Algorithms for Graph Visualization

Flow Methods: Upward Planarity

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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Def: A directed acyclic graph $D = (V, A)$ is called **upward planar**, when D admits a drawing (vertices points, edges simple curves), which is planar and each edge is a monotone curve increasing in y-direction.

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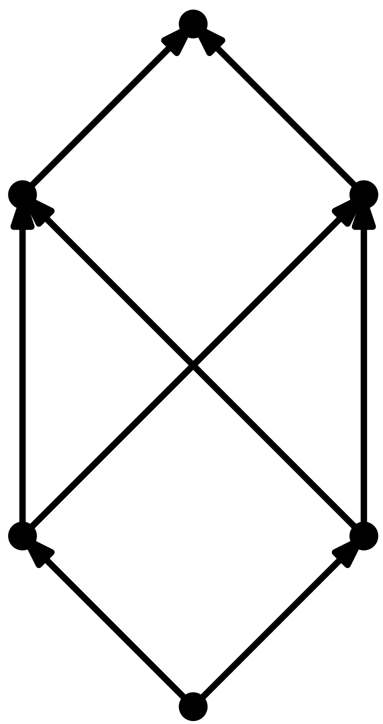

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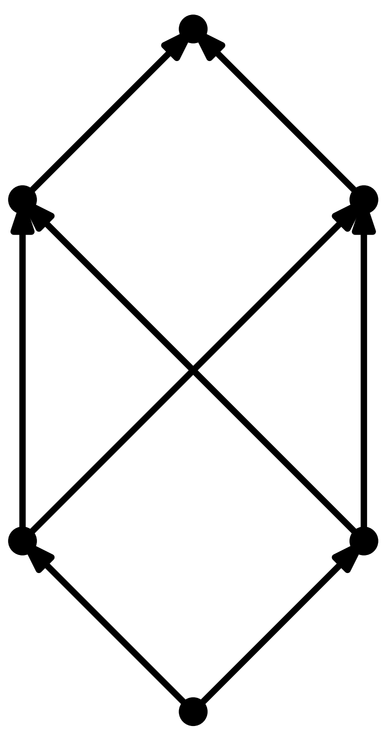

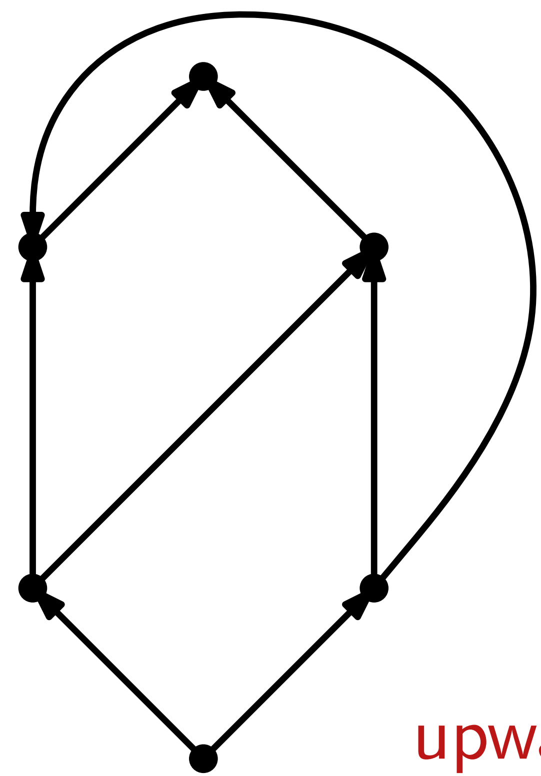

planar!

upward planar? – NO!

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Thm 1: For a directed acyclic graph it is NP-hard to decide whether it is upward planar.

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Characterization

Thm 4: For a directed graph $D = (V, A)$ the following statements are equivalent:

(1) D is upward planar

(2) D admits an upward planar straight-line drawing

(3) D is the spanning subgraph of a planar st-digraph

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Pair, think and share:

How to do the augmentation in case of a disconnected graph?

3 min

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Pair, think and share:
Is the area produced by the algorithm described in the proof of (3) $\Rightarrow$ (2) polynomial?

5 min

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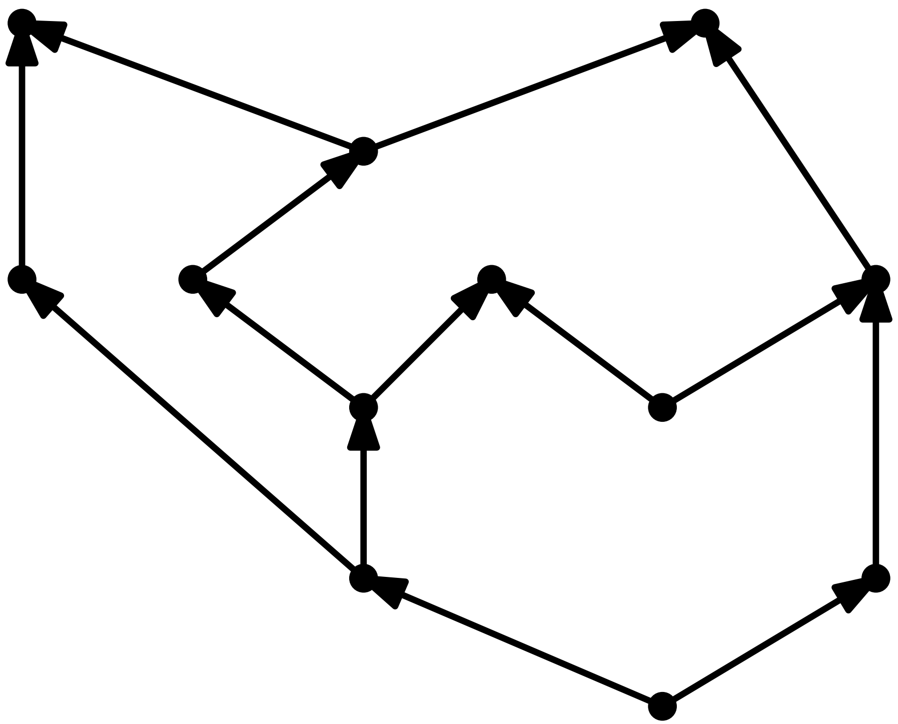
- (3) $\Rightarrow$ (2) triangulation and construction of straight-line drawing (blackboard)

- Step (3) $\Rightarrow$ (2) implies an $O(n)$ algorithm to construct a planar straight-line drawing of an st-digraph.

Fixed Outer Face: Angles

Problem: Consider a directed acyclic graph $D = (V, A)$ with embedding $\mathcal{F}, f_0$. Test whether $D, \mathcal{F}, f_0$ is upward planar and construct corresponding drawing.



