Stability analysis of highly-synchronized periodic railway timetables

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Introduction

Timetable evaluation

- Corridors (capacity, headway, disruptions)
- Stations (capacity, throughput)
- Train network (stability, robustness, reliability, delay propagation)

  $\Rightarrow$ Train interactions and circulations, network properties

PETER: Performance Evaluation of Timed Events in Railways

Benefits of analytical approach

- Explicit model results (instead of black-box)
- Clear problem structure (validation)
- Exact results based on (deterministic) timetable design times
- Fast computation
- Large-scale networks

PETER

Periodic timetable

- Arrival and departure times of train lines at stations
- Same pattern repeats each hour (cycle time)
- Steady-state of train operations

Input (DONS)

- Lines: running times, dwell times, layover times
- Connections: transfer times, (de-)coupling times
- Infrastructure: headway at conflict points

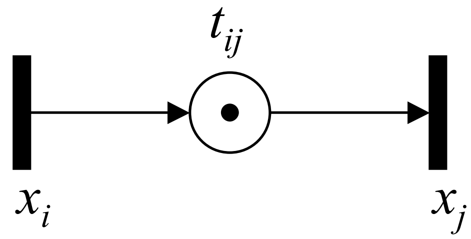
Timetable performance

- Network stability, robustness & throughput
- Critical paths / critical circuits
- Buffer time allocation & sensitivity to delays
- Delay propagation over time and space, settling time

Max-Plus Modelling

Constraint

$$x_j(k) \geq x_i(k-\mu_{ij}) + t_{ij}$$



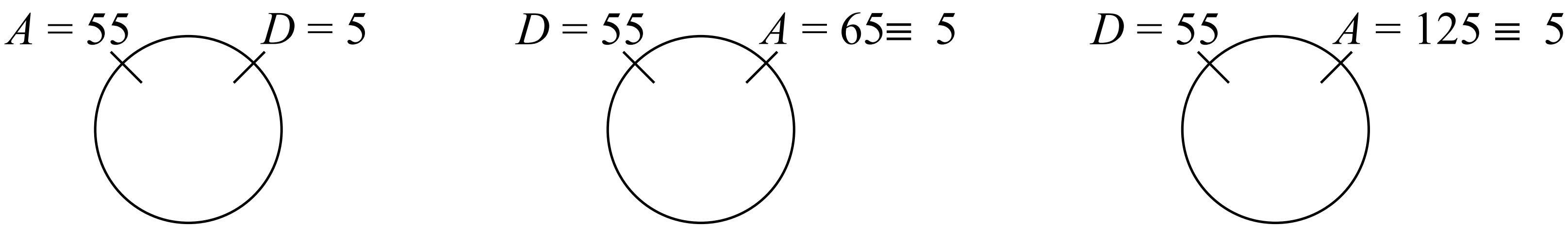
where

$x_i(k)$ = departure time train i in period k

t_{ij} = transportation time from i to j

μ_{ij} = period delay (token) from i to j

Period delay $\mu_{ij} = ceil[(t_{ij} + d_i - d_j)/T], d_i \in [0,60)$



Period delay = 0 Period delay = 1 Period delay = 2

Max-Plus Modelling

Running time constraint

$$D_{L1,S2}(k) \geq D_{L1,S1}(k-\ell) + t_{L1,S1,S2} + t_{L1,S2}$$

where

$D_{L1,S1}(k)$	= k-th departure time of train line L_1 at station S_1
$t_{L1,S1,S2}$	= Running time of a train of L_1 from station S_1 to S_2
$t_{L1,S2}$	= Dwell time of a train of L_1 at station S_2
ℓ	= Period delay, $\ell \in \{0,1,\ldots,p\}$

Transfer constraint

$$D_{L1,S2}(k) \geq D_{L2,S0}(k-\ell) + t_{L2,S0,S2} + t_{L2,L1,S2}$$

$t_{L2,L1,S2}$	= Transfer time from L_2 to L_1 at S_2

Max-Plus Modelling

Timetable constraint

$$D_{L1,S2}(k) \geq d_{L1,S2} + (k-1) \cdot T$$

$d_{L1,S2}$ = Scheduled departure time L_1 from S_2

T = Timetable cycle time or period length (usually T = 60 min)

