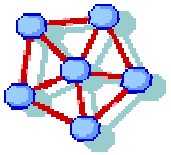

Periodic Time Table Generation: Theory and Models

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Overview

- The basic event-activity model
- Examples of time table problems
- Complexity of constructing (good) time tables
- Graph theoretic background results
- Solution methods

An informal problem description

- Traffic lines are fixed
- Line = sequence of stations
- Want start and departure times (time table)
  - at the stations
  - and possibly other important points
  - subject to temporal side constraints of various kinds
  - time table must be periodic with fixed period T
- Minimize an objective
  - e.g. total change over time

The aperiodic event-activity model

- Directed graph $D = (N, A)$
- Nodes model events, time π_i for node i (node potential)
- arcs use time, they model a time difference
  arc $a = (i, j)$ has difference $x_a = \pi_j - \pi_i$ (tension of arc a)

Feasible potential problem (aperiodic case)

Given constraints on arcs: $x_a \in [l_a, u_a]$, find a
node potential $\pi = (\pi_1,...,\pi_n)$ with $x_a \in [l_a, u_a]$ for every arc a

Introducing periodicity

Serafini & Ukovich '89

- Graph model as before
- Have periodic condition on events (potentials)

periodic potential $\pi = (\pi_1,\ldots,\pi_n)$ for period T fulfils
$\pi_i \in \{\pi_i^0, \pi_i^0 \pm T, \pi_i^0 \pm 2T, \pi_i^0 \pm 3T, \ldots\}$ with $\pi_i^0 \in \{0,1,\ldots,T-1\}$

$\pi = (\pi_1,\ldots, \pi_n)$ is a feasible periodic potential if

$$l_{ij} \le \pi_j - \pi_i \le u_{ij} \text{ with } \pi_j = \pi_j^0 + k_j \cdot T\,,\ \pi_i = \pi_i^0 + k_i \cdot T\,,$$

and k_i, k_j integer (*)

for every arc $a = (i, j)$

$$(*) \Leftrightarrow l_{ij} \le \pi_j - \pi_i + p_a \cdot T \le u_{ij} \text{ for some integer } p_a$$

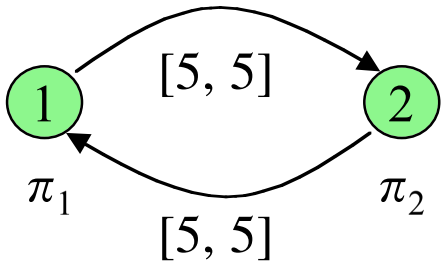
p_a: period offset of arc a

Periodic and aperiodic potentials

every feasible aperiodic potential is a feasible periodic potential

$$l_{ij} \le \pi_j - \pi_i \le u_{ij} \quad \Rightarrow \quad l_{ij} \le \pi_j^0 - \pi_i^0 + p_a \cdot T \le u_{ij} \text{ with integer } p_a$$

the converse is not true



$T = 10$

$$\Rightarrow \pi = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$ is feasible periodic

there is no aperiodic feasible potential

Periodic tensions

periodic tension x for period T
is obtained as potential difference from a periodic potential
i.e., $x_a := \pi_j - \pi_i$

periodic tension x is feasible if it comes from a feasible periodic potential π, i.e., for every arc $a = (i, j)$,
$l_{ij} \le \pi_j - \pi_i + p_a \cdot T \le u_{ij}$ for some integer p_a

$$\Updownarrow$$

$l_{ij} \le x_a + p_a \cdot T \le u_{ij}$ Notation: $x_a \in [l_{ij}, u_{ij}]_T$

feasible periodic tension =
feasible aperiodic tension + integer period offsets p_a for all arcs a

$$l_{ij} \le x_a + p_a \cdot T \le u_{ij} \Leftrightarrow l_{ij} - p_a \cdot T \le x_a \le u_{ij} - p_a \cdot T$$

The feasible periodic potential problem

… or periodic event scheduling problem PESP

Given

- a digraph D with
- periodic side constraints $x_a \in [l_a, u_a]_T$

find a periodic node potential $\pi = (\pi_1,...,\pi_n)$

with $x_a \in [l_a, u_a]_T$ for every arc a

- Is the basic model for periodic time tables
- Can be expressed equivalently via potentials or tensions

