Algorithms for Graph Visualization
Flow Methods: Orthogonal Layouts – Part I

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20.11.2019
Orthogonal Layouts

• Edges consist of vertical and horizontal segments
• Applied in many areas
Orthogonal Layouts

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Orthogonal Layouts

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- Applied in many areas

Aesthetic criteria:
- number of bends
- length of edges
- width, height, area
- monotonicity of edges
- ...
(Planar) Orthogonal Drawings

Three-step approach: *Topology – Shape – Metrics*

\[ \begin{align*}
V &= \{v_1, v_2, v_3, v_4\} \\
E &= \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}
\end{align*} \]

(Planar) Orthogonal Drawings

Three-step approach:  

**Topology – Shape – Metrics**


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Reduce Crossings

combinatorial embedding/planarization
(Planar) Orthogonal Drawings

Three-step approach: **Topology – Shape – Metrics**

\[ V = \{v_1, v_2, v_3, v_4\} \]
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Reduce Crossings

combinatorial embedding/planarization

Bend Minimization

orthogonal representation

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reduce crossings

\begin{align*}
\text{combinatorial embedding/planarization} \\
\text{Bend Minimization} \\
\text{orthogonal representation} \\
\text{planar orthogonal drawing} \\
\text{Area minimization}
\end{align*}
(Planar) Orthogonal Drawings

Three-step approach: **Topology – Shape – Metrics**

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V = \{v_1, v_2, v_3, v_4\} \\
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Reduce Crossings

combinatorial embedding/planarization

Bend Minimization

orthogonal representation

Area minimization

planar orthogonal drawing

Orthogonal Representation

**Given:** planar graph \( G = (V, E) \), set of faces \( \mathcal{F} \), outer face \( f_0 \)

**Find:** orthogonal representation \( H(G) = \{ H(f) \mid f \in \mathcal{F} \} \)

**Face representation** \( H(f) \): of \( f \) is a clockwise ordered sequence of edge descriptions \((e, \delta, \alpha)\) with

- \( e \) edge of \( f \)
- \( \delta \) is sequence of \( \{0, 1\}^* \) (0 = right bend, 1 = left bend)
- \( \alpha \) is angle \( \in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\} \) between \( e \) and next edge \( e' \)
Orthogonal Representation: Example

\[ H(f_0) = ((e_1, 11, \frac{\pi}{2}), (e_5, 111, \frac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \frac{\pi}{2})) \]
\[ H(f_1) = ((e_1, 00, \frac{3\pi}{2}), (e_2, \emptyset, \frac{\pi}{2}), (e_6, 00, \pi)) \]
\[ H(f_2) = ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2})) \]

Combinatorial “drawing” of \( H(G) \)?
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is \( f_0 \) listed wrongly!? 

\[ H(f_2) = ((e_5, 000, \frac{\pi}{2}), (e_6, 11, \frac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \frac{\pi}{2})) \]
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concrete coordinates are not fixed yet!
Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to $\mathcal{F}, f_0$
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(H2) for an edge $\{u, v\}$ shared by faces $f$ and $g$ with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$
sequence $\delta_1$ is reversed and inverted $\delta_2$
Correctness of an Orthogonal Representation

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(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in $\delta$ and $r = (e, \delta, \alpha)$. For $C(r) := |\delta|_0 - |\delta|_1 + 2 - 2\alpha/\pi$ it holds that:

$\sum_{r \in H(f)} C(r) = 4$ for $f \neq f_0$ and $\sum_{r \in H(f_0)} C(r) = -4$
Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to $\mathcal{F}$, $f_0$

(H2) for an edge $\{u, v\}$ shared by faces $f$ and $g$ with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence $\delta_1$ is reversed and inverted $\delta_2$

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\sum_{r \in H(f)} C(r) = 4 \text{ for } f \neq f_0 \text{ and } \sum_{r \in H(f_0)} C(r) = -4
\]

(H4) For each node $v$ the sum of incident angles is $2\pi$
Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to $F, f_0$

(H2) for an edge $\{u, v\}$ shared by faces $f$ and $g$ with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence $\delta_1$ is reversed and inverted $\delta_2$

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(H4) For each node $v$ the sum of incident angles is $2\pi$

Pair, think and share:

What does the condition (H3) mean intuitively?
Bend Minimization with Given Embedding

Problem: Geometric Bend Minimization

Given: • planar Graph $G = (V, E)$ with maximum degree 4
   • combinatorial embedding $\mathcal{F}$ and outer face $f_0$

