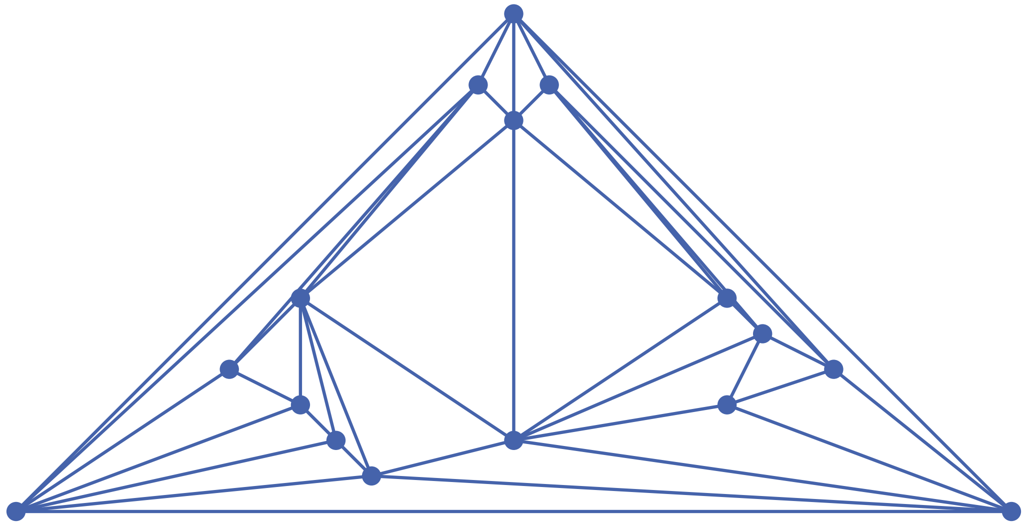
Algorithms for graph visualization

Layouts for planar graphs. Shift method.

Winter semester 2018/2019

Tamara Mchedlidze



KIT – Universität des Landes Baden-Württemberg und
nationales Forschungszentrum in der Helmholtz-Gemeinschaft

Motivation

- Till now we look at planar and straight-line drawings of trees and SP-graphs

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- Bennett, Ryall, Spalteholz and Gooch, 2007 "The Aesthetics of Graph Visualization"

3.2. Edge Placement Heuristics

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

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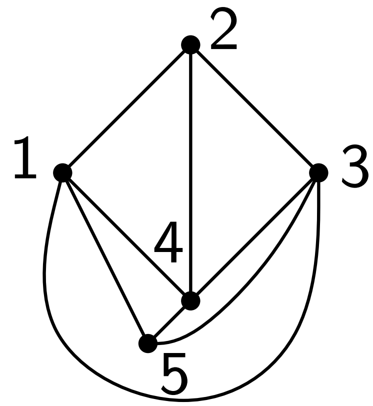
History

- Does every planar graph have a planar straight-line drawing?



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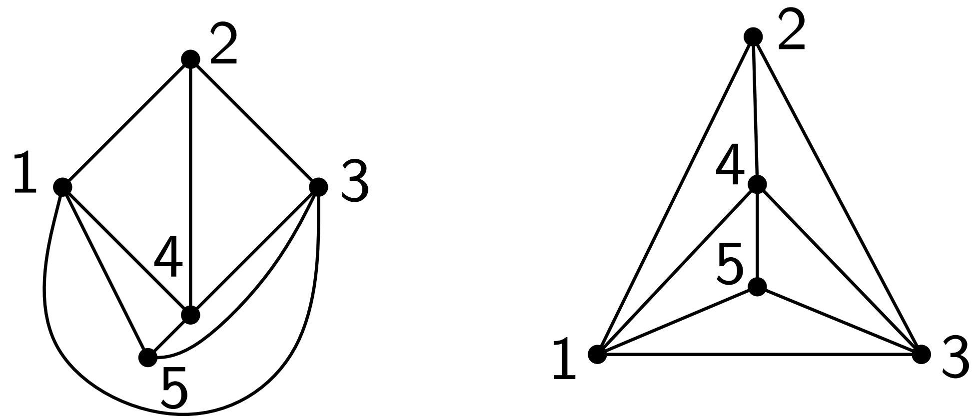

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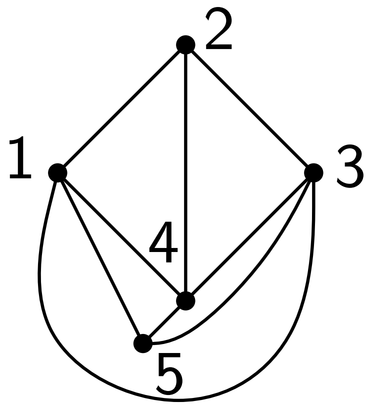

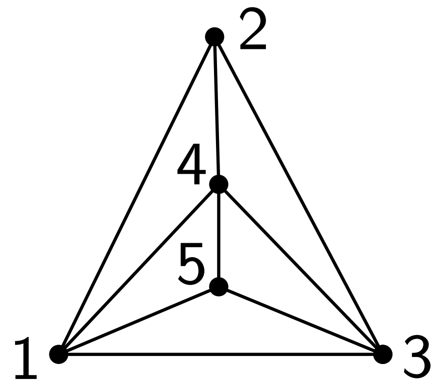

Theorem [Wagner ’36, Fary ’48, Stein ’51]

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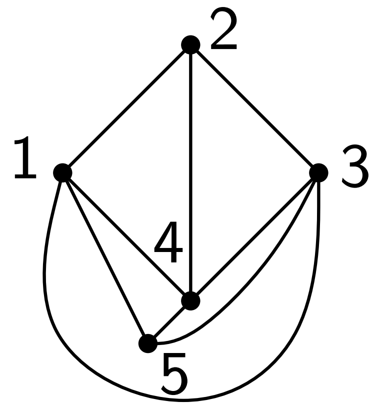

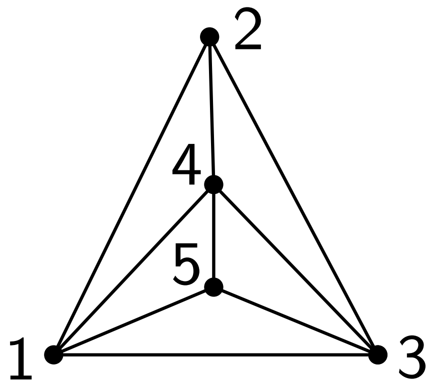

Theorem [Wagner '36, Fary '48, Stein '51]

Every planar graph has a planar straight-line drawing.

- The algorithms implied by this theory produce drawings with area **not bounded** by any polynomial on n.

3 - 4

Next

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This lecture:

Theorem [De Fraysseix, Pach, Pollack '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(2n-4) \times (n-2)$.

Next lecture:

Theorem [Schnyder '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(n-2) \times (n-2)$.

Outline

- Canonical ordering. Existence.
- Canonical ordering. Computation.
- Shift algorithm.
- Proof of planarity.
- Implentational details.

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Canonical Ordering

Definition: Canonical Ordering

Let $G = (V, E)$ be a **triangulated planar embedded graph** of $n \geq 3$ vertices. An ordering $\pi = (v_1, v_2, \ldots, v_n)$ is called a canonical ordering, if the following conditions hold for each k, $3 \leq k \leq n$.

- (C1) Vertices $\{v_1, \ldots v_k\}$ induce a 2-connected internally triangulated graph, call it G_k

Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

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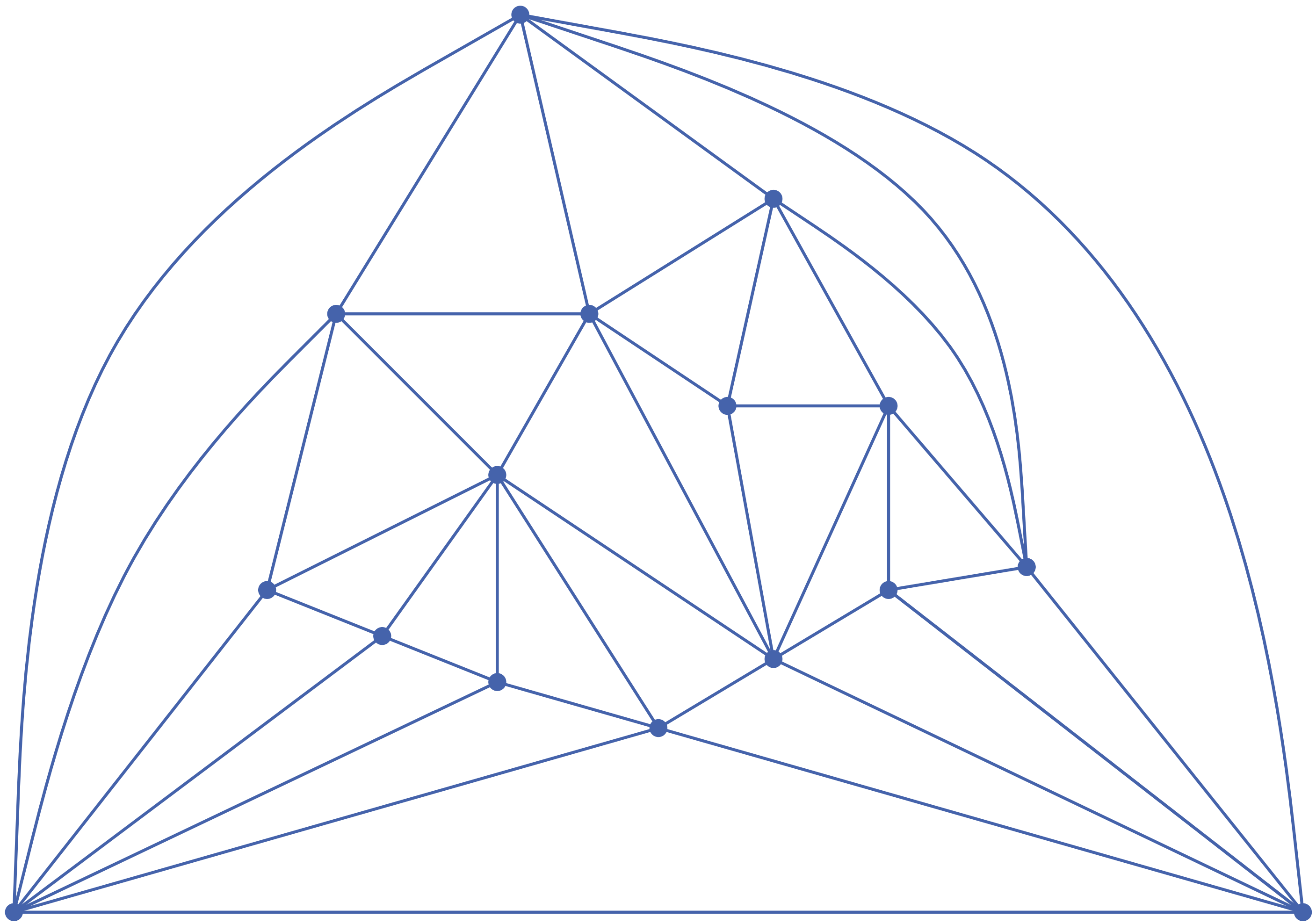
- (C2) Edge (v_1, v_2) belongs to the outer face of G_k
- (C3) If $k < n$ then vertex v_{k+1} lies in the outer face of G_k, and all neighbors of v_{k+1} in G_k appear on the boundary of G_k consecutively.

