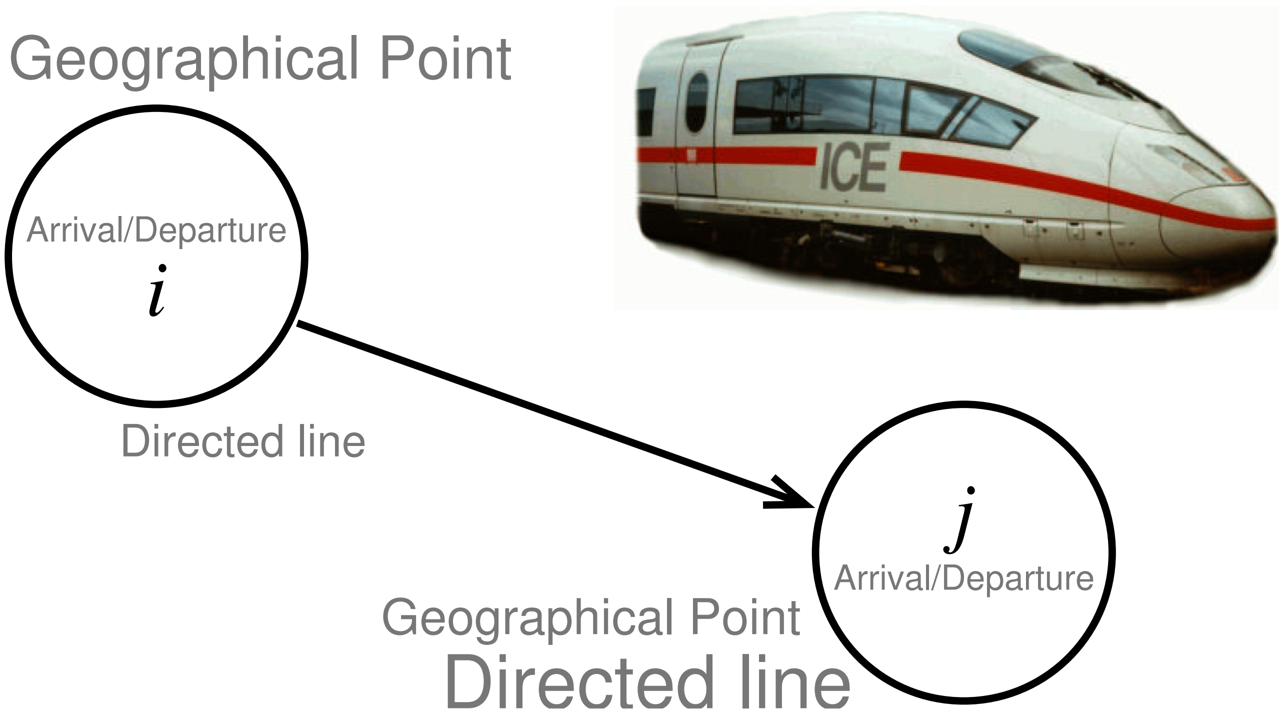
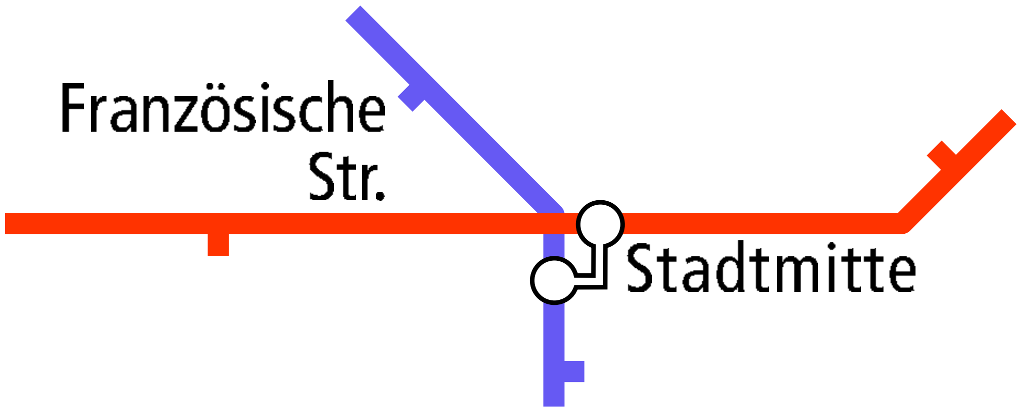


$$\ell_{ij} \le \pi_j - \pi_i + p_{ij}T \le u_{ij}$$

From a Lineplan to its Graph Model



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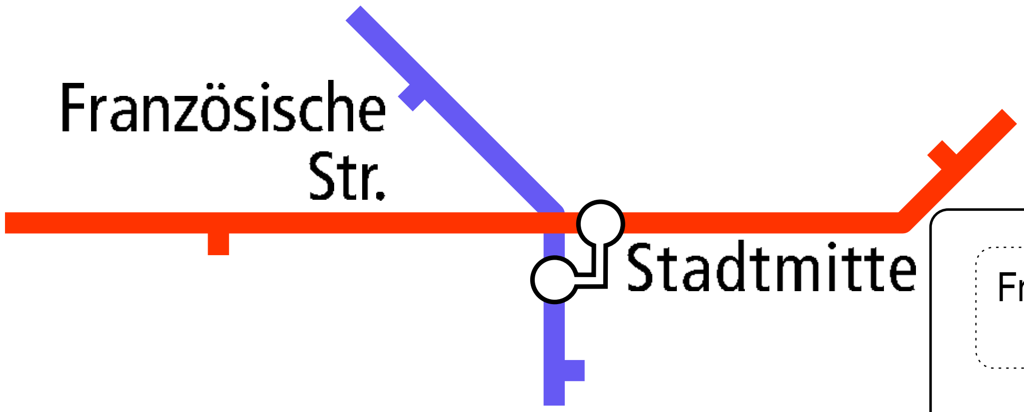
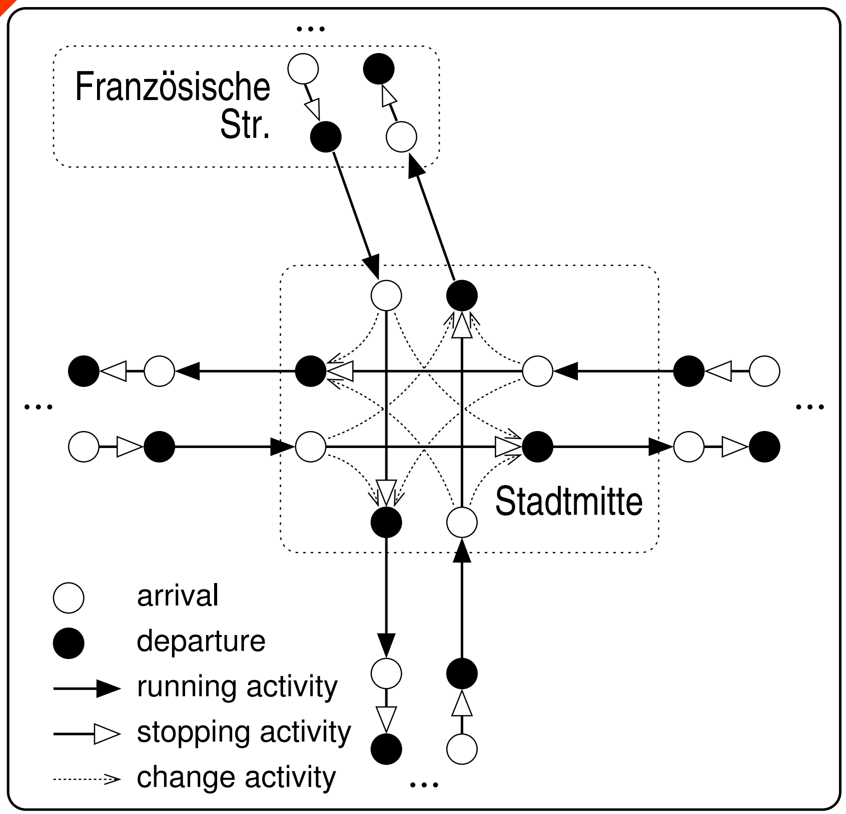