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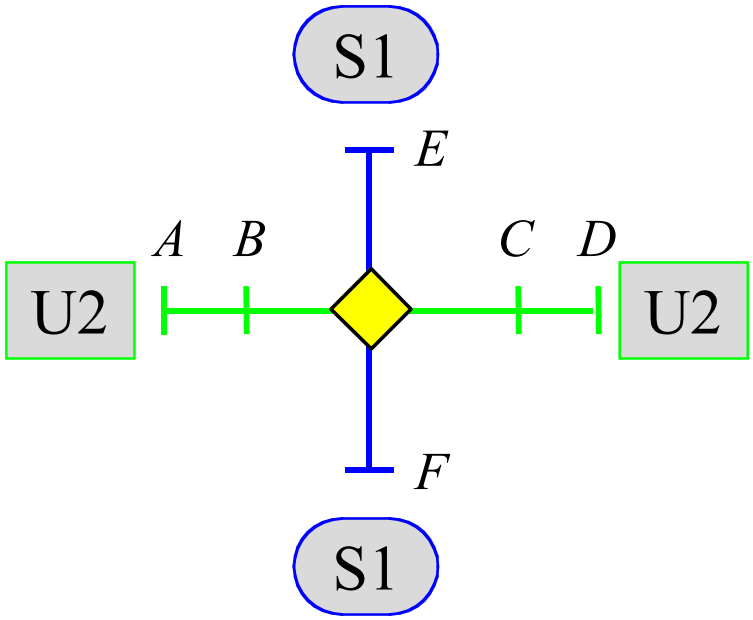
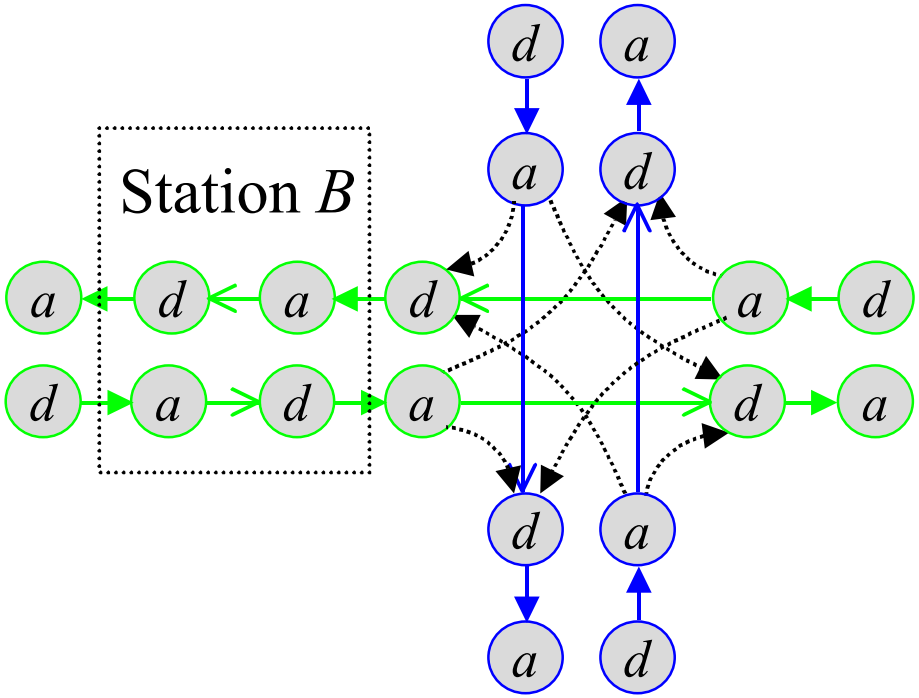
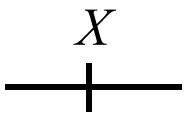
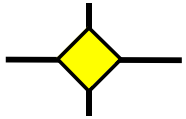
Traffic lines and graphs







line with station *X*



transfer station

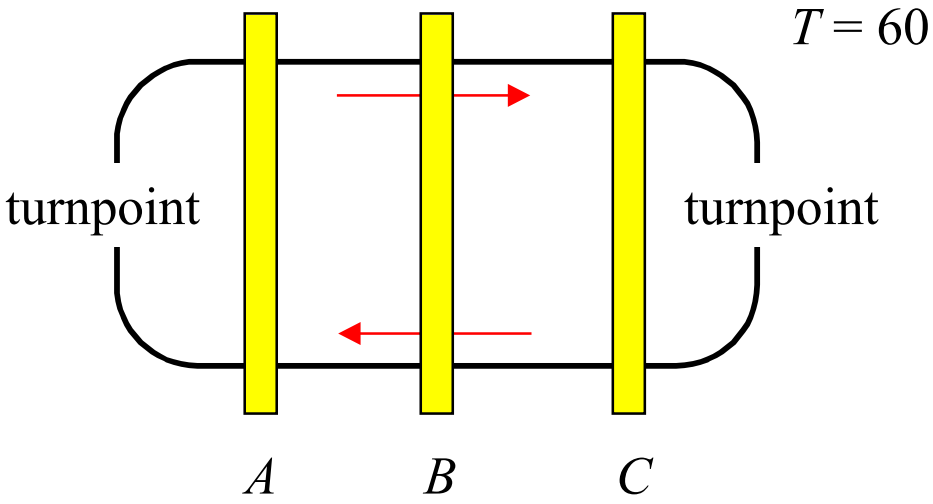
travel arc

stop arc

transfer arc

a arrival *d* departure

Example: Temporal conditions on a single line



Stopping conditions in A

$\pi_{\text{dep in } A} - \pi_{\text{arr in } A} \in [2\ ,\ 4\ ]_T$

xx:59 arr in A

xx:02 dep in A

Travel condition

$\pi_{\text{dep in } B} - \pi_{\text{arr in } A} \in [11\ ,\ 11\ ]_T$

xx:48 dep in B

xx:59 arr in A

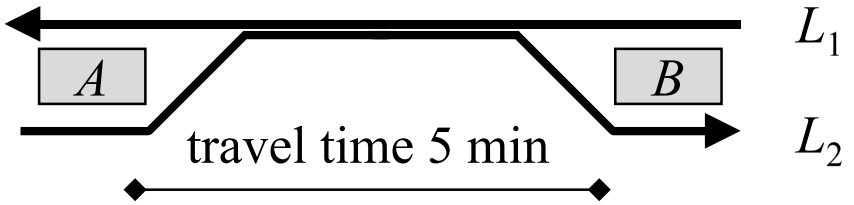
Turning condition

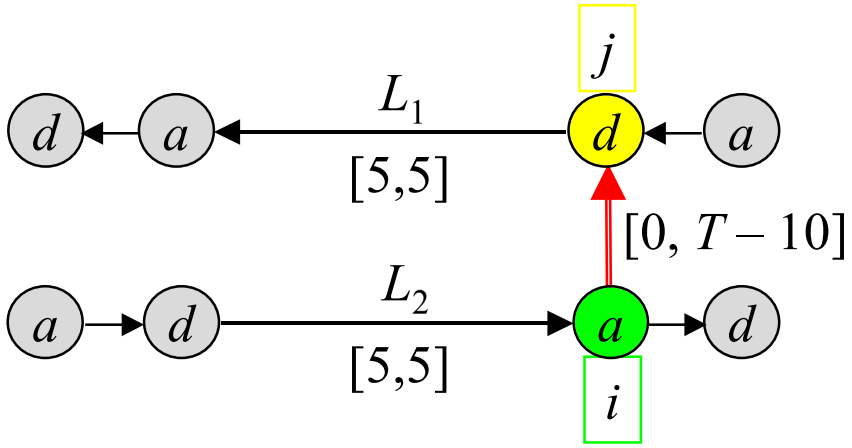
$\pi_{\text{dep in } A} - \pi_{\text{arr in } A} \in [5\ ,\ 10\ ]_T$

xx:02 dep in A

xx:10 arr in A

Example: Temporal conditions on a single track





[0, $T-10$] models one-track conditions:

L_1 may enter track in B only after L_2 has left it

$\Rightarrow \pi_j = \pi_{1 \text{ dep in } B} \geq \pi_{2 \text{ arr in } B} = \pi_i \Rightarrow \pi_i \leq \pi_j$

Next train L_2 may depart in A only after L_1 has left the track

$\Rightarrow \pi_i + T = \pi_{2 \text{ ab } A} + 5 + T \geq \pi_{1 \text{ an } A} + 5 = \pi_j + 10$

$\Rightarrow 0 \leq \pi_j - \pi_i \leq T - 10$

Further modeling possibilities

- rolling stock
- safety conditions (one track and others)
- time between successive trains
- coupling trains (NL)
- disjunctive intervals
- fixed events
- coordinated servicing of central transfer points

all expressible as feasible periodic potential problem

system of inequalities in variables π_i
inequalities correspond to arcs of a digraph,
have periodic condition

Optimization criteria

Linear objective function $\sum_a w_a \cdot x_a$

- Total changeover times
- Turn times at line endpoints (# of trains)
- weak constraints

Side constraints for each arc a

$$l_a \leq x_a + p_a \cdot T \leq u_a$$

$$0 \leq x_a \leq T - 1$$

p_a integer

Mixed integer LP in variables x_a and p_a

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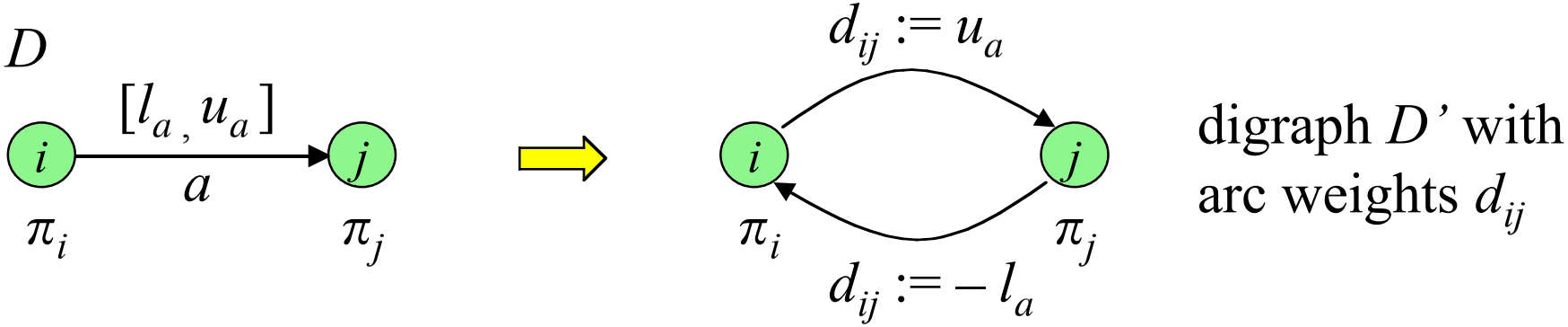
The aperiodic feasible potential problem is easy

can be solved by shortest path methods



digraph D' with arc weights d_{ij}

π is a feasible potential in $D \Leftrightarrow \pi_j - \pi_i \le d_{ij}$ for all arcs in D' (*)

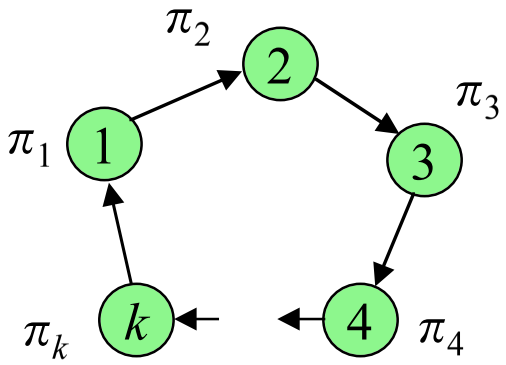
(*) is valid $\Leftrightarrow$ D' has no directed cycle with negative length

If (*) is valid, then the shortest path lengths from node 1 to all other nodes define a feasible potential of D

(*) is valid $\Rightarrow$ *D'* has no directed cycle with negative length

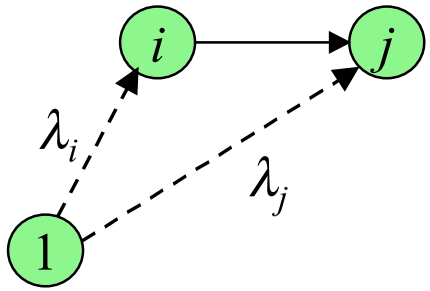
Suppose *D'* has such a cycle $\Rightarrow$

$$\Sigma = 0 \left\{ \begin{array}{l} \pi_2 - \pi_1 \le d_{12} \\ \pi_3 - \pi_2 \le d_{23} \\ \cdots \\ \pi_1 - \pi_k \le d_{k1} \end{array} \right\} \Sigma < 0$$



D' has no directed cycle with negative length $\Rightarrow$ shortest path lengths from node 1 well defined (assume reachibility from 1)

shortest path lengths λ_i fulfil (*)



$$\lambda_j \le \lambda_i + d_{ij} \Rightarrow \lambda_j - \lambda_i \le d_{ij}$$

The feasible periodic potential problem is hard

… even for fixed $T \geq 3$ (reduction from K-colorability [Nachtigall '96])

K-colorability
Given: undirected graph $G = (V, E)$
Question: can G be colored with K colors?

digraph D resulting from an arbitrary orientation of G
arc $a \rightarrow [\, l_a, u_a \,] = [\, 1, T-1 \,]$ with $T = K$

π is feasible periodic potential
$\Leftrightarrow \pi_i \neq \pi_j \bmod T$ for each arc (i, j)
$\Leftrightarrow \pi_i \bmod T \; (i = 1, \ldots, n)$ defines a coloring with at most K colors