Example of Canonical Ordering

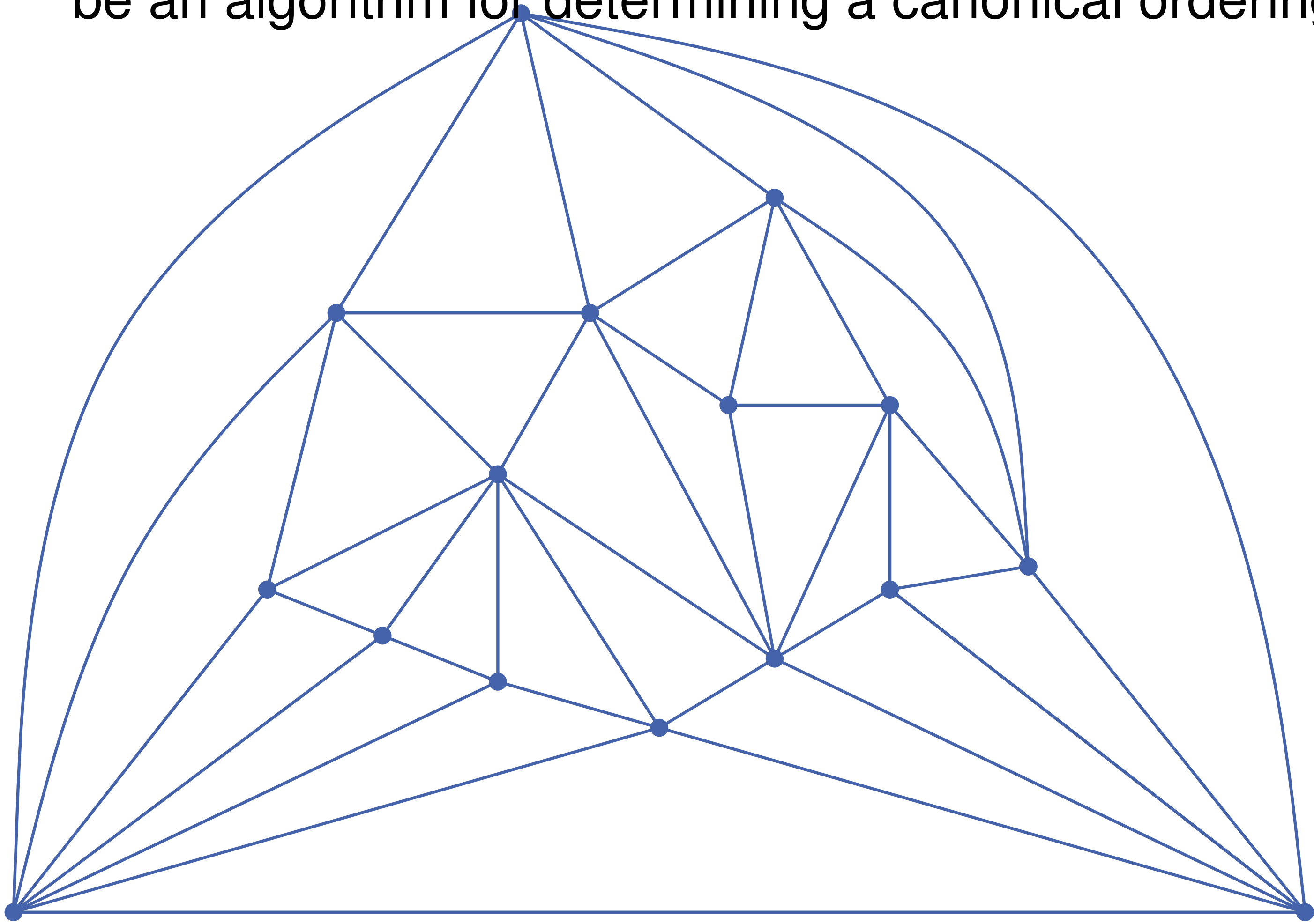


8 - 1

Example of Canonical Ordering

Work with your neighbour(s) and then share **5 min**

- Compute a canonical ordering of this graph. What could be an algorithm for determining a canonical ordering?



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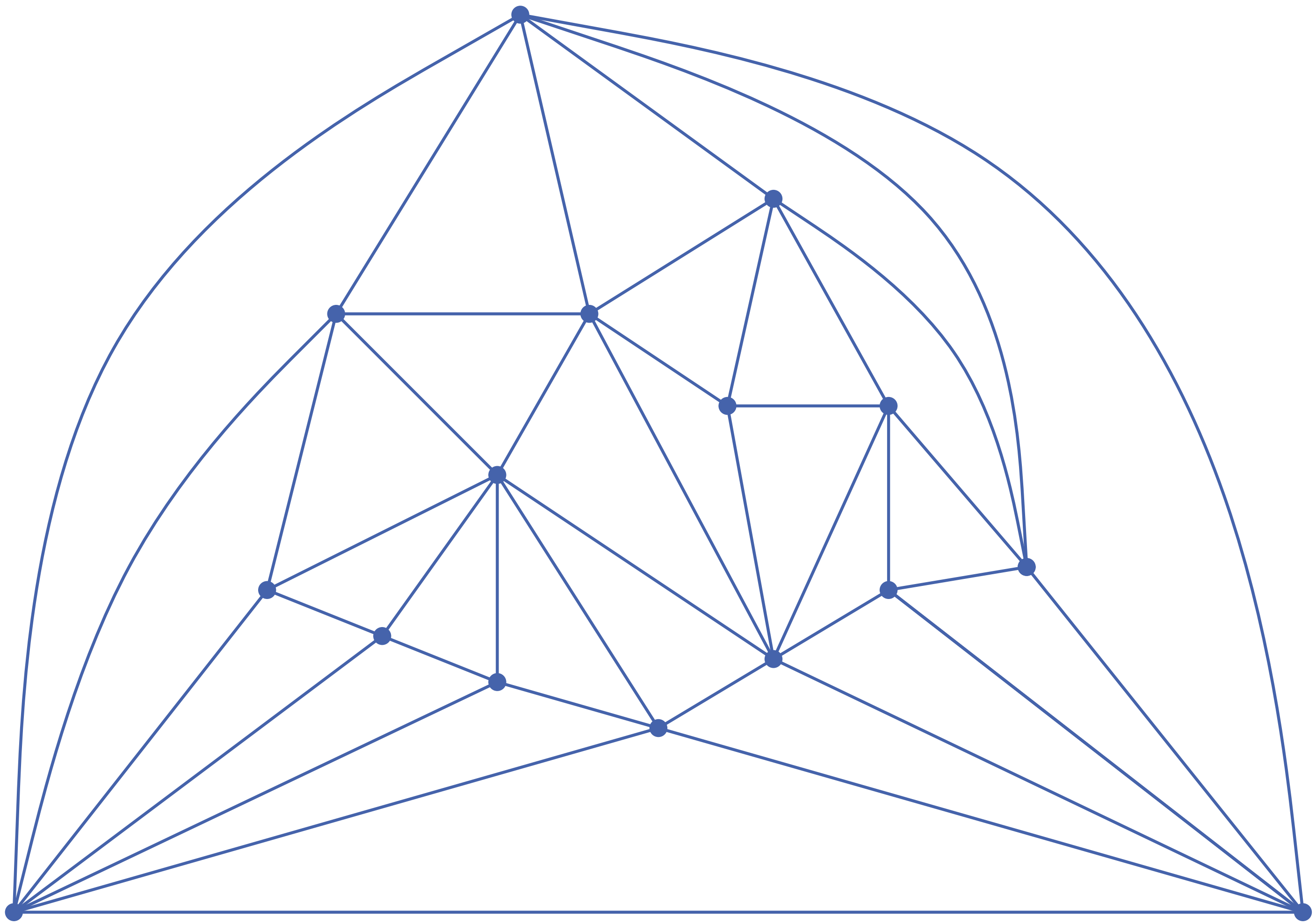

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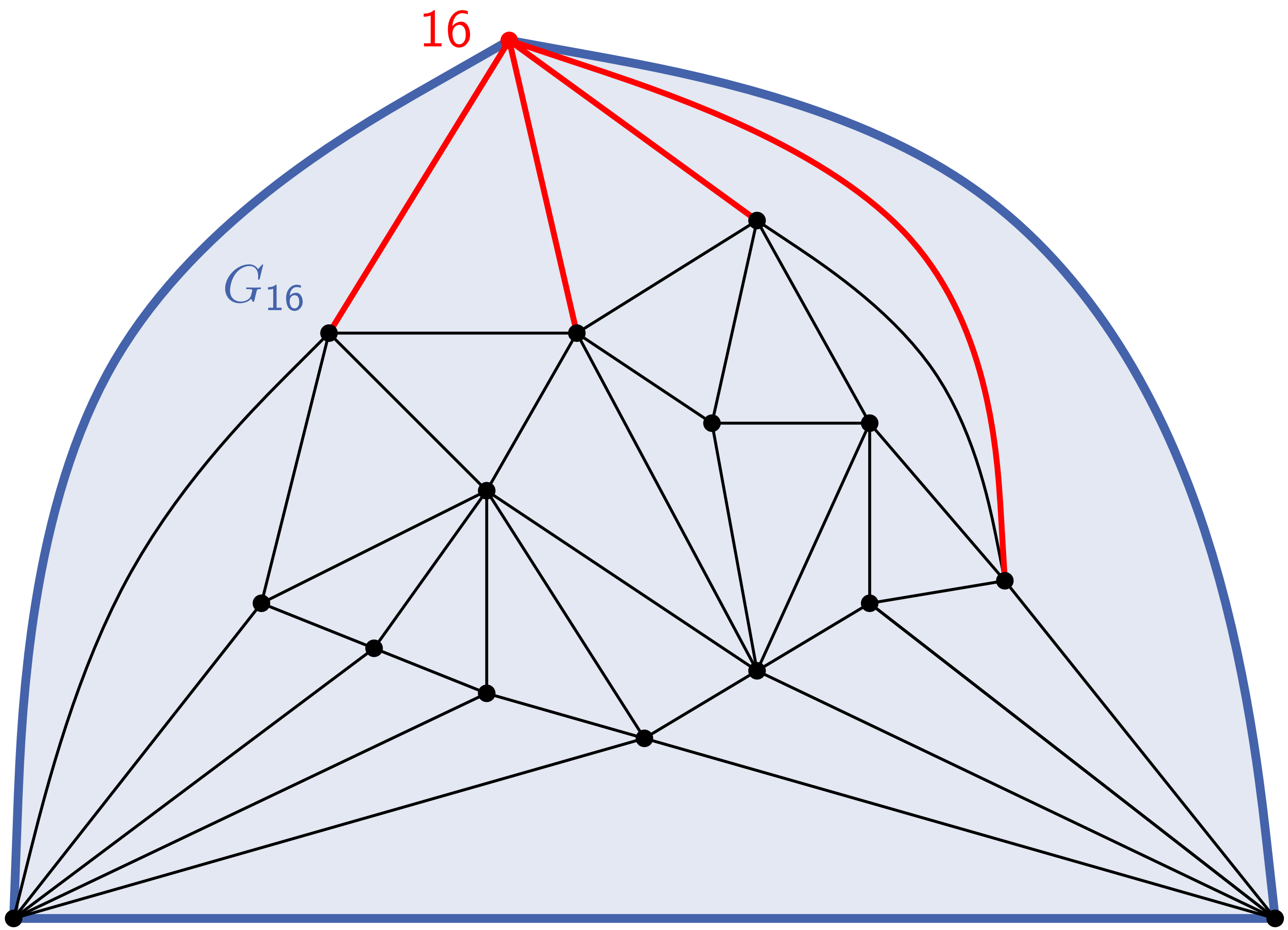

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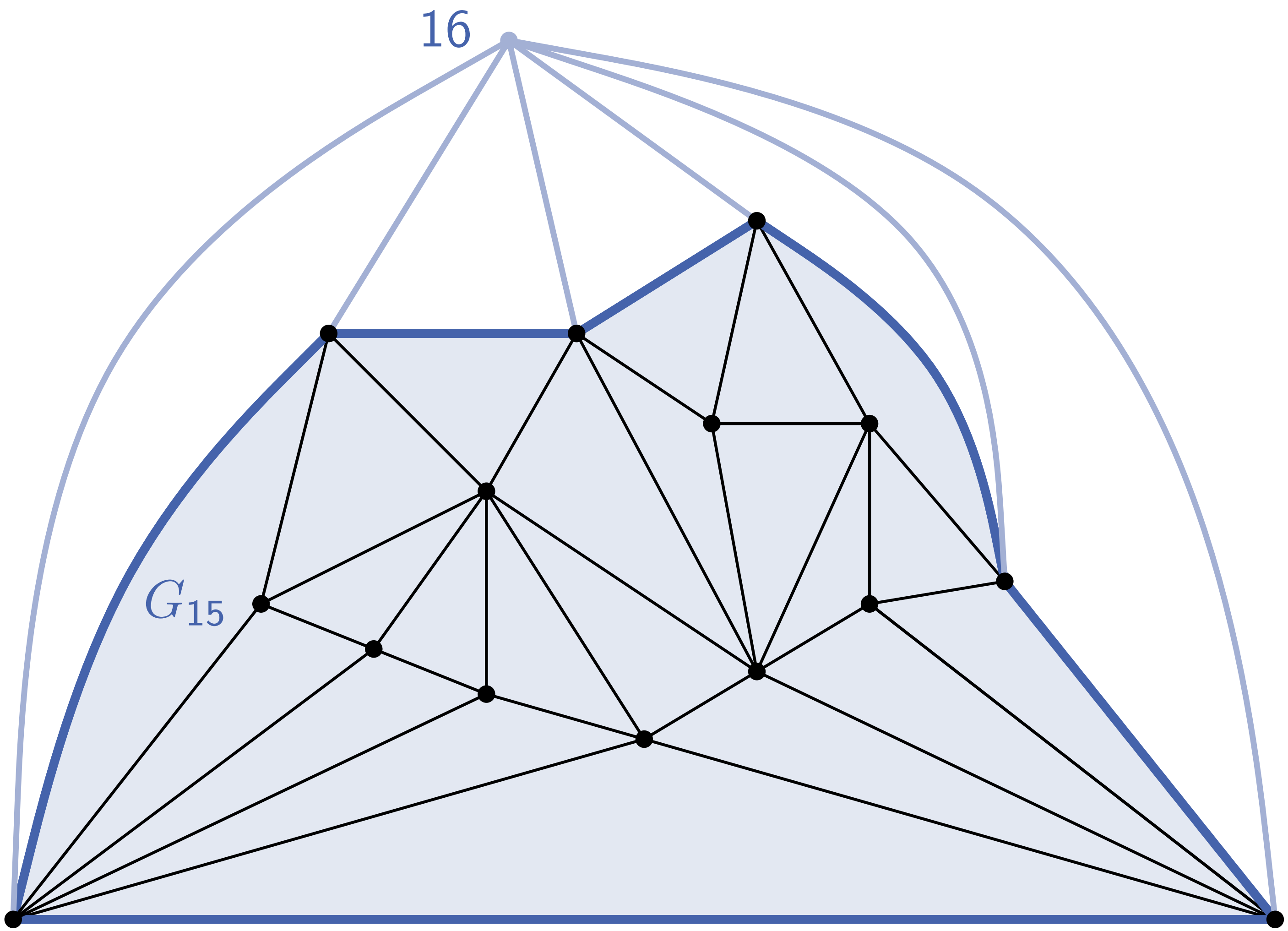