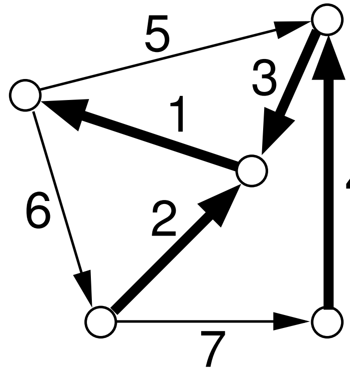
model every arrival/ departure event by an individual node

The Periodic Event Scheduling Problem (PESP)

$$x_{ij} := \ell_{ij} + (\pi_j - \pi_i - \ell_{ij}) \bmod T$$

$$\begin{array}{ll} \text{min} & cx \\ \text{s.t.} & \Gamma x = pT \\ & \ell \leq x \leq u \\ & p \text{ integer} \end{array}$$

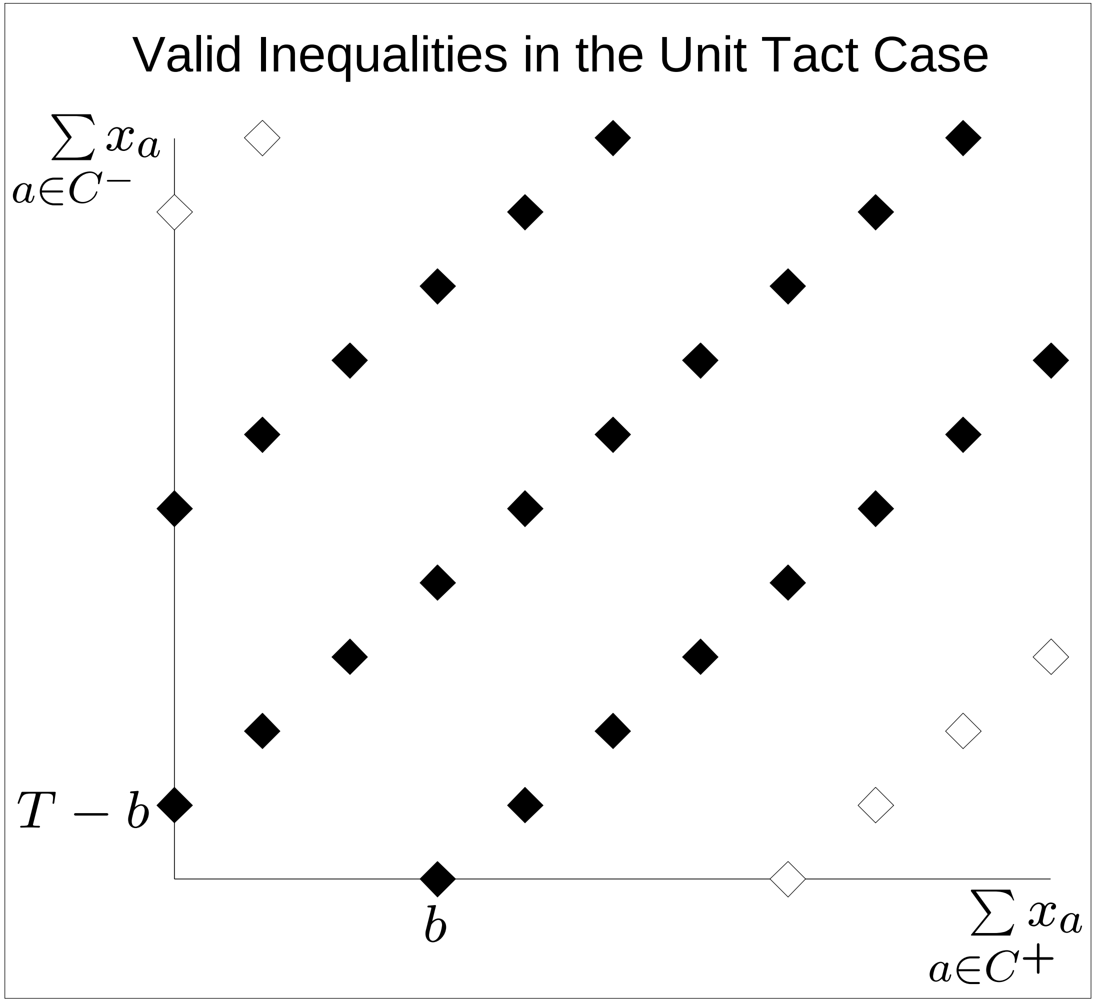
(Serafini & Ukovich 1989)



$$\Gamma = \left(\begin{array}{cccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