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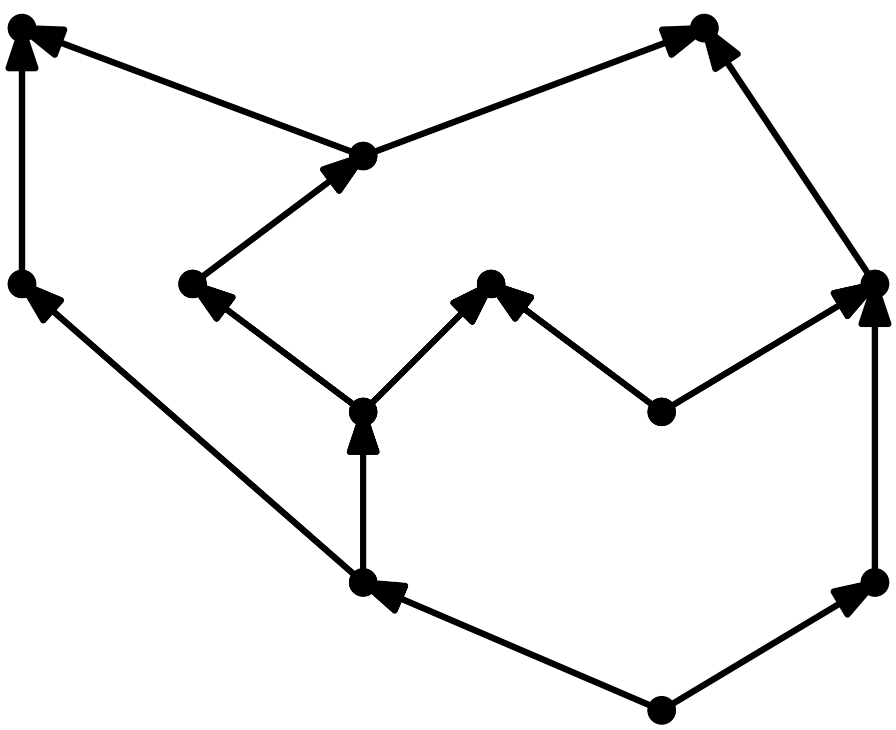
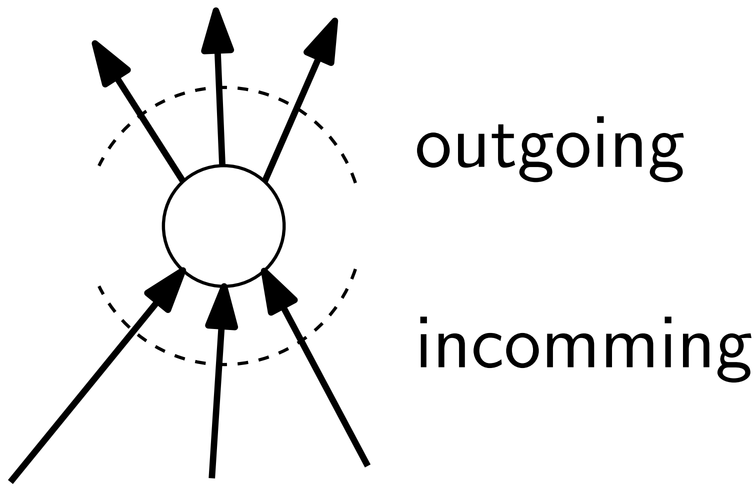


Embedding is **bimodal** if for each node:



Fixed Outer Face: Observations

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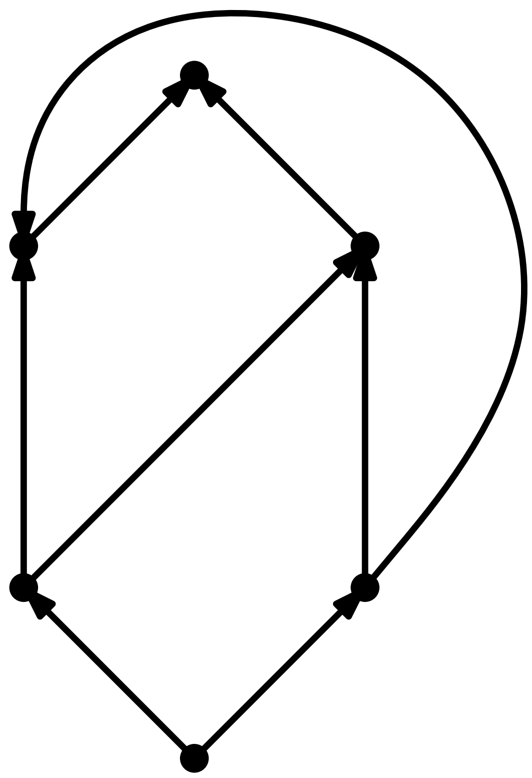

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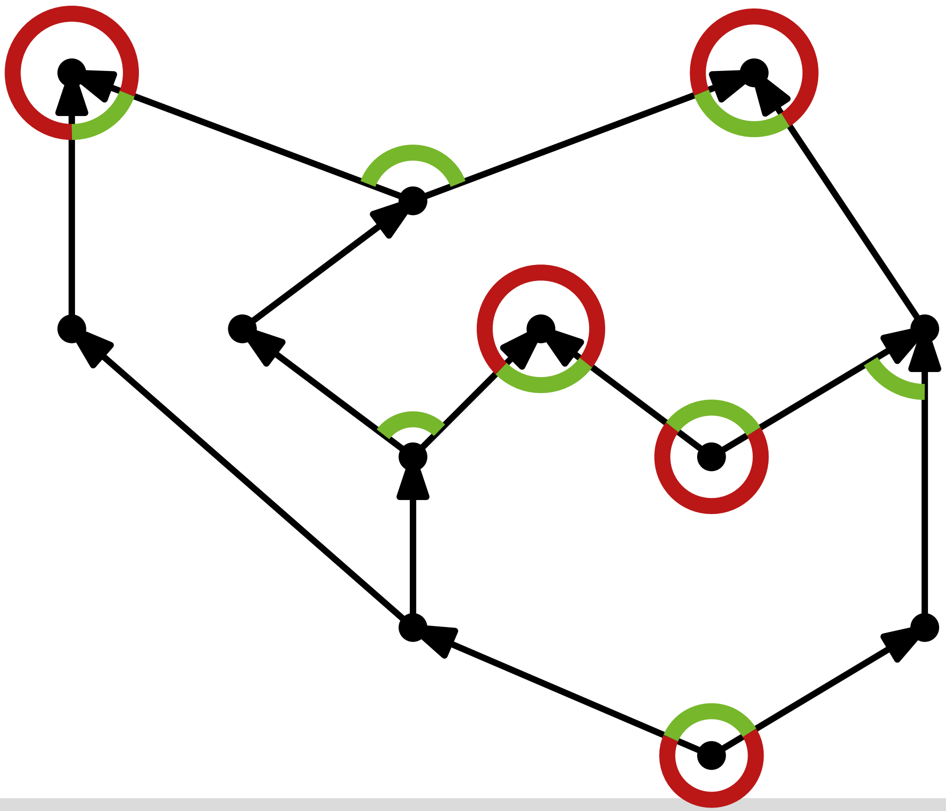
- Bimodality is necessary but not sufficient
- measure angles between two **incoming or two outgoing edges**

Angle α is **large** when $\alpha > \pi$, **small** otherwise
$L(v) :=$ # large angles at node v
$L(f) :=$ # large angles in face f
$S(v)$ resp. $S(f)$: # **small** angles



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(1) $\forall v \in V : L(v) = \begin{cases} 0 & v \text{ not source/sink} \\ 1 & v \text{ source/sink} \end{cases}$

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Pair, think and share:
Think about property (2). Why does it hold?