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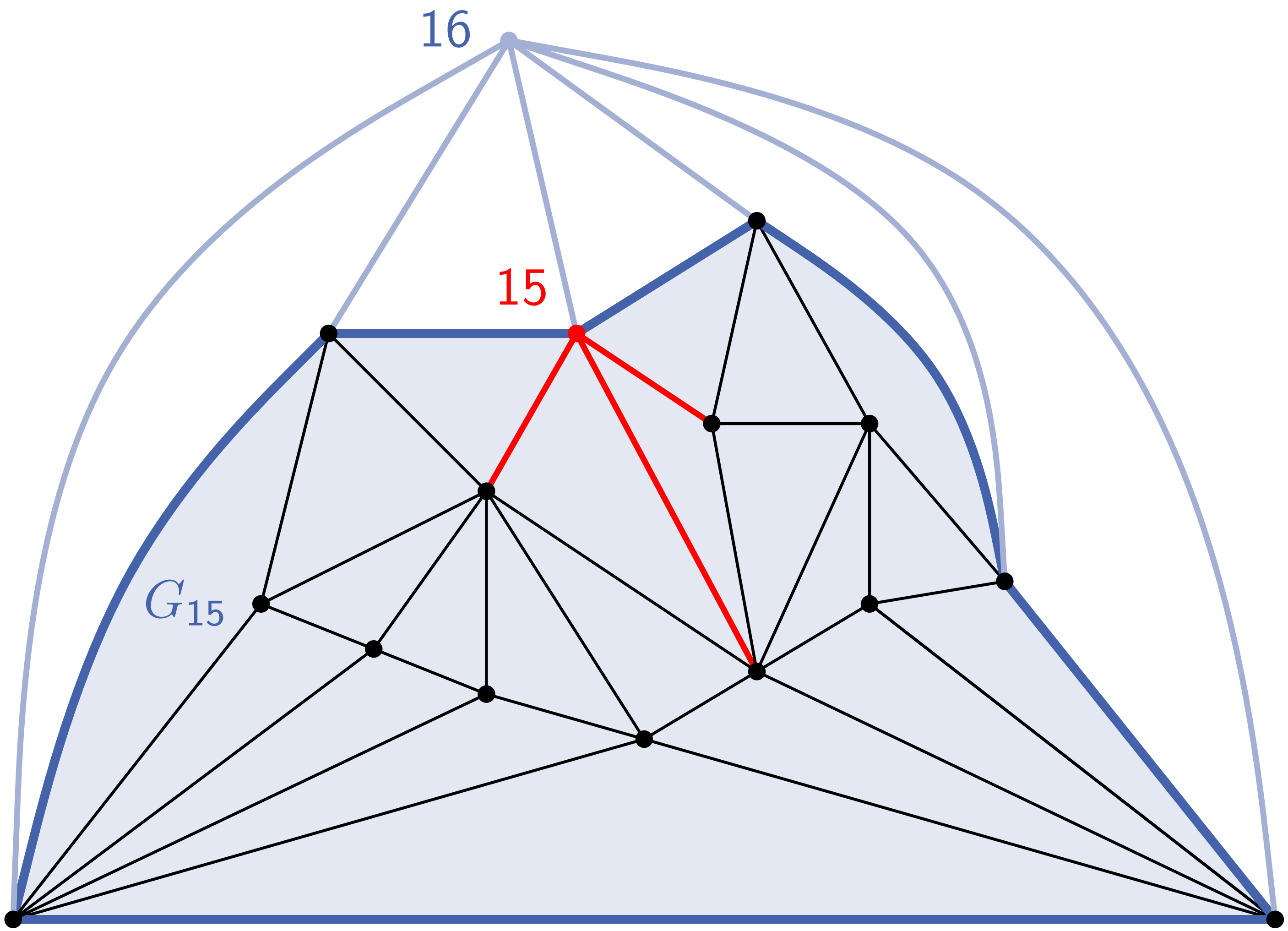

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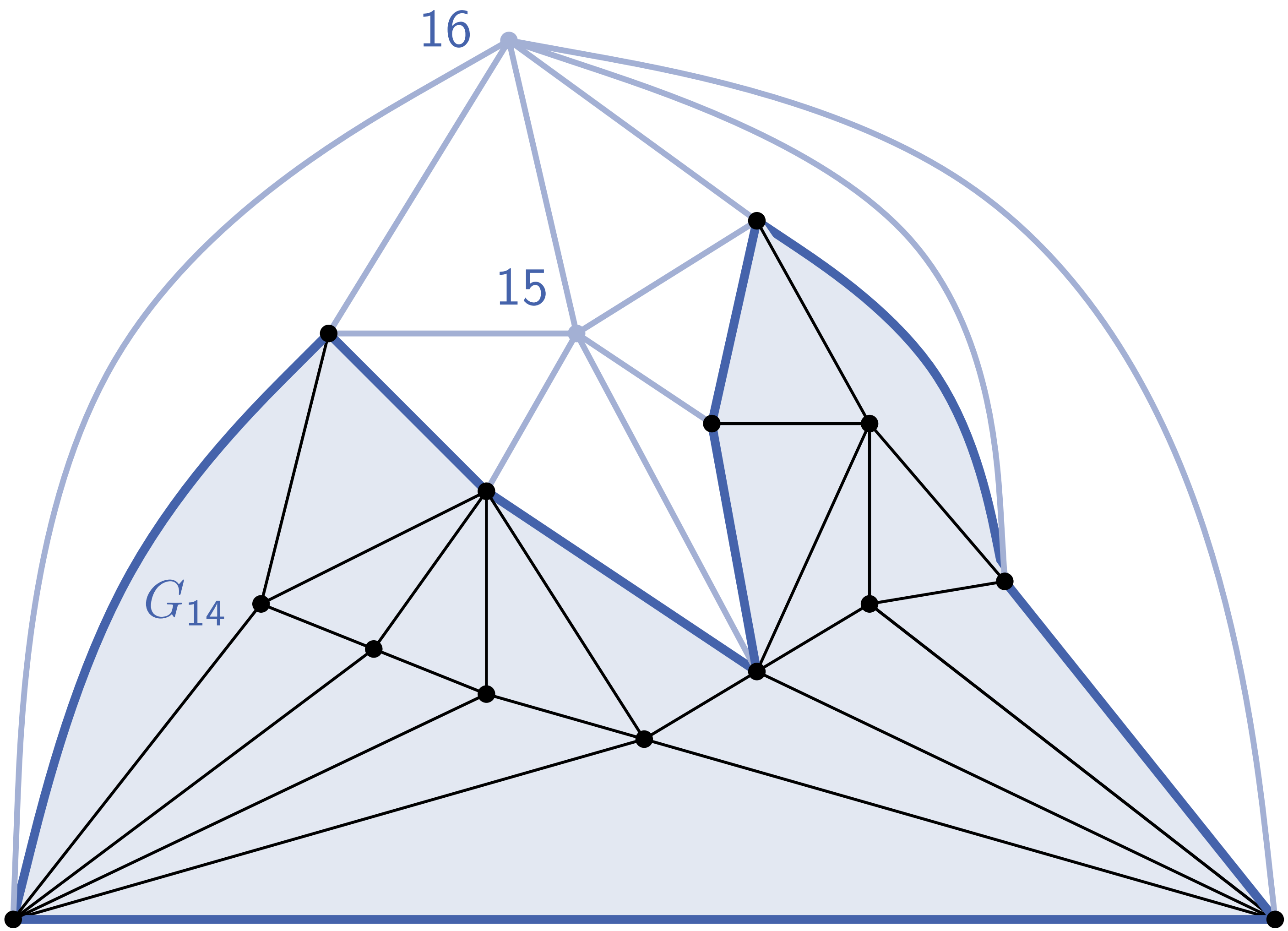

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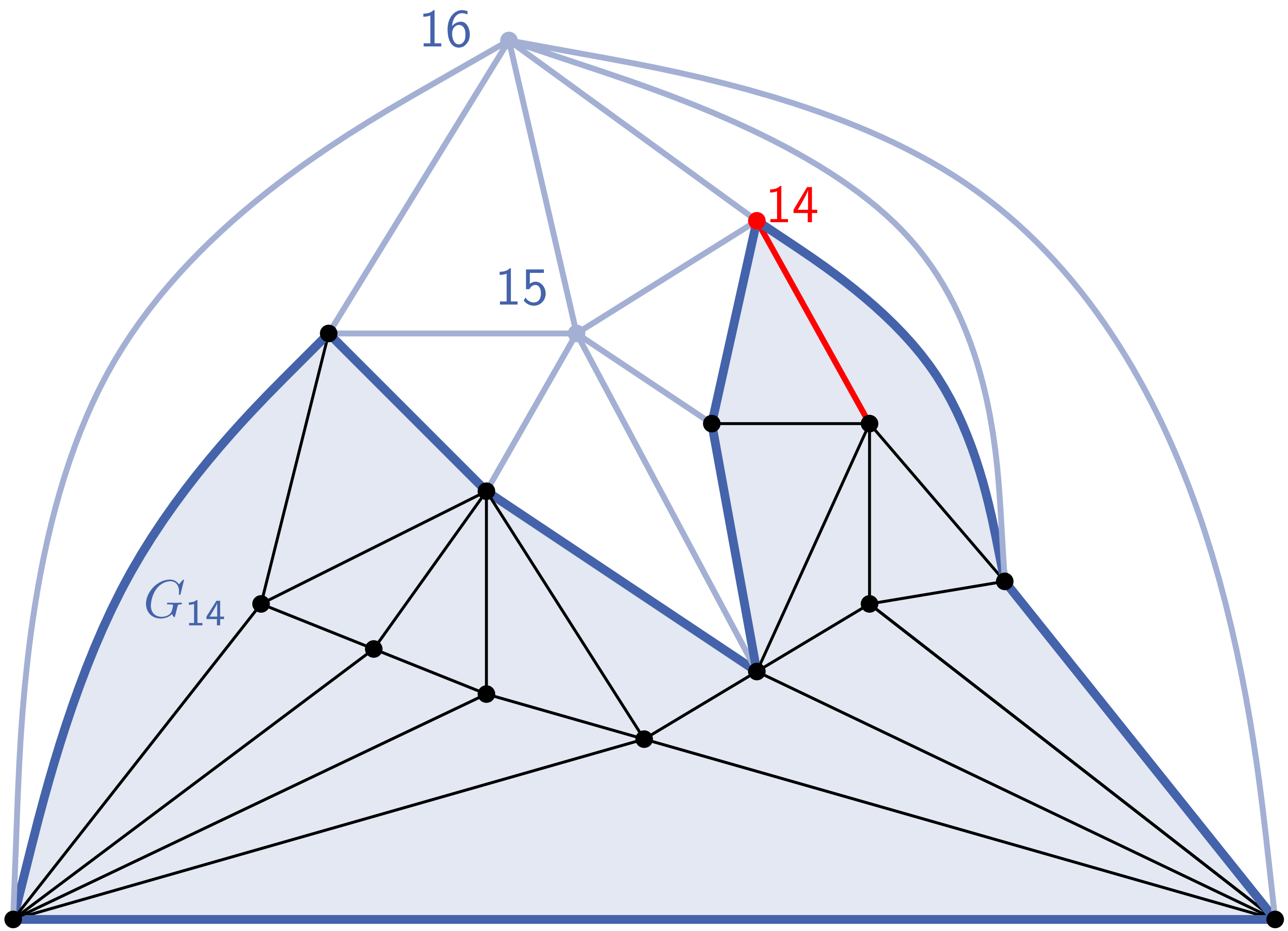

