



# «Magic: The Gathering» is Turing Complete

Alex Churchill, Stella Biderman, Austin Herrick

Presentation by Thomas Bläsius

# What does that even mean?

## **Theorem**

Determining the outcome of a game of **Magic: The Gathering** in which all remaining moves are forced is undecidable.

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Determining the outcome of a game of **Magic: The Gathering** in which all remaining moves are forced is undecidable.

## General plan

- create game state where every move is forced by the game rules
- remaining moves simulate a universal TM with arbitrary input
- first player (Alice) wins if and only if TM halts

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Determining the outcome of a game of **Magic: The Gathering** in which all remaining moves are forced is undecidable.

## General plan

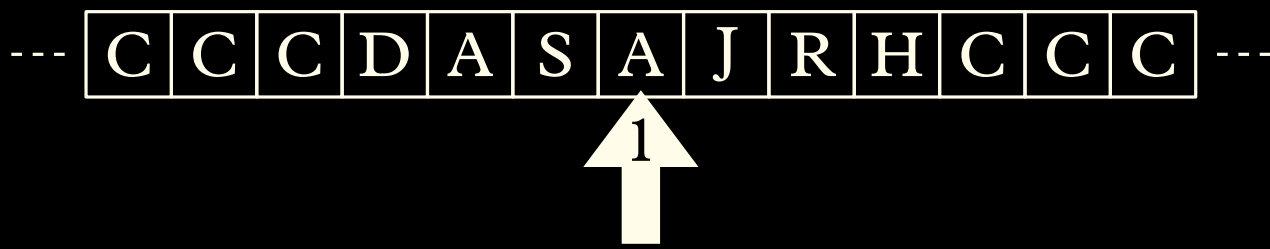
- create game state where every move is forced by the game rules
  - remaining moves simulate a universal TM with arbitrary input
  - first player (Alice) wins if and only if TM halts
- ⇒ deciding whether Alice wins is as hard as the halting problem

# Rogozhin's Universal Turing machine

## UTM(2, 18)

[Rogozhin, 1996]

- has two states: 1 and 2
- alphabet of size 18:  
 {A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, R, S}
- interprets input on the tape as TM and simulates it



	current state	read symbol	write symbol	head movement	new state
1	A	S	←		1
1	B	E	→		1
1	C	S	←		1
1	D	A	→		1
1	E	D	←		1
1	F	H	→		1
1	G	J	→		1
1	H	F	←		1
1	I	F	→		1
1	J	I	←		1
1	K	L	←		2
1	L	I	←		2
1	M	B	←		2
1	N	O	→		1
1	O	P	←		1
1	P	R	→		2
1	R		halt		
1	S	C	→		1
2	A	...			

# Questions to be answered

## Core components

- How to represent the tape?
- How to implement the transition function?
  - How does the head read and write symbols on the tape?
  - How does the head move left or right?
  - How can we represent different states?

## Interplay between components

- How can players trigger the components?
- How to force players to behave as intended?
- How to win the game (a.k.a. the TM halts)?
- What to do, if we are out of tape?

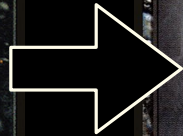
## Setup

- How can one player set up the initial game state?
- Can I do that at a tournament?

# Tape

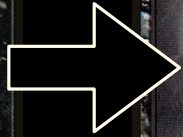


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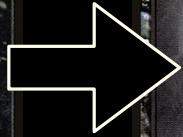




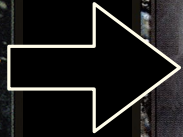
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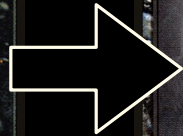


## There are other creature types

- Aetherborn
- Basilisk
- Cephalid
- Demon
- Elf
- Faerie
- Giant
- Harpy
- Illusion
- Juggernaut
- Kavu
- Leviathan
- Myr
- Noggle
- Orc
- Pegasus
- Rhino
- Sliver

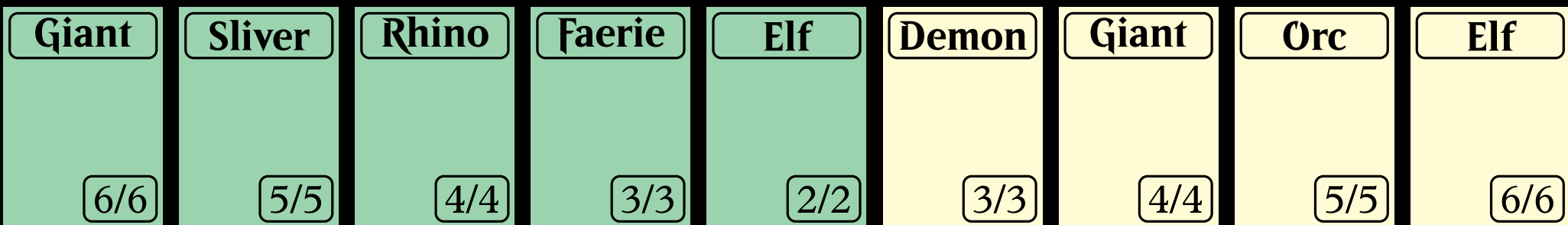


# Tape



## There are other creature types

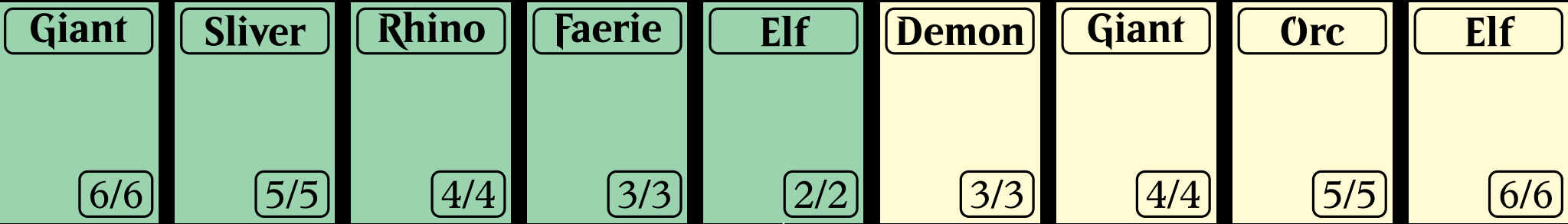
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- Myr
- Noggle
- Orc
- Pegasus
- Rhino
- Sliver



# Transition function: reading and writing



	read symbol	write symbol	head movement
A	S	←	
B	E	→	
C	S	←	
D	A	→	
<b>E</b>	<b>D</b>	<b>←</b>	
F	H	→	
G	J	→	