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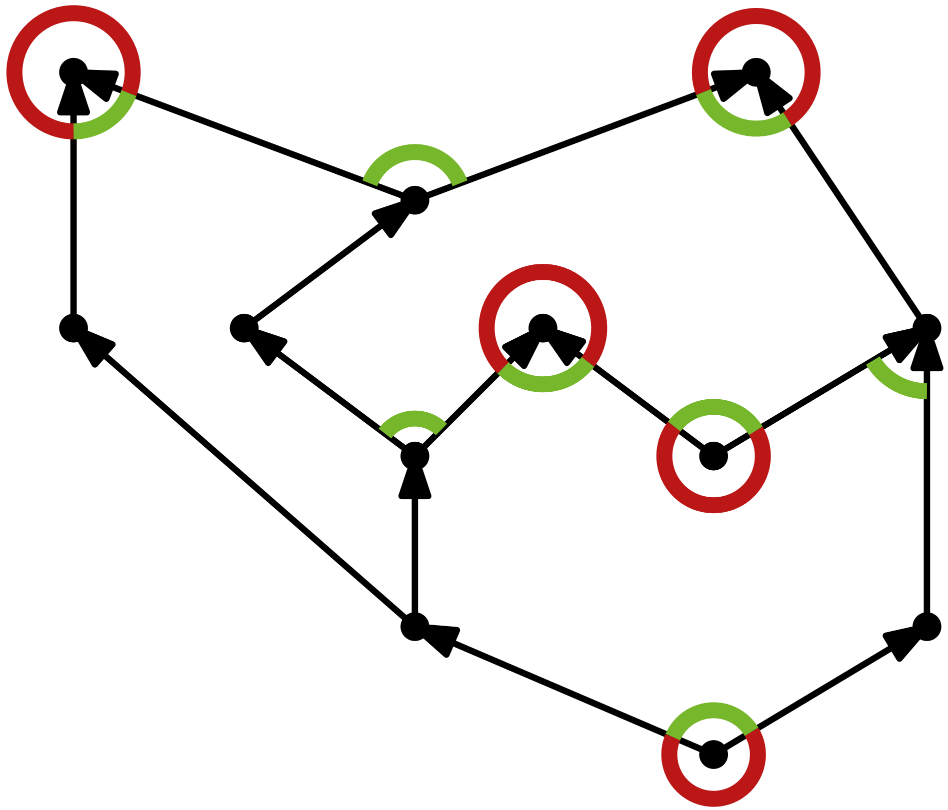

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- $A(f) := \#$ sources in face f (equal to the number of sinks)
  It holds that: $L(f) + S(f) = 2A(f)$ for all faces.

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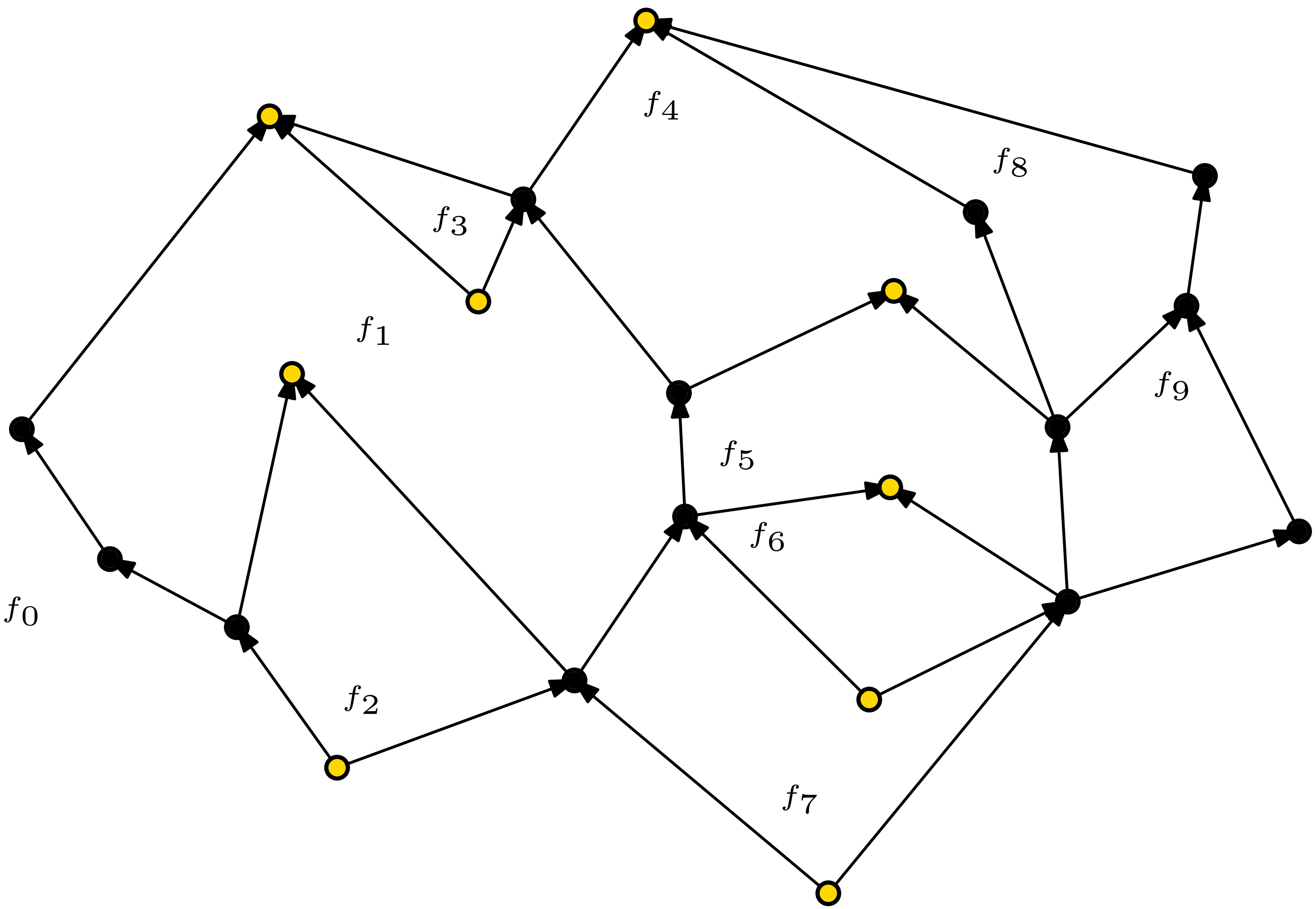
- Define assignment $\Phi : S \cup T \to \mathcal{F}$
  (S set of sources, T sinks), where
  $\Phi : v \mapsto$ incident face, where v is forms large angle

- Φ is called **consistent**, if: $|\Phi^{-1}(f)| = \begin{cases} A(f) - 1 & f \neq f_0 \\ A(f) + 1 & f = f_0 \end{cases}$