The core of the NP-hardness

feasible periodic tension =
feasible aperiodic tension + integer period offsets p_a for all arcs a

$$l_{ij} \le x_a + p_a \cdot T \le u_{ij} \Leftrightarrow \underbrace{l_{ij} - p_a \cdot T \le x_a \le u_{ij} - p_a \cdot T}$$

easy for given period offsets p_a,
just shortest path methods

$\Rightarrow$ Core of hardness lies in finding the period offsets

Period offsets determine the feasible domain

Hardness of optimization

Mixed Integer Linear Program (MIP)

$$\min \sum_a w_a \cdot x_a$$

such that

$$l_a \le x_a + p_a \cdot T \le u_a$$
$$0 \le x_a \le T - 1$$
$$p_a \text{ integer}$$

for each arc a

MIP is easy if period offsets p_a are fixed

Motivates again to look at the domain of the period offsets p_a

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  - The cycle space and the tension space
  - Narrowing the feasibility domain via cycle inequalities
- Solution methods

Different interpretations of the PESP

- potential space
  - node variables π_i
- tension space
  - arc variables $x_a = \pi_j - \pi_i$
- cycle space
  - cycle variables y_C for cycles C from a cycle basis

Look at graph-theoretic relations between them.

Period offsets and spanning trees

feasible periodic tension = feasible aperiodic tension
+ period offsets for all non-tree edges of a spanning tree of D

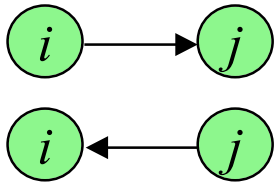
x feasible periodic tension
with period offsets p_a
T spanning tree of D

$\Rightarrow$

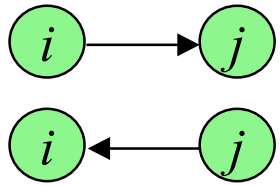
$\exists$ feasible periodic tension y
with period offsets q_a and
$q_a = 0$ for all tree arcs a

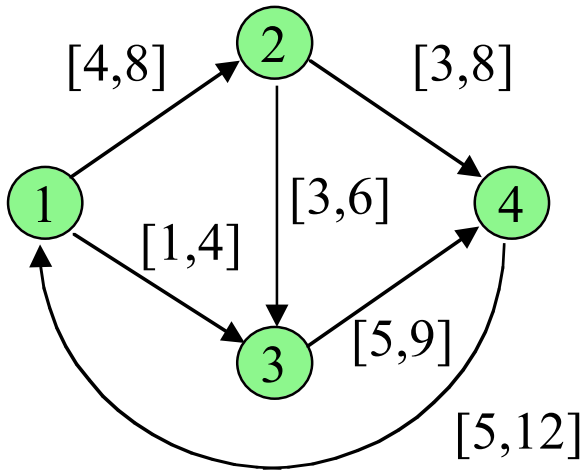
Idea: Let π define x. Choose root w of T.
Construct σ defining y "along the tree". Set $\sigma_w := \pi_w$,

$$\sigma_j := \begin{cases} \sigma_i + l_{ij} + [(\pi_j - \pi_i - l_{ij}) \bmod T] & \text{on a forward arc} \\ \sigma_i - l_{ji} - [(\pi_i - \pi_j - l_{ji}) \bmod T] & \text{on a backward arc} \end{cases}$$

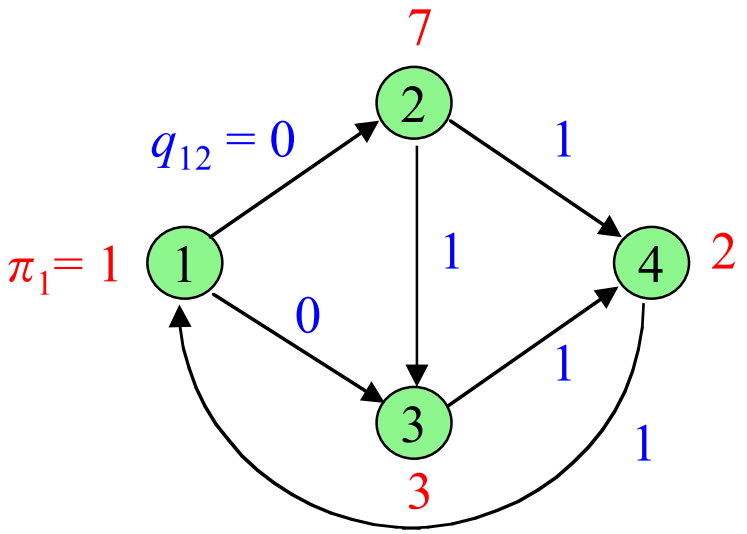


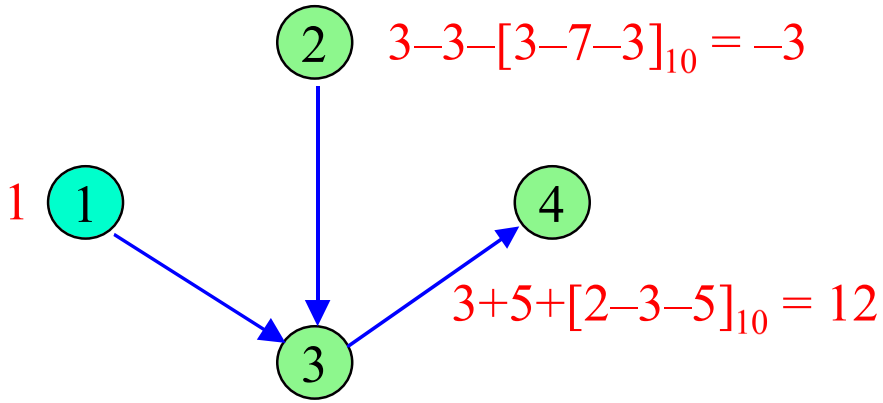
$$\sigma_j := \begin{cases} \sigma_i + l_{ij} + [(\pi_j - \pi_i - l_{ij}) \bmod T\,] & \text{on a forward arc} \\ \sigma_i - l_{ji} - [(\pi_i - \pi_j - l_{ji}) \bmod T\,] & \text{on a backward arc} \end{cases}$$

