Stability analysis of highly-synchronized periodic railway timetables

Rob M.P. Goverde

Delft University of Technology
Faculty of Civil Engineering and Geosciences
Transportation Planning and Traffic Engineering Section

goverde@ct.tudelft.nl
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Introduction

Timetable evaluation
- Corridors (capacity, headway, disruptions)
- Stations (capacity, throughput)
- Train network (stability, robustness, reliability, delay propagation)
  ⇒ Train interactions and circulations, network properties

PETER: Performance Evaluation of Timed Events in Railways

Benefits of analytical approach
- Explicit model results (instead of black-box)
- Clear problem structure (validation)
- Exact results based on (deterministic) timetable design times
- Fast computation
- Large-scale networks
Periodic timetable

- Arrival and departure times of train lines at stations
- Same pattern repeats each hour (cycle time)
- Steady-state of train operations

Input (DONS)

- Lines: running times, dwell times, layover times
- Connections: transfer times, (de-)coupling times
- Infrastructure: headway at conflict points

Timetable performance

- Network stability, robustness & throughput
- Critical paths / critical circuits
- Buffer time allocation & sensitivity to delays
- Delay propagation over time and space, settling time
Max-Plus Modelling

Constraint

\[ x_j(k) \geq x_i(k - \mu_{ij}) + t_{ij} \]

where

- \( x_i(k) \) = departure time train \( i \) in period \( k \)
- \( t_{ij} \) = transportation time from \( i \) to \( j \)
- \( \mu_{ij} \) = period delay (token) from \( i \) to \( j \)

Period delay \( \mu_{ij} = \text{ceil}\left(\frac{(t_{ij} + d_i - d_j)}{T}\right), d_i \in [0,60) \)

\[ A = 55 \quad D = 5 \quad D = 55 \quad A = 65 \equiv 5 \quad D = 55 \quad A = 125 \equiv 5 \]

Period delay = 0 \quad Period delay = 1 \quad Period delay = 2
Max-Plus Modelling

Running time constraint

\[ D_{L_1,S_2}(k) \geq D_{L_1,S_1}(k - \ell) + t_{L_1,S_1,S_2} + t_{L_1,S_2} \]

where

- \( D_{L_1,S_1}(k) \) = \( k \)-th departure time of train line \( L_1 \) at station \( S_1 \)
- \( t_{L_1,S_1,S_2} \) = Running time of a train of \( L_1 \) from station \( S_1 \) to \( S_2 \)
- \( t_{L_1,S_2} \) = Dwell time of a train of \( L_1 \) at station \( S_2 \)
- \( \ell \) = Period delay, \( \ell \in \{0,1,\ldots,p\} \)

Transfer constraint

\[ D_{L_1,S_2}(k) \geq D_{L_2,S_0}(k - \ell) + t_{L_2,S_0,S_2} + t_{L_2,L_1,S_2} \]

- \( t_{L_2,L_1,S_2} \) = Transfer time from \( L_2 \) to \( L_1 \) at \( S_2 \)
Max-Plus Modelling

Timetable constraint

\[ D_{L1,S2}(k) \geq d_{L1,S2} + (k - 1) \cdot T \]

- \( d_{L1,S2} \) = Scheduled departure time \( L_1 \) from \( S_2 \)
- \( T \) = Timetable cycle time or period length (usually \( T = 60 \) min)

Headway constraint \((d_{L1,S2} > d_{L2,S2})\)

\[ D_{L2,S2}(k) \geq D_{L1,S2}(k) + h_{L1,L2,S2} \]
\[ D_{L1,S2}(k) \geq D_{L2,S2}(k - 1) + h_{L2,L1,S2} \]

- \( h_{L1,L2,S2} \) = Minimum departure headway time from \( L_1 \) to \( L_2 \) after \( S_2 \)
Max-Plus Modelling

Precedence graph

- Constraints: \( x_i(k) \geq x_j(k - \mu_{ij}) + t_{ij} \)
- Node for each departure event \( x_i = D_{ll,ss} \)
- Arc \((i,j)\) for each constraint from \( x_i \) to \( x_j \)
- Arc weight: transportation time \( t_{ij} \) in constraint
- 2nd arc weight: period delay \( \mu_{ij} \) in constraint

Timed event graph or timed marked graph

- Transitions: nodes of precedence graph
- Places: arcs of precedence graph
- Holding time of places: transportation times of arcs
- Initial marking (tokens) of places: period delays of arcs
Max-Plus Modelling

The event times satisfy the \((\text{max},+)\) recursion

\[
x_i(k) = \max_{l,j}((A_l)_{ij} + x_j(k - \mu_{ji}), d_i(k))
\]

where

\[
(A_l)_{ij} = \begin{cases} t_{ji} & \text{if } l = \mu_{ji} \\ -\infty & \text{otherwise} \end{cases}
\]

The recursion is **linear** in the **max-plus algebra** where

\[
(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})
\]

\[
(A \otimes B)_{ij} = \bigoplus_{k=1}^{r} (a_{ik} \otimes b_{kj}) = \max(a_{ik} + b_{kj})
\]

\[
\mathbb{R}_{\text{max}}^{n \times n} = (\mathbb{R}^{n \times n} \cup \{-\infty\}, \oplus, \otimes) \text{ is an idempotent semiring (dioid)}
\]
Max-Plus Modelling

Higher \((p\text{-st})\) order \((\text{max},+)\) linear system

\[
x(k) = \bigoplus_{\ell=0}^{p} A_\ell \otimes x(k-\ell) \oplus d(k)
\]

with

\[
(A_\ell)_{ij} = \begin{cases} 
t_{ji} & \text{if } \ell = \mu_{ji} \\
-\infty & \text{otherwise}
\end{cases}
\]