Example: Vertex-Face Assignment



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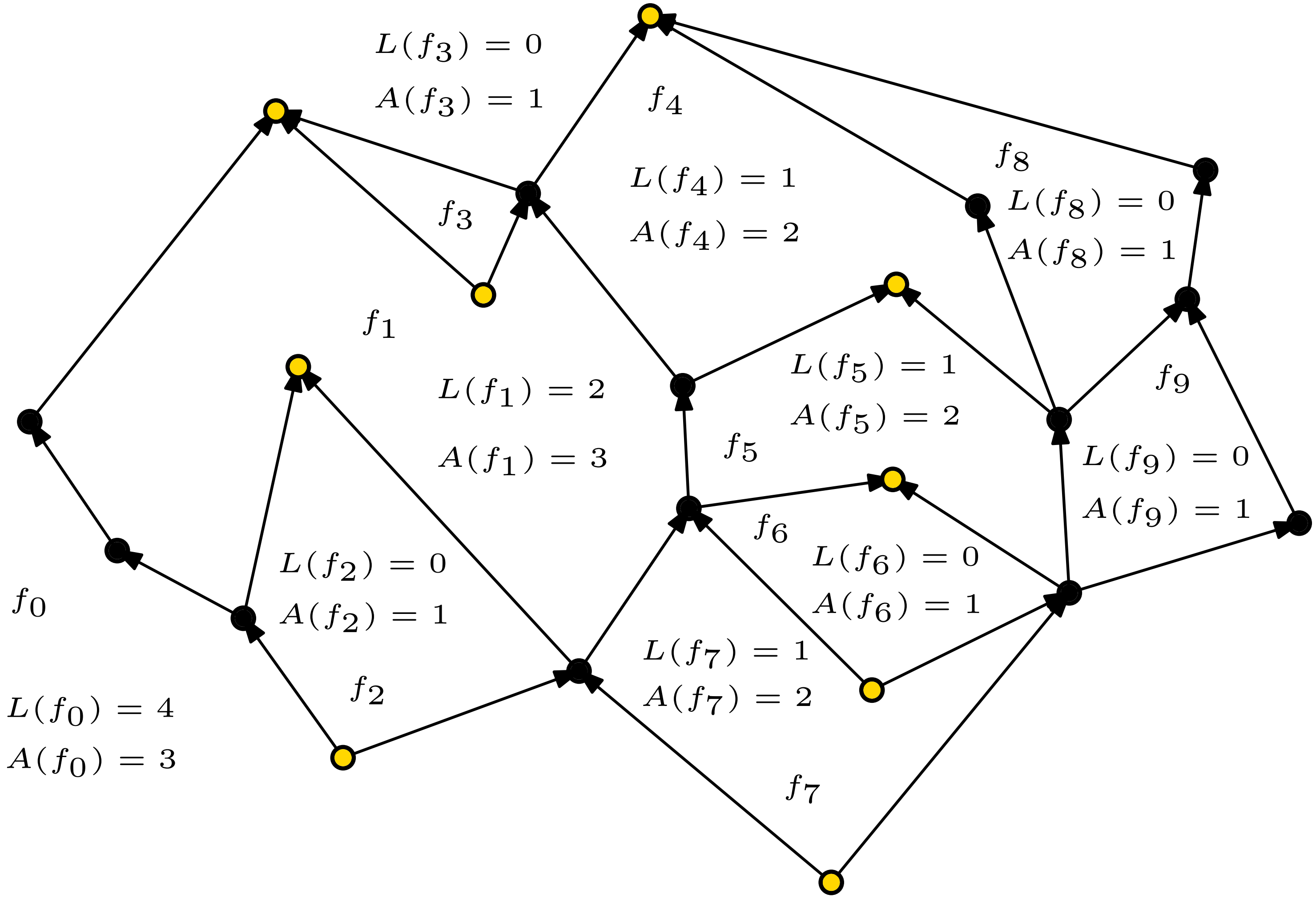

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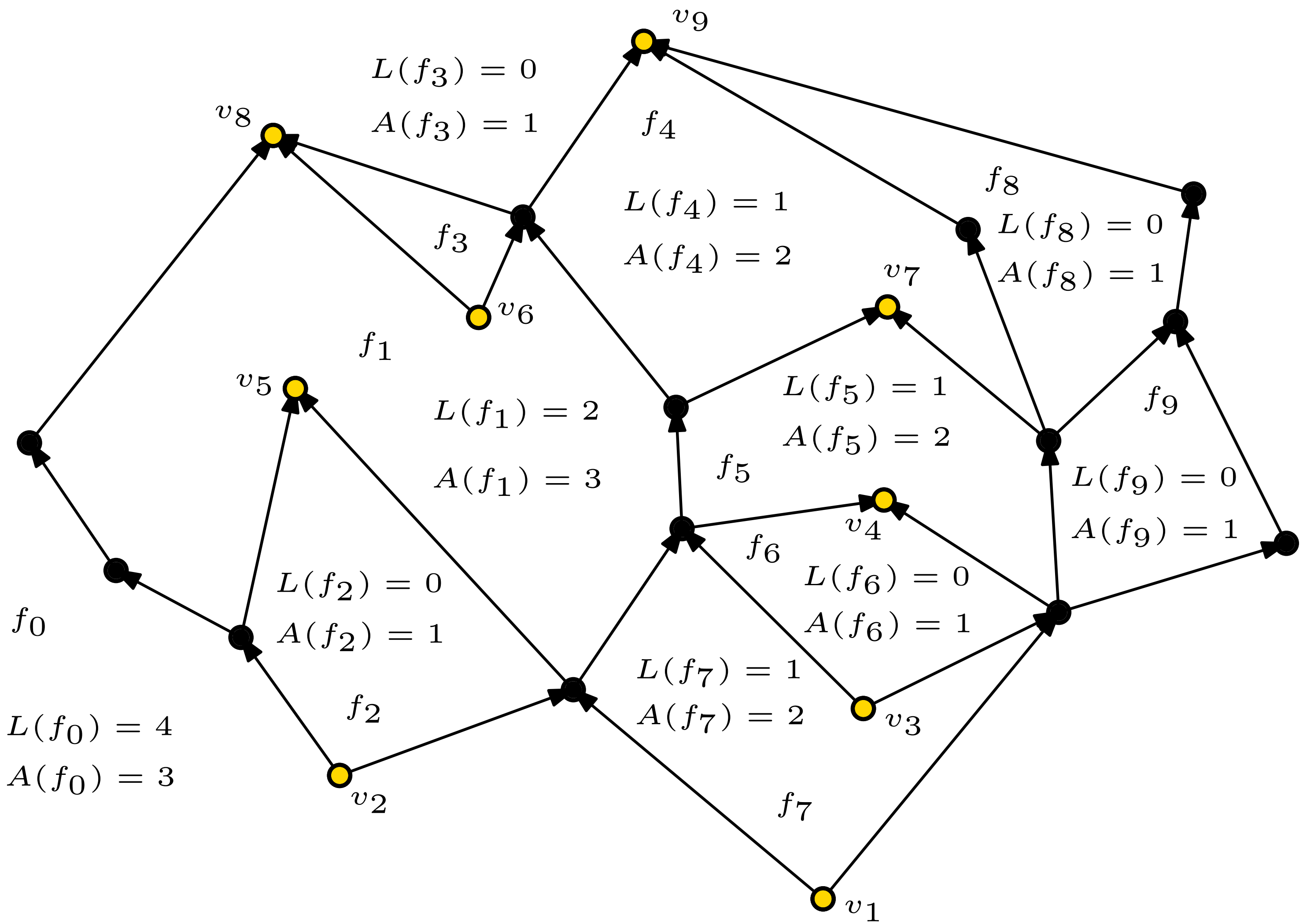