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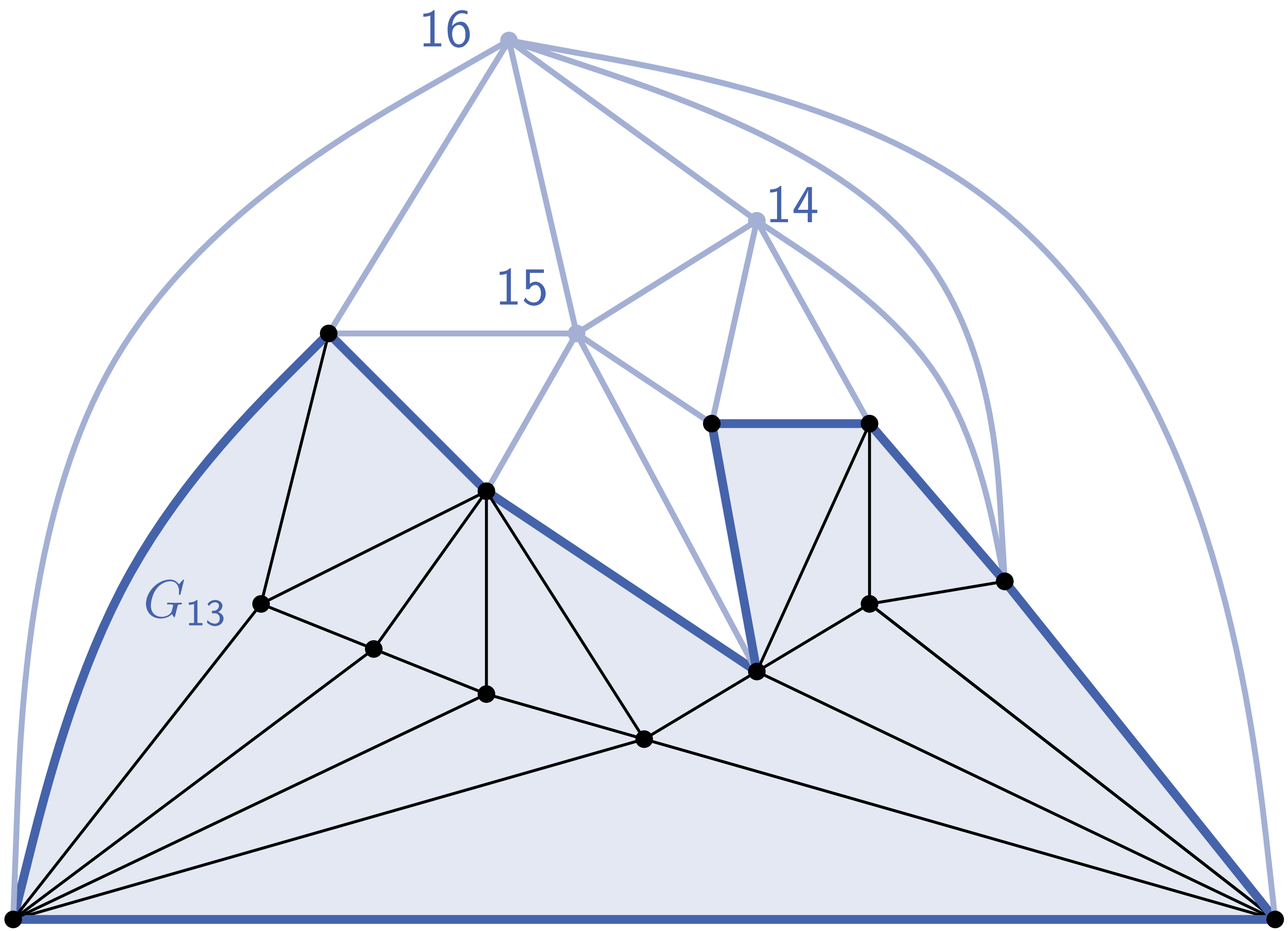

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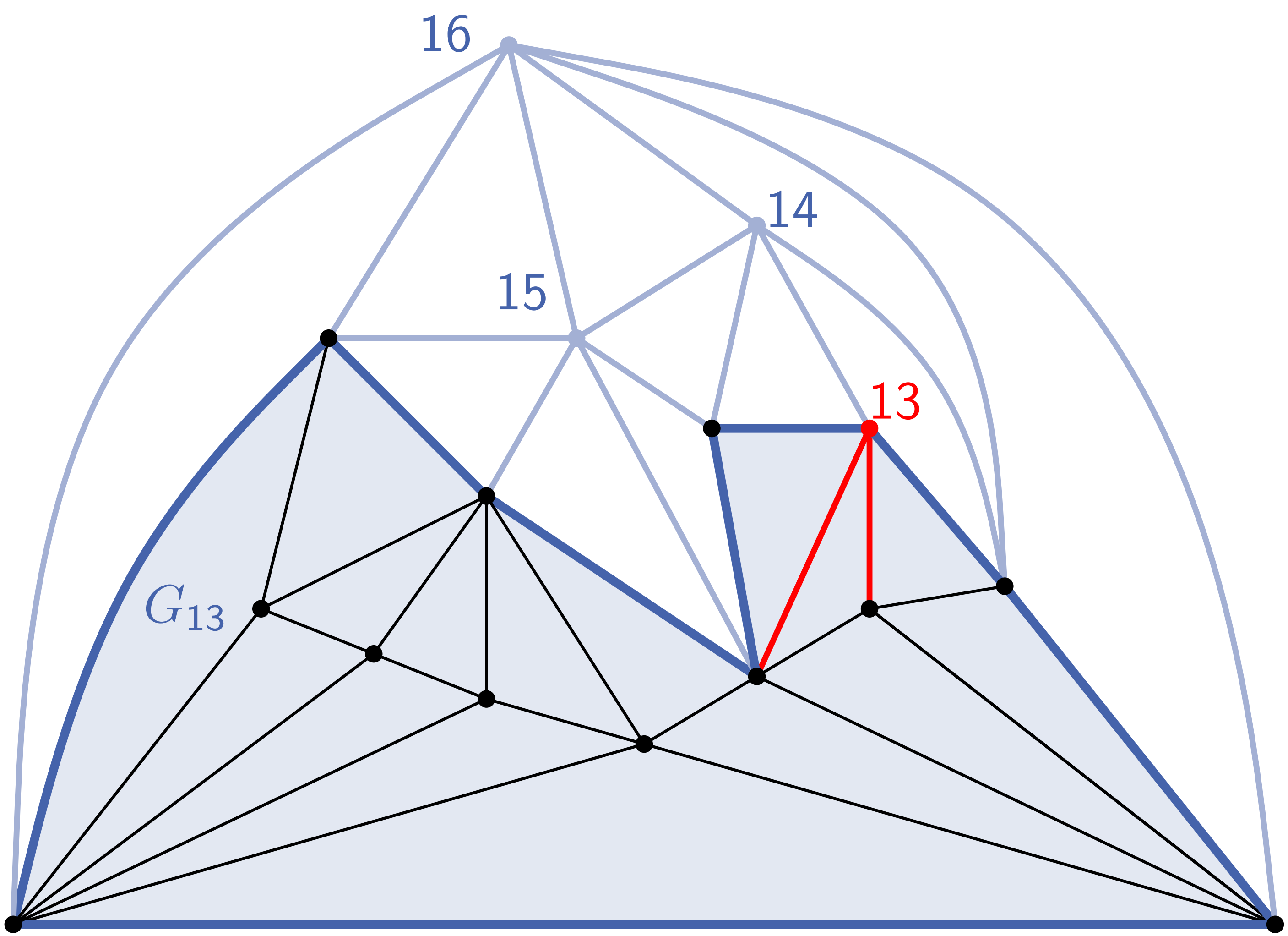

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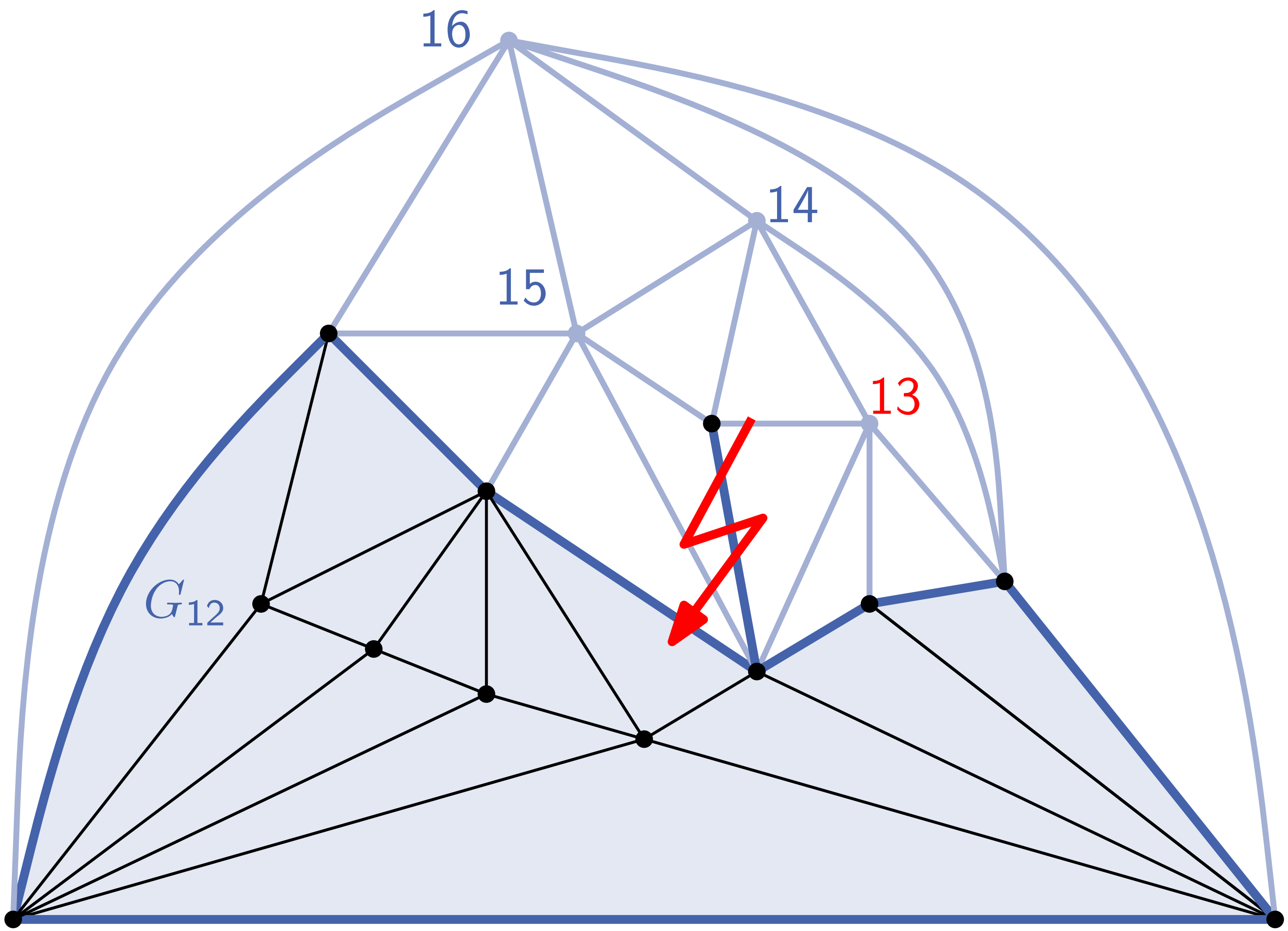

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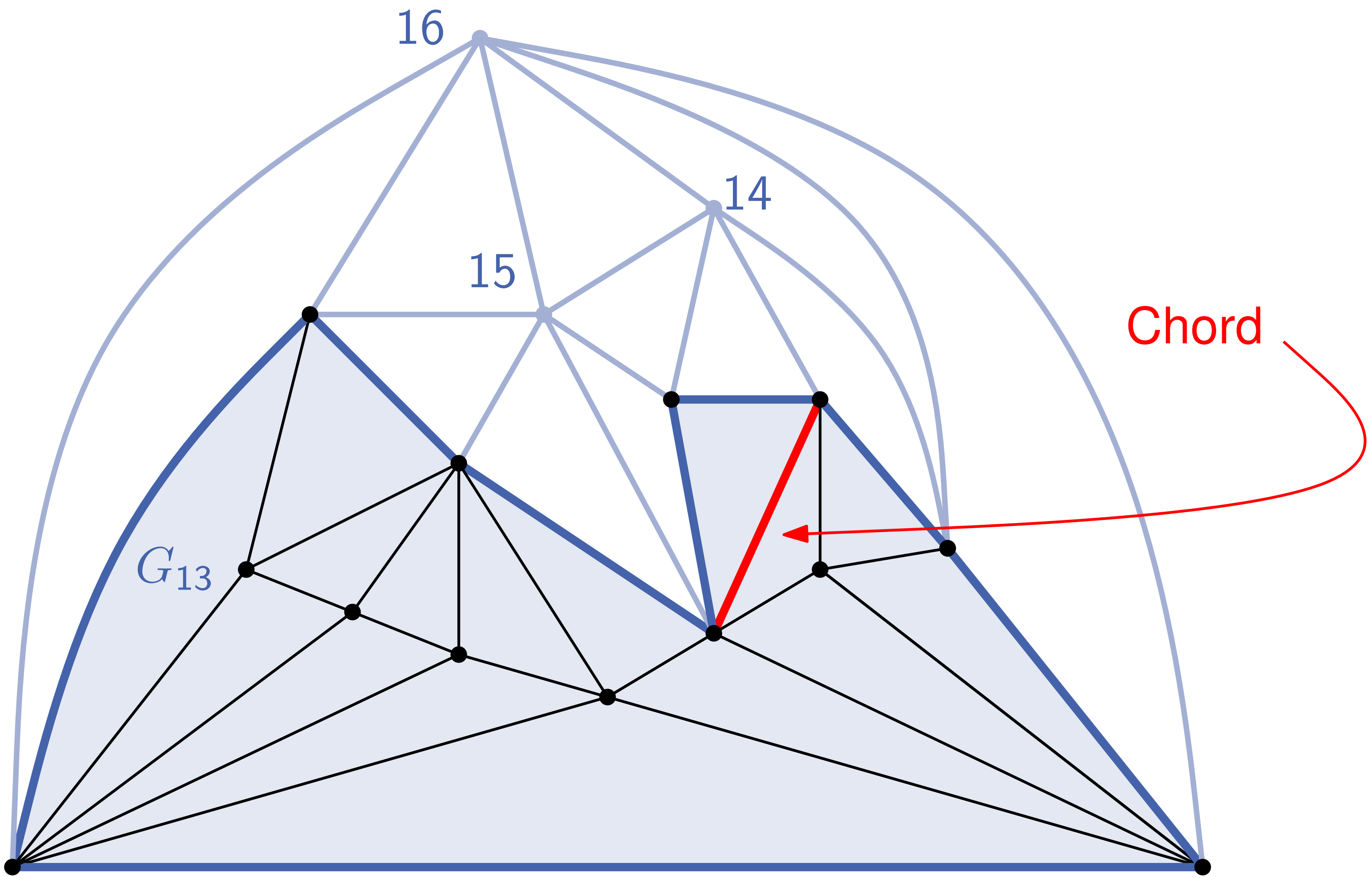