# Transition function: reading and writing



read symbol  
write symbol  
head movement

A	S	←
B	E	→
C	S	←
D	A	→
E	D	←
F	H	→
G	J	→

Giant	Sliver	Rhino	Faerie	Elf	Demon	Giant	Orc	Elf
6/6	5/5	4/4	3/3	2/2	3/3	4/4	5/5	6/6

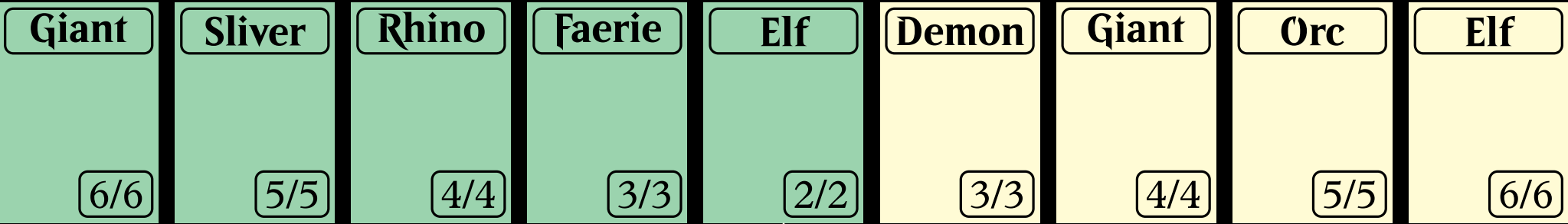


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read symbol  
write symbol  
head movement

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B	E	→
C	S	←
D	A	→
E	D	←
F	H	→
G	J	→





# Transition function: reading and writing



	read symbol	write symbol	head movement
A	S	←	
B	E	→	
C	S	←	
D	A	→	
E	D	←	
F	H	→	
G	J	→	

Giant	Sliver	Rhino	Faerie	Elf	Demon	Giant	Orc	Elf
6/6	5/5	4/4	3/3	2/2	3/3	4/4	5/5	6/6



# Transition function: reading and writing



read symbol	write symbol	head movement	
A	S	←	■
B	E	→	■
C	S	←	■
D	A	→	■
<b>E</b>	<b>D</b>	←	■
F	H	→	■
G	J	→	■

<b>Giant</b>	<b>Sliver</b>	<b>Rhino</b>	<b>Faerie</b>	<b>Elf</b>	<b>Demon</b>	<b>Giant</b>	<b>Orc</b>	<b>Elf</b>
6/6	5/5	4/4	3/3	2/2	3/3	4/4	5/5	6/6

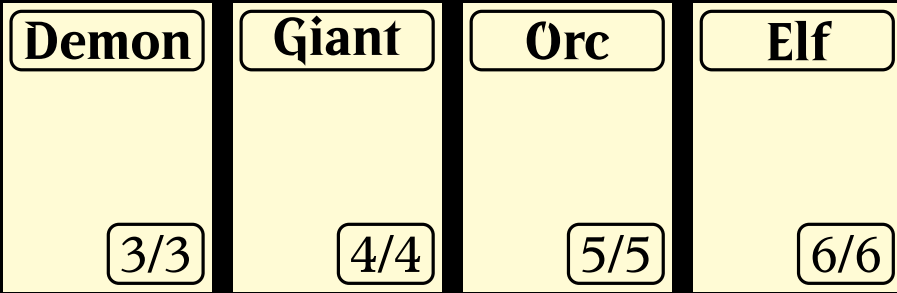
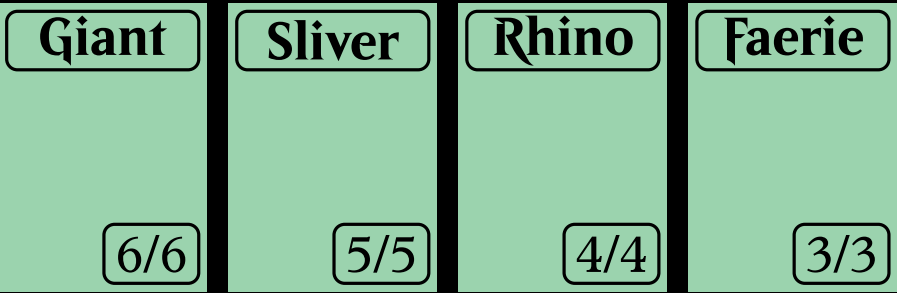


# Transition function: reading and writing



read symbol  
write symbol  
head movement

A	S	←	■
B	E	→	■
C	S	←	■
D	A	→	■
<b>E</b>	<b>D</b>	←	■
F	H	→	■
G	J	→	■



# Transition function: reading and writing

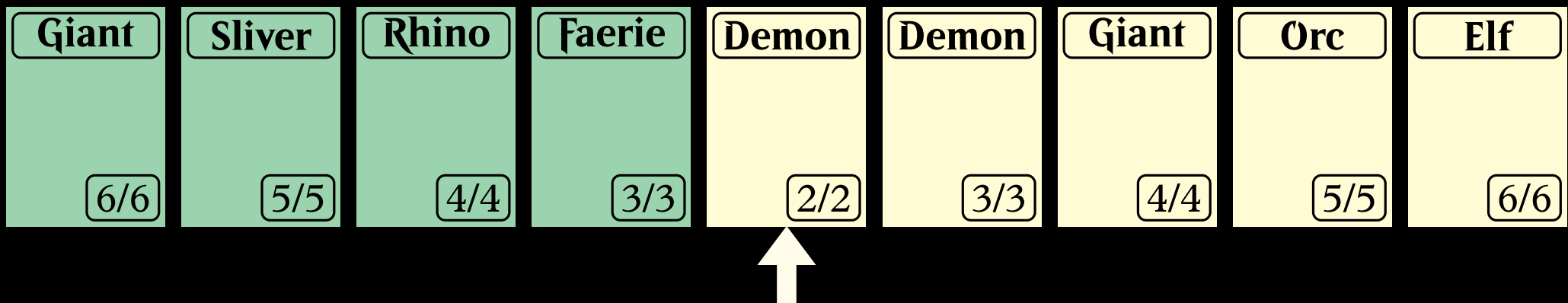


read symbol	write symbol	head movement	
A	S	←	■
B	E	→	■
C	S	←	■
D	A	→	■
<b>E</b>	<b>D</b>	←	■
F	H	→	■
G	J	→	■