Γ network matrix, i.e. cycle-arc incidence matrix

Visualizing Known Valid Inequalities



Visualizing Known Valid Inequalities

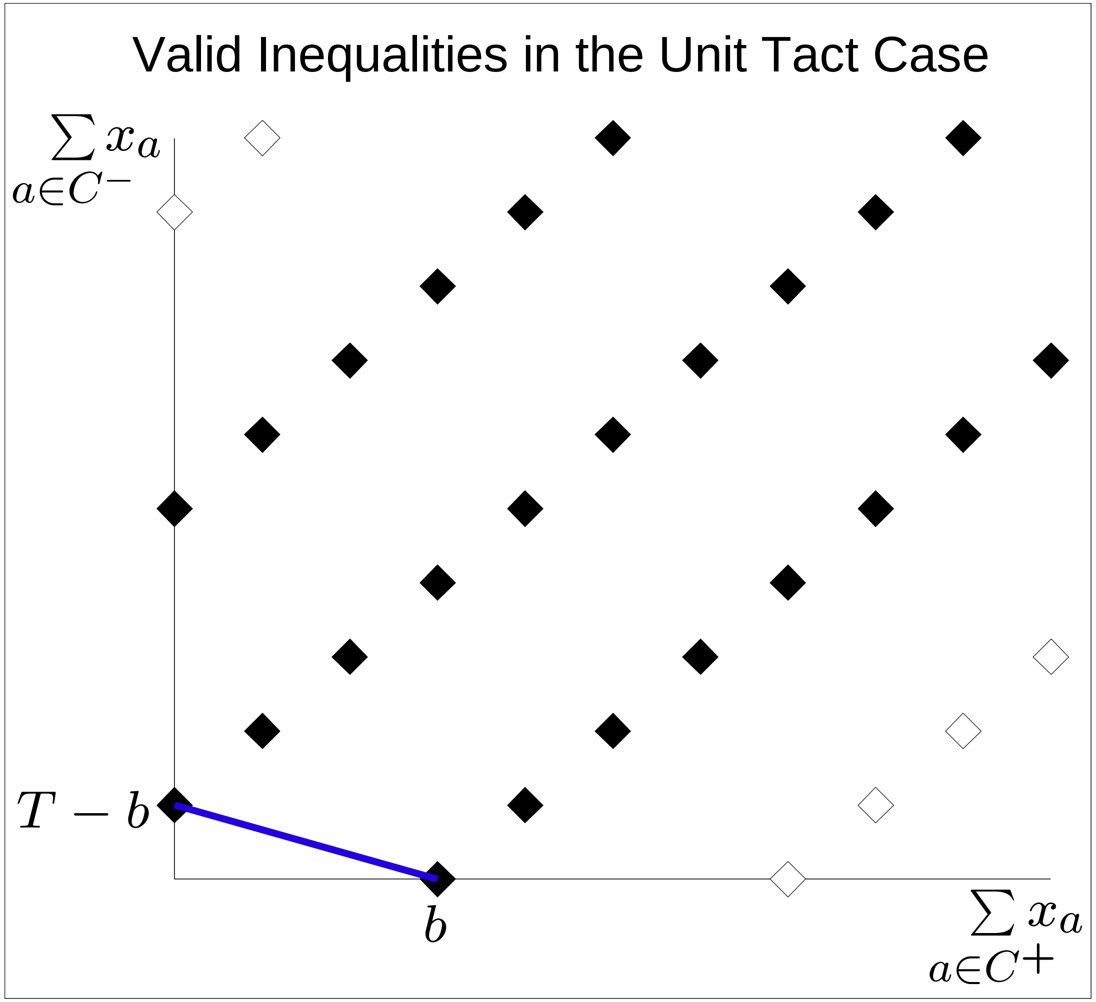
Nachtigall's Inequalities ('96):

$$\frac{b}{T-b}(-\gamma^- x) + \gamma^+ x \geq b$$

with $b := (-\gamma \cdot \ell) \bmod T$

and $\gamma x = \sum_{a \in C^+} x_a - \sum_{a \in C^-} x_a$

and C/γ arbitrary cycle



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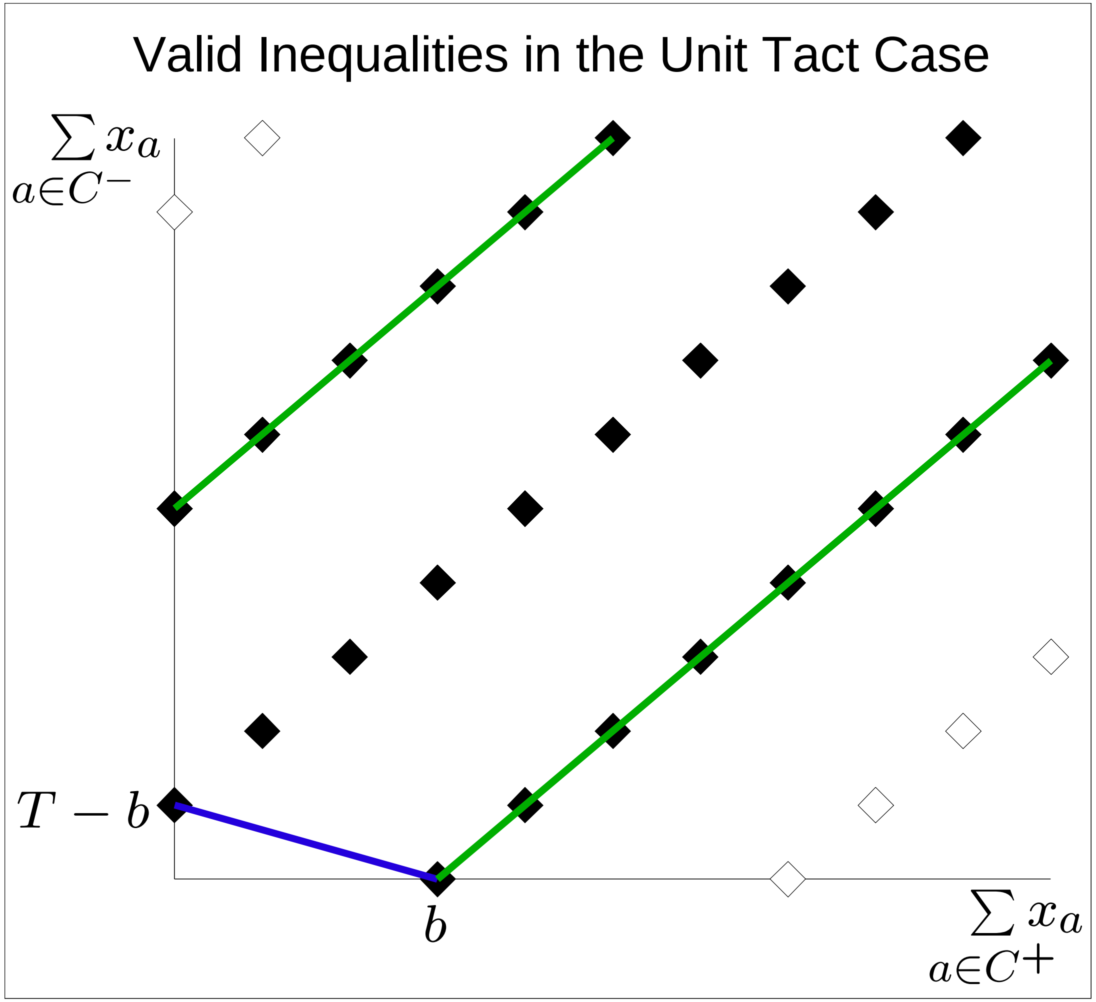
Odijk's Inequalities ('94):

$$T \left\lceil \frac{\gamma^+ \ell + \gamma^- u}{T} \right\rceil \leq \gamma x$$

$$\gamma x \leq T \left\lfloor \frac{\gamma^+ u + \gamma^- \ell}{T} \right\rfloor$$