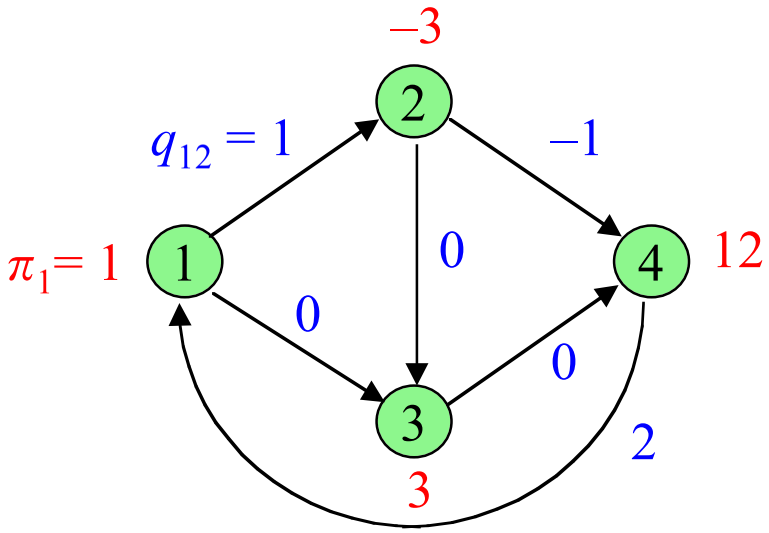




$T = 10$

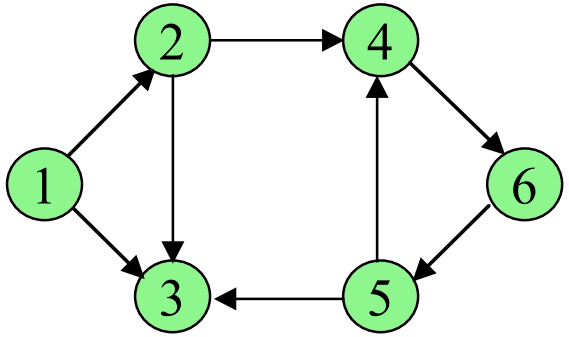






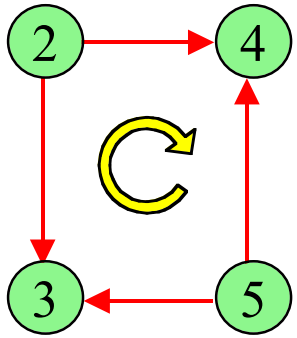
The cycle space (circulation space) of a graph

consists of all incidence vectors χ^C of undirected cycles C and linear combinations thereof



$$\chi^C = \begin{pmatrix} \chi_{12} \\ \chi_{13} \\ \chi_{23} \\ \chi_{24} \\ \chi_{46} \\ \chi_{53} \\ \chi_{54} \\ \chi_{65} \end{pmatrix}$$

$$\chi_{ij} = \begin{cases} 1 & \text{if } (i,j) \text{ forward arc} \\ -1 & \text{if } (i,j) \text{ backard arc} \\ 0 & \text{otherwise} \end{cases}$$



$\Rightarrow$

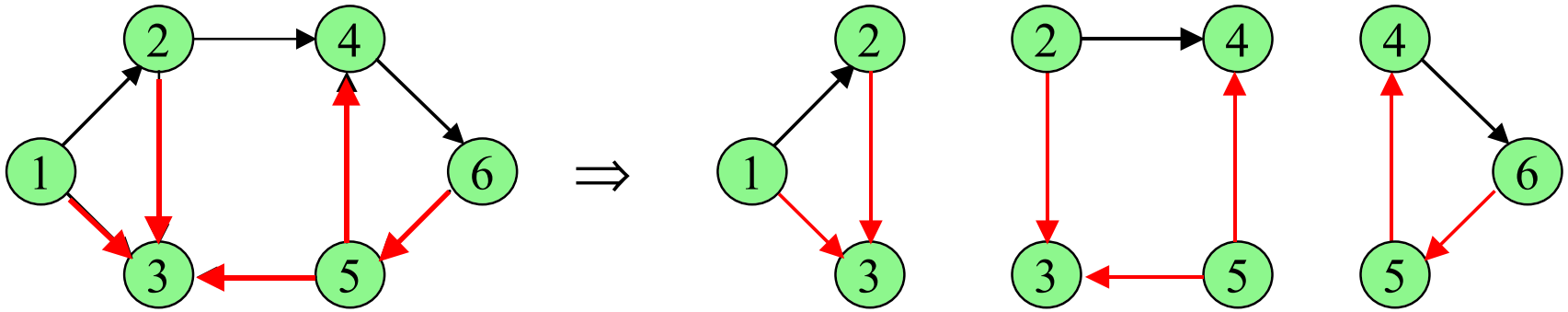
$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

Incidence vectors of cycles generate a vector space, the cycle space

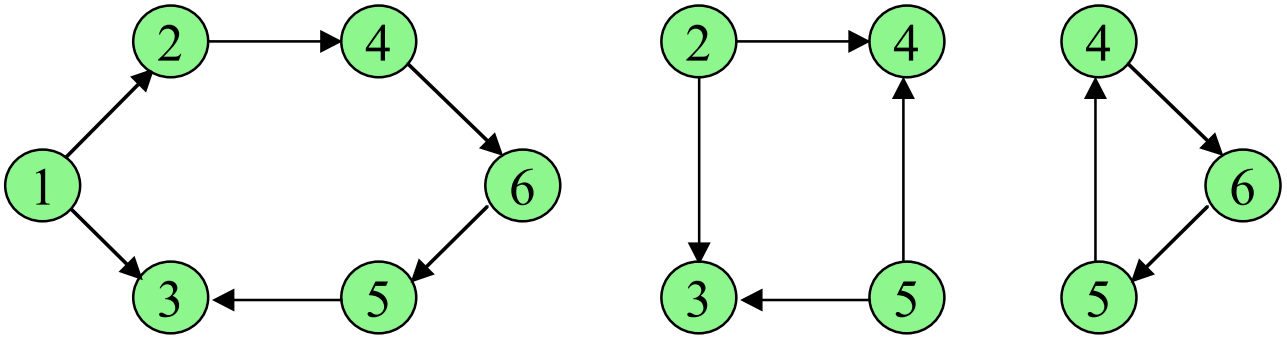
Bases of the cycle space

For every spanning T tree of D, the non-tree arcs induce a cycle basis of the cycle space of D (fundamental cycles w.r.t. T)