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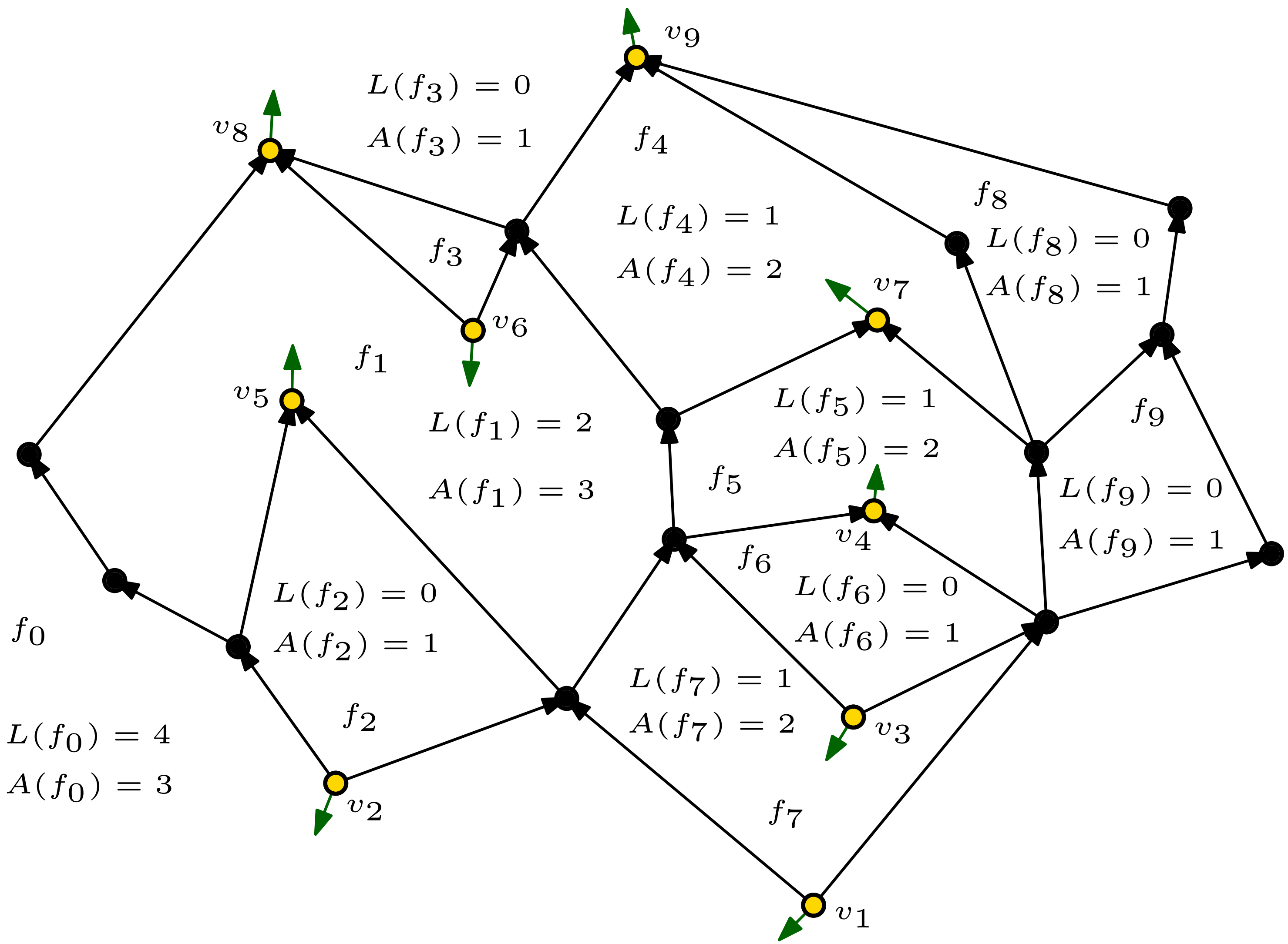

$$\Phi(v_1) = f_0$$
$$\Phi(v_2) = f_0$$
$$\Phi(v_3) = f_7$$
$$\Phi(v_4) = f_5$$
$$\Phi(v_5) = f_1$$
$$\Phi(v_6) = f_1$$
$$\Phi(v_7) = f_4$$
$$\Phi(v_8) = f_0$$
$$\Phi(v_9) = f_0$$

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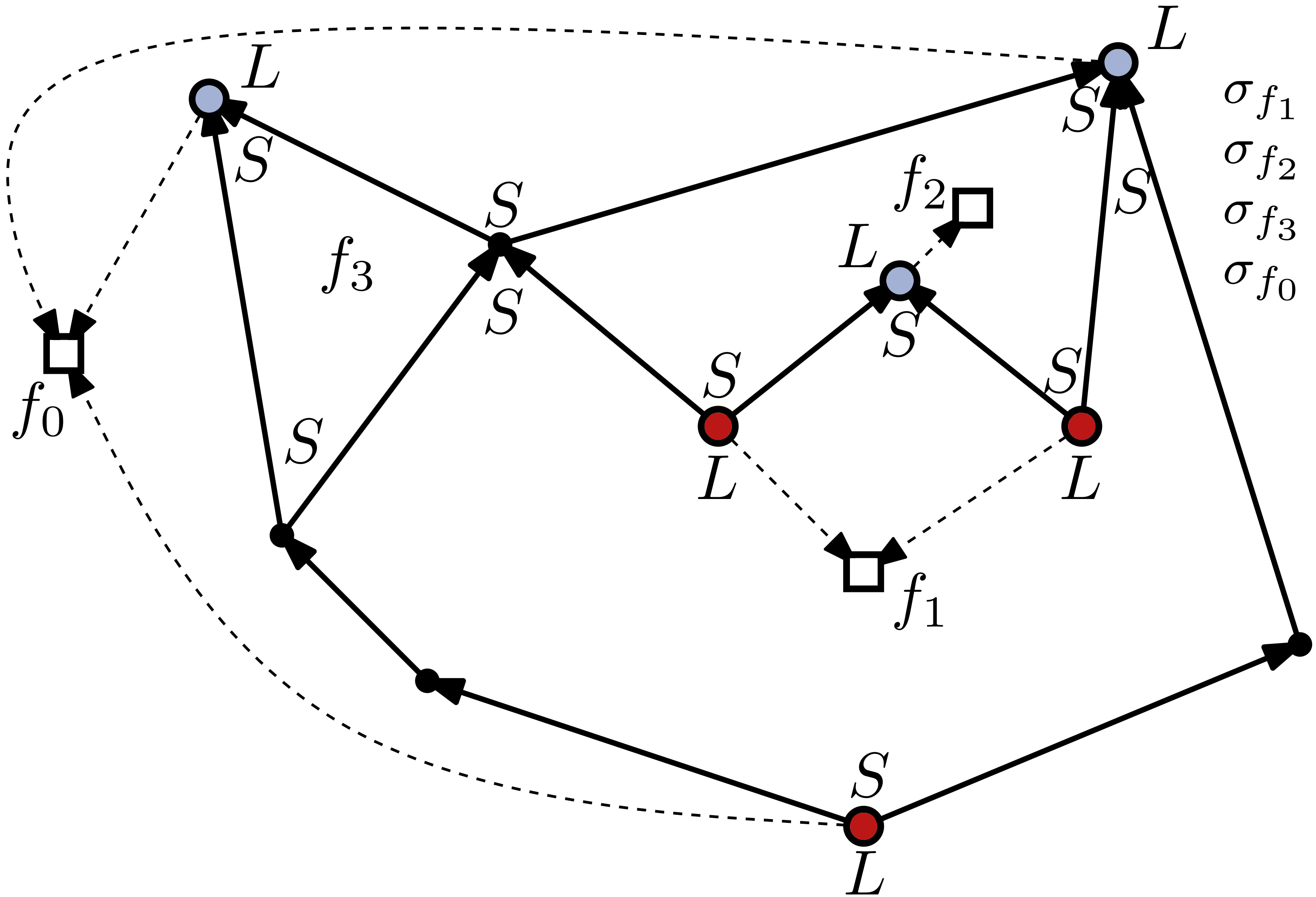
$\Leftarrow$ construct an st-digraph that contains D as spanning subgraph:

- insert edges in faces until they have single source and sink
- prove acyclicity, planarity and bimodality

Proof of Theorem 5

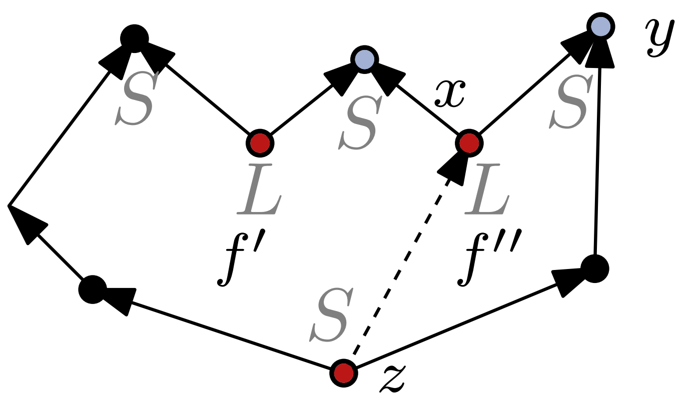
Assign labels s_L, t_L, s_S, t_S to each source/sink of each face f.
Sequence σ_f.