The Two Models Incorporating Different Frequencies

Presence of Periodic Timetables – *Without* Single Period

Tram 5	Montag - Donnerstag						
						F 10	
Zingster Str.	.	**4.41**	**5.01**	**5.21**	**5.34**		**8.04**
Prerower Platz	.	4.45	5.05	5.25	5.38		8.08
Arnimstr.	.	4.47	5.07	5.27	5.40		8.10
Gehrenseestr.	.	4.49	5.09	5.29	5.43		8.13

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Tram **13** Montag - Freitag

							F15	
Zingster Str.	.	.	.	.	**5.18**	**5.32**		**17.32**
Prerower Platz	.	.	.	.	5.22	5.36		17.36
Arnimstr.	.	.	.	.	5.24	5.38		17.38
Gehrenseestr.	.	.	.	**5.06**	**5.26**	**5.41**		**17.41**

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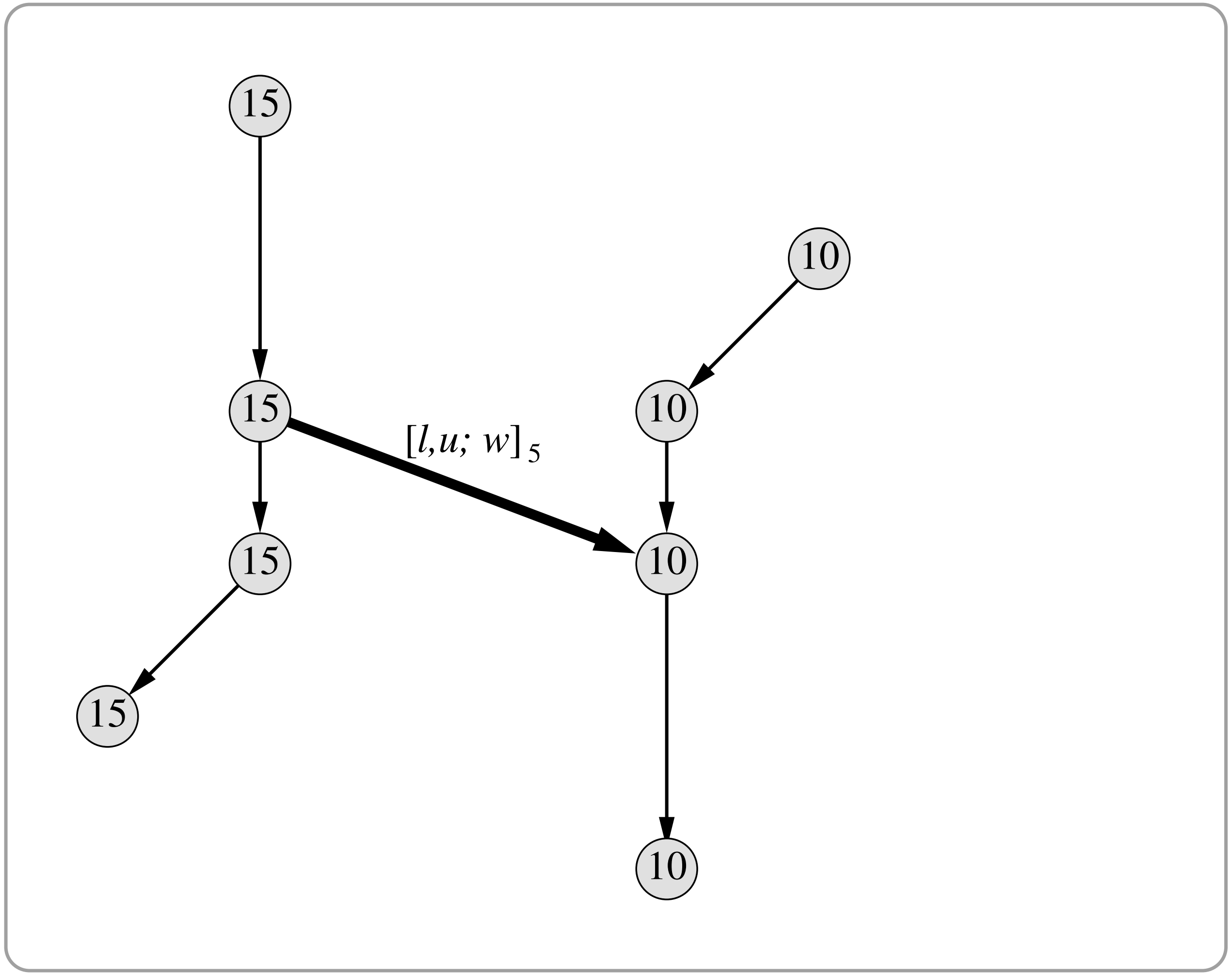
Gehrenseestr.	.	.	.	**5.06**	**5.26**	**5.41**		**17.41**

Tram 26 — **Montag - Freitag**

				F 20			F 20	
Zingster Str.	.	.	**4.28**		**5.28**	**5.46**		**19.26**
Prerower Platz	.	.	4.32		5.32	5.50		19.30
Arnimstr.	.	.	4.34		5.34	5.52		19.32
Gehrenseestr.	.	4.16	4.36		5.36	5.55		19.35

Duplicate Lines Within Single Period Model (Intuition)



