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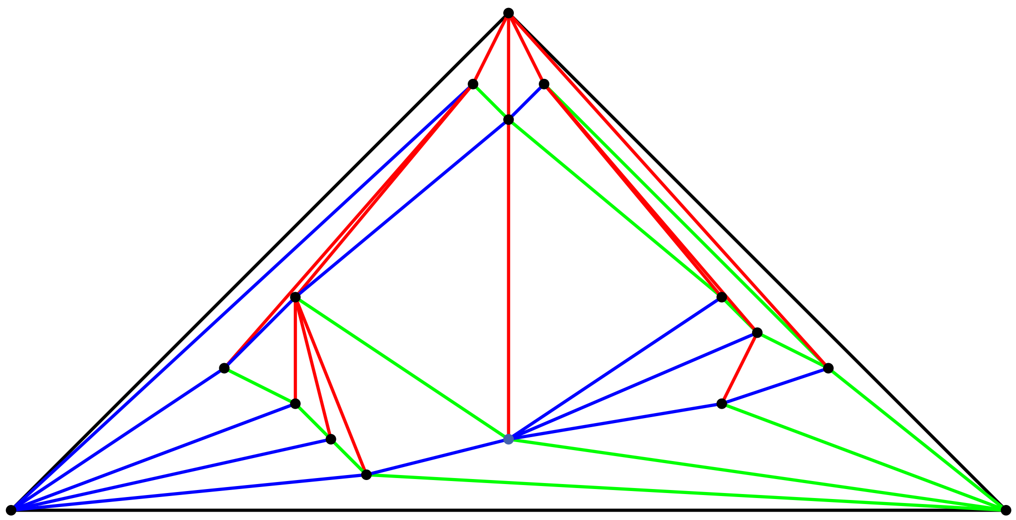
Algorithms for graph visualization

Layouts for planar graphs. Realizer method.

WINTER SEMESTER 2018/2019

Tamara Mchedlidze



Today

Last lecture:

Theorem [De Fraysseix, Pach, Pollack '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(2n-4) \times (n-2)$.

This lecture:

Theorem [Schnyder '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(n-2) \times (n-2)$.

Overview

- Barycentric Coordinates and Barycentric Graph Representation.
- Schnyder Labelling. Existence.
- Schnyder Realizer.
- From Schnyder Realizer to Barycentric Representation.

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Barycentric Representation

Barycentric Coordinates

Let $A, B, C, P \in \mathbb{R}^2$. A triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that:

- $\alpha + \beta + \gamma = 1$
- $P = \alpha A + \beta B + \gamma C$

is called barycentric coordinates of P with respect to $\triangle ABC$.

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Barycentric Representation

A Barycentric Representation of a graph $G = (V, E)$ is an assigment of barycentric coordinates to the vertices of G, i.e. it is an injective function $v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$, such that:

- $v_a + v_b + v_c = 1$ for all $v \in V$
- for each $(x, y) \in E$ and each $z \in V \setminus \{x, y\}$, $\exists k \in \{a, b, c\}$ with $x_k < z_k$ and $y_k < z_k$.

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What does this condition mean?

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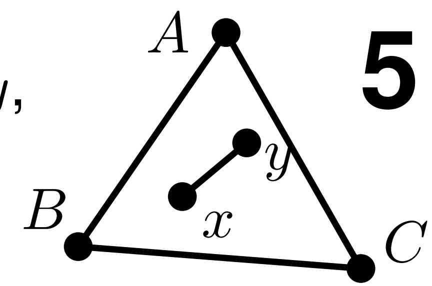
Discuss with your neighbour(s) and then share

- Consider the a-coordinates of points x and y, which is bigger?
- Interpret the condition below.



5 min

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Barycentric Representation

Lemma [Schnyder '90]

Let $v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$ be a barycentric representation of a graph $G = (V, E)$ an let $A, B, C \in \mathbb{R}^2$. The function

$$f : v \in V \mapsto v_a A + v_b B + v_c C$$

gives a planar drawing of G inside triangle $\triangle ABC$.

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Schnyder Labeling

Definition: Schnyder-Labeling

A Schnyder-Labeling of a planar trianglulated graph G is a labeling of all internal angles with labels $1, 2$ and 3 such that:

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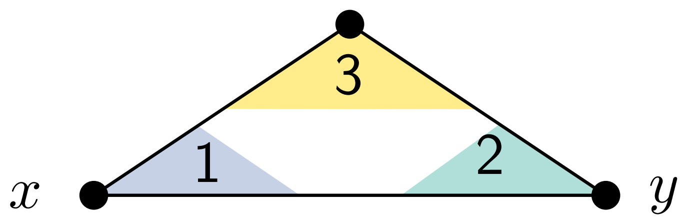
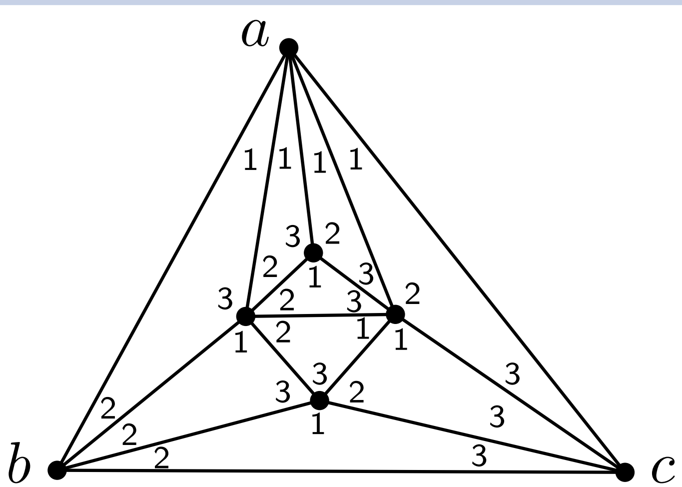
Face Each internal face contain vertices with all three labels 1, 2 and 3, appearing in a counterclockwise order.





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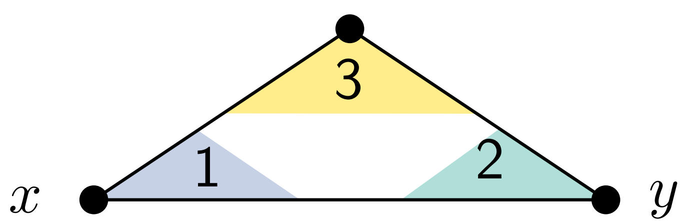
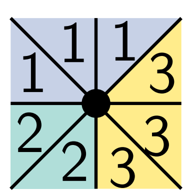
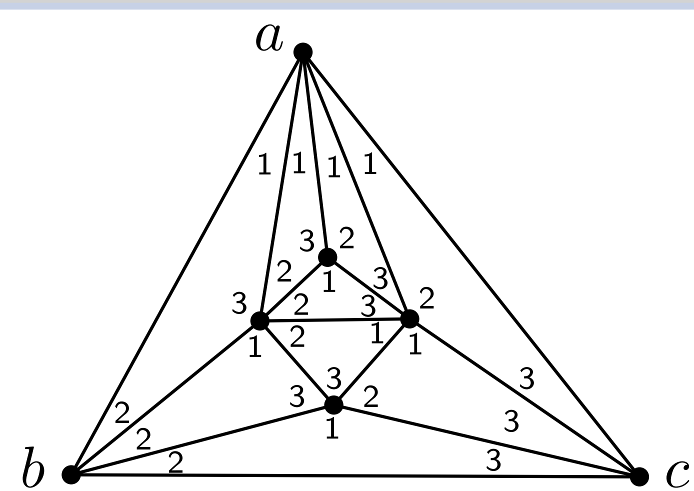
Vertex The counterclockwise ordering of the labels around each vertex consists of a nonempty interval of 1's followed by a nonempty interval of 2's followed by a nonempty interval of 3's.







Schnyder Labeling

Theorem [Schnyder ’90]

Every triangulated plane graph has a Schnyder labeling.