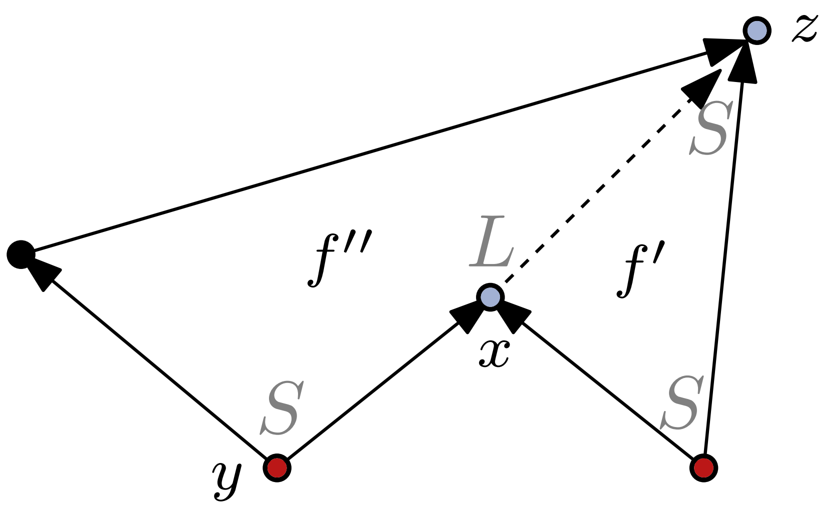


$$\sigma_{f_1} := (S, S, L, S, L, S)$$
$$\sigma_{f_2} := (L, S, S, S)$$
$$\sigma_{f_3} := (S, S)$$
$$\sigma_{f_0} := (L, L, S, L)$$

Proof of Theorem 5

- Cancel all sources and sinks: search for subsequence LSS.





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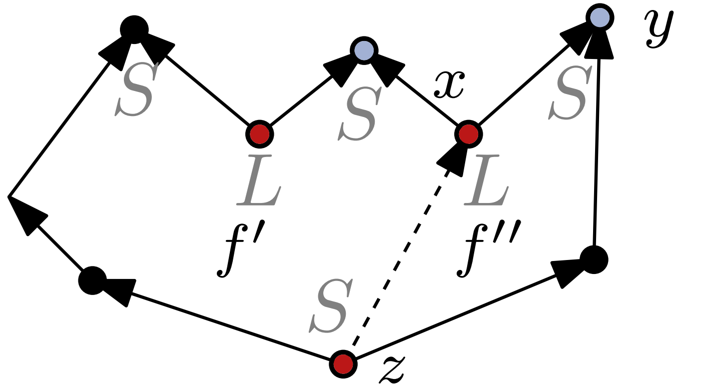

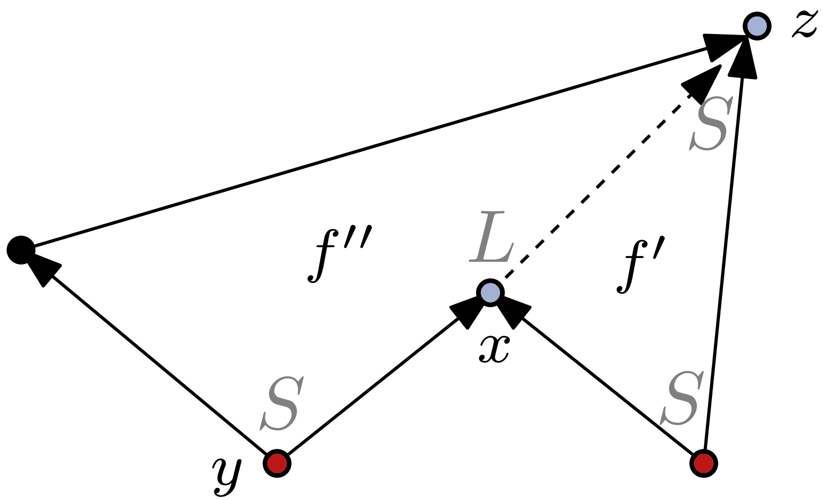

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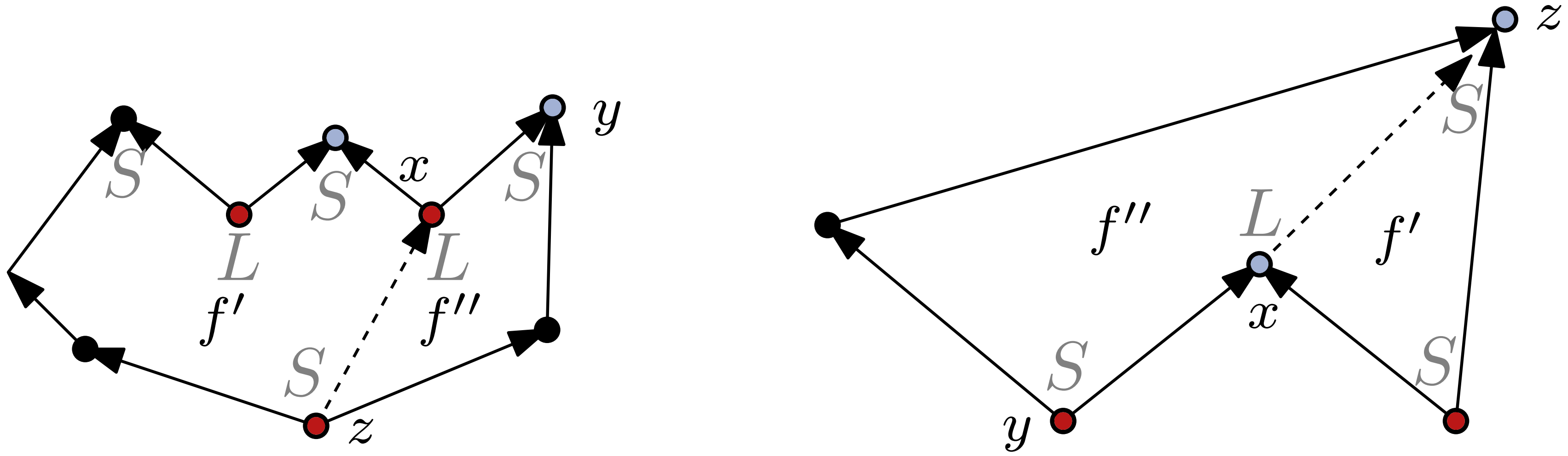

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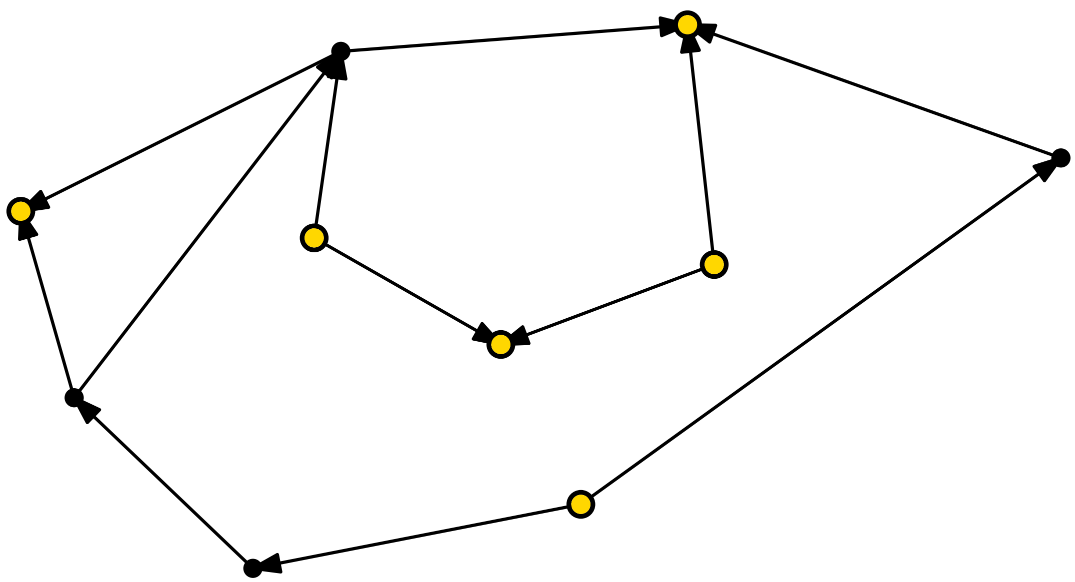
How to check whether a consistent assignment exists?