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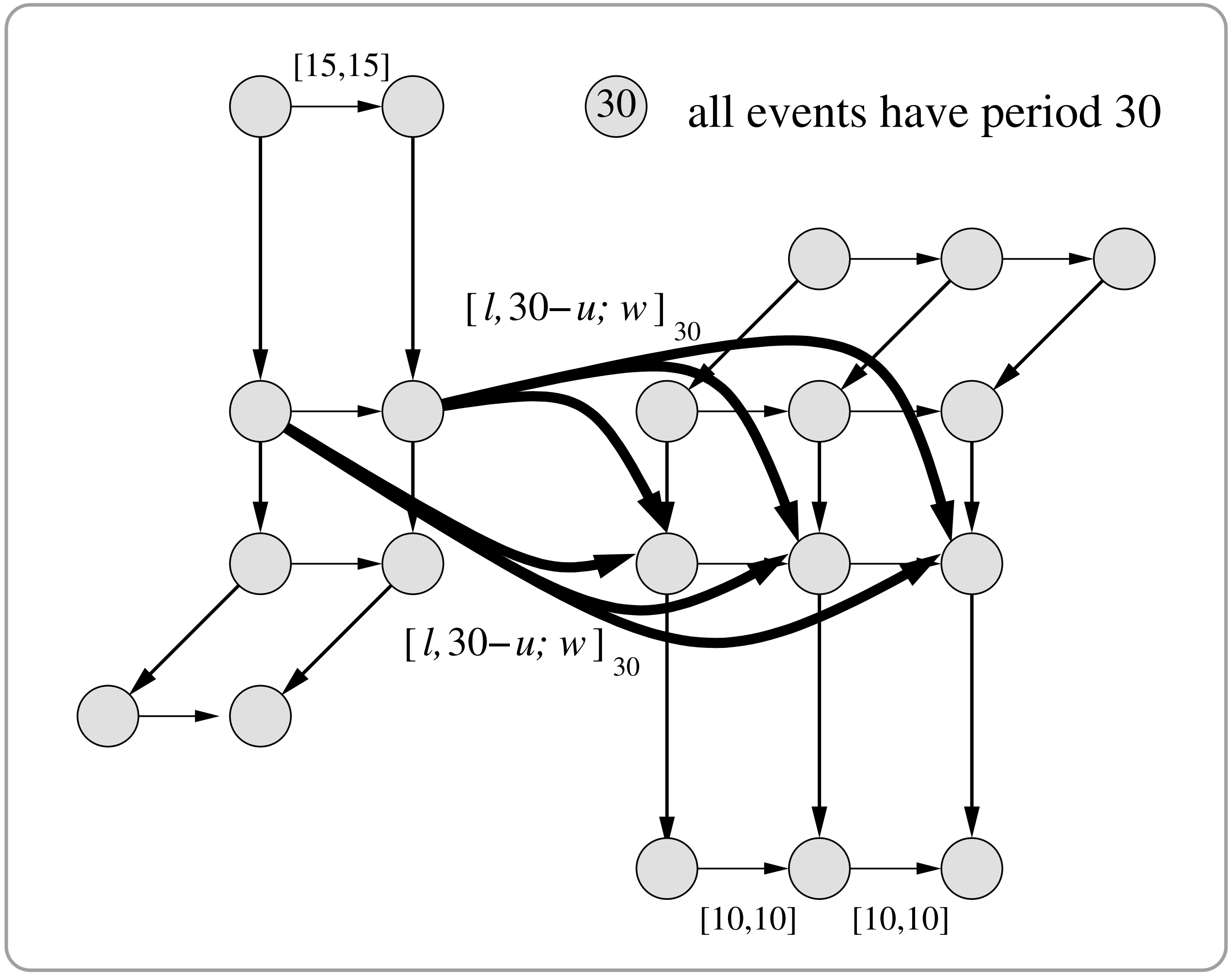

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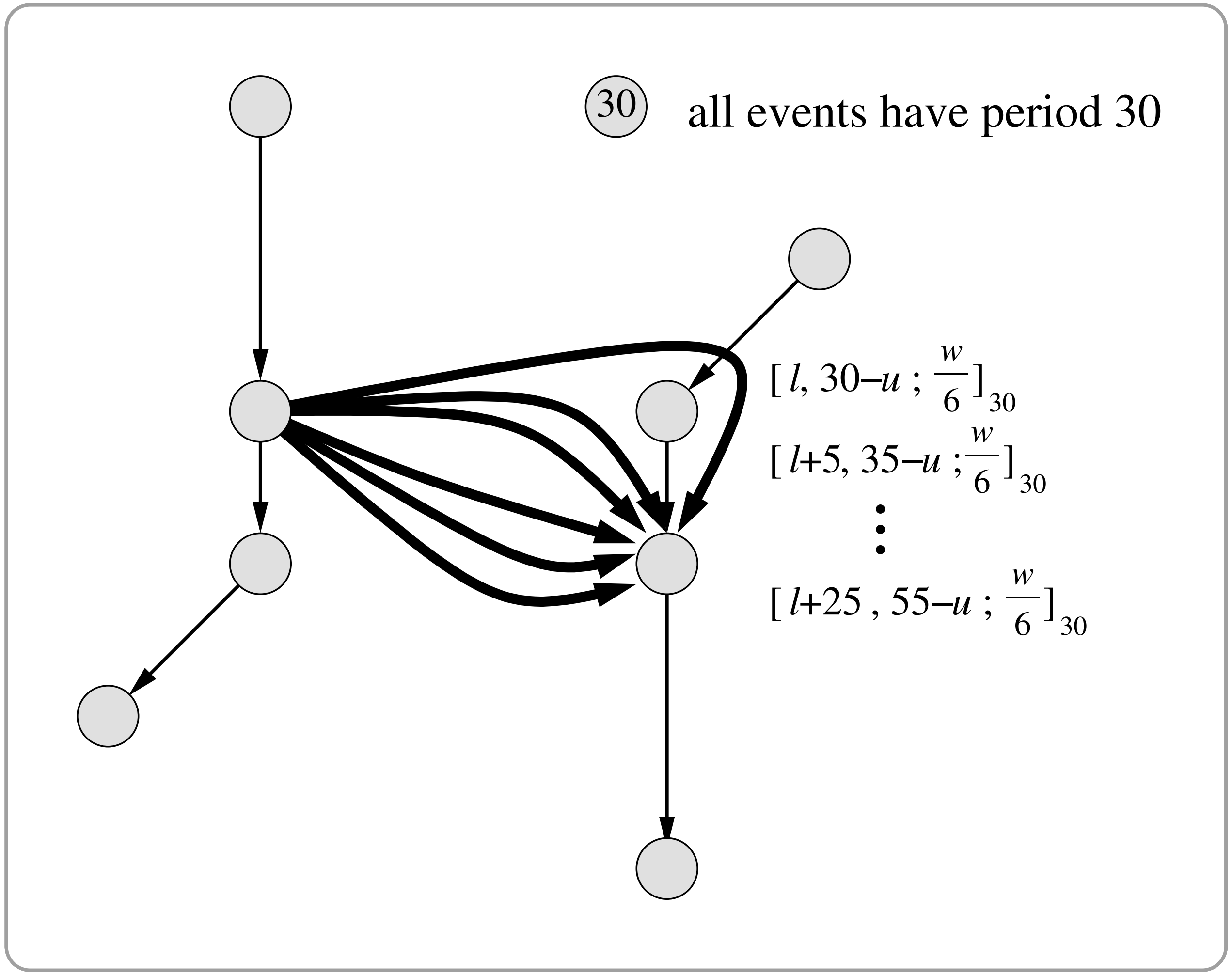