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Lehrstuhl Algorithmik I

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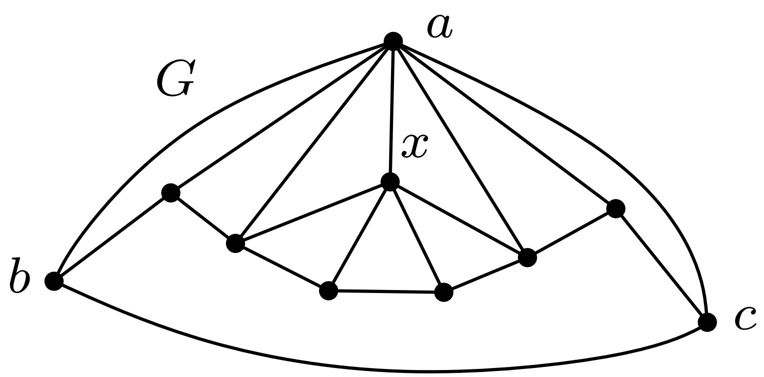

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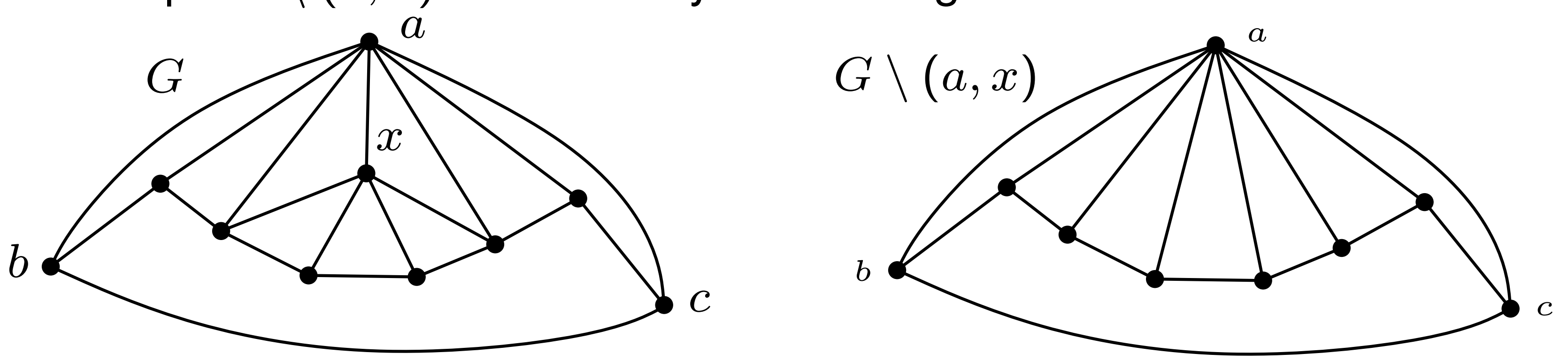

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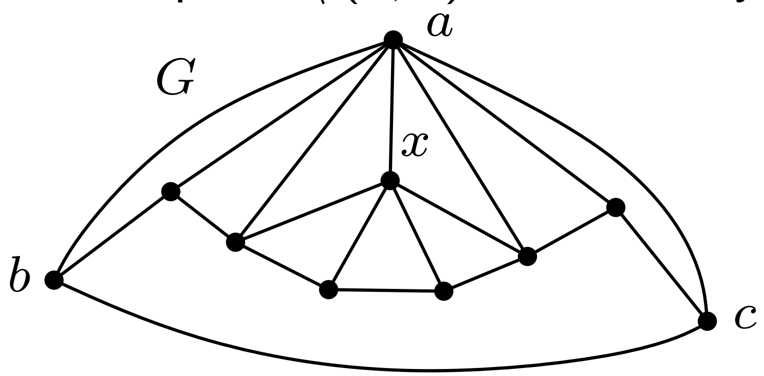

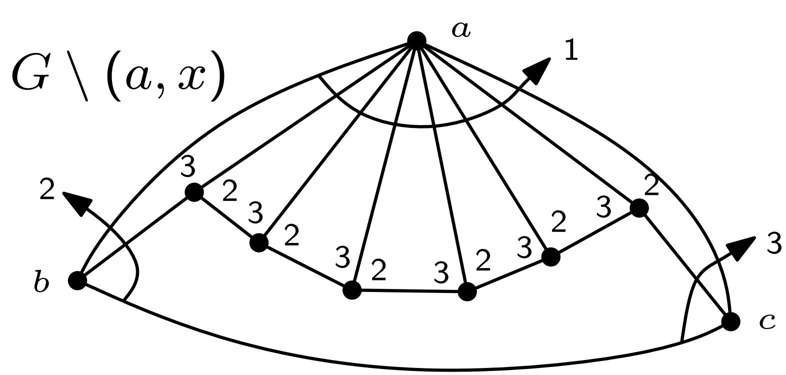

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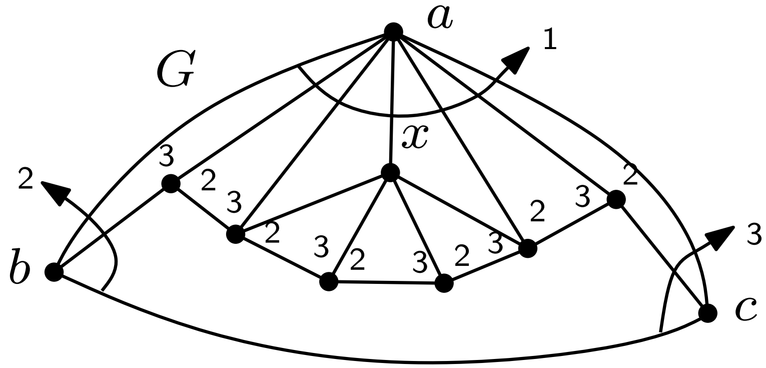

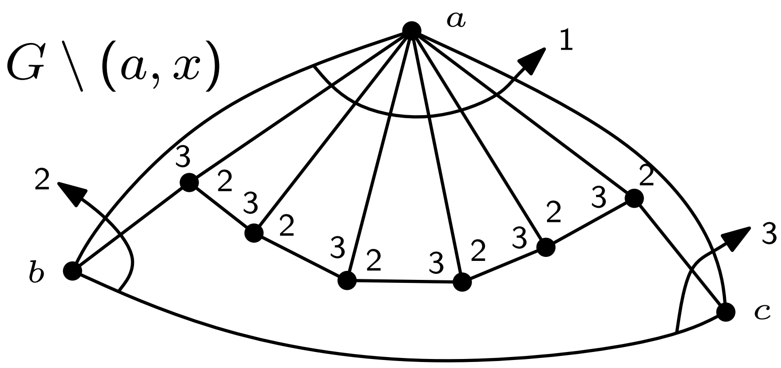

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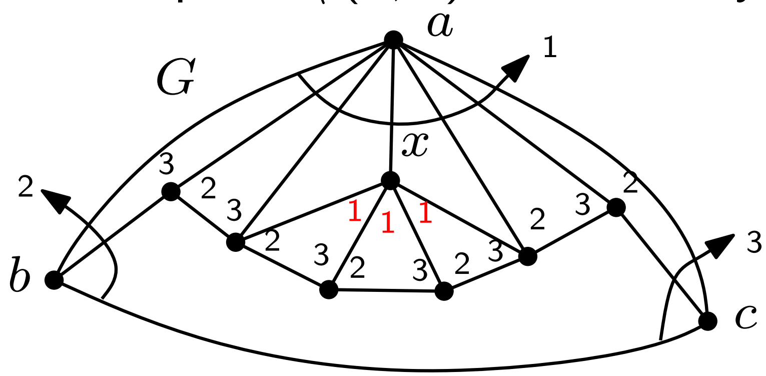