$\Rightarrow$ dimension $m - n + 1$

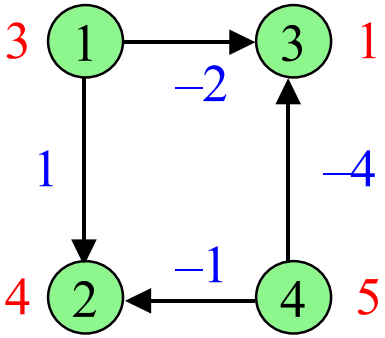


some bases do not come from a tree, e.g.



The tension space of a graph

Aperiodic tensions form a vector space.

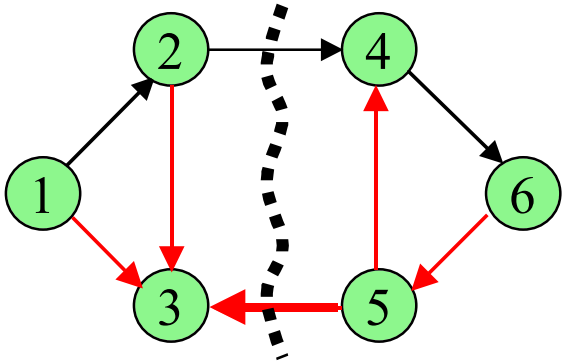


$x \cdot \chi^C = 0$ for every tension x and cycle C

$(-2)\cdot 1 + (-4)\cdot(-1) + (-1)\cdot 1 + 1\cdot(-1) = 0$

The tension space is orthogonal to the cycle space

The incidence vectors of cocycles of a spanning tree form a basis



A characterization of periodic tensions

Not quite orthogonal to the cycle space, but

The following are equivalent

(1) x is a periodic tension

(2) $x \cdot \chi^C$ is a multiple of T for every cycle C

(3) $x \cdot \chi^C$ is a multiple of T for every cycle C from a tree basis of D

Call $q_C := x \cdot \chi^C$ the cycle multiple of cycle C

Proof of the characterization

(1) $\Rightarrow$ (2): Let π be the periodic potential associated with x.

$$x \cdot \chi^C = \sum_{a \in C+}(\pi_j - \pi_i + p_a T) + \sum_{a \in C-}(\pi_i - \pi_j + p_a T) = x' \cdot \chi^C + zT$$

$= 0$ since x' is a tension

(2) $\Rightarrow$ (3): obvious

(3) $\Rightarrow$ (1): Let $C_1, \ldots, C_r$ be the basis fulfilling (3).

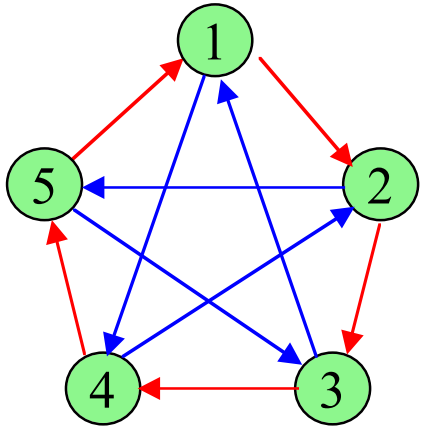
Every cycle C is a linear combination of $C_1, \ldots, C_r$ with integer coefficients. *

$\Rightarrow$ this special linear combination preserves multiples of T.

* can be proved directly

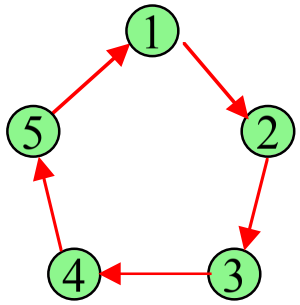
Deeper reason: Γ = matrix with incidence vectors of $C_1, \ldots, C_r$ as rows is totally unimodular for a tree basis

Characterization is not true for arbitrary basis



Outer 3-cycles and inner 5-cycle form a basis
Matrix Γ of incidence vectors is

	(1,2)	(2,3)	(3,4)	(4,5)	(5,1)	(3,1)	(4,2)	(5,3)	(1,4)	(2,5)
C_1	1	1				1				
C_2		1	1				1			
C_3			1	1				1		
C_4				1	1				1	
C_5	1				1					1
C_6						1	1	1	1	1



$$= \frac{1}{2}C_1 + \frac{1}{2}C_2 + \frac{1}{2}C_3 + \frac{1}{2}C_4 + \frac{1}{2}C_5 - \frac{1}{2}C_6$$

Example by B. Gerards

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- The basic event-activity model
- Examples of time table problems
- Complexity of constructing (good) time tables
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  - The cycle space and the tension space
  - Narrowing the search via cycle inequalities
- Solution methods

Period offsets and cycle multiples

Remember: feasible periodic tension = feasible tension
+ integer period offsets p_a for all arcs a

Now: feasible periodic tension = feasible tension
+ integer cycle multiples q_C for cycles from a cycle basis

Necessary and sufficient for tree basis
Necessary for arbitrary basis

Calculating period offsets from cycle multiples

- Can efficiently determine the q_C for a tree basis from an arbitrary basis (linear algebra)
- Can check efficiently if they are integer
- Can determine the period offsets p_a from the tree basis q_C

Let $C_1, \ldots, C_r$ be a tree basis and Γ the associated matrix

Then $\begin{pmatrix} q_{C_1} \\ \vdots \\ q_{C_r} \end{pmatrix} = \Gamma \cdot p$

totally unimodular $\Rightarrow$ linear equations with p_a as variables have integer solutions

Two natural search spaces

feasible periodic tension = feasible tension
+ integer period offsets p_a for all arcs a

Search through space of "possible" period vectors in $\mathbb{R}^m$

feasible periodic tension = feasible tension
+ integer cycle multiples q_C for cycles from a cycle basis

Search through space of "possible" cycle multiples in $\mathbb{R}^{m-n+1}$

Which one is better? Smaller?