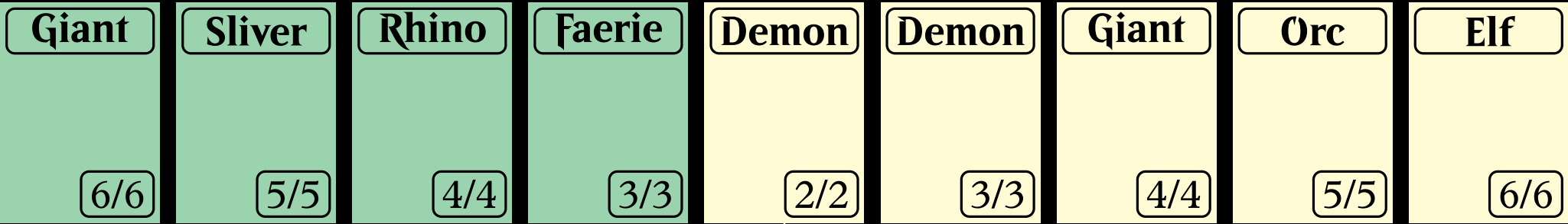
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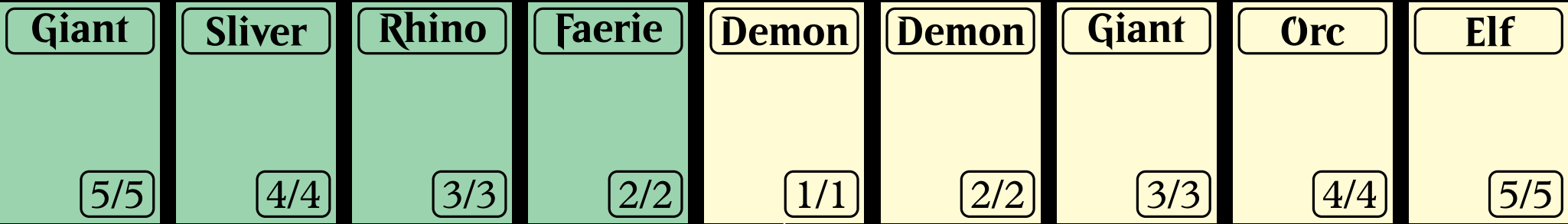
# Transition function: moving the head (left)



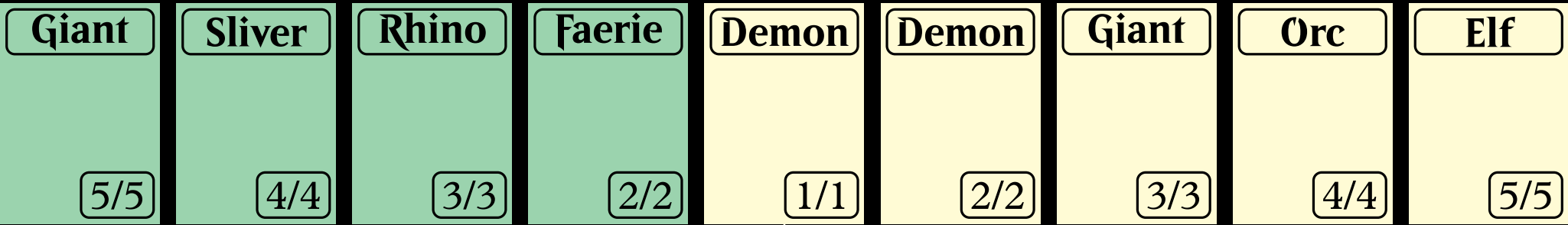
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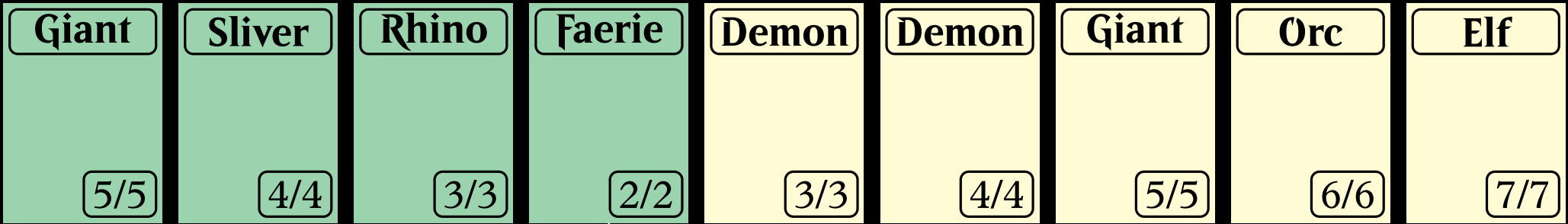


# Transition function: moving the head (left)





# Transition function: moving the head (left)



# Transition function: changing state

	current state	read symbol	write symbol	head movement	new state
	⋮	⋮	⋮	⋮	⋮
	1	E	D	←	1
	1	F	H	→	1
	⋮	⋮	⋮	⋮	⋮
	2	E	A	←	2
	2	F	K	→	1
	⋮	⋮	⋮	⋮	⋮

# Transition function: changing state



1

current state	read symbol	write symbol	head movement	new state
:	:	:	:	:
1	E	D	←	1
1	F	H	→	1
:	:	:	:	:
2	E	A	←	2
2	F	K	→	1
:	:	:	:	:

# Transition function: changing state



1

current state	read symbol	write symbol	head movement	new state
:	:	:	:	:
1	E	D	←	1
1	F	H	→	1
:	:	:	:	:
2	E	A	←	2
2	F	K	→	1
:	:	:	:	:



2

# Transition function: changing state



1

current state	read symbol	write symbol	head movement	new state
⋮	⋮	⋮	⋮	⋮
1	E	D	←	1
1	F	H	→	1
⋮	⋮	⋮	⋮	⋮
2	E	A	←	2
2	F	K	→	1



2



# Transition function: changing state



1

current state	read symbol	write symbol	head movement	new state
:	:	:	:	:
1	E	D	←	1
1	F	H	→	1
:	:	:	:	:
2	E	A	←	2
2	F	K	→	1



2



# Transition function: changing state



1

current state	read symbol	write symbol	head movement	new state
⋮	⋮	⋮	⋮	⋮
1	E	D	←	1
1	F	H	→	1
⋮	⋮	⋮	⋮	⋮
2	E	A	←	2
2	F	K	→	1



2



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## Setup

- How can one player set up the initial game state?
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# What the players do

## Two players: Alice and Bob

- Bob can't do anything
- Alice repeatedly takes four turns
- she sometimes skips turn 3 to change state (remember: phasing)

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## Turn 1



# What the players do

## Two players: Alice and Bob

- Bob can't do anything
- Alice repeatedly takes four turns
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## Turn 1