Headway constraint $(d_{L1,S2} > d_{L2,S2})$

$$D_{L2,S2}(k) \geq D_{L1,S2}(k) + h_{L1,L2,S2}$$

$$D_{L1,S2}(k) \geq D_{L2,S2}(k-1) + h_{L2,L1,S2}$$

$h_{L1,L2,S2}$ = Minimum departure headway time from L_1 to L_2 after S_2

Max-Plus Modelling

Precedence graph

- Constraints: $x_i(k) \geq x_j(k - \mu_{ij}) + t_{ij}$
- Node for each departure event $x_i = D_{Ll,Ss}$
- Arc (i,j) for each constraint from x_i to x_j
- Arc weight: transportation time t_{ij} in constraint
- 2nd arc weight: period delay μ_{ij} in constraint

Timed event graph or timed marked graph

- Transitions: nodes of precedence graph
- Places: arcs of precedence graph
- Holding time of places: transportation times of arcs
- Initial marking (tokens) of places: period delays of arcs

Max-Plus Modelling

The event times satisfy the (max,+) recursion

$$x_i(k) = \max_{l,j}((A_l)_{ij} + x_j(k - \mu_{ji}), d_i(k))$$

where

$$(A_l)_{ij} = \begin{cases} t_{ji} & \text{if } l = \mu_{ji} \\ -\infty & \text{otherwise} \end{cases}$$

The recursion is **linear** in the **max-plus algebra** where

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij})$$

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{r} (a_{ik} \otimes b_{kj}) = \max_{k=1,\ldots,r}(a_{ik} + b_{kj})$$

$\mathbf{R}_{\max}^{n\times n} = (\mathbf{R}^{n\times n} \cup \{-\infty\}, \oplus, \otimes)$ is an idempotent semiring (dioid)

Max-Plus Modelling

Higher (p-st) order (max,+) linear system

$$x(k) = \bigoplus_{\ell=0}^{p} A_l \otimes x(k-\ell) \oplus d(k)$$

with

$$(A_\ell)_{ij} = \begin{cases} t_{ji} & \text{if } \ell = \mu_{ji} \\ -\infty & \text{otherwise} \end{cases}$$

x = departure time vector

d = scheduled departure time vector

t_{ji} = holding time of arc (place) from j to i

μ_{ji} = period delay (token) of arc from j to i

First-order representation

- by state augmentation

$$\widetilde{x}(k) = A \otimes \widetilde{x}(k-1) \oplus \widetilde{d}(k)$$

Max-Plus System Analysis

- **Stability**: the ability to return to schedule (the steady-state) after disruptions

- The **minimum cycle time** equals the **maximum cycle mean**

$$\lambda = \max_{c \in C} \sum_c t_{ij} \Big/ \sum_c \mu_{ij}$$

and this equals the (max-plus) **eigenvalue** λ of (irreducible) A

$$A \otimes v = \lambda \otimes v$$

- Circuits with maximum cycle mean are **critical circuits**
- Stability test $\lambda < T$
- The maximum mean cycle problem / maximum profit-to-time ratio cycle problem is solvable in *O(nm)* time
- Algorithms: Karp, power algorithm, Howard's policy iteration, LP

Max-plus System Analysis

- **Stability margin**: maximum delay of all holding times that can be settled within one timetable period (cycle time)

- A formal **polynomial matrix** of a finite matrix series $\{A_\ell\}_{\ell=0}^{p}$ is

$$\mathrm{A}(\gamma) = \bigoplus_{\ell} (A_\ell \otimes \gamma^\ell)$$

which defines a matrix for given $\gamma \in \Re$

- Stability margin is $\Delta = -\mu$ where $B \otimes v = \mu \otimes v$ with

$$B = \mathrm{A}(T^{-1}) = \bigoplus_{\ell=0}^{p} A_\ell \otimes T^{-\ell} := \max_{\ell=0,\ldots,p} (A_\ell - \ell \cdot T)$$

- Eigenvalue problem for matrix $\mathrm{A}(T^{-1})$

Max-Plus System Analysis

- **Recovery matrix**: the matrix R where the ij-th entry is the maximum delay of $x_j(k)$ such that $x_i(m)$ is not delayed for all $m \geq k$

- Recovery matrix is given by

$$(R)_{ij} = d_i - d_j - (\mathrm{A}(T^{-1}))^+_{ij}$$

where A^+ is the **longest path matrix** $A^+ = \bigoplus_{k=1}^{n} A^k$

- Note A^k is the matrix of the largest-weight path of length k
e.g. $(A^2)_{ij} = \max_{k=1,\ldots,n}(a_{ik} + a_{kj})$