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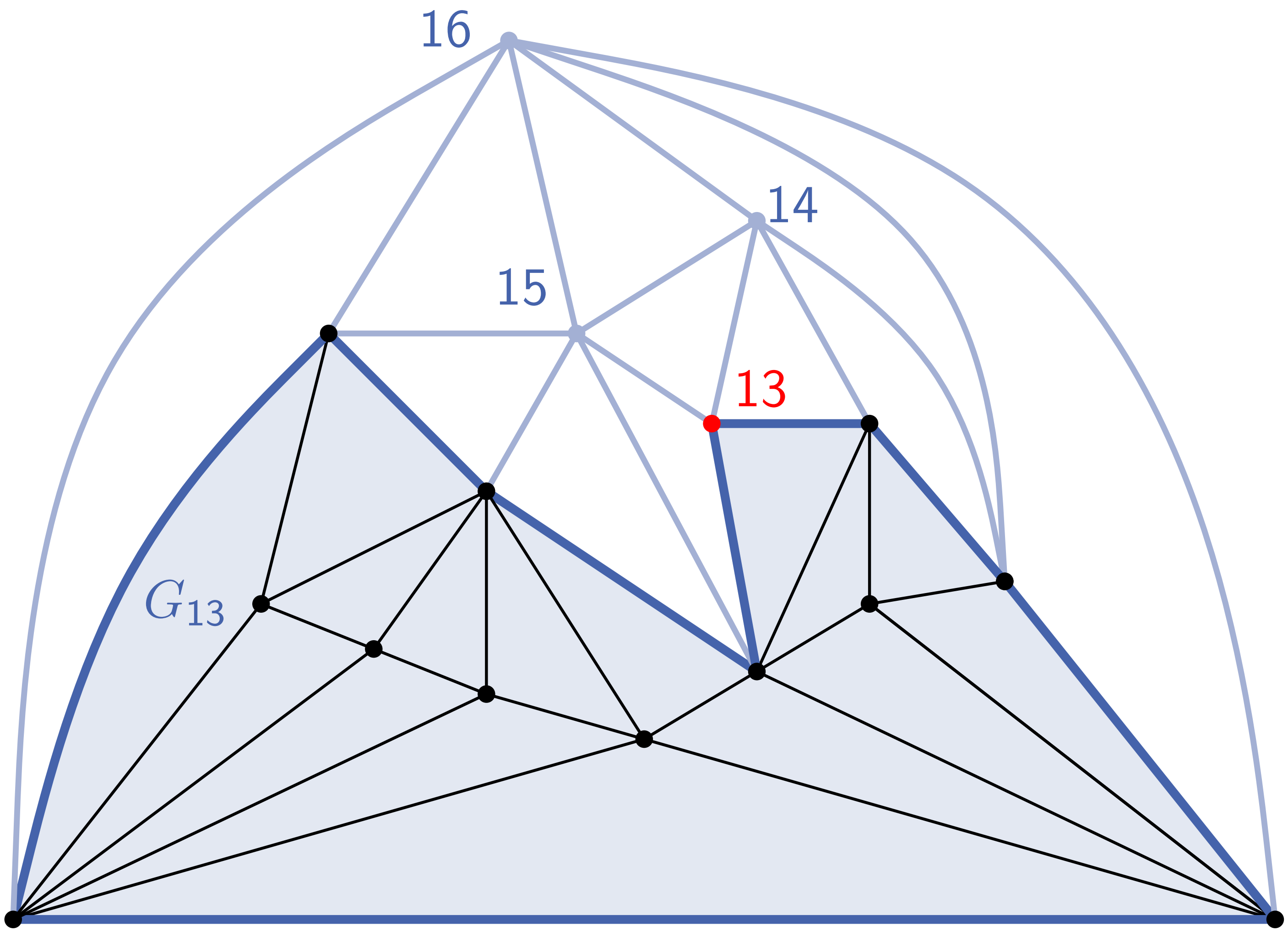

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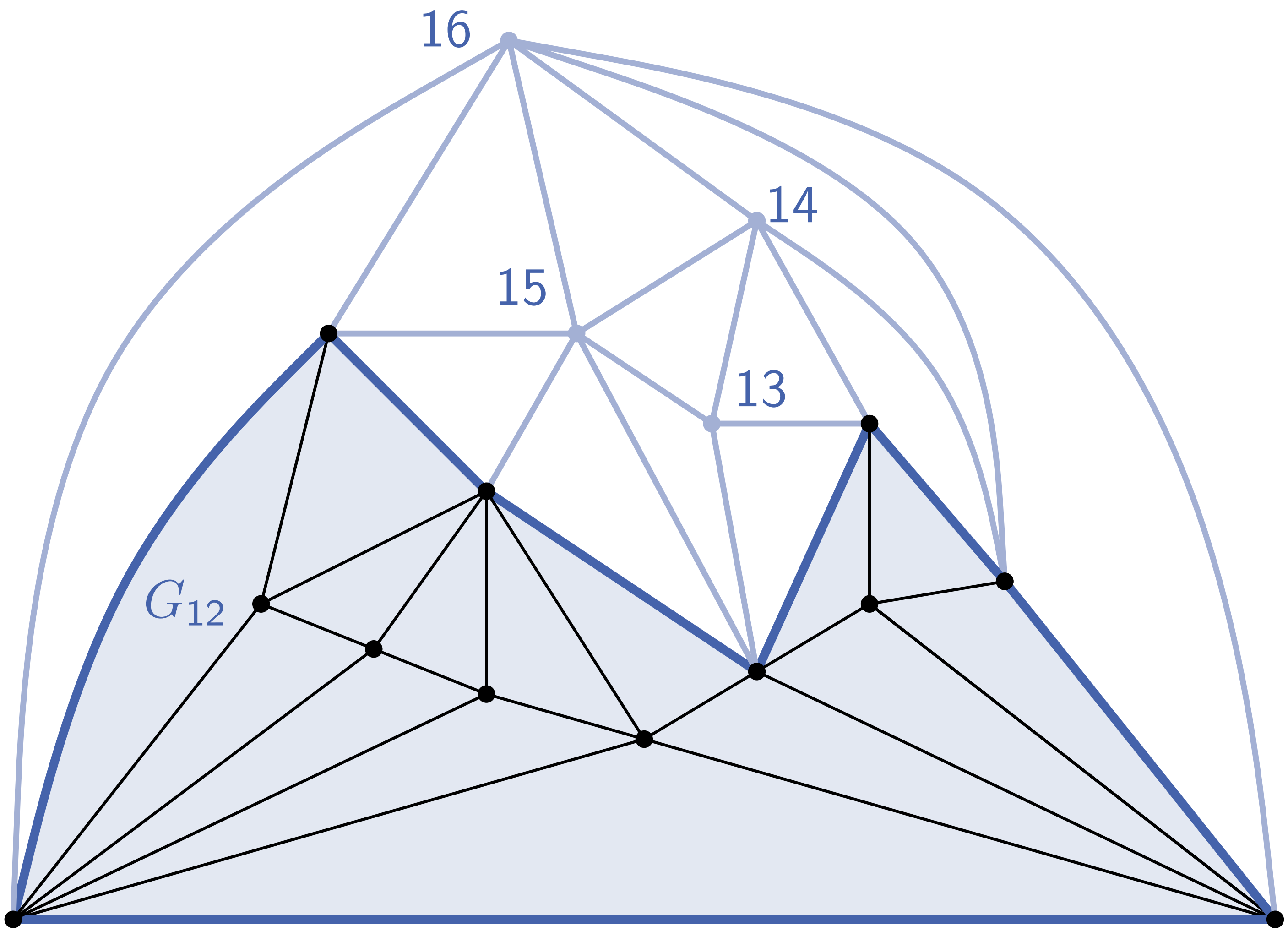

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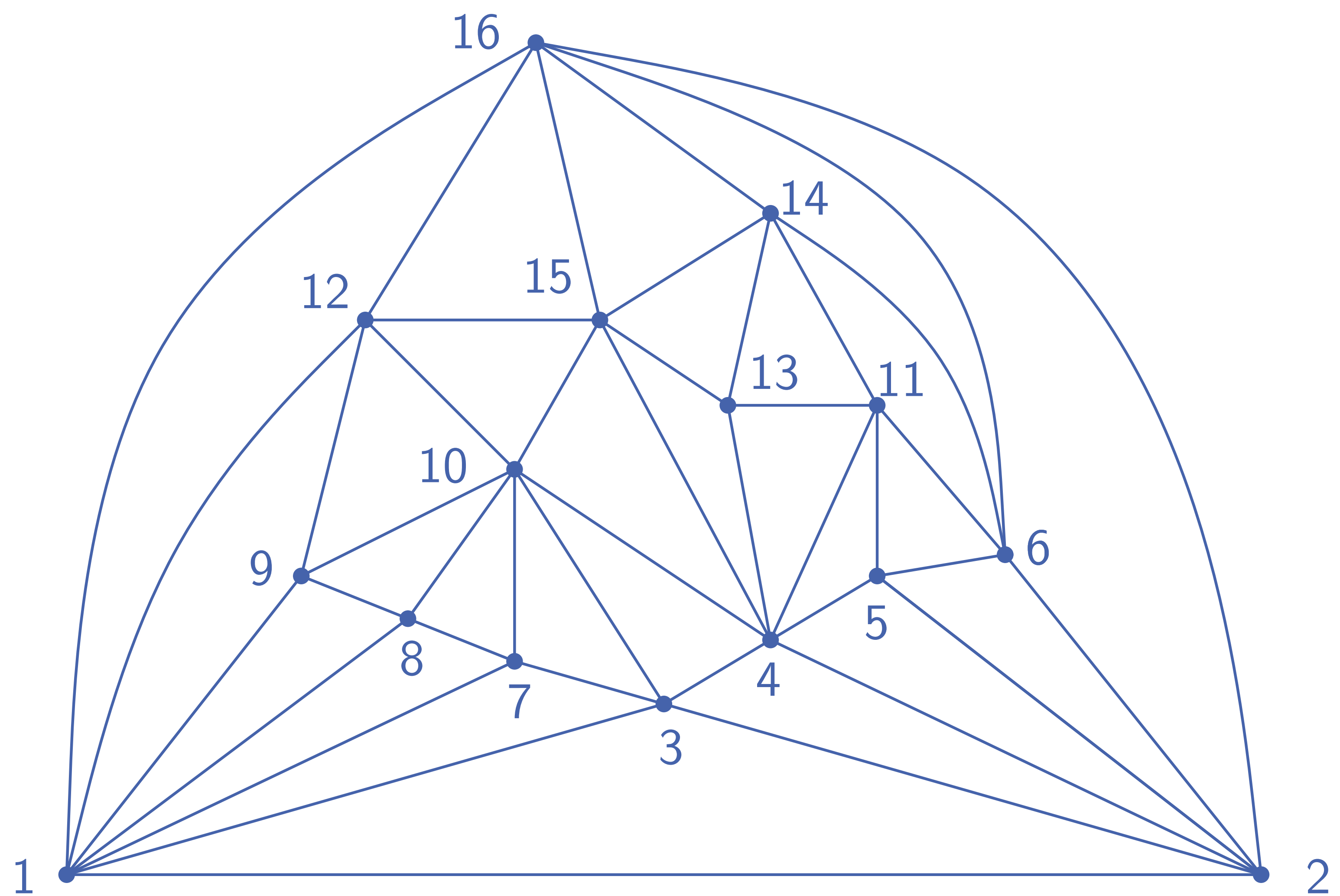

8 - 15

Canonical Ordering. Existence.

Lemma

Every triangulated plane graph has a canonical ordering.

- Let $G_n = G$, and let v_1, v_2, v_n be the vertices of the outer face of G_n. Conditions C1-C3 hold.

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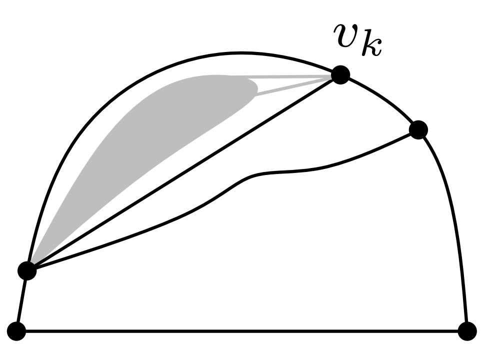
- Consider G_k. We search for v_k.