Narrowing the search via cycle inequalities

There exists a feasible periodic tension with cycle multiples q_C

$\Leftrightarrow \quad q_C \le \frac{1}{T}\left(\sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a\right)$ for every elementary cycle C

Odijk '94

Traversing C in the opposite direction gives

$$-q_C \le \frac{1}{T}\left(\sum_{a\in C^-} u_a - \sum_{a\in C^+} l_a\right) \Leftrightarrow \frac{1}{T}\left(\sum_{a\in C^+} l_a - \sum_{a\in C^-} u_a\right) \le q_C$$

Rounding gives lower and upper bounds for q_C

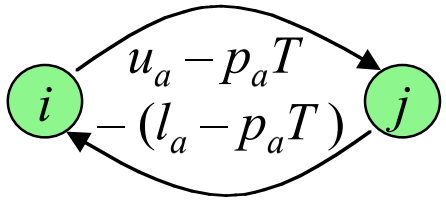
$$\left\lceil \frac{1}{T}\left(\sum_{a\in C^+} l_a - \sum_{a\in C^-} u_a\right)\right\rceil \le q_C \le \left\lfloor \frac{1}{T}\left(\sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a\right)\right\rfloor$$

Proving the validity of cycle inequalities

$\exists$ feasible periodic tension π with cycle multiples q_C

$\Leftrightarrow \exists$ feasible aperiodic potential for constraints $[l_a - p_a T\,,\, u_a - p_a T\,]$

with p a solution of $q = \Gamma p$

$\Leftrightarrow$ digraph D' with arc length 

has no negative directed cycle

$\Leftrightarrow$ every elementary cycle C in D fulfils

$$\sum_{a\in C^+}(u_a - p_a T) + \sum_{a\in C^-} -(l_a - p_a T) \geq 0$$

$$\Leftrightarrow \sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a - p\cdot\chi^C \cdot T \geq 0 \quad \text{for every elementary cycle } C$$

$$\Leftrightarrow p\cdot\chi^C \leq \frac{1}{T}\left(\sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a\right) \quad \text{for every elementary cycle } C$$

Bounds for a tree basis

Consider tree with zero period offsets on tree arcs
and a cycle C corresponding to a non-tree arc a

$$\Rightarrow q_C = p \cdot \chi^C = p_a$$

$$\underbrace{\left\lceil \frac{1}{T}\left(\sum_{a\in C^+} l_a - \sum_{a\in C^-} u_a\right)\right\rceil}_{\underline{p}_a} \le p_a \le \underbrace{\left\lfloor \frac{1}{T}\left(\sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a\right)\right\rfloor}_{\overline{p}_a}$$

Number of feasible periodic currents is bounded by
$\prod_{a \text{ non-tree arc}} \left(\overline{p}_a - \underline{p}_a\right)$ for every tree

One step further: bounds for an arbitrary basis

Let $B = \{ C_1, \ldots, C_r \}$ be a cycle basis.
Cycle multiples q_C of B determine period offsets

Cycles C in basis B have upper and lower bound

$$\underbrace{\left\lceil \frac{1}{T}\left(\sum_{a\in C^+} l_a - \sum_{a\in C^-} u_a\right)\right\rceil}_{\underline{q}_C} \le q_C \le \underbrace{\left\lfloor \frac{1}{T}\left(\sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a\right)\right\rfloor}_{\overline{q}_C}$$

Number of feasible periodic currents is bounded by
$\prod_{C\in B}\left(\overline{q}_C - \underline{q}_C\right)$ for every basis B

Can this be used?

Narrowing the search further

$$\bar{q}_C - \underline{q}_C = \left\lfloor \frac{1}{T}\left(\sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a\right)\right\rfloor - \left\lceil \frac{1}{T}\left(\sum_{a\in C^+} l_a - \sum_{a\in C^-} u_a\right)\right\rceil$$

$$\leq \frac{1}{T}\left(\sum_{a\in C^+} u_a - \sum_{a\in C^-} l_a\right) - \frac{1}{T}\left(\sum_{a\in C^+} l_a - \sum_{a\in C^-} u_a\right) + 1$$

$$= \frac{1}{T}\underbrace{\sum_{a\in C}(u_a - l_a)}_{\text{length of the underlying undirected cycle for arc weights } u_a - l_a} + 1$$

Narrowing the search = Finding a short cycle basis B in the underlying undirected graph for objective

$$\prod_{C\in B}\left(\bar{q}_C - \underline{q}_C\right) \leftrightarrow \sum_{C\in B}\log\left(\bar{q}_C - \underline{q}_C\right)$$

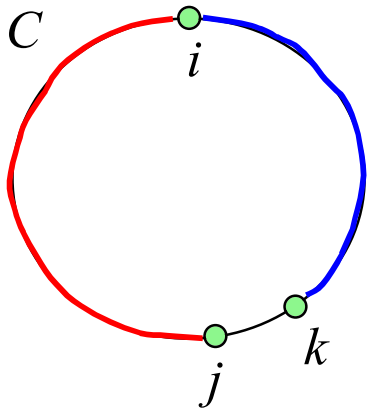
Finding a shortest cycle basis

- Polynomial for arbitrary cycle bases [Horton '87]
- NP-hard for tree bases [Deo, Prabhu, Krishnamoorthy '82]

Idea of the polynomial algorithm

- cycles of an undirected graph form a matroid
- so greedy algorithm can be used
- show that only a polynomial number of cycles need to be considered in the greedy algorithm