- Algorithms of all-pair shortest paths: Floyd-Warshall (dense networks), Johnson (sparse networks)

Max-Plus System Analysis

Interpretation of recovery matrix entries

Delay impact (vector) departing train

- Minimum delay that reaches subsequent trains
- Columns of R

Delay sensitivity (vector) waiting train

- Minimum delay of preceding trains that reach waiting train
- Rows of R

Feedback delay time (vector)

- Minimum delay that returns to current station
- Diagonal of R

Max-Plus System Analysis

Delay propagation

- Given (initial) delays at reference time
- Given timetable
- Deterministic dynamic system:

$$\begin{cases} x(k) = \bigoplus_{\ell=0}^{p} A_l \otimes x(k-\ell) \oplus d(k), \quad k = 1, \ldots \\ d(k) = T \otimes d(k-1), \quad k = 1, \ldots \\ d(0), x(1), x(0), \ldots, x(1-p) \text{ given} \\ z(k) = x(k) - d(k), \quad k = 1, \ldots \end{cases}$$

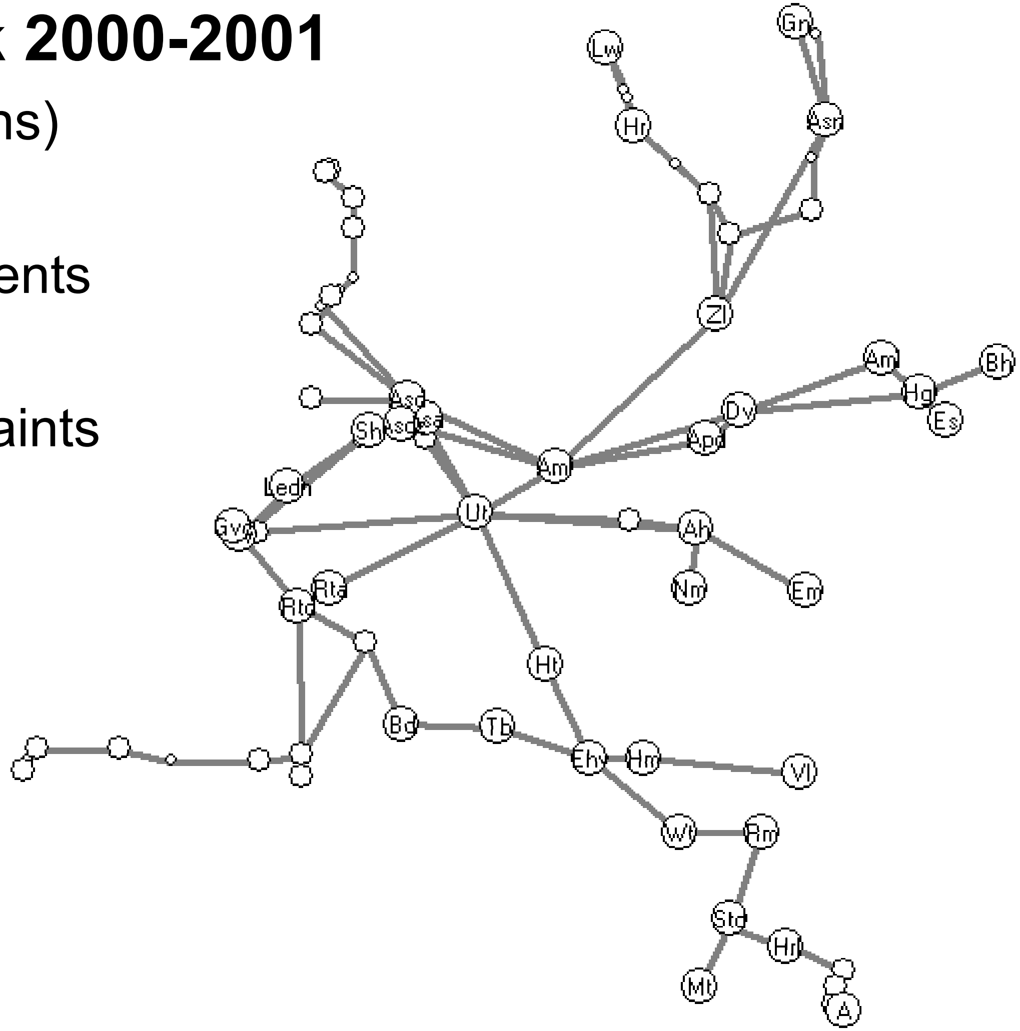
Output

- Delay vectors $z(1), \ldots, z(K)$, where K is settling period
- Aggregated output: cumulative (secondary) delay, average (secondary) delay, settling time, # reached trains, # reached stations