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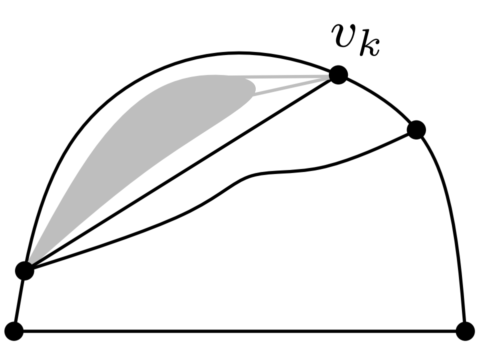

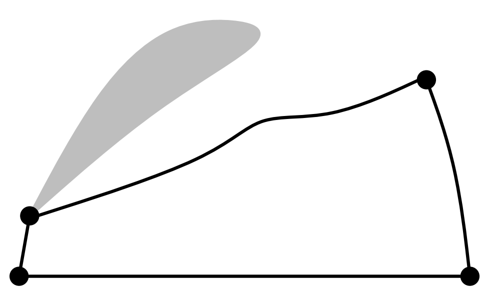

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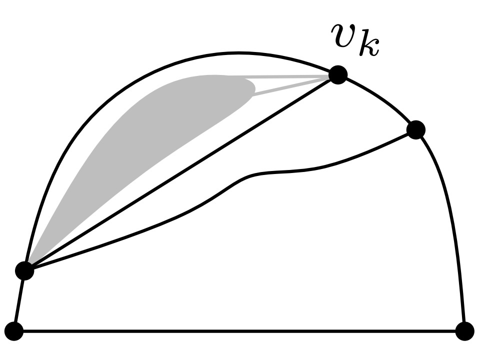

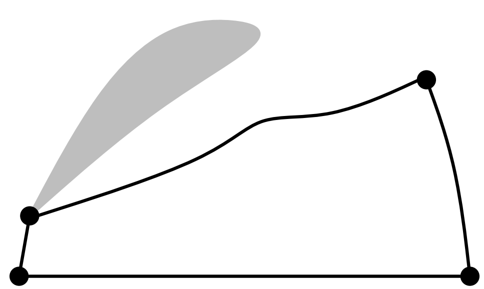

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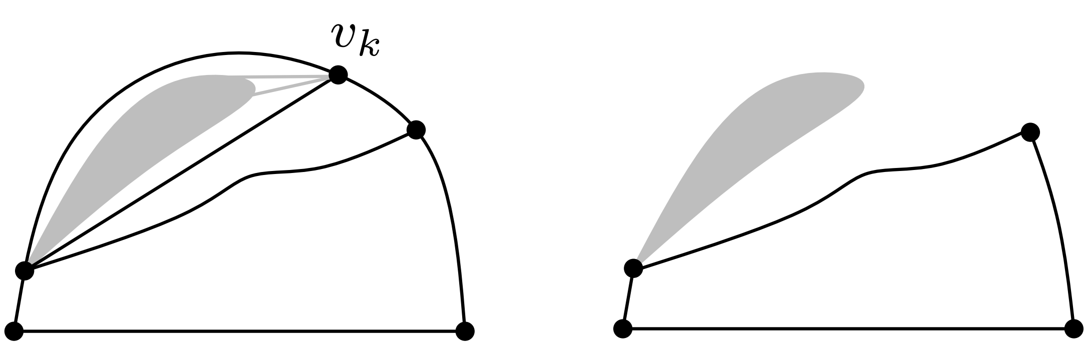

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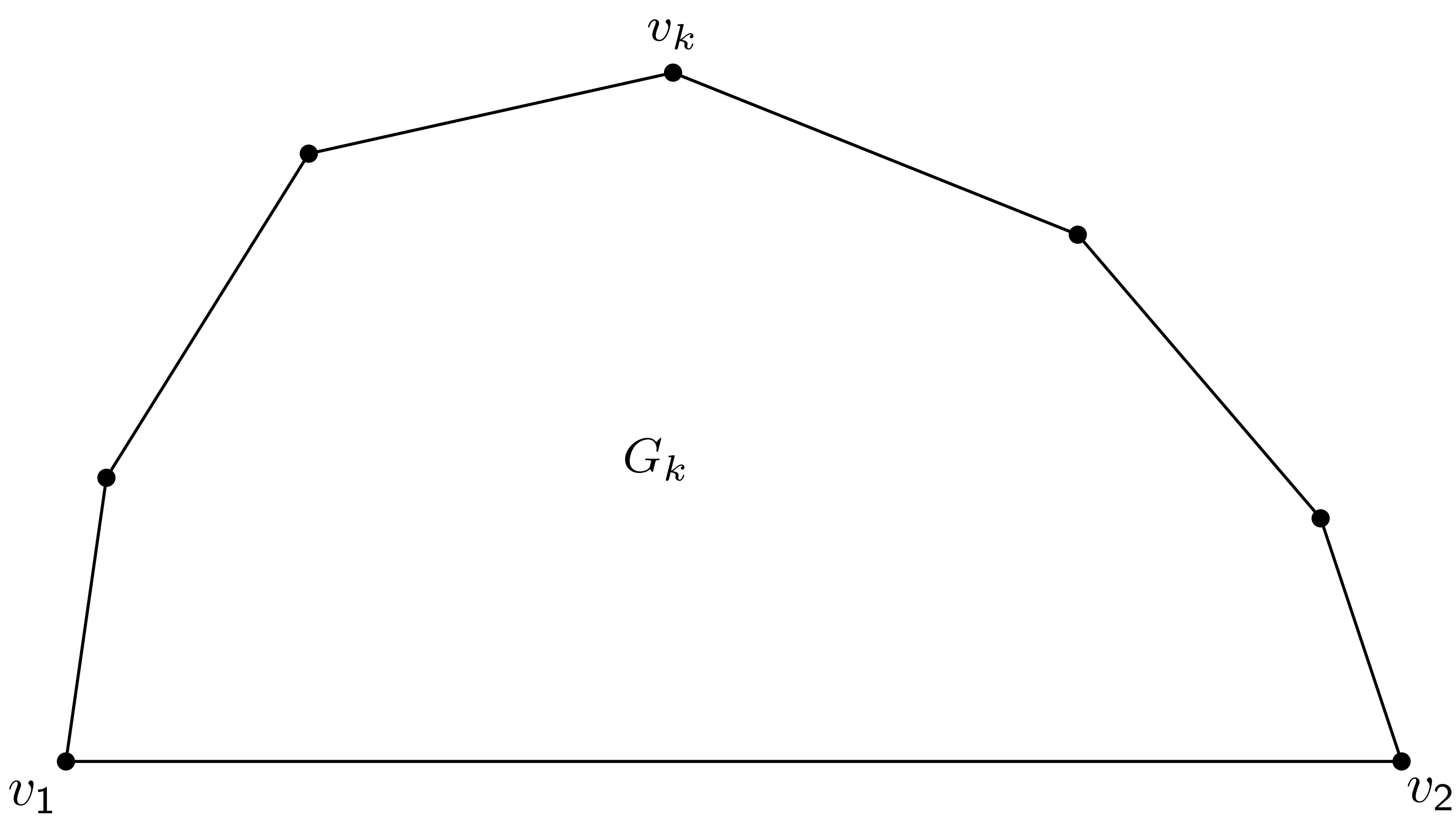
Is it sufficient?

Canonical Ordering. Existence.

Statement If v_k is not adjacent to a chord then removal of v_k leaves the graph biconnected.



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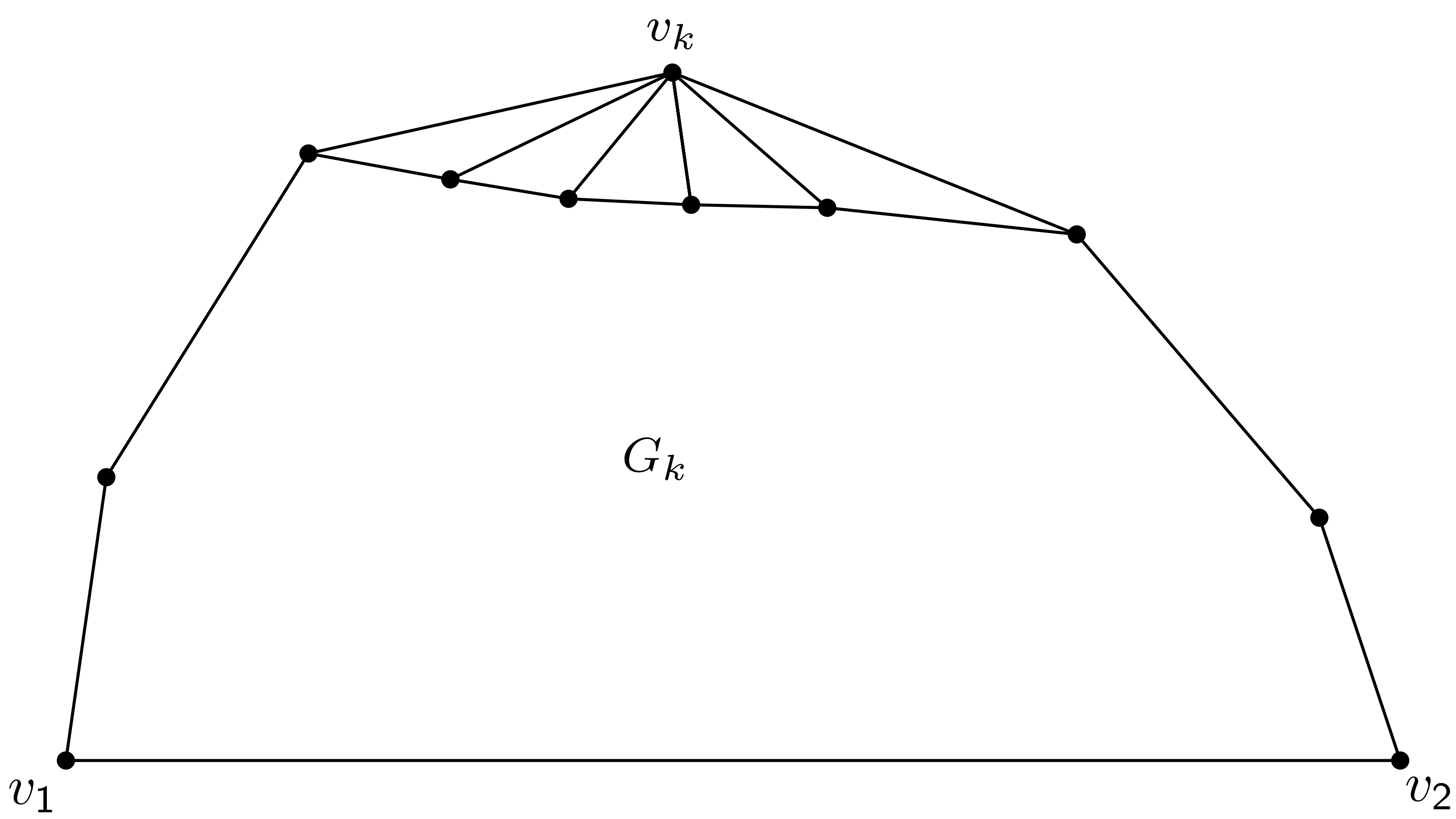