Duplicate Lines Within Single Period Model (Formalism)

$T \leftarrow$ lcm$\{ T_i \mid i \in V \}$ {artificial single period length}
for every arc $a = (i, j)$ **do**
  $g \leftarrow$ gcd$\{ T_i, T_j \}$ {arc's periods' gcd}
  $n \leftarrow \frac{T}{g}$ {number of new arcs}
  $w' \leftarrow \frac{w}{n}$ {weight of new arcs}
  for $k = 0$ to $n - 1$ **do**
    $\ell_k \leftarrow \ell_a + k \cdot g$ {lower bound of current new arc}
    $u_k \leftarrow u_a + (n-1)g + k \cdot g$ {upper bound of current new arc}
    INSERT_ARC(i, j, ℓ_k, u_k, w')
  end for
  DELETE_ARC(a)
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$\hookrightarrow$ unavoidable **base** weight $g \cdot w' \cdot \frac{n(n-1)}{2}$.

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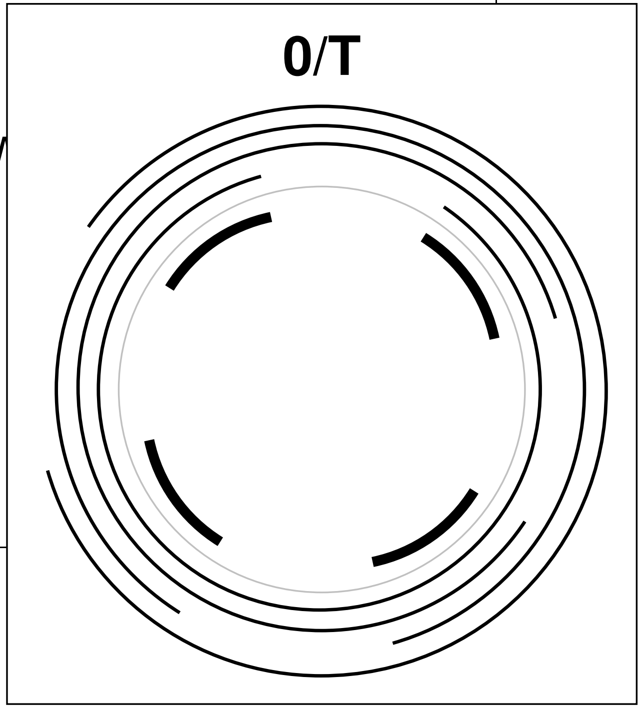

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- The Feasibility Problem may be stated as:

$$\Gamma x = \Gamma \begin{pmatrix} z_{a_1} T_{a_1} \\ \vdots \\ z_{a_m} T_{a_m} \end{pmatrix}$$

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Γ given in HNF again permits reduction to $m - n + 1$ integer variables.

Objective Function for the EPESP

The waiting times that occur are exactly
$\{x_0 + i \cdot T_a \mid i = 0, \ldots, \frac{T_j}{T_a} - 1\}$, with $x_0 := (\pi_j - \pi_i - \ell_a) \bmod T_a$.
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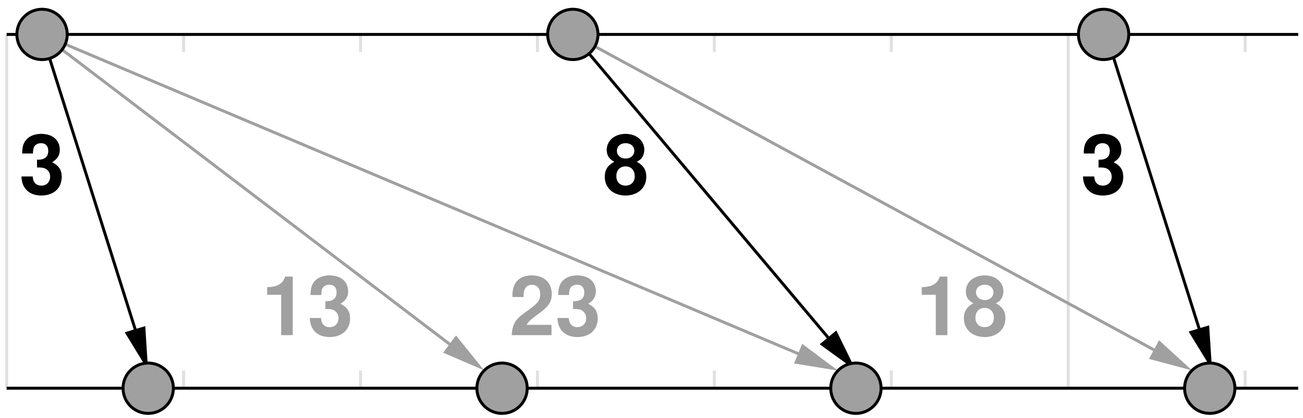