## Turn 2



# What the players do

## Two players: Alice and Bob

- Bob can't do anything
- Alice repeatedly takes four turns
- she sometimes skips turn 3 to change state (remember: phasing)

### Turn 1



### Turn 2



### Turn 3



# What the players do

## Two players: Alice and Bob

- Bob can't do anything
- Alice repeatedly takes four turns
- she sometimes skips turn 3 to change state (remember: phasing)

### Turn 1



### Turn 2



### Turn 3



### Turn 4



# Forcing Alice to cast spells



(controlled by Bob)

# Forcing Alice to cast spells



(controlled by Bob)

# Forcing Alice to cast spells



(controlled by Bob)

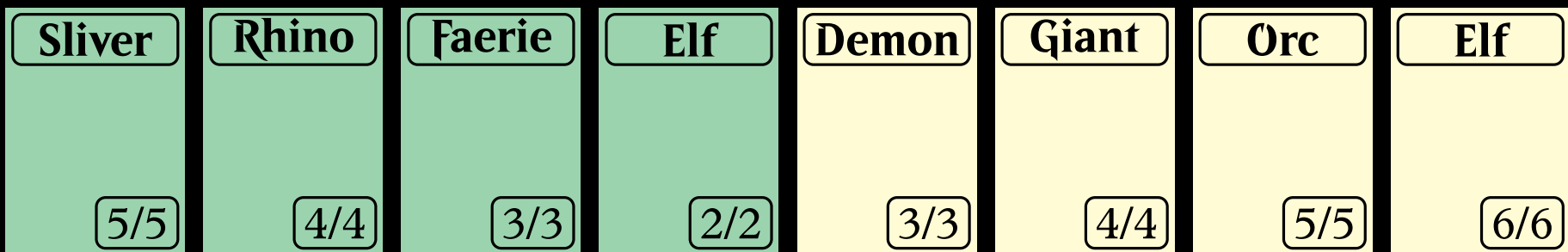


# Changing the state by skipping turn 3



(controlled by Alice)

# Forcing Alice to pick the right target



# Forcing Alice to pick the right target

protecting Bob's  
creatures



protecting Alice's creatures



Sliver	Rhino	Faerie	Elf	Demon	Giant	Orc	Elf
5/5	4/4	3/3	2/2	3/3	4/4	5/5	6/6

# Forcing Alice to pick the right target

protecting Bob's creatures



protecting Alice's creatures



only legal target:  
newest creature



controller:

<b>Sliver</b>	<b>Rhino</b>	<b>Faerie</b>	<b>Elf</b>	<b>Demon</b>	<b>Giant</b>	<b>Orc</b>	<b>Elf</b>
Bob	Bob	Bob	Alice	Bob	Bob	Bob	Bob
5/5	4/4	3/3	2/2	3/3	4/4	5/5	6/6



# Keeping your engine alive

**Vigor** 3 



**Creature — Elemental Incarnation** 

Trample  
If damage would be dealt to another creature you control, prevent that damage. Put a +1/+1 counter on that creature for each 1 damage prevented this way.  
When Vigor is put into a graveyard from anywhere, shuffle it into its owner's library.

6/6 

215 / 254 R  
BBD • EN  JIM MURRAY  
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**Prismatic Lace** 



**Instant** 

Target permanent becomes the color(s) of your choice. Costs to tap, maintain, or use an ability of that permanent remain unchanged.

Illus. David O'Connor  
© 1996 Wizards of the Coast, Inc. All rights reserved.

**Fungus Sliver** 3 



**Creature — Fungus Sliver** 

All Slivers have “Whenever this creature is dealt damage, put a +1/+1 counter on it.” (The damage is dealt before the counter is put on.)  
“When a sliver of this breed enters the hive, the others claw each other in frenzied fits, thereby ensuring their rapid growth.”  
—Rukarumel, field journal

Daniel Gelon  
™ & © 1993-2006 Wizards of the Coast, Inc. 195/301

2/2

# Locking everything else down



(controlled by Bob)



(controlled by Alice)



(controlled by both)

# Halting

	current state	read symbol	write symbol	head movement	new state
:	:	:	:	:	:
1	R		halt		
:	:	:	:	:	:

goal: TM halts  $\Leftrightarrow$  Alice wins with:



# Halting

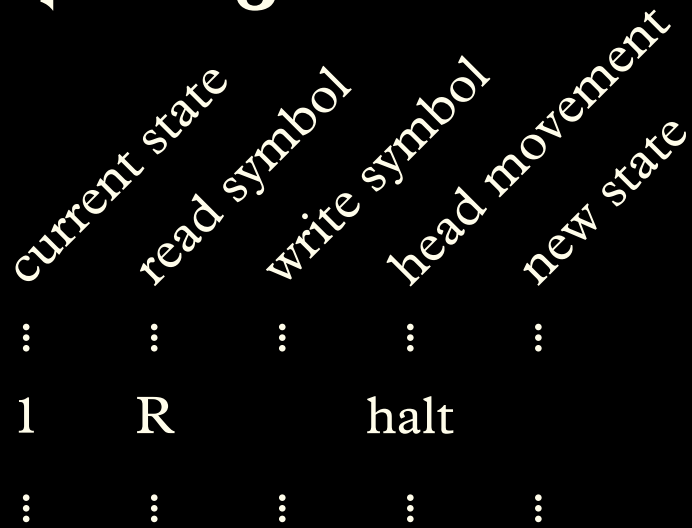
current state	read symbol	write symbol	head movement	new state
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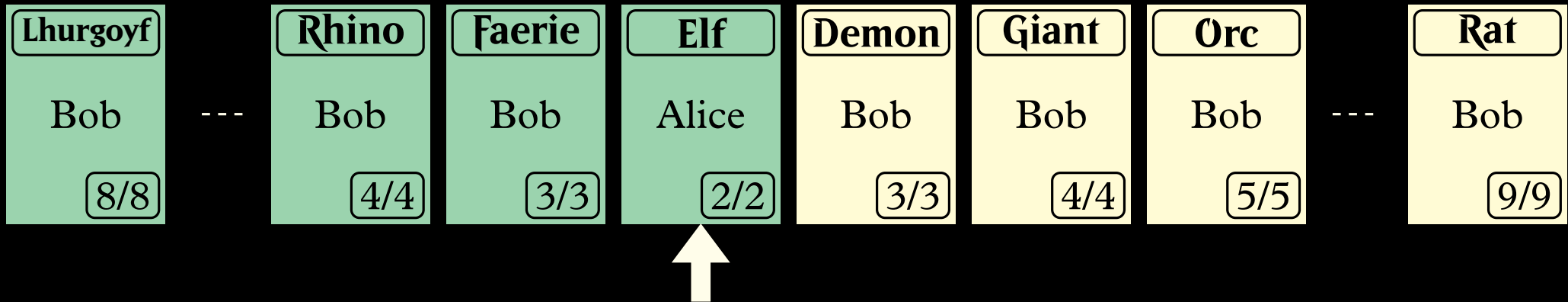
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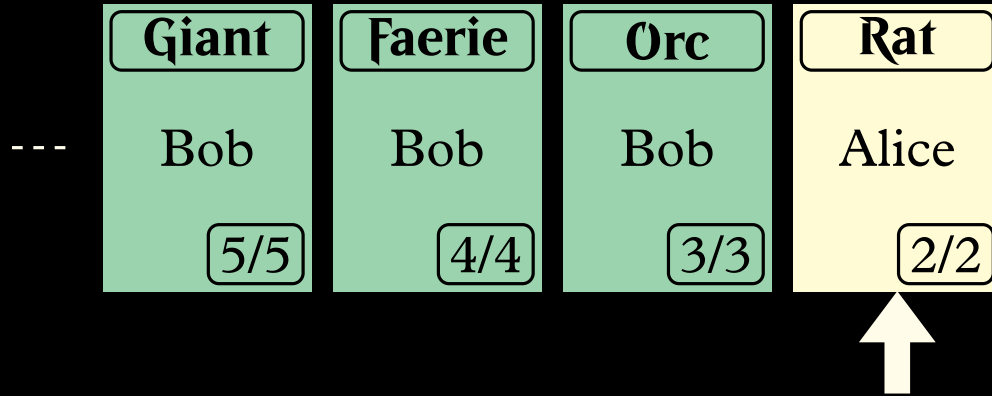
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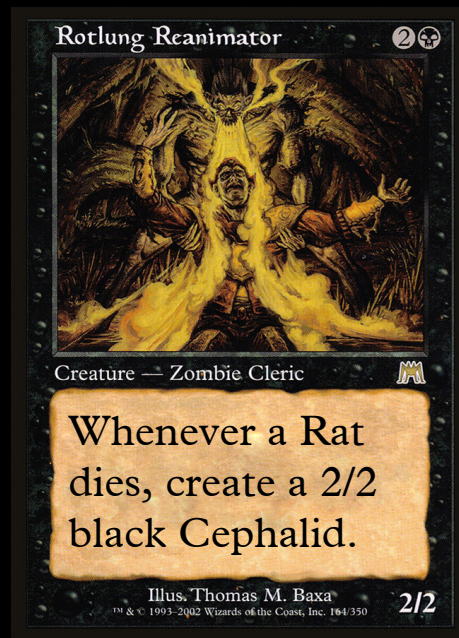
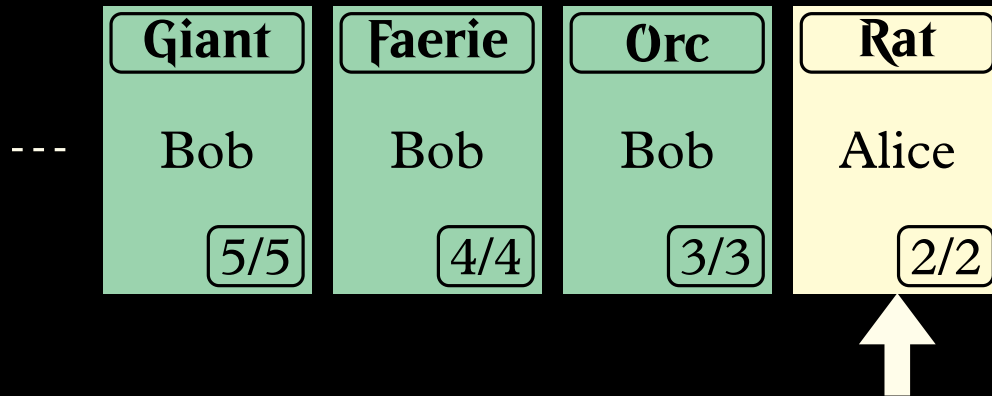
# Out of tape exception



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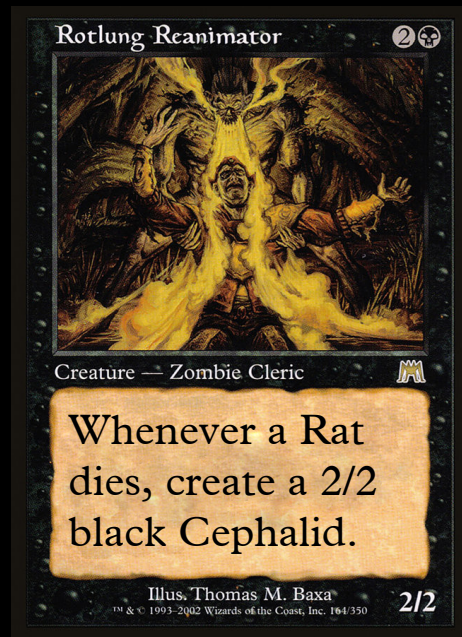
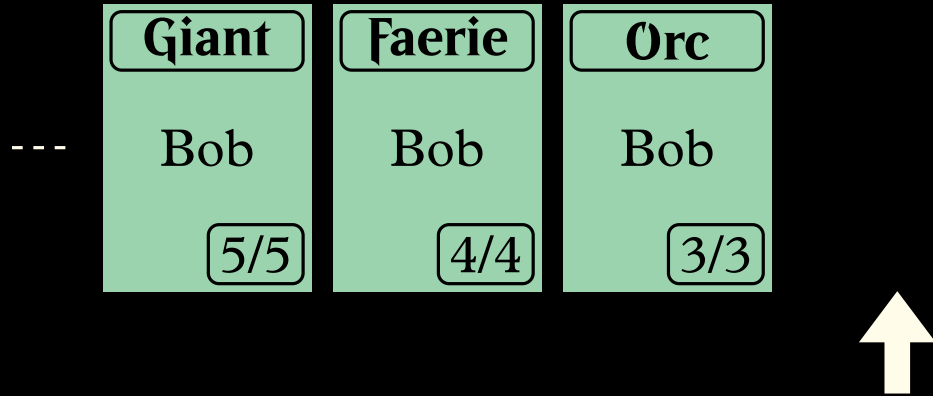