Find: orthogonal drawing with minimum number of bends that preserves the embedding
Bend Minimization with Given Embedding

**Problem: Geometric Bend Minimization**

Given: • planar Graph $G = (V, E)$ with maximum degree 4  
• combinatorial embedding $F$ and outer face $f_0$

Find: orthogonal drawing with minimum number of bends that preserves the embedding

compare with the following variation

**Problem: Combinatorial Bend Minimization**

Given: • planar Graph $G = (V, E)$ with maximum degree 4  
• combinatorial embedding $F$ and outer face $f_0$

Find: **orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding
Combinatorial Bend Minimization

Problem Combinatorial Bend Minimization

Given: • Graph $G = (V, E)$ with maximum degree 4
  • combinatorial embedding $\mathcal{F}$ and outer face $f_0$

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Combinatorial Bend Minimization

Problem Combinatorial Bend Minimization

Given: • Graph $G = (V, E)$ with maximum degree 4
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Find: **orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding

**Idea:** formulate as a network flow problem
  • a unit of flow represents an angle of $\pi/2$
  • flow from vertices to faces represents the angles at the vertices
  • flow between adjacent faces represent the bends at the edges
Reminder: \( s-t \) Flow Network

**Flow network** \( (D = (V, A); s, t; u) \) with

- directed graph \( D = (V, A) \)
- edge capacity \( u: A \rightarrow \mathbb{R}_0^+ \)
- source \( s \in V \), sink \( t \in V \)

A function \( X: A \rightarrow \mathbb{R}_0^+ \) is called **s-t-flow**, if:

\[
0 \leq X(i, j) \leq u(i, j) \quad \forall (i, j) \in A \tag{1}
\]

\[
\sum_{(i, j) \in A} X(i, j) - \sum_{(j, i) \in A} X(j, i) = 0 \quad \forall i \in V \setminus \{s, t\} \tag{2}
\]
Reminder: General Flow Network

**Flow network** \((D = (V, A); \ell; u; b)\) with
- directed graph \(D = (V, A)\)
- edge **lower bound** \(\ell: A \to \mathbb{R}_0^+\)
- edge **capacity** \(u: A \to \mathbb{R}_0^+\)
- node **production/consumption** \(b: V \to \mathbb{R}\) with \(\sum_{i \in V} b(i) = 0\)

A function \(X: A \to \mathbb{R}_0^+\) is called **valid flow**, if:

\[
\ell(i, j) \leq X(i, j) \leq u(i, j) \quad \forall (i, j) \in A \tag{3}
\]

\[
\sum_{(i, j) \in A} X(i, j) - \sum_{(j, i) \in A} X(j, i) = b(i) \quad \forall i \in V \tag{4}
\]
Problems for Flow Networks

(A) Valid Flow Problem:
Find a valid flow $X : A \rightarrow \mathbb{R}_0^+$, i.e., such that
- lower bounds and capacities $\ell(e), u(e)$ are respected
  (inequalities (3))
- consumption/production $b(i)$ satisfied (inequalities (4))
Problems for Flow Networks

(A) Valid Flow Problem:
Find a valid flow $X : A \rightarrow \mathbb{R}_0^+$, i.e., such that
- lower bounds and capacities $\ell(e), u(e)$ are respected (inequalities (3))
- consumption/production $b(i)$ satisfied (inequalities (4))

Additionally provided: **Cost function** $\text{cost} : A \rightarrow \mathbb{R}_0^+$

**Def:** $\text{cost}(X) := \sum_{(i,j) \in A} \text{cost}(i,j) \cdot X(i,j)$
Problems for Flow Networks

(A) Valid Flow Problem:
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Additionally provided: Cost function $\text{cost}: A \rightarrow \mathbb{R}_0^+$

Def: $\text{cost}(X) := \sum_{(i,j) \in A} \text{cost}(i, j) \cdot X(i, j)$

(B) Minimum Cost Flow Problem:
Find a valid flow $X: A \rightarrow \mathbb{R}_0^+$, that minimizes cost function $\text{cost}(X)$ (over all valid flows)
Flow Network for Bend Minimization

Define flow network \( N(G) = ((V \cup F, A); \ell; u; b; \text{cost}) \):

- \( A = \{(v, f) \in V \times F \mid v \text{ incident to } f\} \cup \{(f, g) \in F \times F \mid f, g \text{ adjacent through edge } e\} \)
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- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2(d_G(f) - 2) \quad \forall f \in F \setminus \{f_0\}$
- $b(f_0) = -2(d_G(f_0) + 2)$
Flow Network for Bend Minimization