Flow Network

Def: Flow network $N(D, \mathcal{F}, f_0) = ((W, A_N); \ell; u; b)$

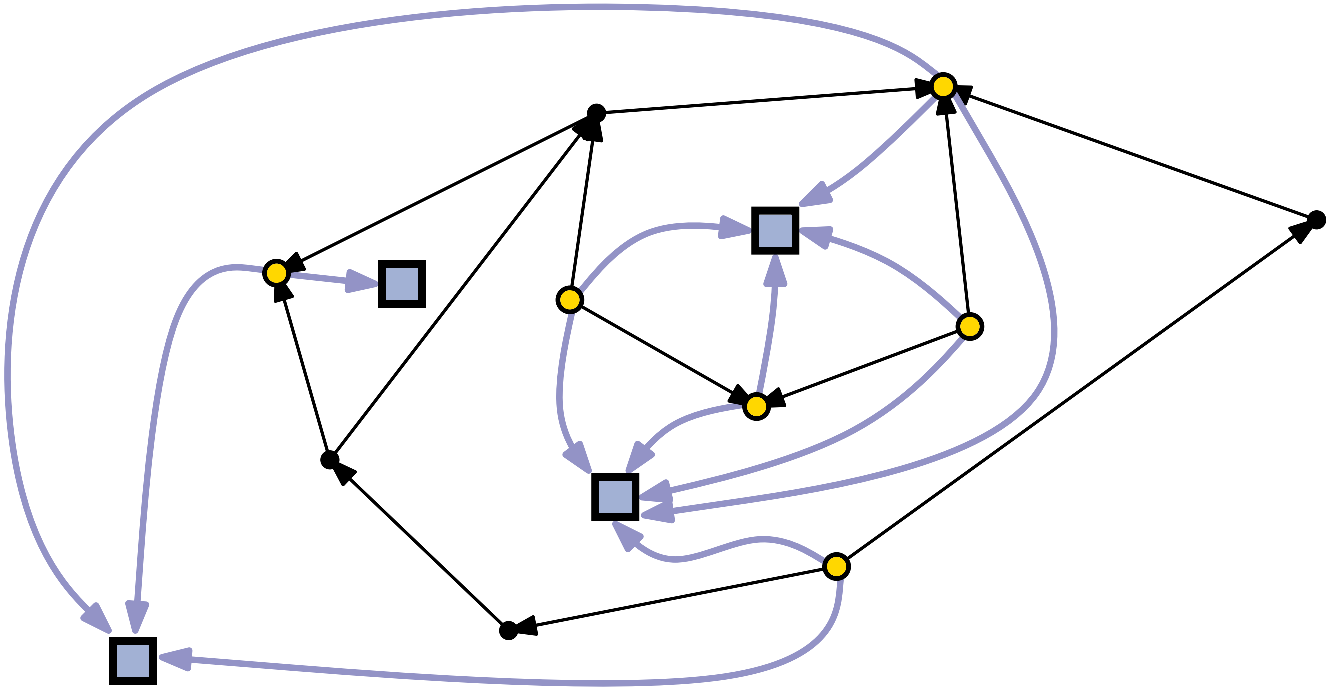
- $W = \{v \in V \mid v \text{ is source or sink } \} \cup \mathcal{F}$
- $A_N = \{(v, f) \mid v \text{ incident to } f\}$
- $\ell(a) = 0 \quad \forall a \in A_N$
- $u(a) = 1 \quad \forall a \in A_N$
- $b(q) = \begin{cases} 1 & \forall q \in W \cap V \\ -(A(q) - 1) & \forall q \in \mathcal{F} \setminus \{f_0\} \\ -(A(q) + 1) & q = f_0 \end{cases}$

Example



- normal nodes
- sources/sinks

Example



- normal nodes
- sources/sinks
- face nodes

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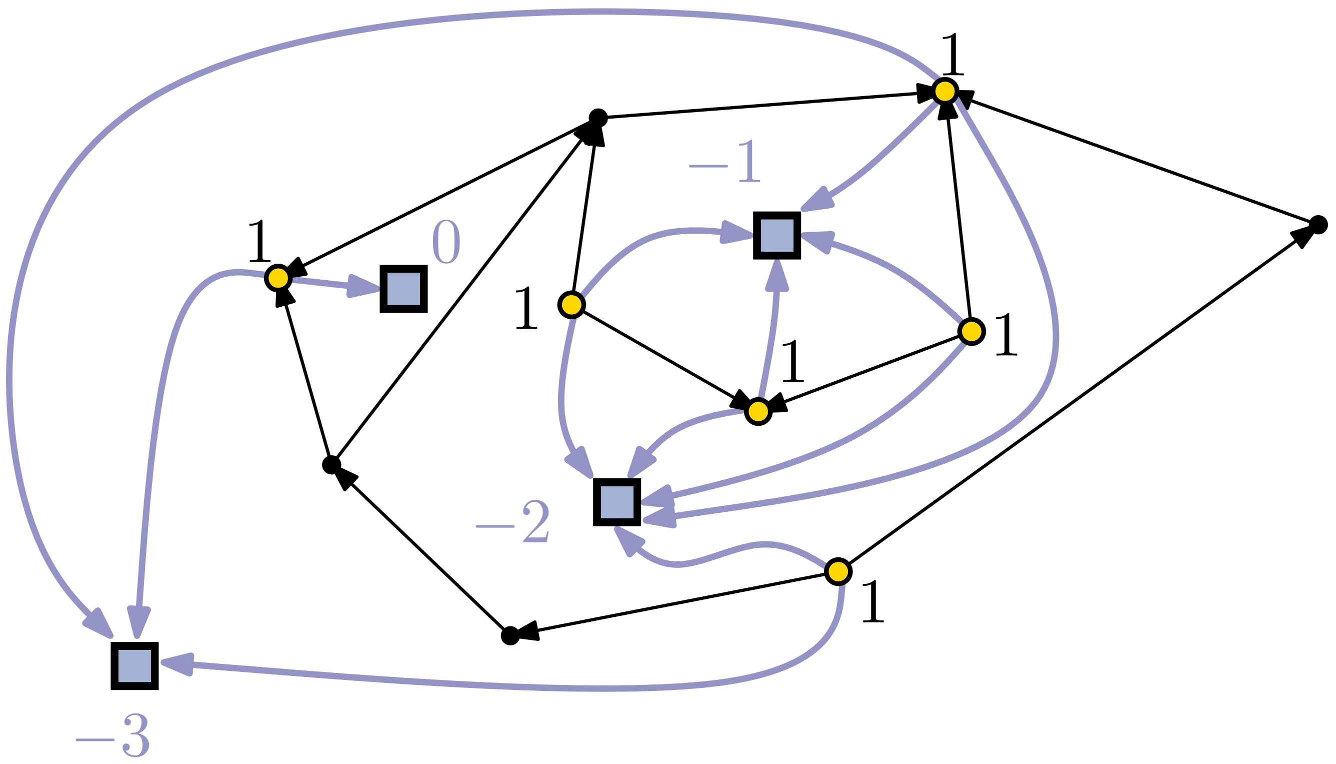