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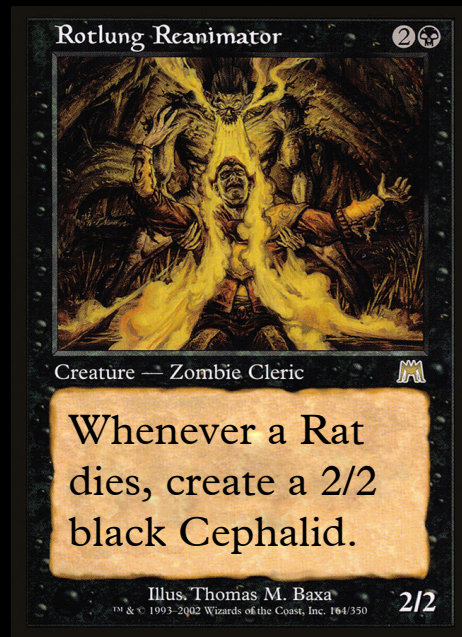
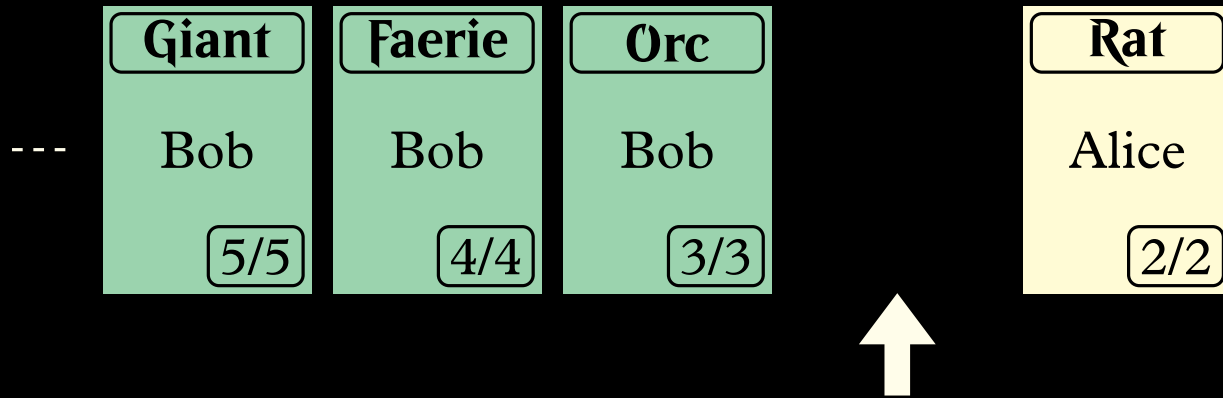
(controlled by Bob) (controlled by Alice)