\(x\) = departure time vector

\(d\) = scheduled departure time vector

\(t_{ji}\) = holding time of arc (place) from \(j\) to \(i\)

\(\mu_{ji}\) = period delay (token) of arc from \(j\) to \(i\)

First-order representation

- by state augmentation

\[
\tilde{x}(k) = A \otimes \tilde{x}(k-1) \oplus \tilde{d}(k)
\]
Max-Plus System Analysis

- **Stability**: the ability to return to schedule (the steady-state) after disruptions

- The **minimum cycle time** equals the **maximum cycle mean**
  \[ \lambda = \max_{c \in C} \frac{\sum_c t_{ij}}{\sum_c \mu_{ij}} \]
  and this equals the (max-plus) **eigenvalue** \( \lambda \) of (irreducible) \( A \)
  \[ A \otimes v = \lambda \otimes v \]

- Circuits with maximum cycle mean are **critical circuits**
- Stability test \( \lambda < T \)
- The maximum mean cycle problem / maximum profit-to-time ratio cycle problem is solvable in \( O(nm) \) time
- Algorithms: Karp, power algorithm, Howard’s policy iteration, LP
Max-plus System Analysis

- **Stability margin**: maximum delay of all holding times that can be settled within one timetable period (cycle time)

- A formal **polynomial matrix** of a finite matrix series \( \{A\ell\}_{\ell=0}^p \) is

  \[
  A(\gamma) = \bigoplus_{\ell} (A\ell \otimes \gamma^\ell)
  \]

  which defines a matrix for given \( \gamma \in \mathbb{R} \)

- Stability margin is \( \Delta = -\mu \) where \( B \otimes \nu = \mu \otimes \nu \) with

  \[
  B = A(T^{-1}) = \bigoplus_{\ell=0}^p A\ell \otimes T^{-\ell} := \max_{\ell=0,\ldots,p} (A\ell - \ell \cdot T)
  \]

- Eigenvalue problem for matrix \( A(T^{-1}) \)
Max-Plus System Analysis

- **Recovery matrix**: the matrix $R$ where the $ij$-th entry is the maximum delay of $x_j(k)$ such that $x_i(m)$ is not delayed for all $m \geq k$

- Recovery matrix is given by

\[
(R)_{ij} = d_i - d_j - (A(T^{-1}))_{ij}^+
\]

where $A^+$ is the **longest path matrix** $A^+ = \bigoplus_{k=1}^{n} A^k$

- Note $A^k$ is the matrix of the largest-weight path of length $k$
eq G_{15}

  e.g. $(A^2)_{ij} = \max_{k=1,...,n} (a_{ik} + a_{kj})$

- Algorithms of all-pair shortest paths: Floyd-Warshall (dense networks), Johnson (sparse networks)
Max-Plus System Analysis

Interpretation of recovery matrix entries

Delay impact (vector) departing train
- Minimum delay that reaches subsequent trains
- Columns of $R$

Delay sensitivity (vector) waiting train
- Minimum delay of preceding trains that reach waiting train
- Rows of $R$

Feedback delay time (vector)
- Minimum delay that returns to current station
- Diagonal of $R$
Max-Plus System Analysis

Delay propagation
- Given (initial) delays at reference time
- Given timetable
- Deterministic dynamic system:

\[
x(k) = \bigoplus_{\ell=0}^{p} A_{\ell} \otimes x(k-\ell) \oplus d(k), \quad k = 1, \ldots
\]

\[
d(k) = T \otimes d(k-1), \quad k = 1, \ldots
\]

\[
d(0), x(1), x(0), \ldots, x(1-p) \text{ given}
\]

\[
z(k) = x(k) - d(k), \quad k = 1, \ldots
\]

Output
- Delay vectors \( z(1), \ldots, z(K) \), where \( K \) is settling period
- Aggregated output: cumulative (secondary) delay, average (secondary) delay, settling time, # reached trains, # reached stations
Case Study (PETER)

Dutch Intercity Network 2000-2001
- 26 train lines (both directions)
- 74 stations
- 328 departures / line segments
- 51 connections
- 597 (dual) headway constraints

Model
- 328 transitions (nodes)
- 981 places (arcs)
- 441 tokens
  - 114 trains
  - 301 ordering tokens
IC network: Critical Circuit
### IC Network: Stability Analysis

<table>
<thead>
<tr>
<th>Network</th>
<th>Cycle time</th>
<th>Throughput</th>
<th>Margin</th>
<th>Route</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Complete</td>
<td>57:24</td>
<td>0.96</td>
<td>0:30</td>
<td>Shl-Lw</td>
</tr>
<tr>
<td>2. Excl. transfer</td>
<td>57:00</td>
<td>0.95</td>
<td>0:30</td>
<td>Sgn-Hdr</td>
</tr>
<tr>
<td>3. Excl. infra</td>
<td>56:48</td>
<td>0.95</td>
<td>1:00</td>
<td>Shl-Es</td>
</tr>
<tr>
<td>4. Excl. infra/transfer</td>
<td>56:10</td>
<td>0.94</td>
<td>1:03</td>
<td>Gvc-Hrl</td>
</tr>
</tbody>
</table>
IC Network: Delay Impact 1900 VI-Gvc
IC Network: Delay Sensitivity 500 Amf-ZI
IC Network: Delay Propagation

Scenario: during one hour all trains in Utrecht depart 10 min late
Conclusions

• PETER is a software tool based on max-plus algebra to help railway planners

• PETER computes network performance indicators for evaluation and comparison of timetable structures

• Bottlenecks (critical circuits) with the tightest schedule are identified

• Robustness to delays through buffer times are clearly detailed by recovery times

• Delay forecasting by propagation of initial delays over time and network

• PETER gives results of large-scale networks in real-time