Case Study (PETER)

Dutch Intercity Network 2000-2001

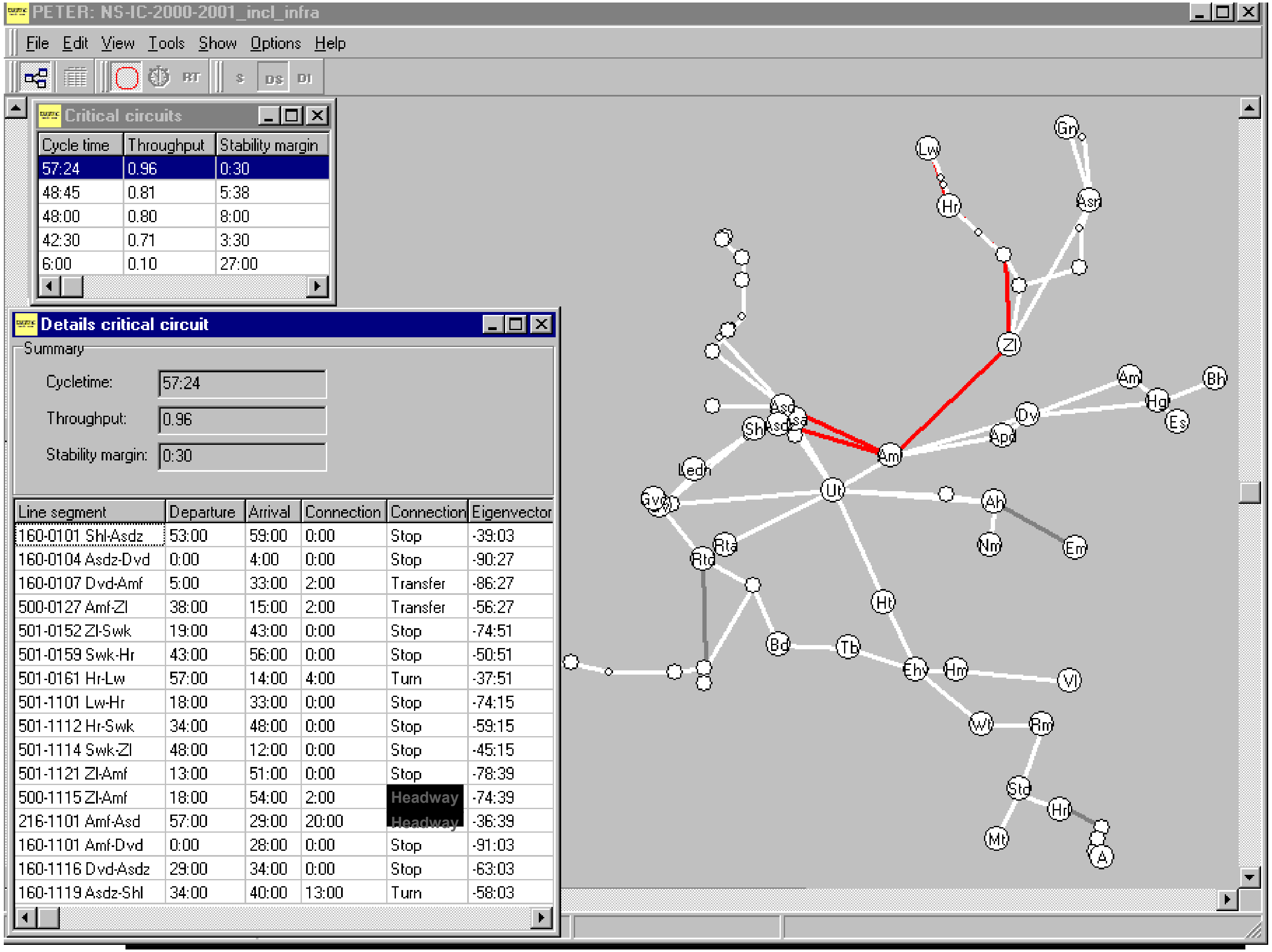
- 26 train lines (both directions)
- 74 stations
- 328 departures / line segments
- 51 connections
- 597 (dual) headway constraints