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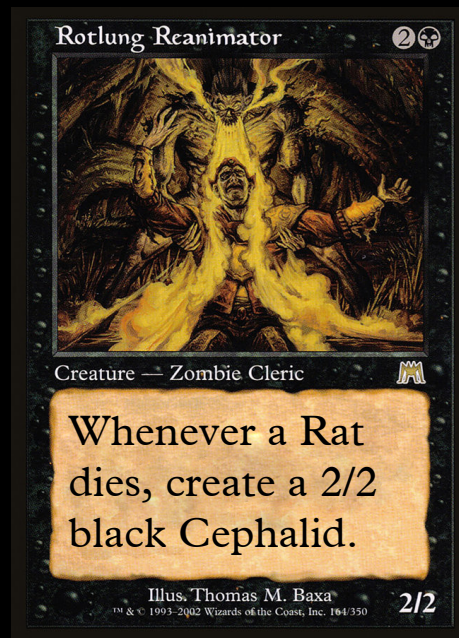
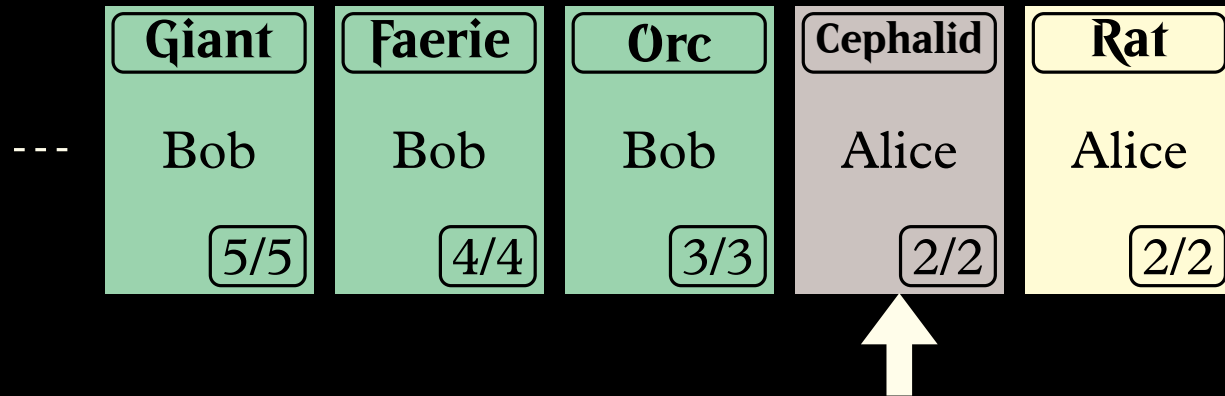
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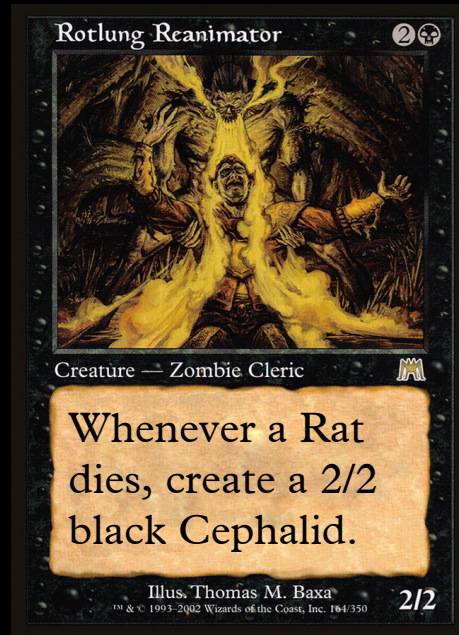
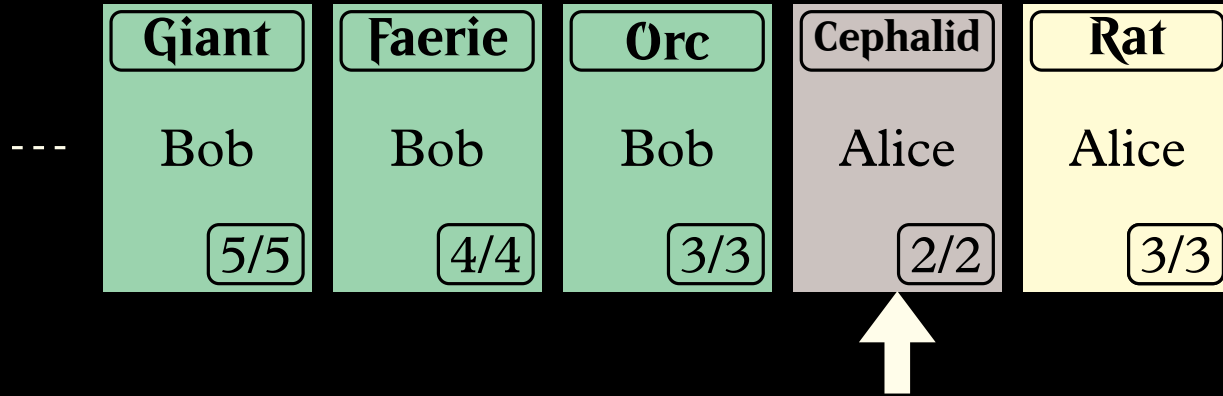
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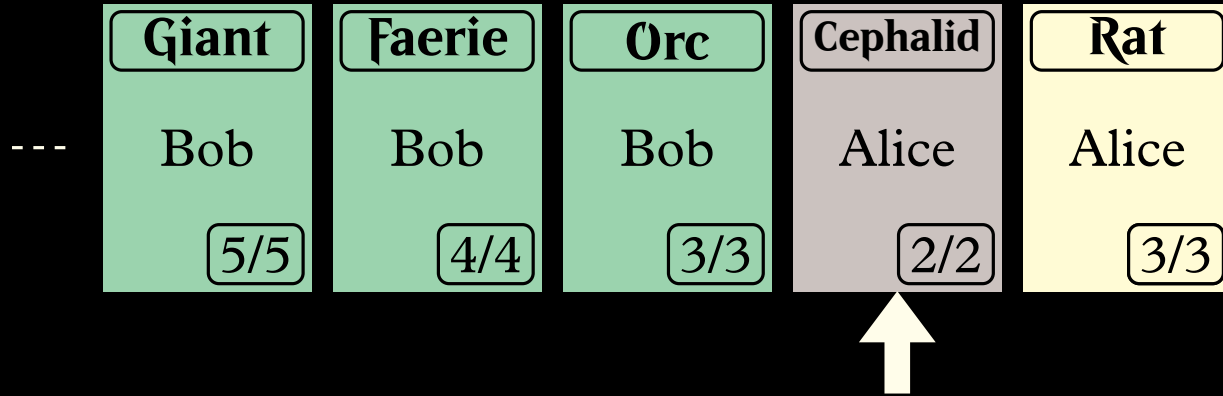
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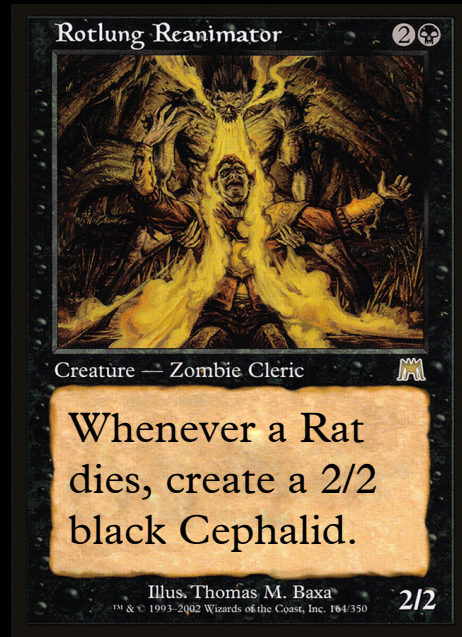
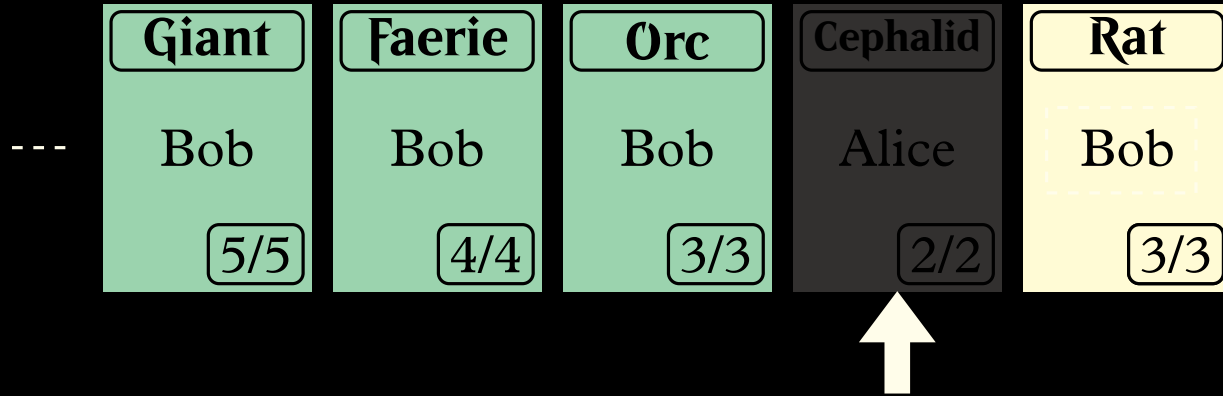


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(controlled by Bob) (controlled by Alice)

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(controlled by Bob) (controlled by Alice)

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## Setup

- How can one player set up the initial game state?
- Can I do that at a tournament?

# Setup: getting started

BATTLEFIELD

MANA	colorless	white	blue
	black	red	green

Ancient Tomb



Land

☞: Add two colorless mana to your mana pool. Ancient Tomb deals 2 damage to you.

*There is no glory to be gained in the kingdom of the dead.*  
 —Vec tomb inscription

Illus. Colin MacNeil  
 ©1997 Wizards of the Coast, Inc.

HAND

Ancient Tomb



Land

☞: Add two colorless mana to your mana pool. Ancient Tomb deals 2 damage to you.

*There is no glory to be gained in the kingdom of the dead.*  
 —Vec tomb inscription

Illus. Colin MacNeil  
 ©1997 Wizards of the Coast, Inc.

Grim Monolith 2



Artifact

Grim Monolith does not untap during your untap phase.

☞: Add three colorless mana to your mana pool. Play this ability as a mana source.