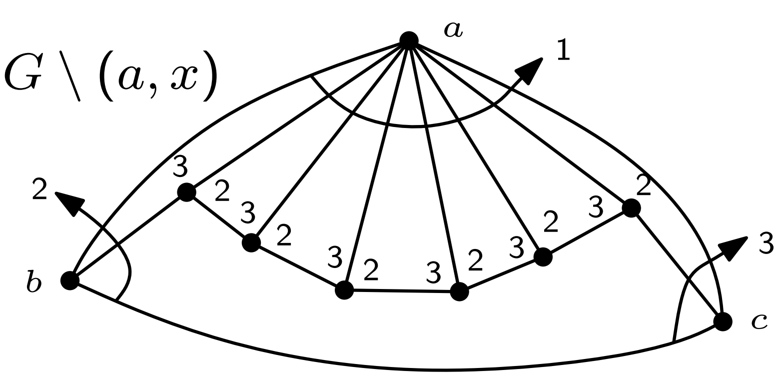

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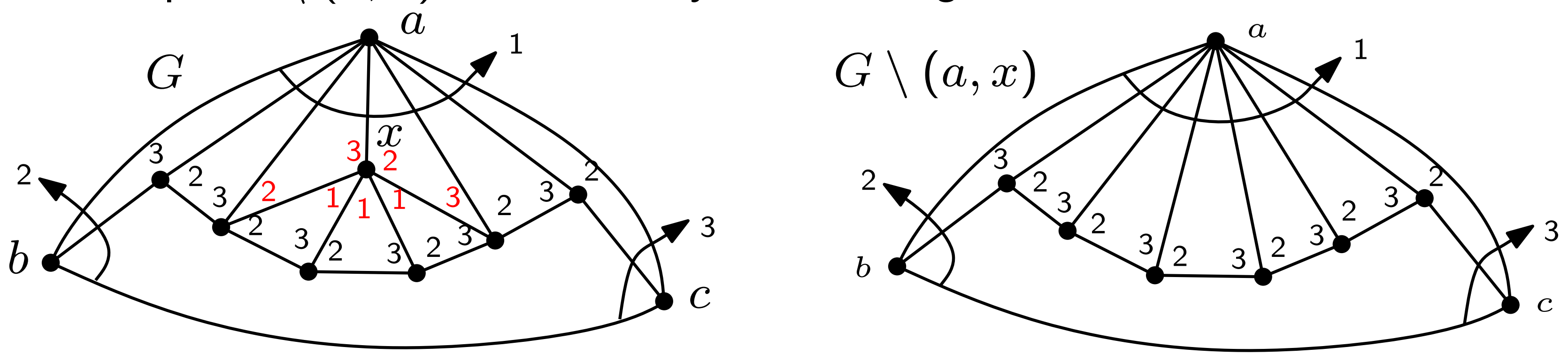

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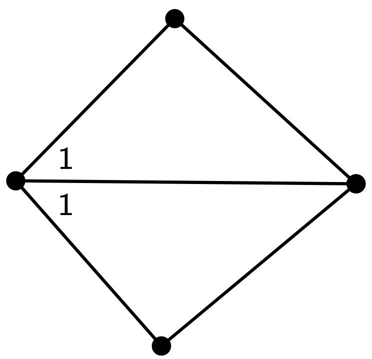
- Schnyder labeling induces an edge labeling.



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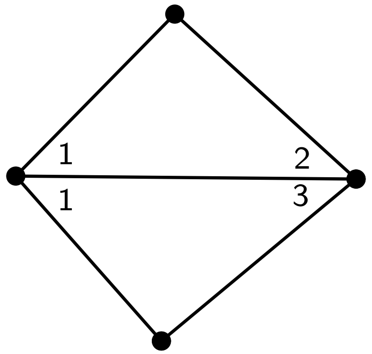

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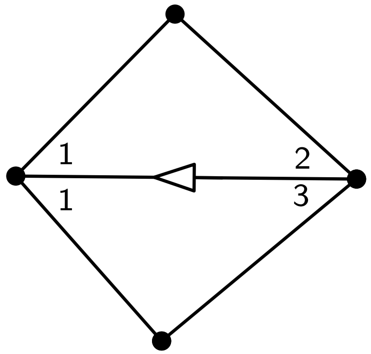

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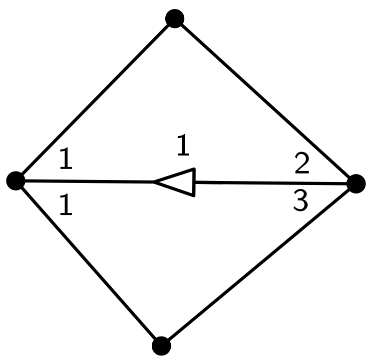

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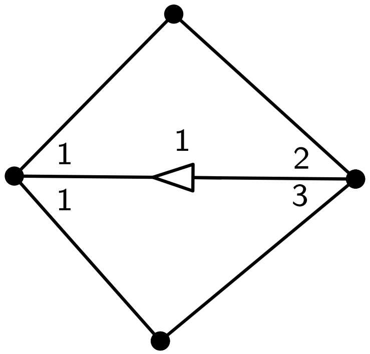

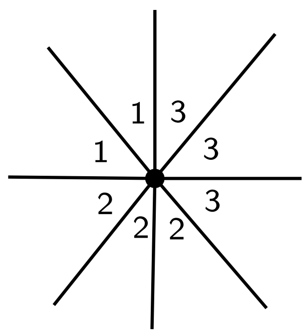

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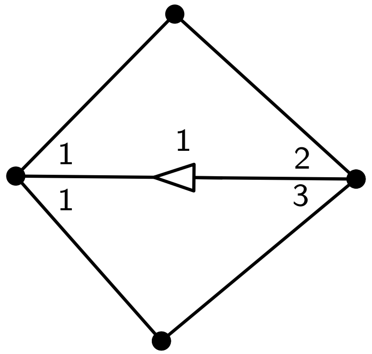

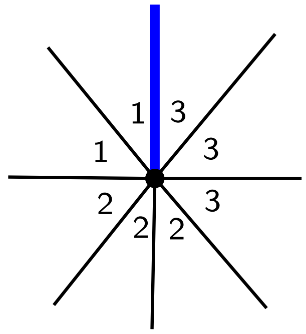

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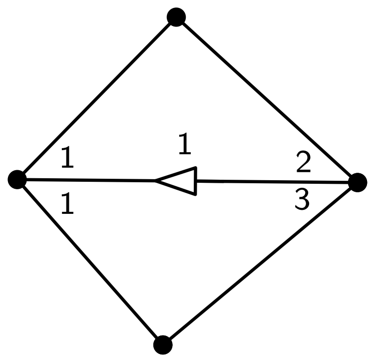

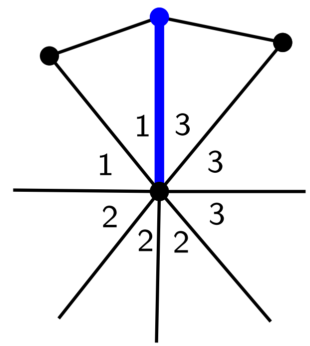

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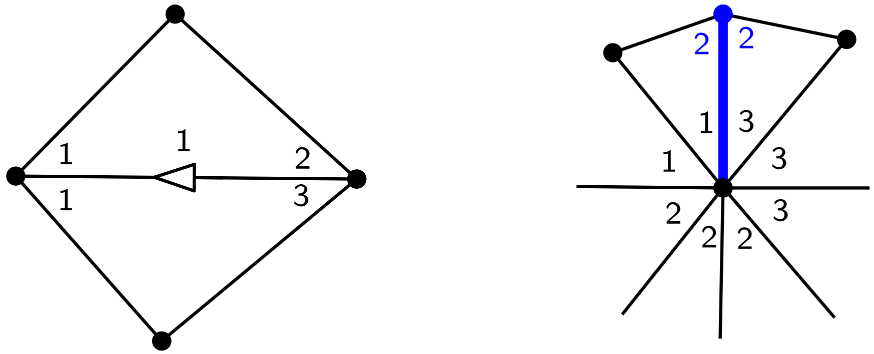

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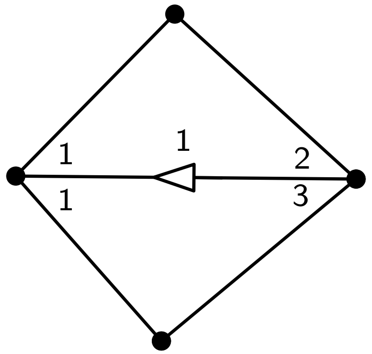

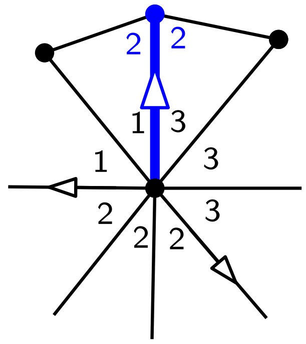

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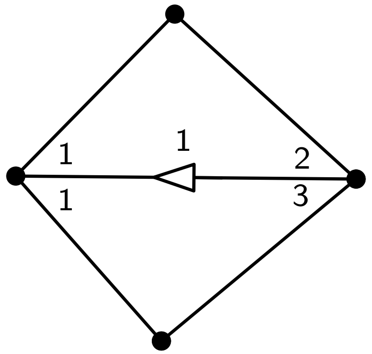