Model

- 328 transitions (nodes)
- 981 places (arcs)
- 441 tokens
  - 114 trains
  - 301 ordering tokens

IC network: Critical Circuit

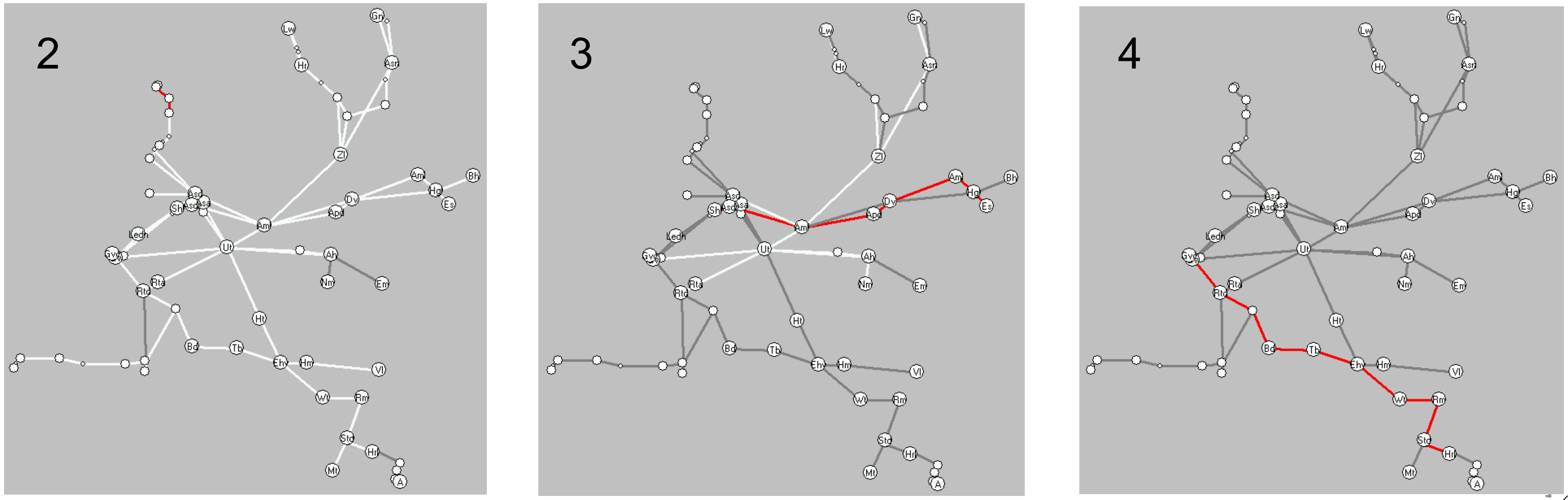


Cycle time	Throughput	Stability margin
57:24	0.96	0:30
48:45	0.81	5:38
48:00	0.80	8:00
42:30	0.71	3:30
6:00	0.10	27:00