Define flow network $N(G) = ((V \cup \mathcal{F}, A); \ell; u; b; \text{cost})$:

- $A = \{(v, f) \in V \times \mathcal{F} \mid v \text{ incident to } f\} \cup \{(f, g) \in \mathcal{F} \times \mathcal{F} \mid f, g \text{ adjacent through edge } e\}$
- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2(d_G(f) - 2) \quad \forall f \in \mathcal{F} \setminus \{f_0\}$
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$\Rightarrow \sum_{w} b(w) = 0$
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\[ \sum_i b(i) = 0 \quad \text{(Euler)} \]
Flow Network for Bend Minimization

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- $b(f_0) = -2(d_G(f_0) + 2)$

$$\forall (f, g) \in A, f, g \in F \quad \ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$$

$$\forall (v, f) \in A, v \in V, f \in F \quad \ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$$

$$\sum_i b(i) = 0 \quad \text{(Euler)}$$
Example Flow Network

$\mathbf{f}_0$

\begin{tikzpicture}
  \node (v1) at (0,0) [fill=black]{$v_1$};
  \node (v2) at (2,2) [fill=black]{$v_2$};
  \node (v3) at (4,2) [fill=black]{$v_3$};
  \node (v4) at (4,0) [fill=black]{$v_4$};
  \node (v5) at (0,2) [fill=black]{$v_5$};

  \node (e1) at (0.5,1) [left]{$e_1$};
  \node (e2) at (1.5,1.5) [left]{$e_2$};
  \node (e3) at (3.5,2) [right]{$e_3$};
  \node (e4) at (3.5,0) [right]{$e_4$};
  \node (e5) at (2,0) [below]{$e_5$};
  \node (e6) at (0.5,0.5) [below]{$e_6$};

  \draw (v1) -- (v2);
  \draw (v2) -- (v3);
  \draw (v3) -- (v4);
  \draw (v4) -- (v5);
  \draw (v5) -- (v1);

  \draw (v1) to [out=90,in=180] (v2);
  \draw (v2) to [out=0,in=90] (v3);
  \draw (v3) to [out=270,in=90] (v4);
  \draw (v4) to [out=270,in=270] (v5);
  \draw (v5) to [out=90,in=270] (v1);

  \draw (v1) -- (v5);

  \draw (v2) -- (v3);

  \draw (v3) -- (v4);

  \draw (v4) -- (v5);

  \draw (v5) -- (v1);

  \node (f1) at (1,1) [above]{$f_1$};
  \node (f2) at (2.5,1) [above]{$f_2$};

  \node (f0) at (1.5,0.5) [below]{$f_0$};
\end{tikzpicture}
Example Flow Network

\[ f_0 \]

\[ V \]

\[ F \]
Example Flow Network

\[ V \times \mathcal{F} \supseteq \]
Example Flow Network

\[ V \times \mathcal{F} \supseteq \]

\[ \mathcal{F} \times \mathcal{F} \supseteq \]
Example Flow Network

\[ f_0 \]

\[ f_1 \]

\[ f_2 \]

\[ e_1 \]

\[ e_2 \]

\[ e_3 \]

\[ e_4 \]

\[ e_5 \]

\[ e_6 \]

\[ V \times \mathcal{F} \supseteq \]

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Example Flow Network

\[ f_0 \]

\[ f_1 \]

\[ f_2 \]

\[ f_3 \]

\[ f_4 \]

\[ f_5 \]

\[ v_1 \]

\[ v_2 \]

\[ v_3 \]

\[ v_4 \]

\[ v_5 \]

\[ e_1 \]

\[ e_2 \]

\[ e_3 \]

\[ e_4 \]

\[ e_5 \]

\[ e_6 \]

\[ \ell / u / \text{cost} \quad 1/4/0 \]

\[ V \times \mathcal{F} \supseteq \]

\[ \mathcal{F} \times \mathcal{F} \supseteq \]
Example Flow Network
Example Flow Network

\[ f_0 \]

\[ v_1 \]

\[ e_1 \]

\[ v_2 \]

\[ e_2 \]

\[ v_3 \]

\[ e_3 \]

\[ v_4 \]

\[ e_4 \]

\[ v_5 \]

\[ e_5 \]

\[ e_6 \]

\[ f_1 \]

\[ f_2 \]

\[ f_0 \]

\[ \ell/u/cost \]

\[ 1/4/0 \]

\[ V \times \mathcal{F} \supseteq \]

\[ 0/\infty/1 \]

\[ \mathcal{F} \times \mathcal{F} \supseteq \]
Example Flow Network

cost = 1
bend! outside
Main Statement

**Thm 1:** A planar embedded graph $(G, \mathcal{F}, f_0)$ has a valid orthogonal representation $H(G)$ with $k$ bends if and only if the flow network $N(G)$ has a valid flow $X$ with cost $k$. 
Main Statement