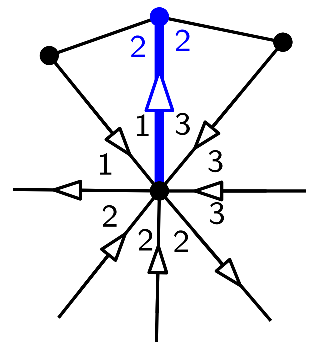

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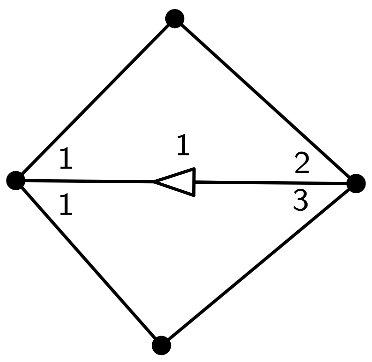

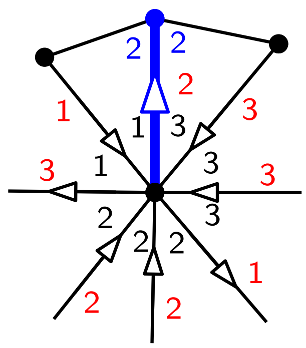

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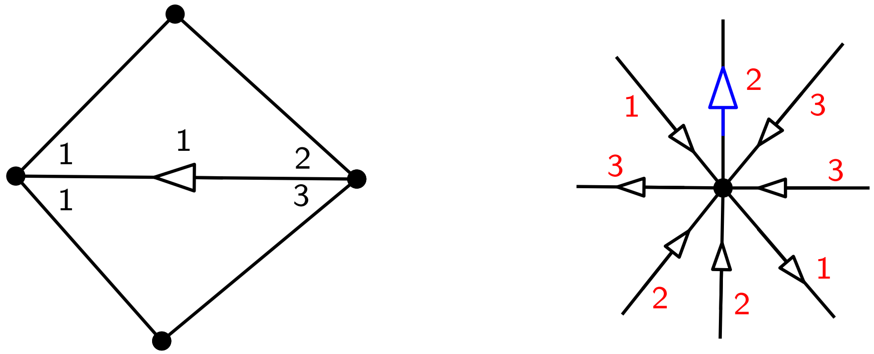

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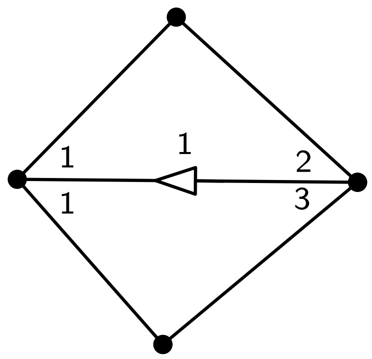

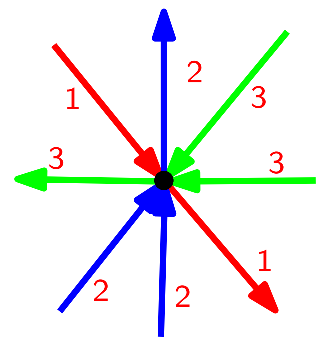

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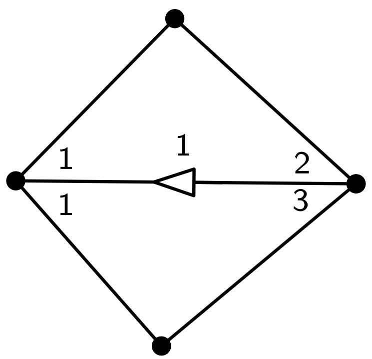

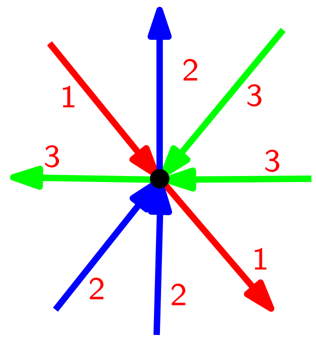

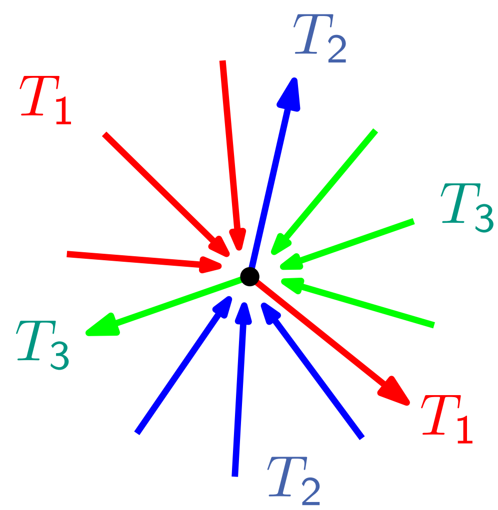

Definition: Schnyder Forest

A Schnyder Forest or a Realizer of a planar triangulated graph $G = (V, E)$ is a partition of the inner edges of E into three sets of oriented edges T_1, T_2, T_3 such that for each inner vertex $v \in V$ hold:

- v has an outgoing edge in each of T_1, T_2, T_3
- The counterclockwise order of the edges around v is as follows: edges leaving in T_1, entering in T_3, leaving in T_2, entering in T_1, leaving in T_3, entering in T_2.

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Theorem [Schnyder '90]

Every triangulated plane graph has a Schnyder labeling.

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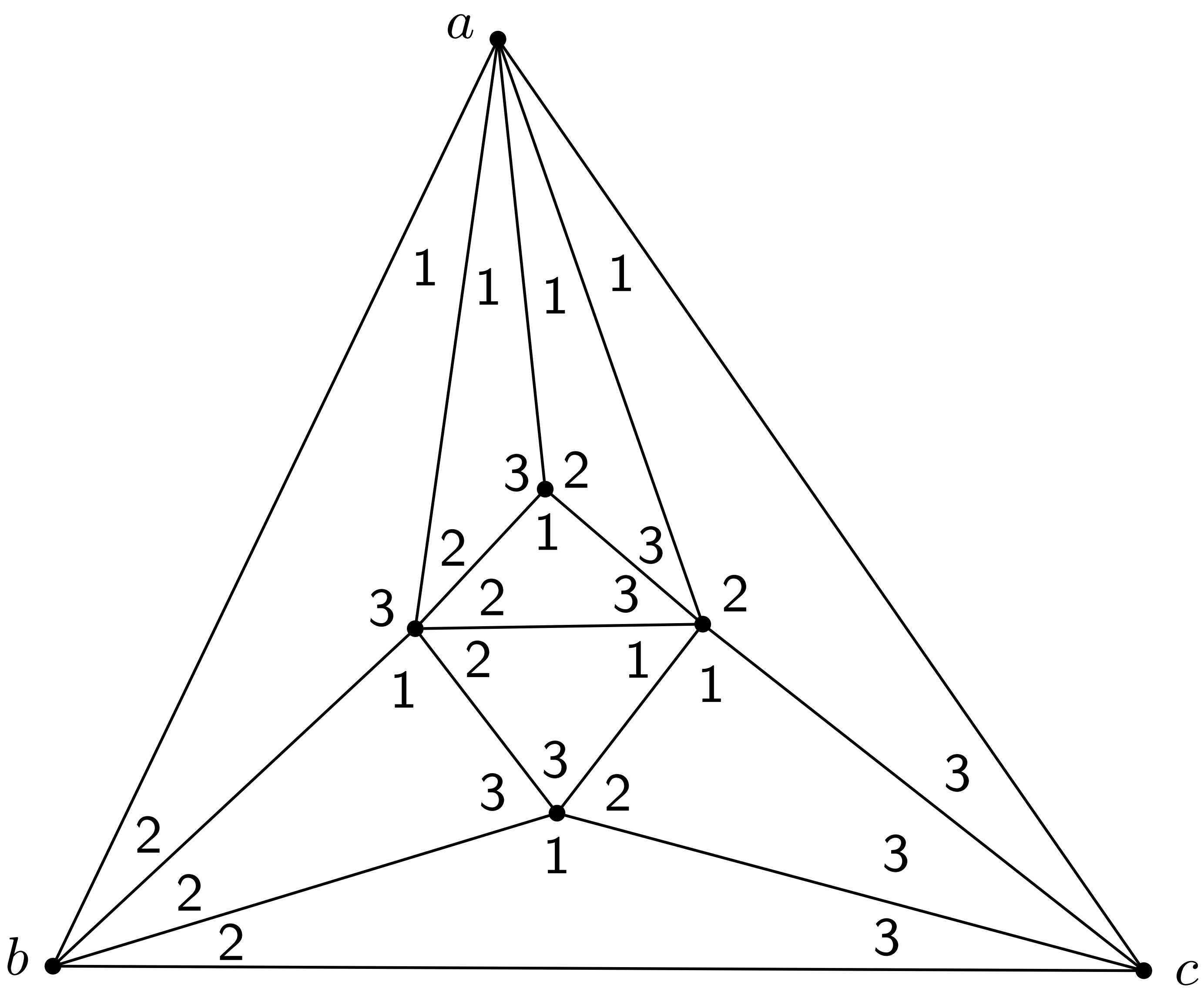
By this theorem and by previous construction:

Theorem [Schnyder ’90]

Every triangulated plane graph has a Schnyder realizer.