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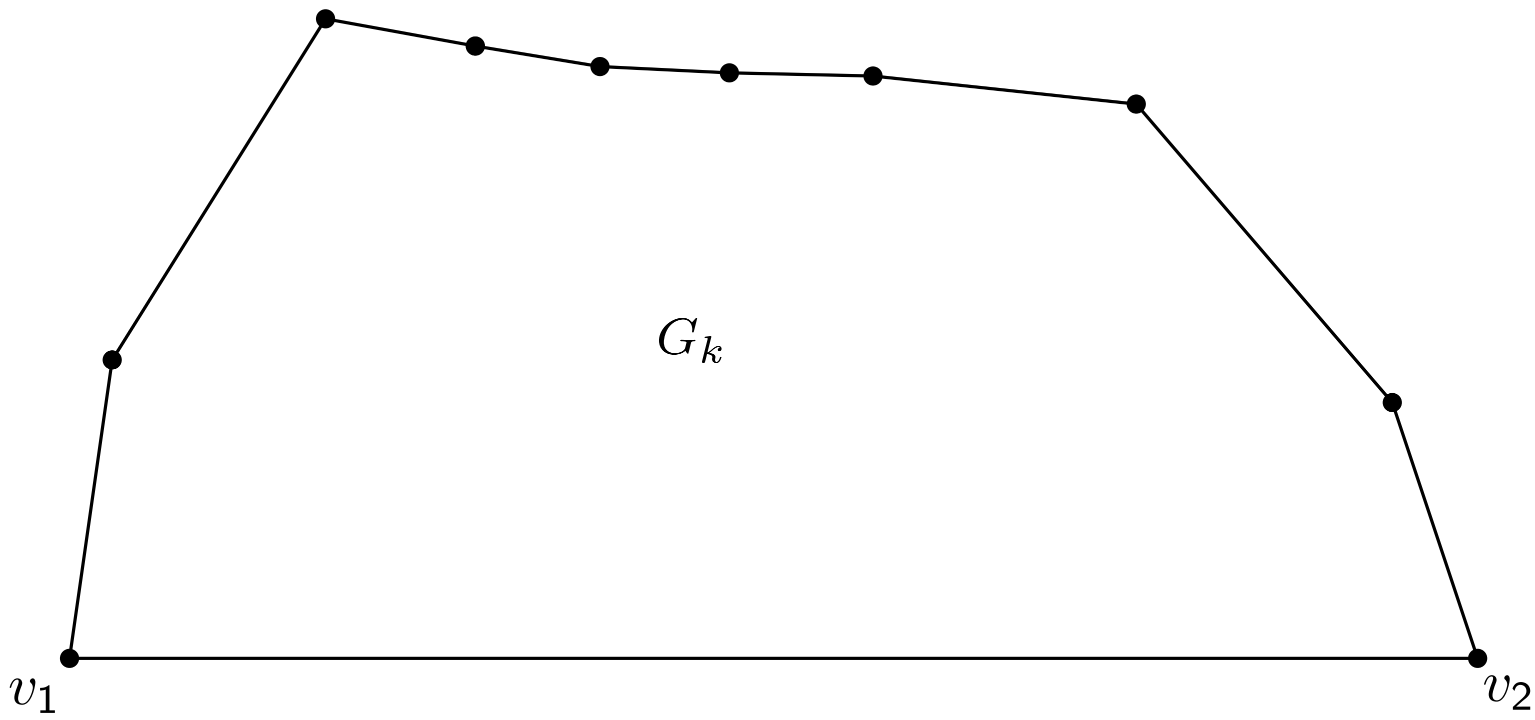

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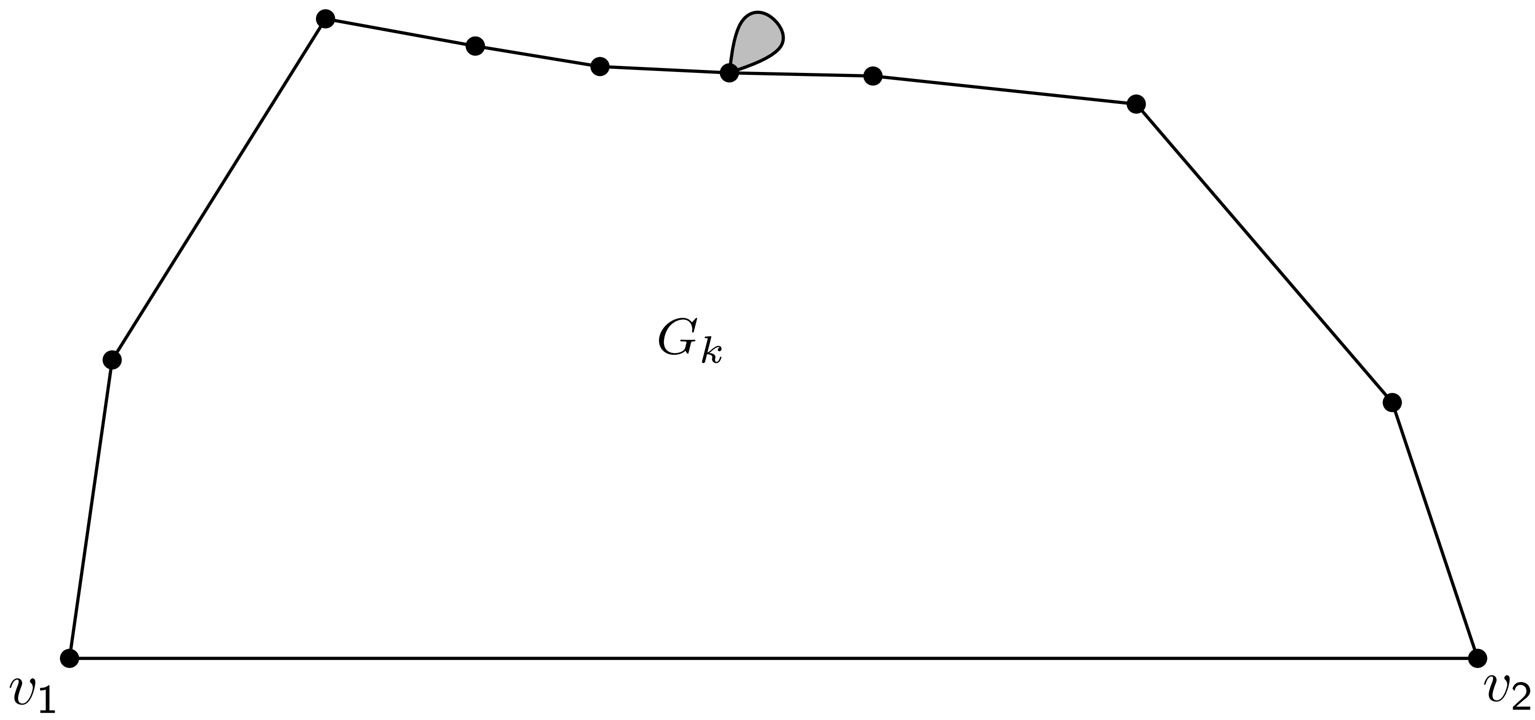

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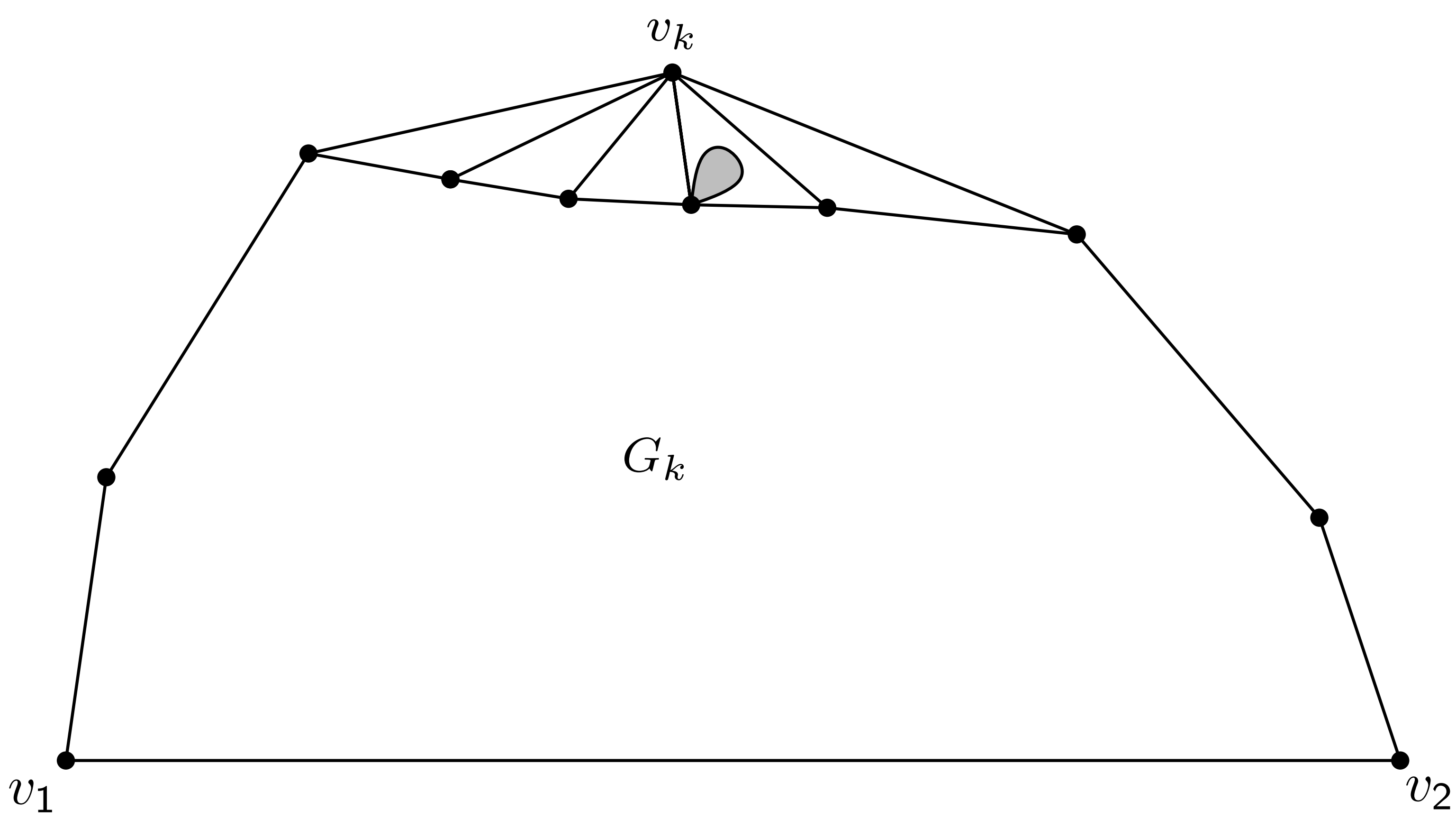

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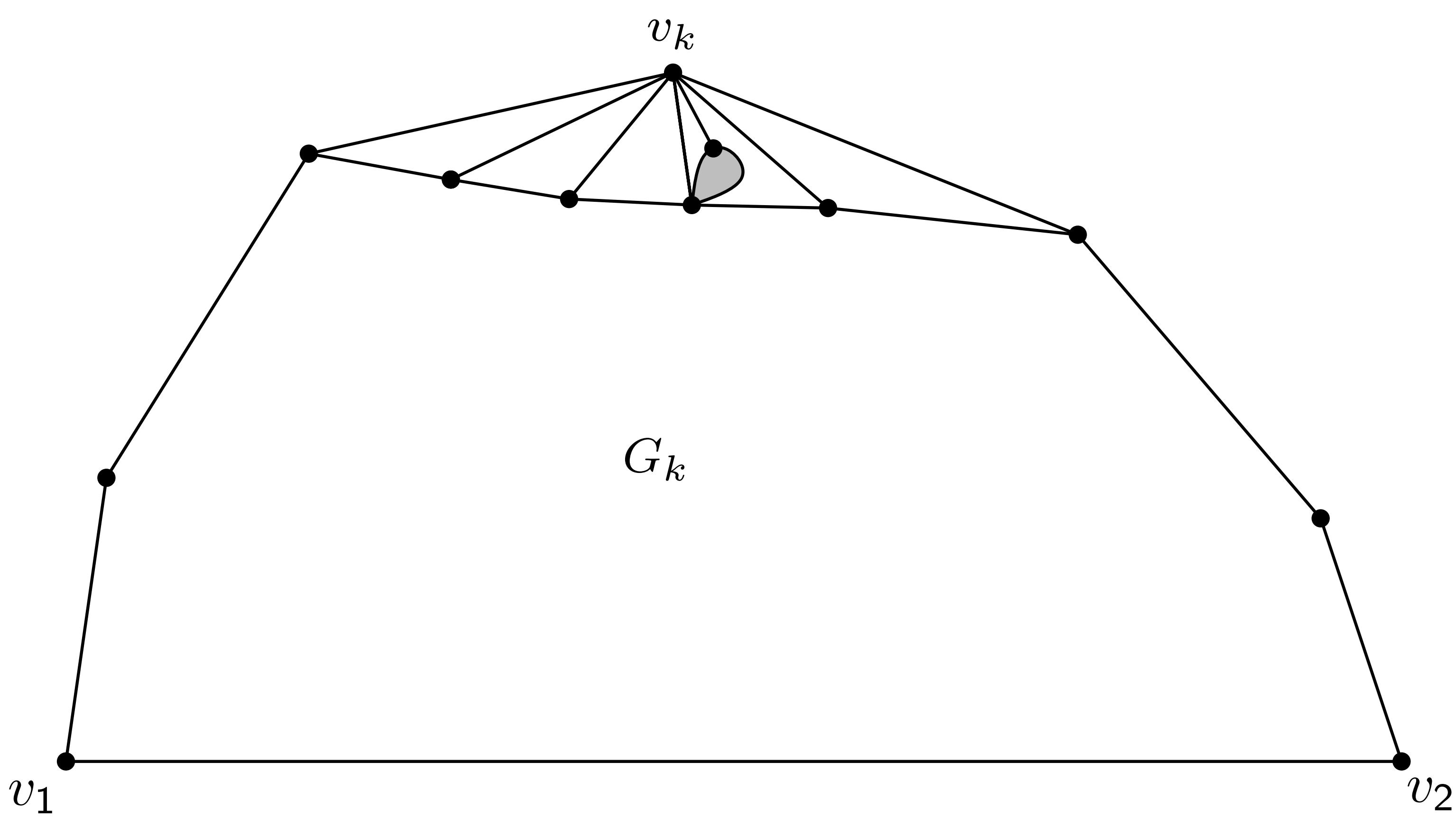