⚡: Untap Grim Monolith.  
*Part prison, part home.*

Illus. Chippy  
 ©1993-1999 Wizards of the Coast, Inc. 120/143

Lotus Petal 0



Artifact

☞, Sacrifice Lotus Petal: Add one mana of any color to your mana pool.

*"Hard to imagine," mused Hanna, stroking the petal, "such a lovely flower inspiring such greed."*

228/269 U  
 17W EN APRIL LEE      TM & © 2015 Wizards of the Coast

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Power Artifact 0



Enchantment — Aura

Enchant artifact

Enchanted artifact's activated abilities cost 2 less to activate. This effect can't reduce the amount of mana an ability costs to activate to less than one mana.

Illus. Douglas Shuler  
 TM & © 1993-2011 Wizards of the Coast, Inc. 126/802

Staff of Domination 3



Artifact

1: Untap Staff of Domination.  
2, ☞: You gain 1 life.  
3, ☞: Untap target creature.  
4, ☞: Tap target creature.  
5, ☞: Draw a card.

Illus. Ben Thompson  
 TM & © 1993-2011 Wizards of the Coast, Inc. 150/165

# Setup: getting started

BATTLEFIELD



MANA	colorless	white	blue
	black	red	green



HAND



# Setup: getting started

BATTLEFIELD



HAND



MANA	colorless	white	blue
	2		
	black	red	green



# Setup: getting started

BATTLEFIELD



MANA	colorless	white	blue
	2		
	black	red	green



HAND



# Setup: getting started

BATTLEFIELD



**MANA**

colorless	white	blue
black	red	green



HAND





# Setup: getting started

BATTLEFIELD



MANA	colorless	white	blue
	black	red	green



HAND



# Setup: getting started

BATTLEFIELD



MANA	colorless	white	blue
	black	red	green



HAND



# Setup: getting started

BATTLEFIELD



MANA	colorless	white	blue 2
	black	red	green



HAND



# Setup: getting started

BATTLEFIELD



MANA	colorless	white	blue 2
	black	red	green



HAND



# Setup: getting started

BATTLEFIELD



HAND



MANA

colorless	white	blue
black	red	green



# Setup: getting started

BATTLEFIELD



HAND



MANA

colorless	white	blue
black	red	green



# Setup: getting started

BATTLEFIELD



HAND



MANA	colorless	white	blue
	3		
	black	red	green



# Setup: getting started

BATTLEFIELD



HAND



MANA

colorless	white	blue
1		
black	red	green





# Setup: getting started

BATTLEFIELD



HAND



MANA	colorless	white	blue
	4		
	black	red	green





# Setup: getting started

BATTLEFIELD



HAND



MANA	colorless $n$	white	blue
	black	red	green



# Setup: getting started

BATTLEFIELD



HAND



MANA	colorless $n$	white	blue
	black	red	green



# Setup: getting started

BATTLEFIELD



HAND

MANA	colorless $n$	white	blue
	black	red	green



# Setup: getting started

BATTLEFIELD



MANA	colorless <i>n</i>	white	blue
	black	red	green



HAND



# Setup: getting started

BATTLEFIELD



**MANA**

colorless <i>n</i>	white	blue
black	red	green

**Gemstone Array** 4



**Artifact**

2: Put a charge counter on Gemstone Array.  
Remove a charge counter from Gemstone Array: Add one mana of any color to your mana pool.

*Outside, it reflects the suns' light. Inside, it harvests the suns' power.*

—Ittoka

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HAND



# Setup: getting started

BATTLEFIELD



MANA	colorless <i>n</i>	white <i>n</i>	blue <i>n</i>
	black <i>n</i>	red <i>n</i>	green <i>n</i>



HAND





# Setup: getting started

BATTLEFIELD



MANA	colorless	white	blue
	$n$	$n$	$n$
	black	red	green
	$n$	$n$	$n$



HAND



# Setup: reusing cards

**Stolen Identity** 4



**Sorcery**

Put a token onto the battlefield that's a copy of target artifact or creature.

*Cipher (Then you may exile this spell card encoded on a creature you control. Whenever that creature deals combat damage to a player, its controller may cast a copy of the encoded card without paying its mana cost.)*

Clint Cearley

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**Memnarch** 7



**Legendary Artifact Creature — Wizard**

1 : Target permanent becomes an artifact in addition to its other types.

3 : Gain control of target artifact.

*In the blur between metal and flesh, Memnarch found madness.*

4/5

Carl Critchlow

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**Reito Lantern** 2



**Artifact**

3 : Put target card from a graveyard on the bottom of its owner's library.

*Lanterns carved from the mystic stones of the Reito Mines were said to light the way of lost souls.*

Greg Hildebrandt

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**Capsize** 1



**Instant**

Buyback 3 (You may pay an additional 3 as you cast this spell. If you do, put this card into your hand as it resolves.)

Return target permanent to its owner's hand.

042/269 U

TPR • EN • TOM WÄNERSTRAND

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# Setup: creating the tape



# Setup: cleaning up

Donate 2 



Sorcery 

Target player gains control of target permanent you control.

*Campus pranksters initiate new students with the old "beeble bomb" routine.*

Illus. Jeff Miracola  
© 1993-1999 Wizards of the Coast, Inc. 31/143

Reality Ripple 1 



Instant 

Target artifact, creature, or land phases out.

*"Can you prepare for the unexpected? No, you cannot. You can prepare only to be surprised."*

*—Teferi*

Illus. Alan Rabinowitz  
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# Setup: cleaning up

Donate 2 U



Sorcery U

Target player gains control of target permanent you control.

*Campus pranksters initiate new students with the old "beeble bomb" routine.*

Illus. Jeff Miracola  
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Reality Ripple 1 U



Instant U

Target artifact, creature, or land phases out.

*"Can you prepare for the unexpected? No, you cannot. You can prepare only to be surprised."*

—Teferi

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Karn Liberated 7



Legendary Planeswalker — Karn U

**+4** : Target player exiles a card from their hand.

**-3** : Exile target permanent.

**-14** : Restart the game, leaving in exile all non-Aura permanent cards exiled with Karn Liberated. Then put those cards onto the battlefield under your control.

**6**

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UMA • EN JASON CHAN

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# Conclusion

## Theorem

Determining the outcome of a game of **Magic: The Gathering** in which all remaining moves are forced is undecidable.

## Note

- a player can set up the corresponding game state in the first turn
- with a tournament legal deck (in the Legacy format)

# Conclusion

## Theorem

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## Note

- a player can set up the corresponding game state in the first turn
- with a tournament legal deck (in the Legacy format)

## Related fun fact

- deciding whether a game action (blocking) is legal is coNP-complete  
[Chatterjee and Ibsen-Jensen, 2016]