Schnyder Realizer



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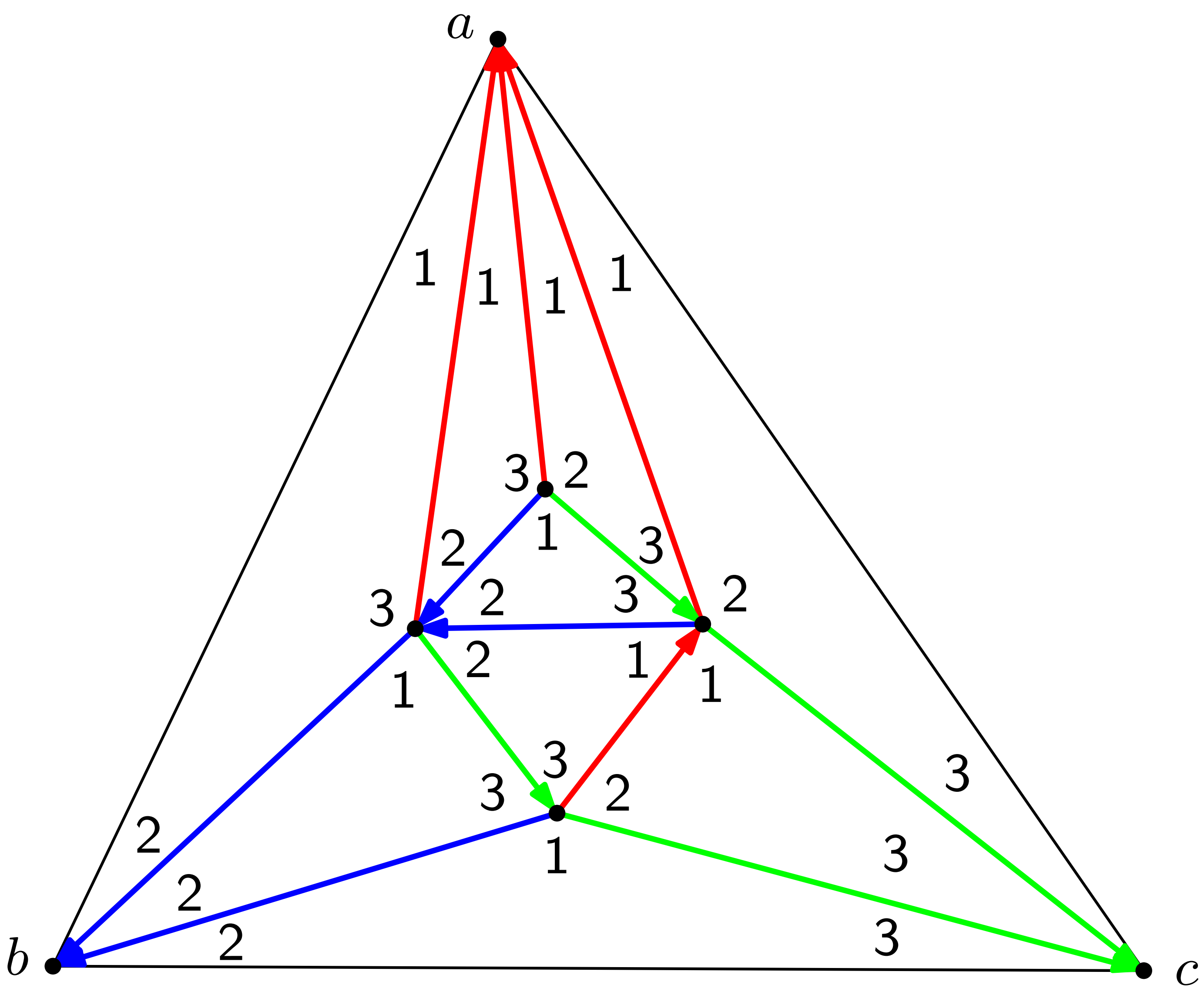
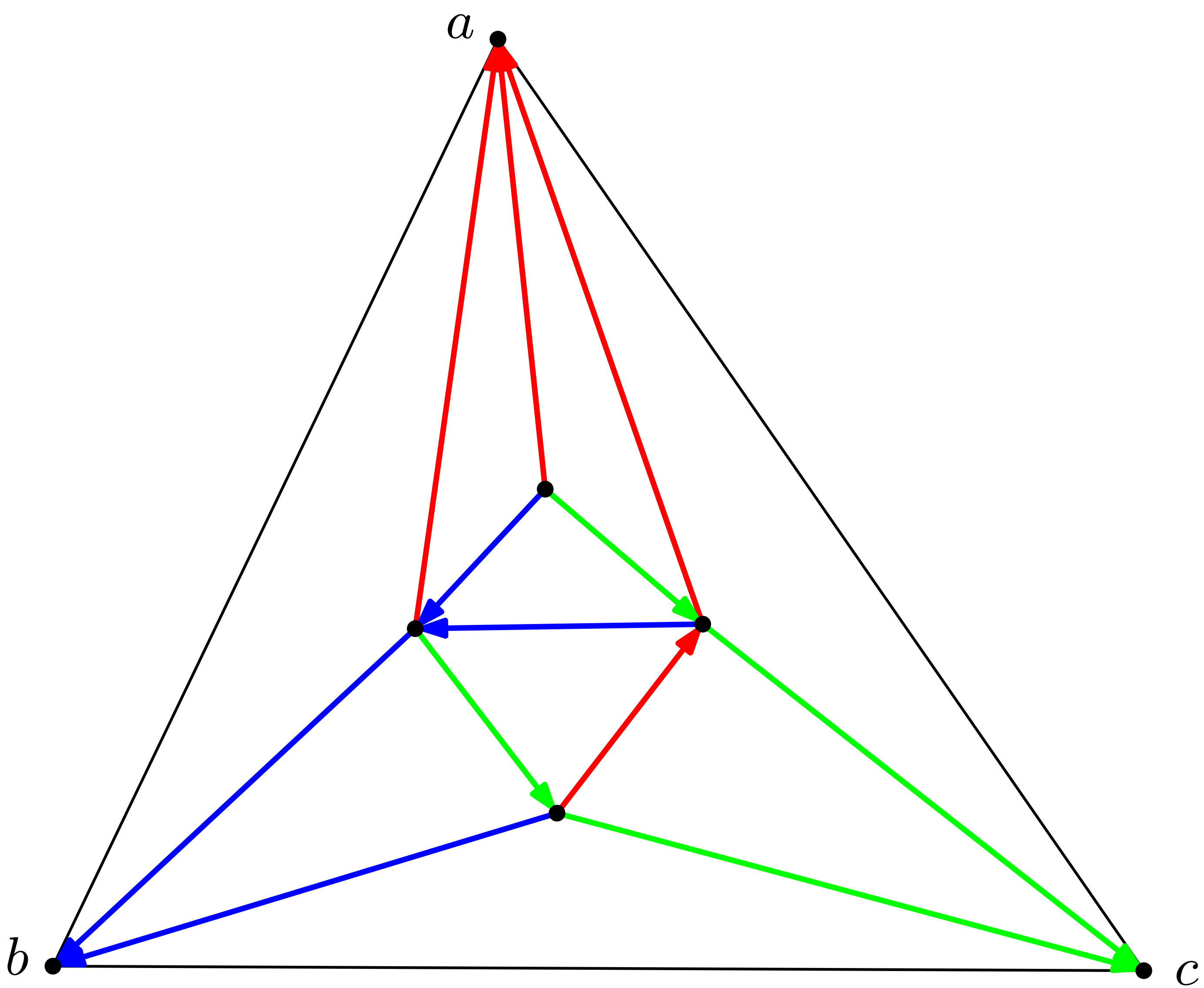