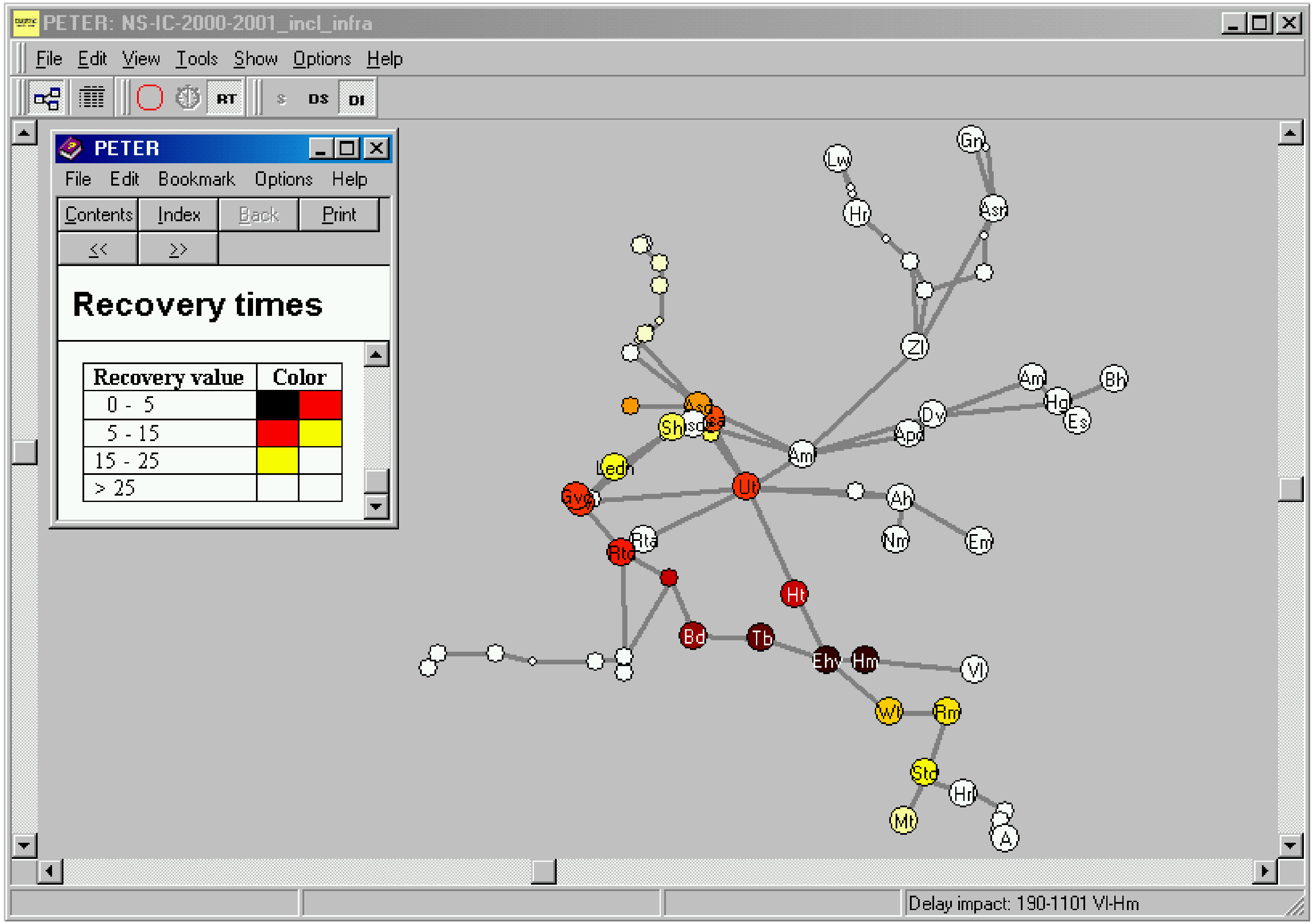
Line segment	Departure	Arrival	Connection	Connection	Eigenvector
160-0101 Shl-Asdz	53:00	59:00	0:00	Stop	-39:03
160-0104 Asdz-Dvd	0:00	4:00	0:00	Stop	-90:27
160-0107 Dvd-Amf	5:00	33:00	2:00	Transfer	-86:27
500-0127 Amf-Zl	38:00	15:00	2:00	Transfer	-56:27
501-0152 Zl-Swk	19:00	43:00	0:00	Stop	-74:51
501-0159 Swk-Hr	43:00	56:00	0:00	Stop	-50:51
501-0161 Hr-Lw	57:00	14:00	4:00	Turn	-37:51
501-1101 Lw-Hr	18:00	33:00	0:00	Stop	-74:15
501-1112 Hr-Swk	34:00	48:00	0:00	Stop	-59:15
501-1114 Swk-Zl	48:00	12:00	0:00	Stop	-45:15
501-1121 Zl-Amf	13:00	51:00	0:00	Stop	-78:39
500-1115 Zl-Amf	18:00	54:00	2:00	Headway	-74:39
216-1101 Amf-Asd	57:00	29:00	20:00	Headway	-36:39
160-1101 Amf-Dvd	0:00	28:00	0:00	Stop	-91:03
160-1116 Dvd-Asdz	29:00	34:00	0:00	Stop	-63:03
160-1119 Asdz-Shl	34:00	40:00	13:00	Turn	-58:03