Vertex i is in a cycle C from a shortest basis
$\Rightarrow$ C contains edge (j,k) such that
the pieces of C from i to j and from i to k
are shortest paths from i to j and k



The algorithm by Horton

- Find a shortest path $P(j,k)$ between each pair j, k of vertices
- For each vertex i and edge (j,k), create the cycle $C(i,j,k) = P(i,j) + P(i,k) + \{j,k\}$ and calculate its length *
  (Omit degenerate cases leading to non-elementary cycles)
- Order these cycles by length
- Use the greedy algorithm to find a shortest cycle basis from this set of cycles

* If arc weights are ≥ 0, then the choice of paths is irrelevant

Run time $O(n^3) + O(m\ n) + O(c \log c) + O(m\ r)$

c = # of generated cycles $\leq n\ m$

r = time to test independence of a cycle from a given set of cycles
= Gaussian elimination in an $m \times m$ matrix $\leq m^3$

Overview

- The basic event-activity model
- Examples of time table problems
- Complexity of constructing (good) time tables
- Graph theoretic background results
- Solution methods

Solution methods

- Mixed integer programming
  - enhanced by Branch & Bound or Branch & Cut
  - e.g. by using rounded cycle inequalities [Odijk '94]
  - or other inequalities [Nachtigall '98, Nachtigall & Lindner '00]
- Searching the feasible domain of period offsets or cycle multiples
  - constraint propagation [Schrijver, Voorhoeve '93/94]
  - simulated annealing [many]
  - genetic algorithms [Kolonko, Nachtigall & Voget '96]
- Lower bounds
  - LP relaxation, enhanced by cycle inequalities
- Preprocessing
- Decomposition, hierarchies, …

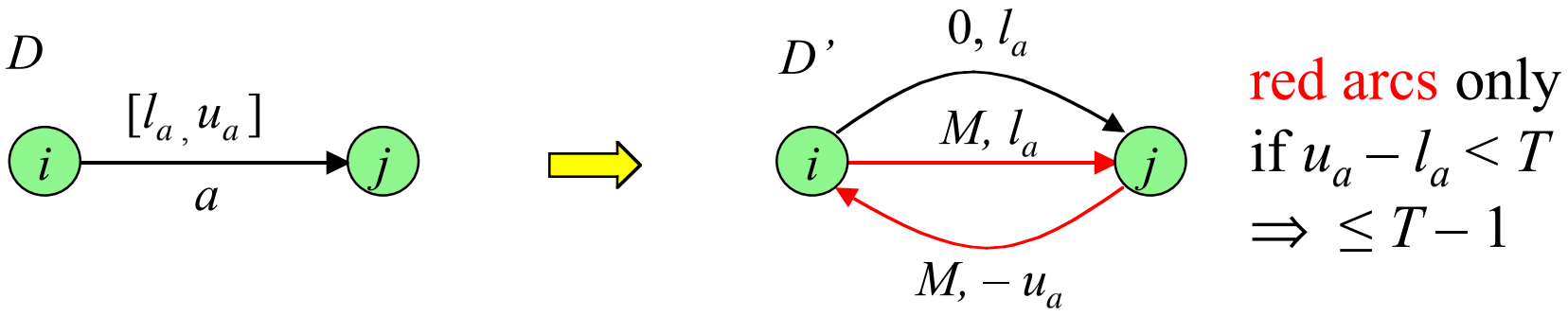
A genetic algorithm: the graph transformation

Nachtigall & Voget '96

Observe

$$[u_a - x_a]_T + [x_a - l_a]_T + l_a - u_a = \begin{cases} 0 & \text{if } l_a \le [x_a]_T \le u_a \\ T & \text{otherwise} \end{cases}$$

Model this "feasibility indicator" by a transformed graph D'



red arcs only
if $u_a - l_a < T$
$\Rightarrow \le T - 1$

first number = weight w' with M big
second number = lower bound l_a

A genetic algorithm: the modified problem

Consider the following problem in D'

$$\min \sum_{a'} w'_a [x_{a'} - l_{a'}]_T$$
$$\text{such that } \Gamma x = 0 \mod T$$

← modulo expression in objective, tests feasibility via objective value

← only periodic current required in side constraint

D has feasible periodic current $\Leftrightarrow$

D' has solution with value w^* such that

$$w^* + \sum_{a, u_a - l_a < T} M \cdot (l_a - u_a) < M \cdot T$$

A genetic algorithm: the ingredients

- population = potential vectors
  (also periodic currents or cycle multiples would work)
- Crossover: take randomly potentials from one of the parents
  (uniform distribution)
- Mutation: change a few node potentials randomly
- Reduce: survival of the fittest
  use objective here
- Improve the fittest by local search

Implementation at TU Berlin

Christian Liebchen

Java, SUN, Solaris, uses CPLEX

Optimization

- Integer Solver from CPLEX
- Integer solver and cutting planes

Solutions with quality guarantee

- Genetic algorithm with local search enhancements
- Lower bounds by LP-relaxation

Use in practice - 1998

- U-Bahn Berlin in low traffic hours
  - 18 lines, 190 stations
  - 93 nodes, 246 arcs (after preprocessing)
  - solved in 13 min with 16% optimality gap

- S-Bahn Berlin
  - Berliner Stadtbahn: 18 lines
  - 29 nodes, 194 arcs (after preprocessing)
  - solved optimally in 70 sec

lines = bidirectional

Use in practice - 2000

PTV test sets

DBRegio (RegionalExpress and RegionalBahn in Berlin-Brandenburg)

- 74 lines, 307 stations
- 1465 / 73 nodes, 1846 / 317 arcs (before / after preprocessing)

- HAVAG (Halle)
  - 79 lines, 461 stations
  - 2521 / 78 nodes, 1846 / 3480 arcs (before / after preproc.)
- SBB (Stuttgart)
  - 160 lines, 922 stations
  - 5791 / 159 nodes, 6776 / 968 arcs (before / after preproc.)

lines = bidirectional