Schnyder Realizer



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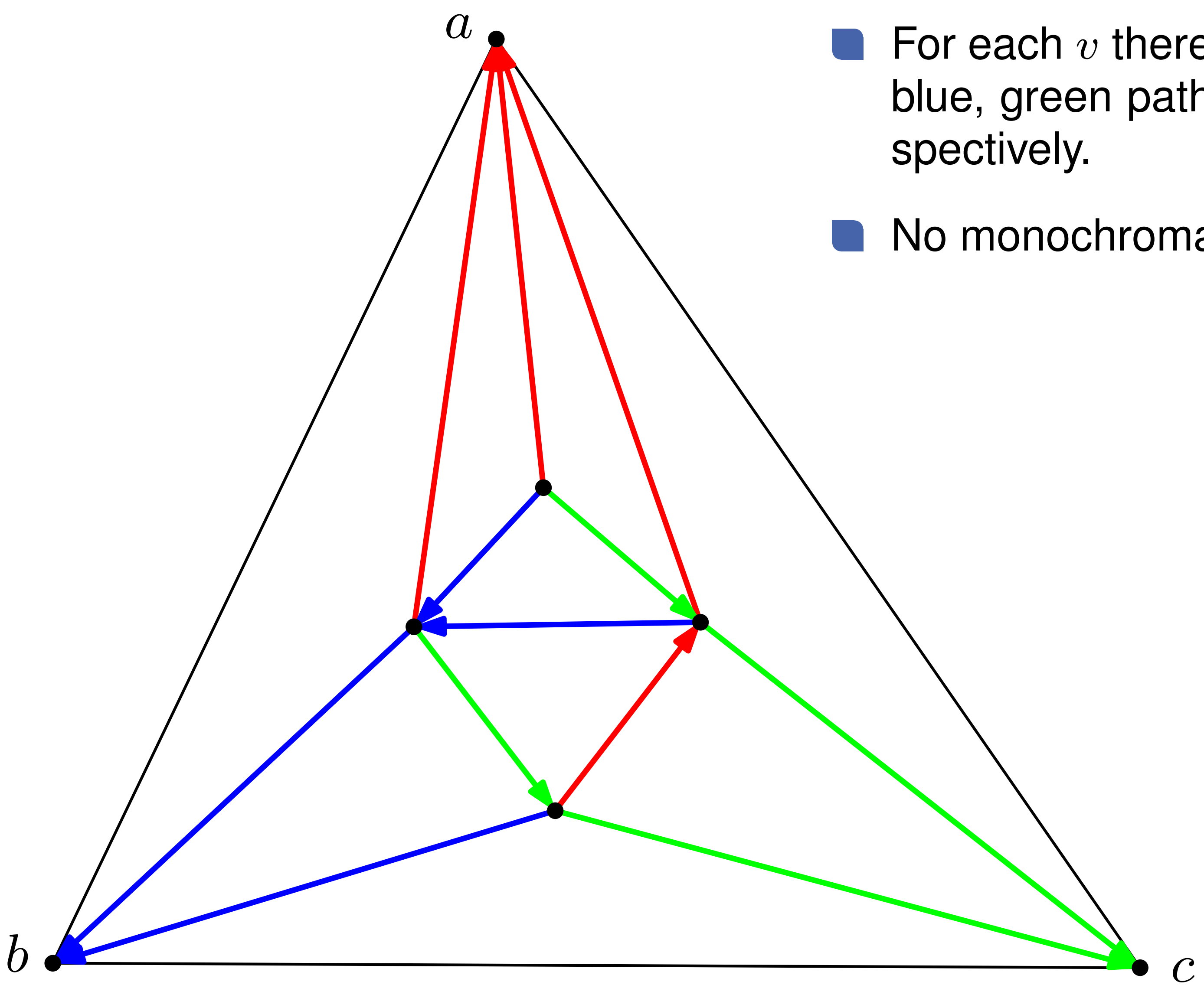

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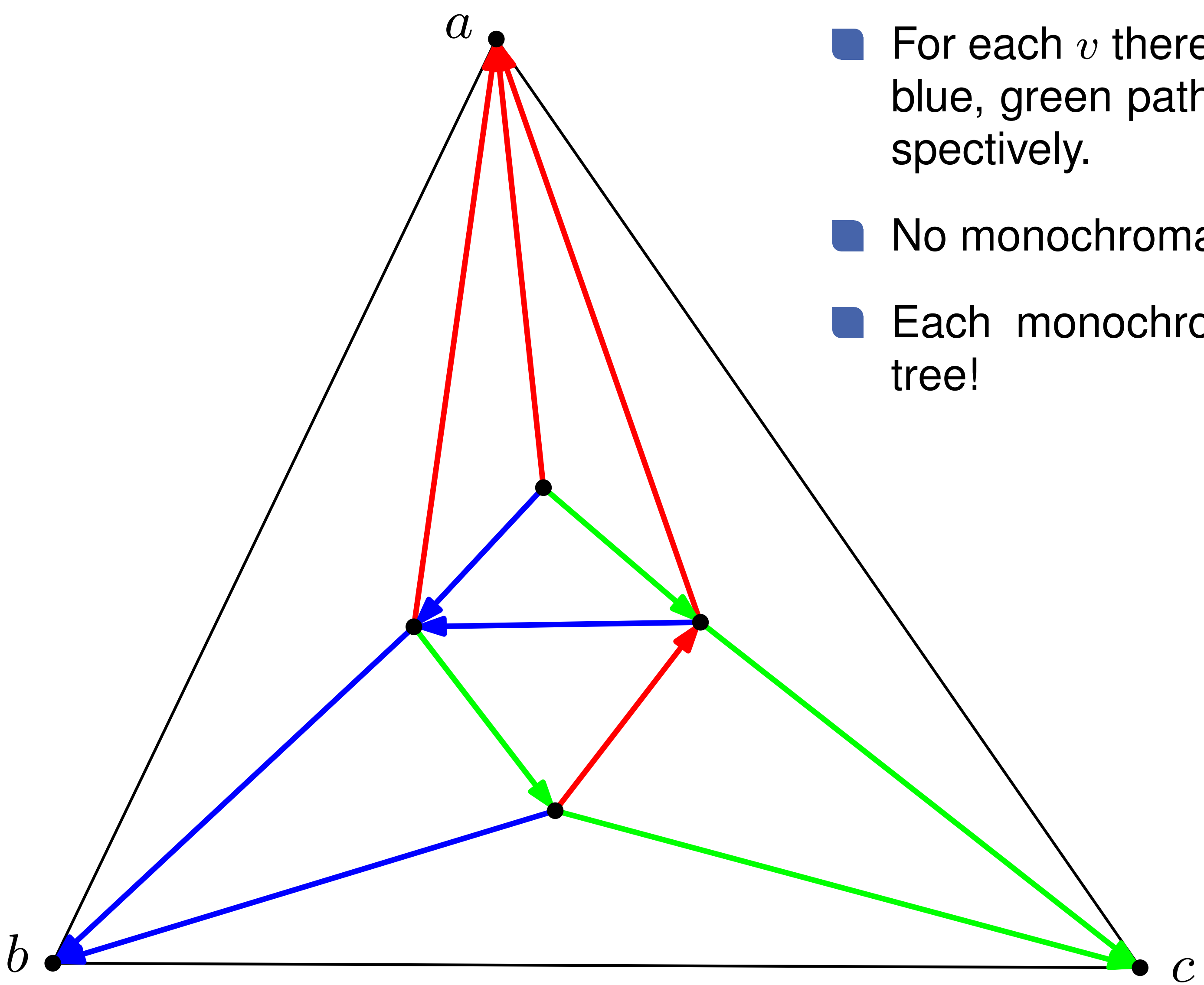