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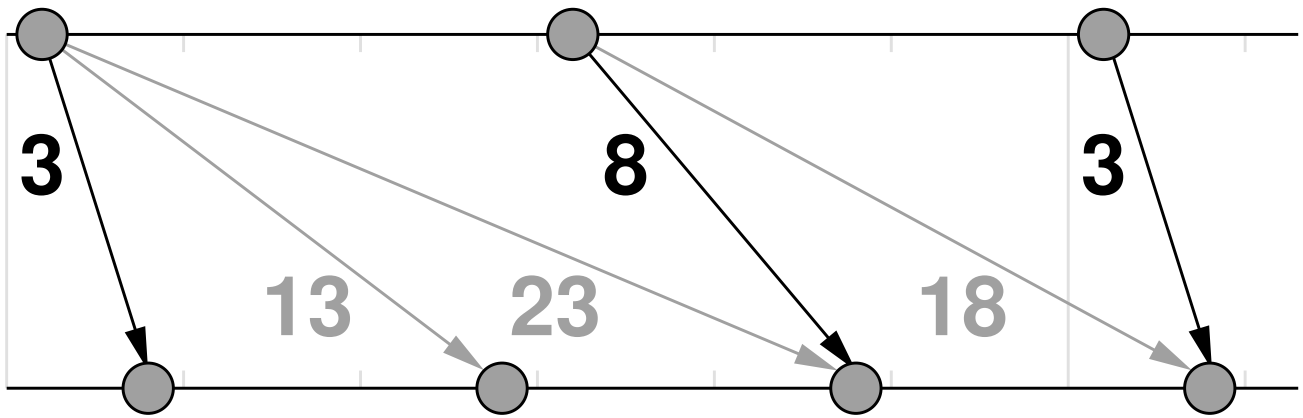
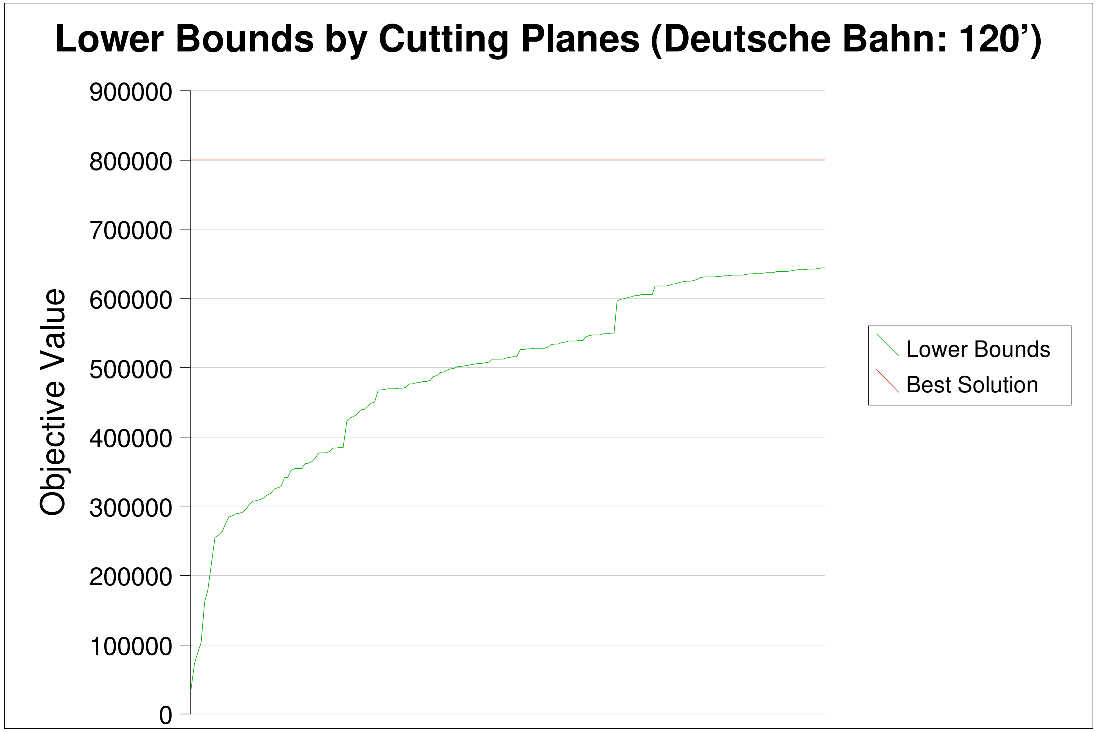
$\hookrightarrow$ we may restict ourselves on penalizing only x_0