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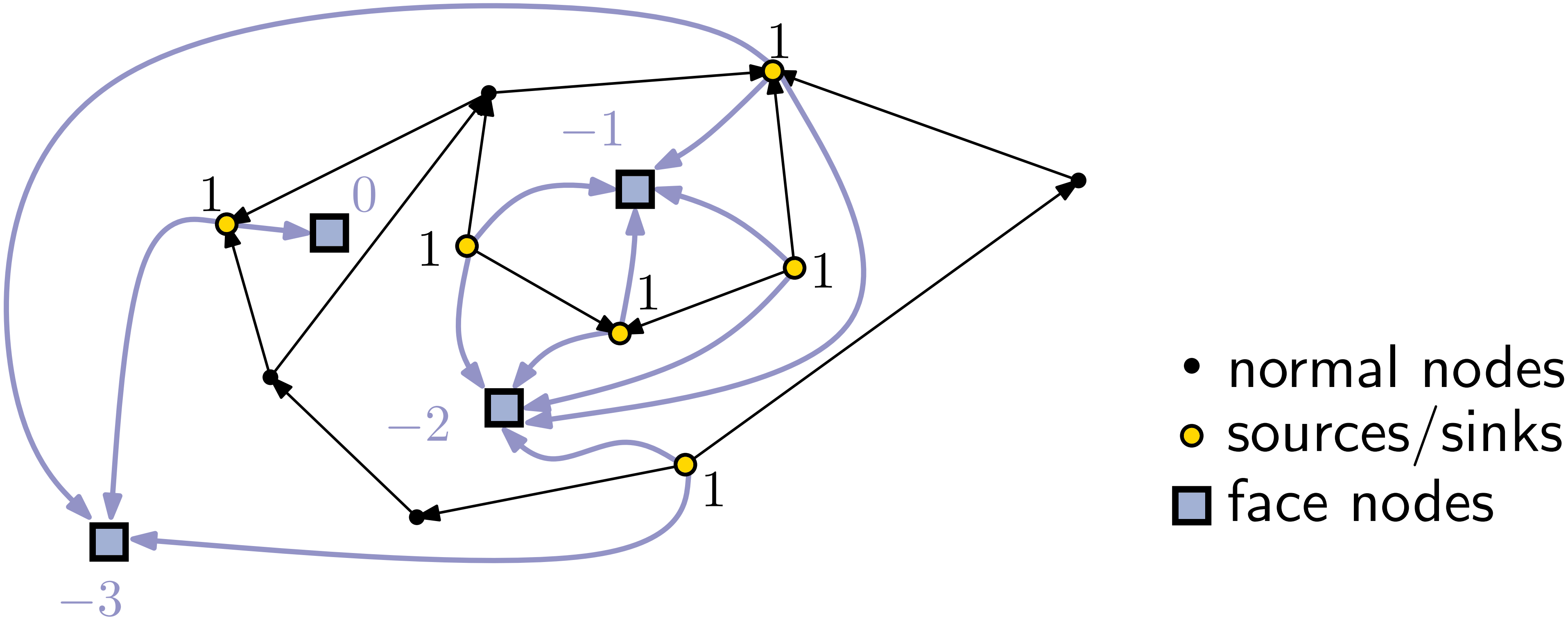
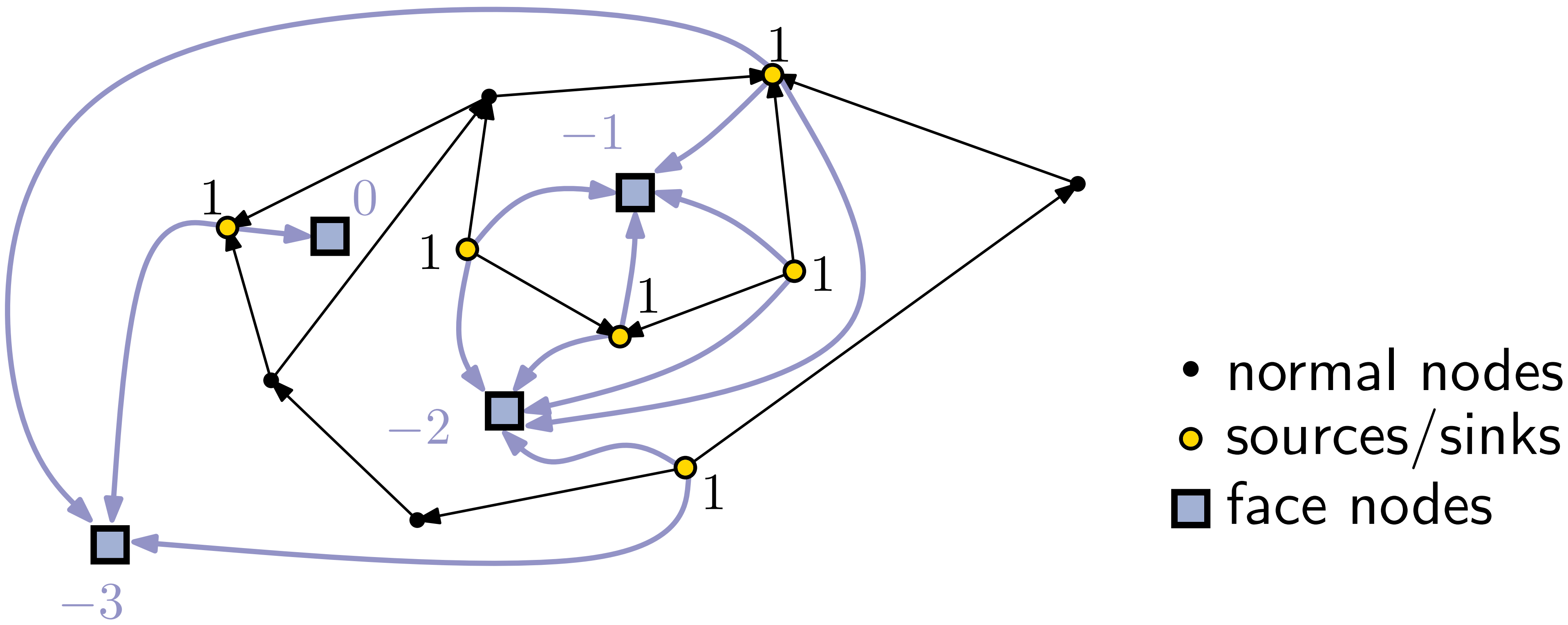


Thm 6: Let G be a directed acyclic digraph with embedding F and outer face f_0. The bipartite flow network $N(D, \mathcal{F}, f_0)$ admits a valid flow of value r ($\#$ of sources/sinks) if and only if G has a consistent assignment of sources and sinks to faces.

Example



- start with zero flow
- search for augmenting path (r times for total of r sources and sinks)

Final Remarks

- $O(rn)$ to decide whether consistent assignment exists
- works also without fixed outer face f_0: first compute all faces as internal and then add two units of demand to a face vertex and test whether the total flow can be augmented by two units. Do it for every face.

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(recall) combinatorial embedding $\mathcal{F}, f_0$ it holds:
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The layout can be constructed in the same time: $O(n)$ to augment to st-digraphs and $O(n)$ to draw the st-digraph

Discussion

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- There exists a fixed-parameter tractable algorithm to test upward planarity, with parameter being the number of triconnected components [Healy, Lynch SOFSEM 2005]
- The decision of Theorem 2 can be done in $O(n + r^{1.5})$ time where $r = \#$ sources/sinks [Abbasi, Healy, Rextin IPL 2010]
- many related concepts have been studied recently: quasi-planarity, upward drawings of mixed graphs