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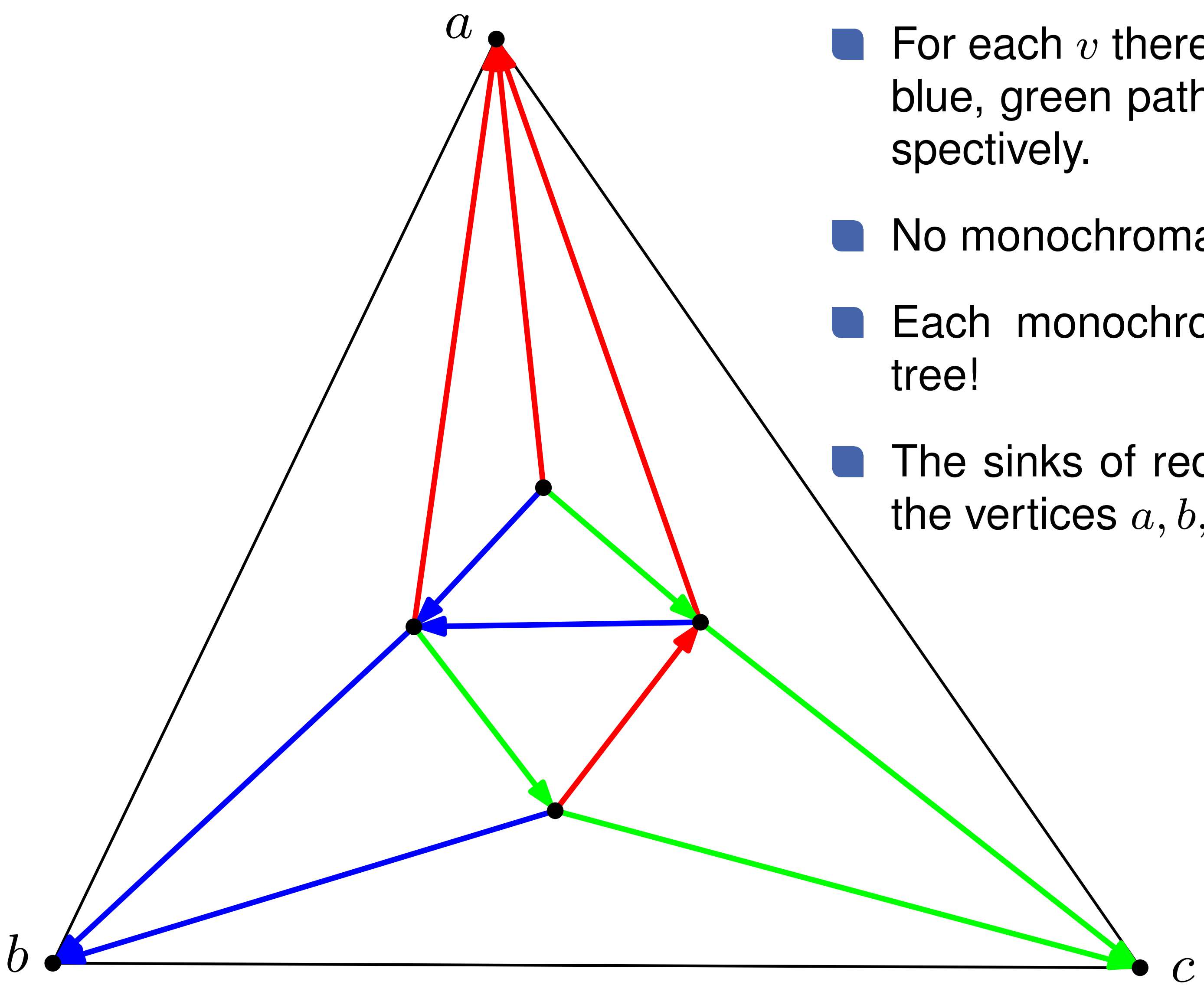

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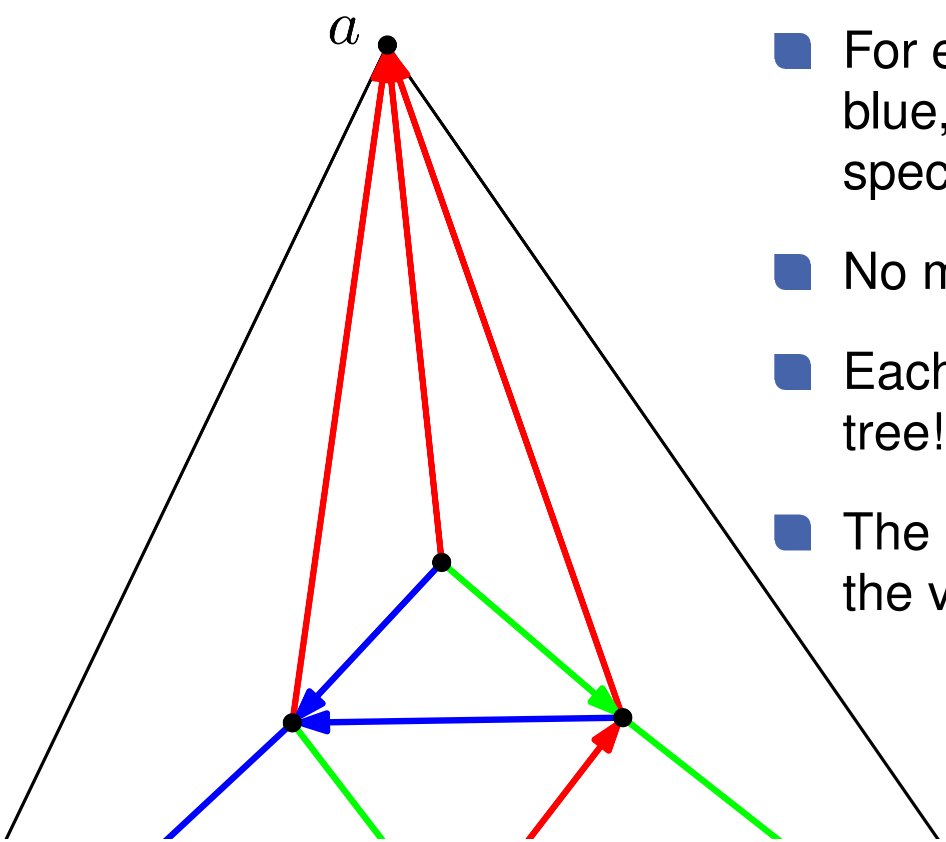

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Work with your neighbour(s) and then share

- Prove that each monochromatic graph is a tree.

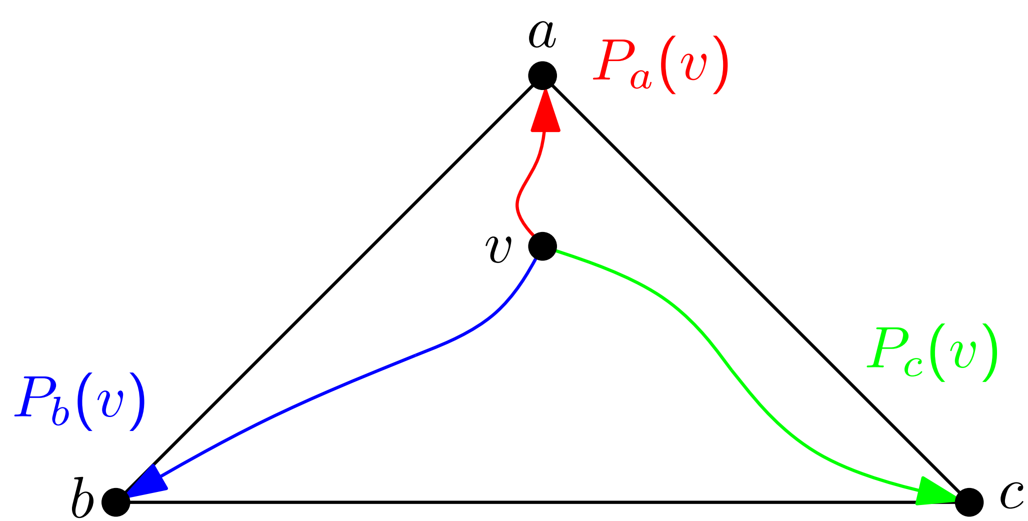
5 min

Overview

- Baricentric Coordinates and Barycentric GraphRepresentation.
- Schnyder Labelling. Existence.
- Schnyder Realizer.
- From Schnyder Realizer to Barycentric Representation.