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**Proof:**

\(\Leftarrow\) Given valid flow \(X\) in \(N(G)\) with cost \(k\)

Construct orthogonal representation \(H(G)\) with \(k\) bends
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• show properties (H1)–(H4)
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\(\Rightarrow\) Given an orthogonal representation \(H(G)\) with \(k\) bends

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**Proof:**

\(\Leftarrow\) Given valid flow \(X\) in \(N(G)\) with cost \(k\)

   Construct orthogonal representation \(H(G)\) with \(k\) bends
   
   • transform from flow to orthogonal description
   • show properties (H1)–(H4)

\(\Rightarrow\) Given an orthogonal representation \(H(G)\) with \(k\) bends

   Construct valid flow \(X\) in \(N(G)\) with cost \(k\)
   
   • define flow \(X : A \rightarrow \mathbb{R}_0^+\)
   • show that \(X\) is a valid flow and has cost \(k\)
Summary of Bend Minimization

• From Theorem 1 it follows that the combinatorial orthogonal bend minimization problem for embedded planar graphs can be solved using an algorithm for the Min-Cost-Flow Problem.
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• This special flow problem for a planar network $N(G)$ can be solved in $O(n^{3/2})$ time. [Cornelsen, Karrenbauer GD 2011]
Summary of Bend Minimization

- From Theorem 1 it follows that the combinatorial orthogonal bend minimization problem for embedded planar graphs can be solved using an algorithm for the Min-Cost-Flow Problem.

- This special flow problem for a planar network $N(G)$ can be solved in $O(n^{3/2})$ time. [Cornelsen, Karrenbauer GD 2011]

(Planar) Orthogonal Drawings

Three-step approach:  

Topology – Shape – Metrics

\[ V = \{v_1, v_2, v_3, v_4\} \]
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Bend Minimization

Reduce Crossings

Area minimization

Orthogonal representation

Combinatorial embedding/plannerization
(Planar) Orthogonal Drawings

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\[ V = \{v_1, v_2, v_3, v_4\} \]
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Reduce Crossings

combinatorial embedding/planarization

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Compaction

Compaction Problem:
Given: • planar graph $G = (V, E)$ with maximum degree 4
  • orthogonal representation $H(G)$
Find: compact orthogonal layout of $G$ that realizes $H(G)$
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Special Case: all faces are rectangles
→ Guarantees possible
  • minimum total edge length
  • minimum area
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We will formulate a flow network for (horizontal) compaction
Flow Network for Edge Length Computation

**Def:** Flow Network $\mathcal{N}_{\text{hor}} = (((W_{\text{hor}}, A_{\text{hor}}); \ell; u; b; \text{cost}))$

- $W_{\text{hor}} = \mathcal{F} \setminus \{f_0\} \cup \{s, t\}$
- $A_{\text{hor}} = \{(f, g) \mid f, g \text{ share a horizontal segment and } f \text{ lies below } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in A_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in A_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in A_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$

![Flow Network Diagram]
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$s$ and $t$ represent lower and upper side of $f_0$
Flow Network for Edge Length Computation

Def: Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, A_{\text{ver}}); \ell; u; b; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$
- $A_{\text{ver}} = \{(f, g) | f, g \text{ share a vertical segment and } f \text{ lies to the left of } g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \ \forall a \in A_{\text{ver}}$
- $u(a) = \infty \ \forall a \in A_{\text{ver}}$
- $\text{cost}(a) = 1 \ \forall a \in A_{\text{ver}}$
- $b(f) = 0 \ \forall f \in W_{\text{ver}}$
Flow Network for Edge Length Computation

**Def:** Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, A_{\text{ver}}); \ell; u; b; \text{cost})$

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What values of the drawing represent the following?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$?
- $\sum_{a \in A_{\text{hor}}} X_{\text{hor}}(a) + \sum_{a \in A_{\text{ver}}} X_{\text{ver}}(a)$