Properties of the Two Models

Benefit of Cutting Planes in the Single Period Length Case



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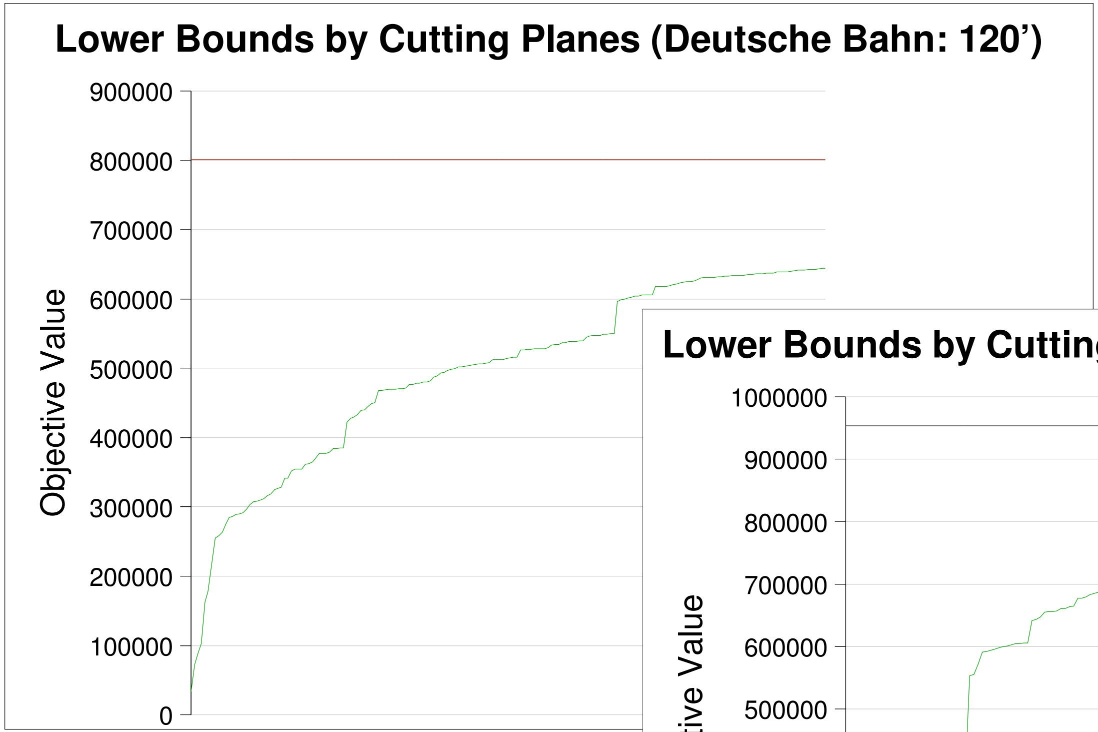
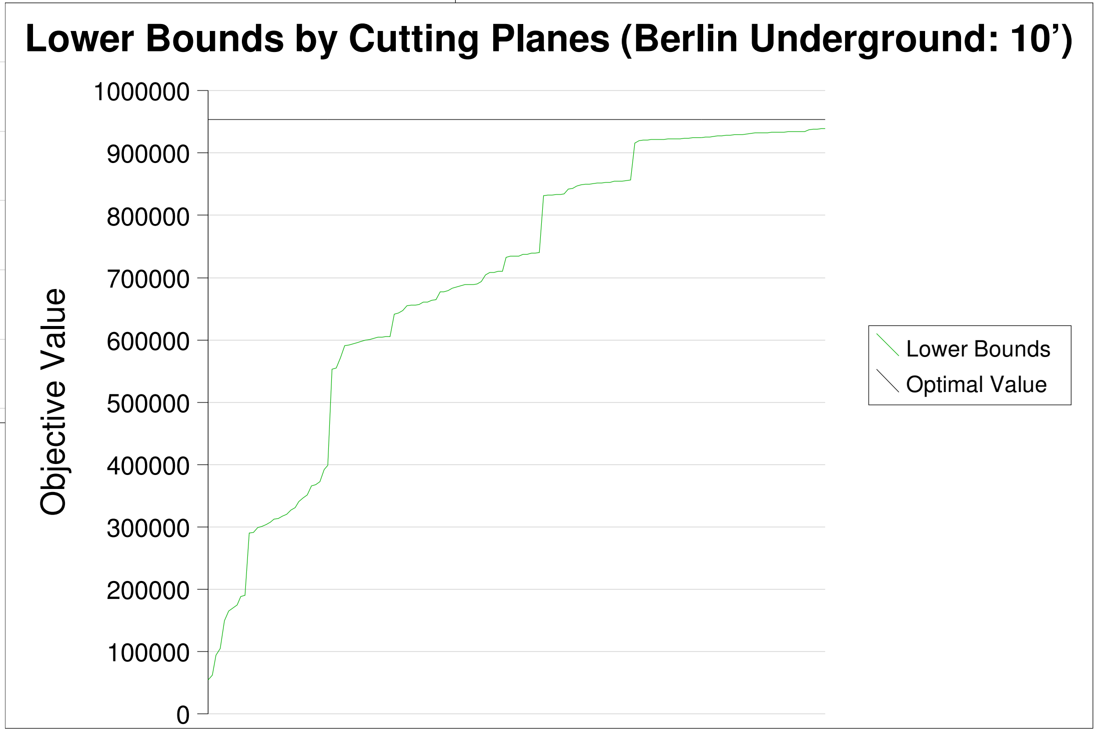
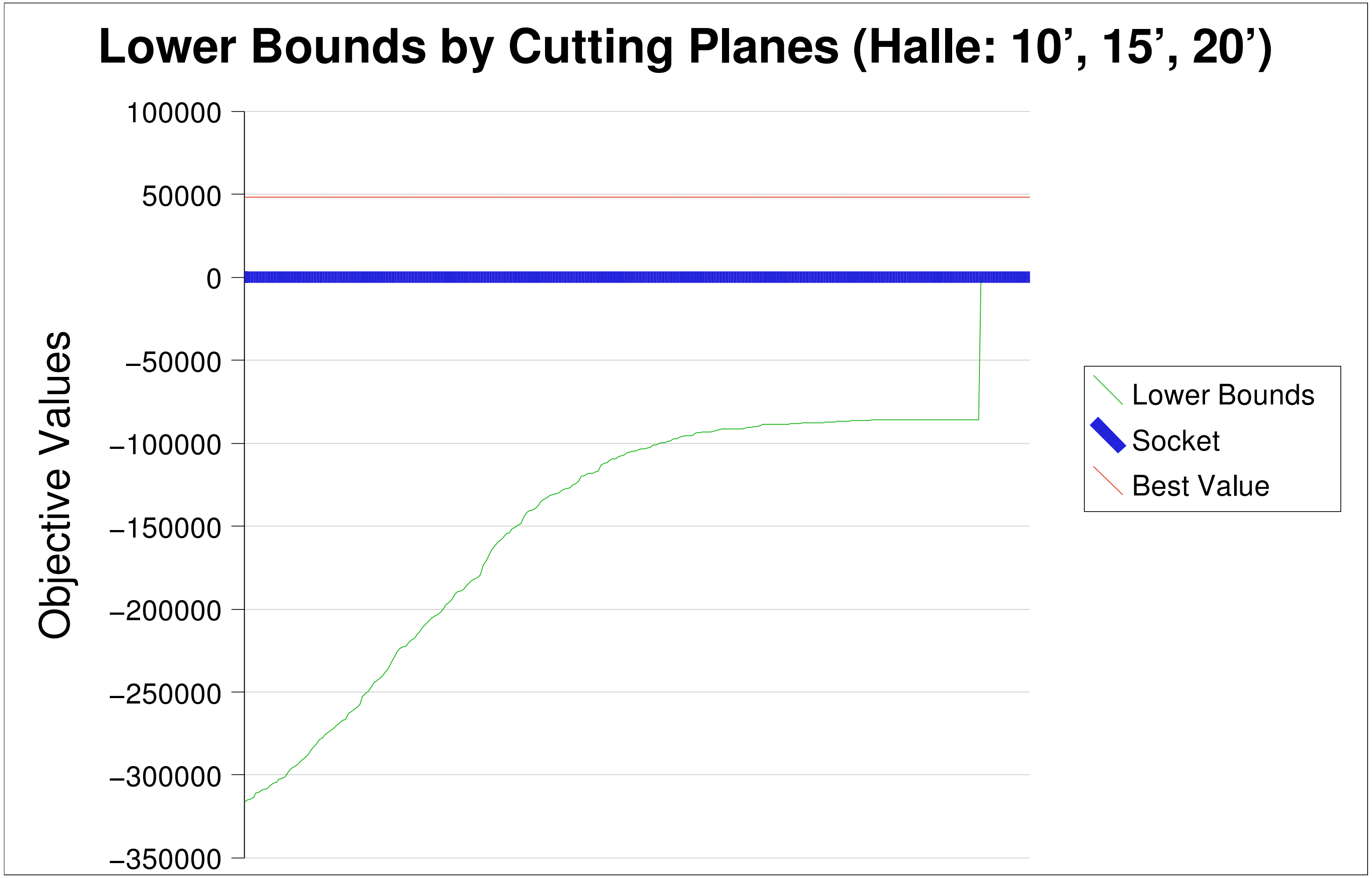




No Profit in Duplicated Lines Model by Standard Cuts



Generalize Valid Inequalities to EPESP

- For every feasible solution, we know
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- Normalizing the tensions' sum $\sum_{a\in C^+} x_a - \sum_{a\in C^-} x_a$ to **zero**, we obtain

$$\frac{\gamma^+\ell + \gamma^- u}{T_C} \leq \sum_{a\in C^+} z_a \frac{T_a}{T_C} - \sum_{a\in C^-} z_a \frac{T_a}{T_C} \leq \frac{\gamma^+ u + \gamma^- \ell}{T_C}, \qquad T_C := \text{gcd}\{\, T_a \,|\, a \in C \,\}.$$

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- Hope: These inequalities provide **intrinsic lower bounds**!

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Provides ability to model easily different arc frequencies Cutting planes commonly used are defined exclusively in the single period length model	Really takes advantage of different event frequencies

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Provides ability to model easily different arc frequencies Cutting planes commonly used are defined exclusively in the single period length model	Really takes advantage of different event frequencies
Standard cutting planes do not necessarily enrich the arc-expanded model	Improve EPESP by generalized inequalities!