Face Regions

- Paths $P_a(v)$, $P_b(v)$, $P_c(v)$ cross only at vertex v.
- $R_a(v), R_b(v), R_c(v)$ are sets of faces.



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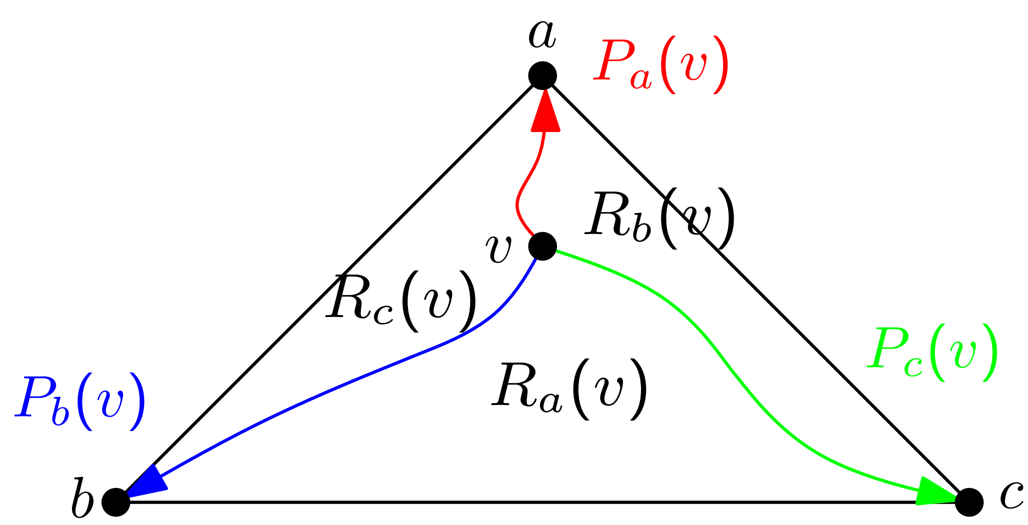

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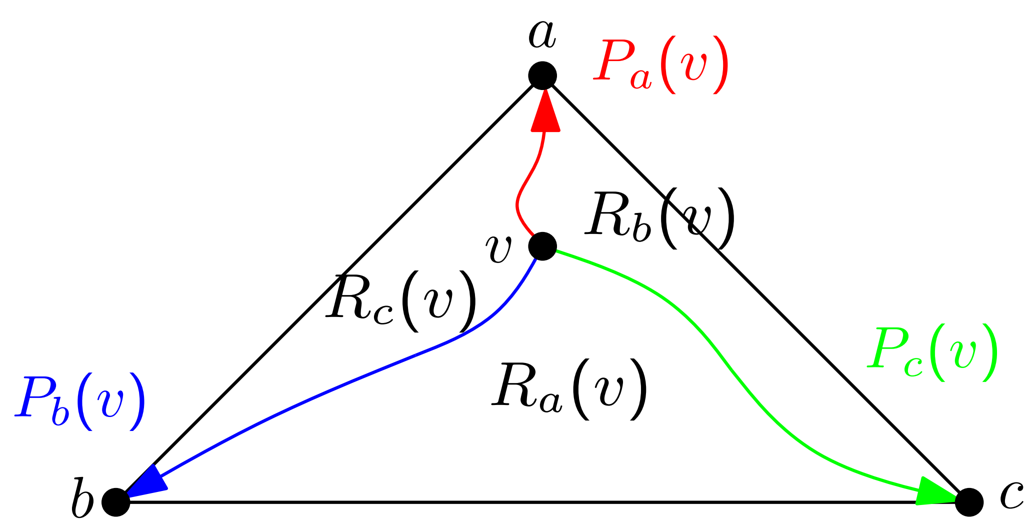

Lemma [Schnyder '90]

For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.

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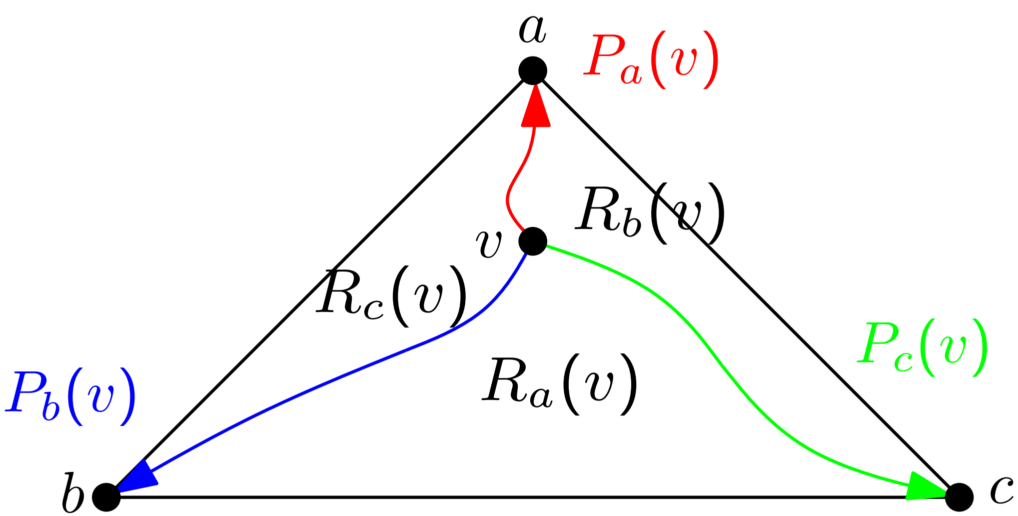

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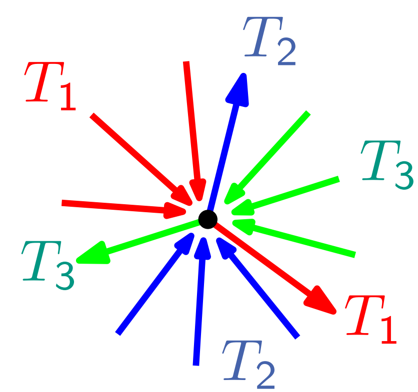
Proof ...



Barycentric Representation

- Let barycentric coordinates of $v \in G \setminus a, b, c$ be (v_a, v_b, v_c), where $v_a = |R_a(v)|/(2n-5)$, $v_b = |R_b(v)|/(2n-5)$ and $v_c = |R_c(v)|/(2n-5)$.
- We set: $A = (2n-5, 0)$, $B = (0, 2n-5)$, $C = (0, 0)$.

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Theorem [Schnyder '90]

The function

$$f : v \mapsto (v_a, v_b, v_c) = \frac{1}{2n-5}(|R_a(v)|, |R_b(v)|, |R_c(v)|)$$

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Proof

- Condition1 : $v_a + v_b + v_c = 1$.
- Condition 2: For each edge (u, v) and vertex $w \neq u, v$ at least one of three is true: $w_a > u_a, v_a, \quad w_b > u_b, v_b, \quad w_c > u_c, v_c$.

Final Remarks

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- The resulting drawing is a grid drawing.
- It is bounded by the triangle $\triangle ABC$ with $A = (2n-5, 0)$, $B = (0, 2n-5)$, $C = (0, 0)$.
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How to obtain area $n - 2 \times n - 2$?

- Use weak barycentric coordinates $\frac{1}{n-1}(n_1(v), n_2(v), n_3(v))$, $n_i(v) = |\text{vertices in } R_i(v)| - |P_{i-1}(v)|$ with respect to $A = (n-1, 0)$, $B = (0, n-1)$, $C = (0, 0)$.

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- Weak barycentric coordinates: Triple (v_a, v_b, v_c) such that
  - $v_a + v_b + v_c = 1$
  - For each edge (u, v) and vertex $w \neq u, v$, $\exists k \in \{a, b, c\}$, such that $(u_k, u_{k+1}) <_{lex} (w_k, w_{k+1})$, and $(v_k, v_{k+1}) <_{lex} (w_k, w_{k+1})$.
  - Here we say that $(a, b) <_{lex} (c, d)$ iff $a < c$ or $a = c$ and $b < d$.

Literature

Schnyder Realizer Method

- [NR04] Book T. Nishizeki, Md. S. Rahman “’Planar Graph Drawing. Chapter 4.3.