IC Network: Stability Analysis

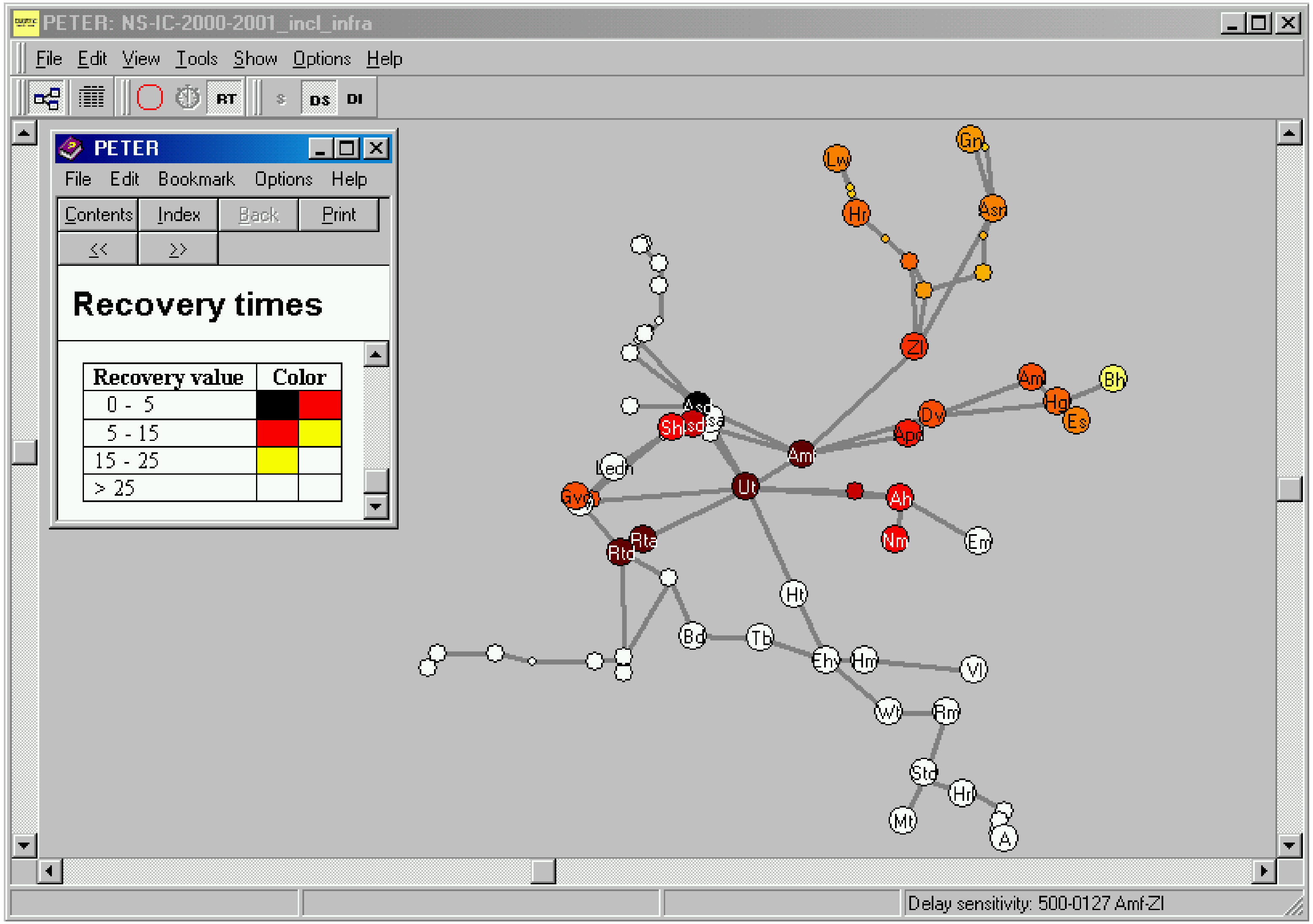
Network	Cycle time	Throughput	Margin	Route
1. Complete	57:24	0.96	0:30	Shl-Lw
2. Excl. transfer	57:00	0.95	0:30	Sgn-Hdr
3. Excl. infra	56:48	0.95	1:00	Shl-Es
4. Excl. infra/transfer	56:10	0.94	1:03	Gvc-Hrl



IC Network: Delay Impact 1900 Vl-Gvc

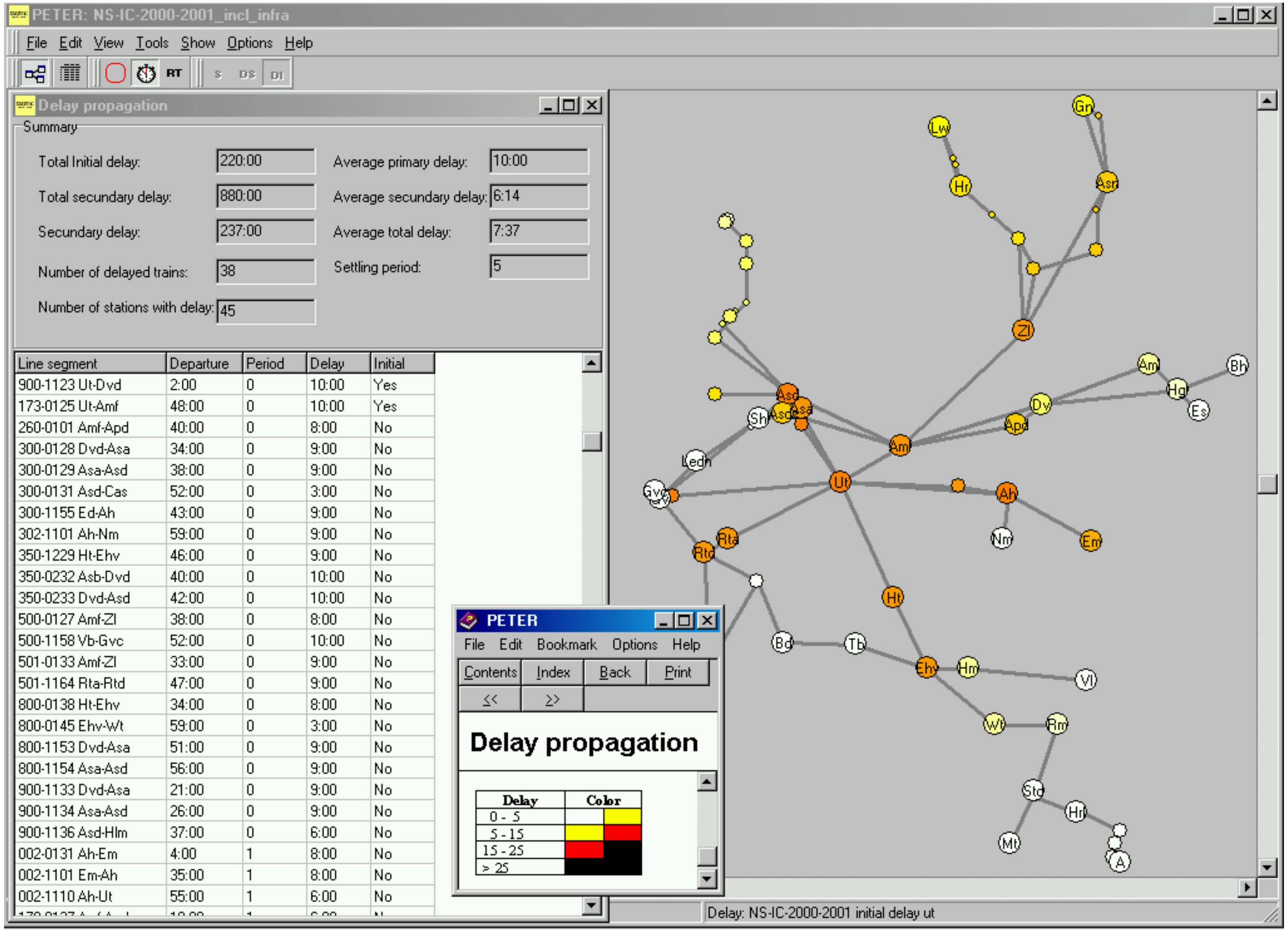


IC Network: Delay Sensitivity 500 Amf-Zl



IC Network: Delay Propagation

Scenario: during one hour all trains in Utrecht depart 10 min late



Conclusions

- PETER is a software tool based on max-plus algebra to help railway planners
- PETER computes **network perfomance indicators** for evaluation and comparison of timetable structures
- **Bottlenecks** (critical circuits) with the tightest schedule are identified
- **Robustness to delays** through buffer times are clearly detailed by recovery times
- **Delay forecasting** by propagation of initial delays over time and network
- PETER gives results of **large-scale networks** in **real-time**