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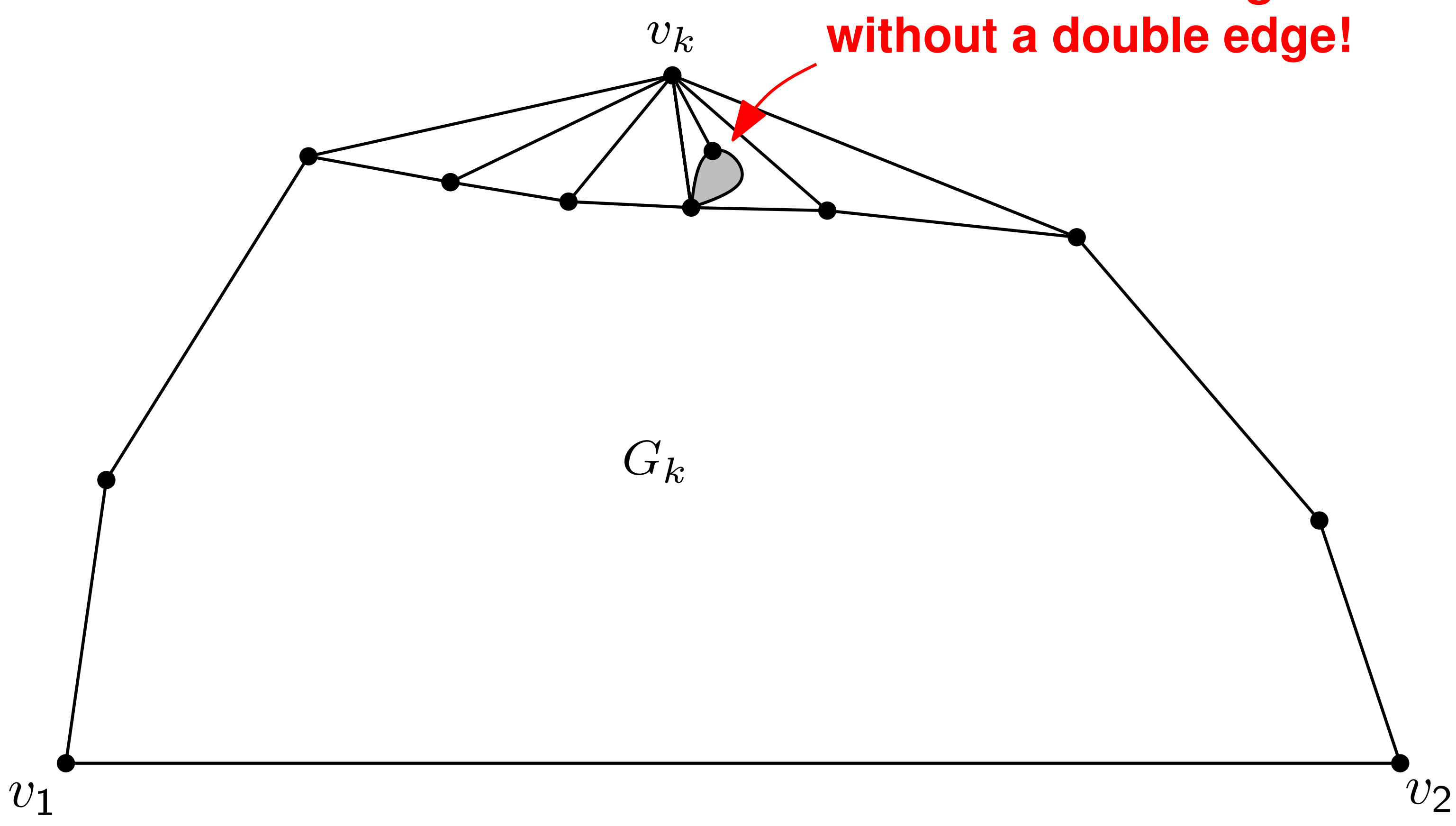
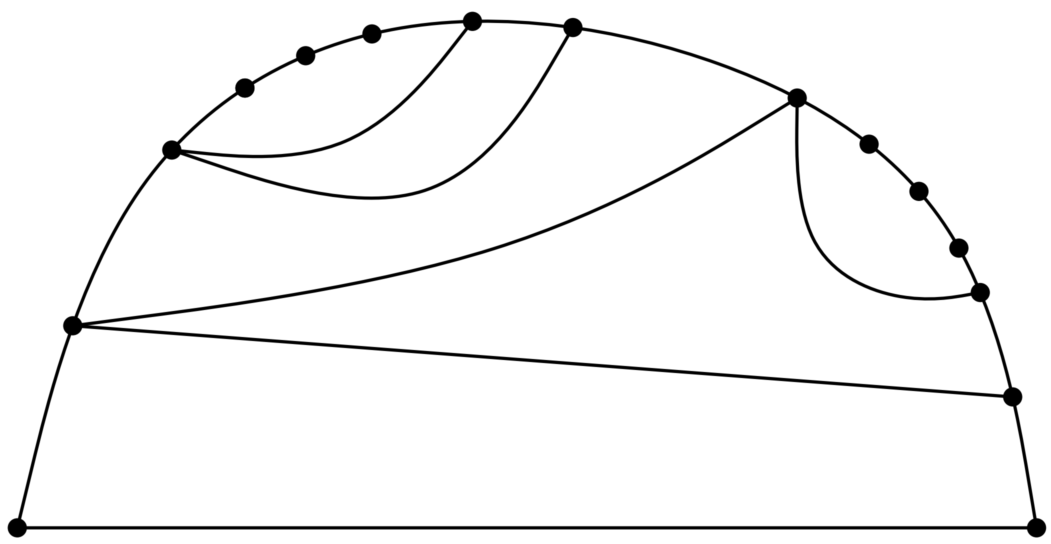
We can not triangulate this face without a double edge!



Canonical Ordering. Existence.

- Why a vertex not adjacent to a chord exists?



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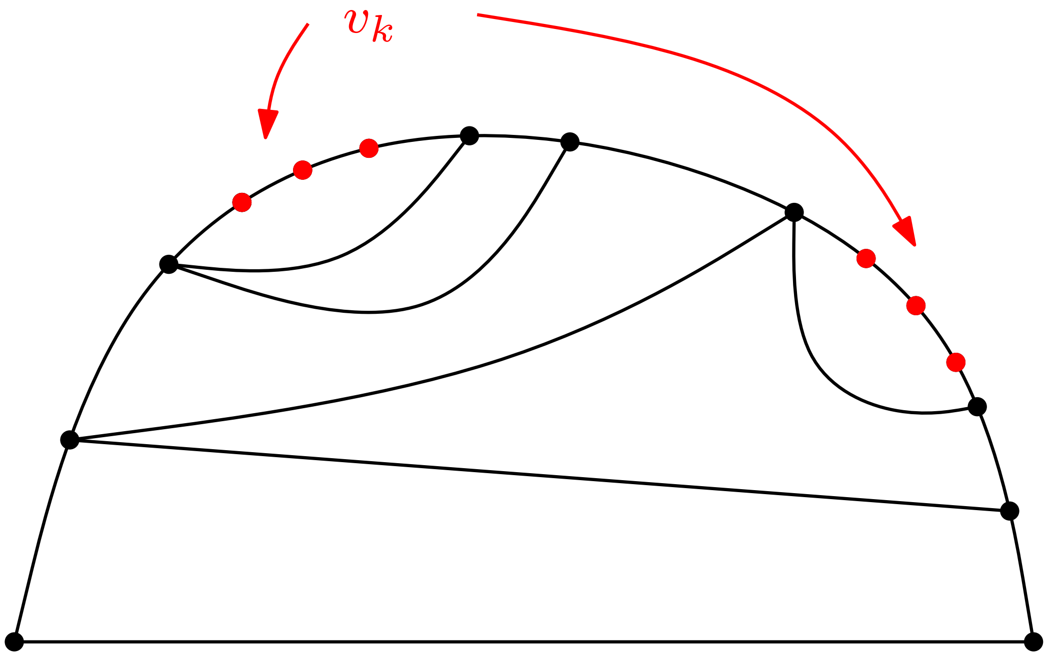

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- Implentational details.

Computing Canonical Ordering

Algorithm CO

forall $v \in V$ **do**
  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false;

out(v_1), out(v_2), out(v_n) $\leftarrow$ true;
for $k = n$ **to** 3 **do**
  choose $v \neq v_1, v_2$ such that mark(v) = false, out(v) = true,
    chords(v) = 0;
  $v_k \leftarrow v$; mark(v) $\leftarrow$ true;
  // Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{k-1};
  and let $w_p, \ldots, w_q$ be the unmarked neighbors v_k;
  out(w_i) $\leftarrow$ true for all $p < i < q$;
  update number of chords for w_i and its neighbors;

- chord(v) - number of chords adjacent to v
- mark(v) = true iff vertex v was numbered
- out(v)=true iff v is the outer vertex of current plane graph

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Lemma

Algorithm CO computes a canonical ordering of a graph in $O(n)$ time.

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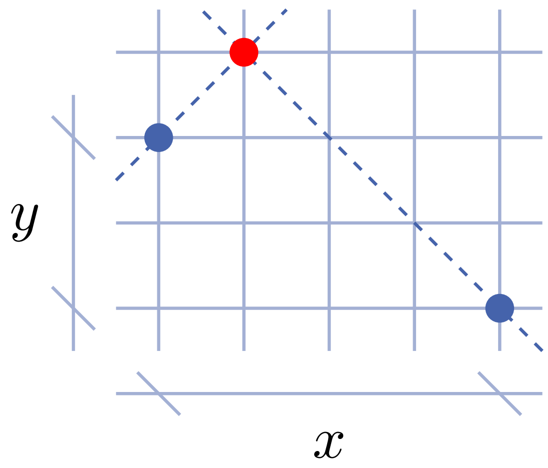
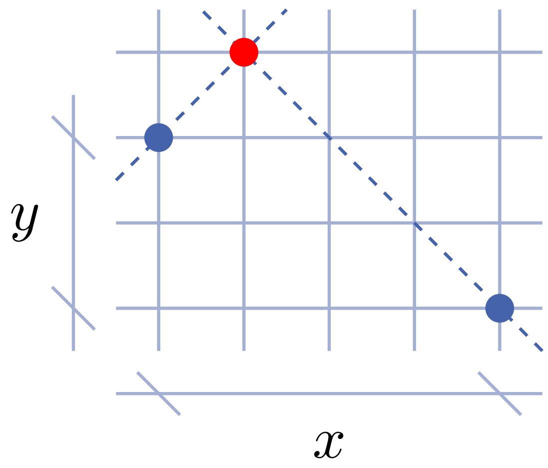
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



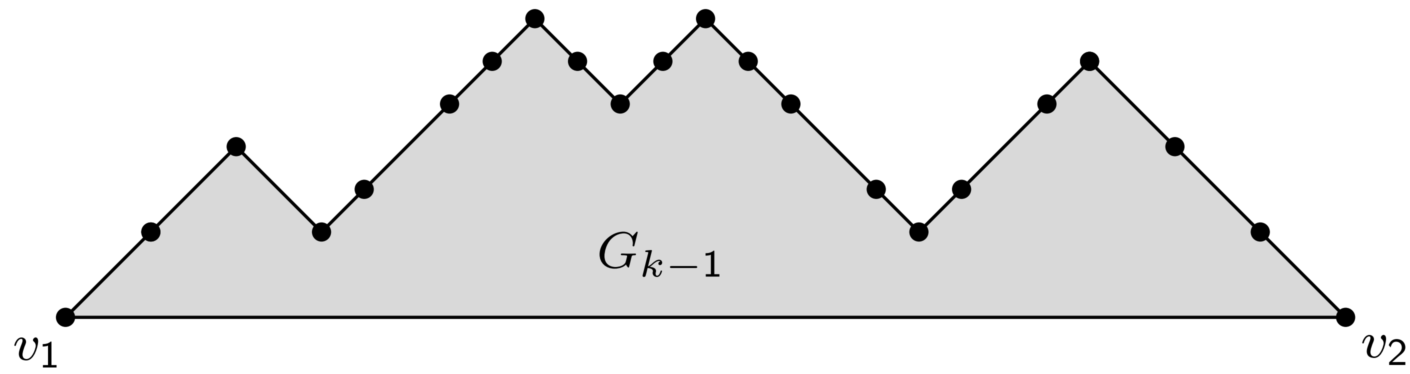
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



Algorithm invariants: G_{k-1} is drawn such that

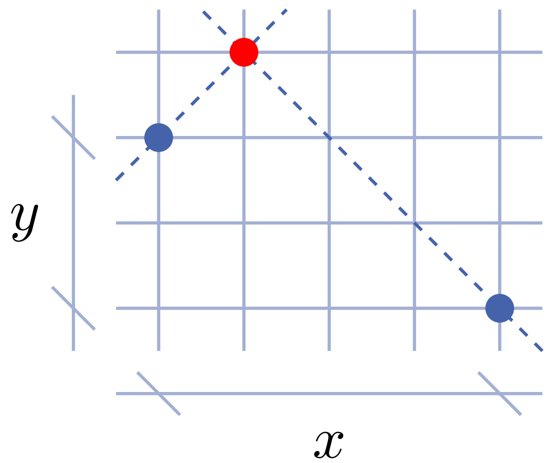
- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$
- Boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn with slopes ± 1



15 - 2

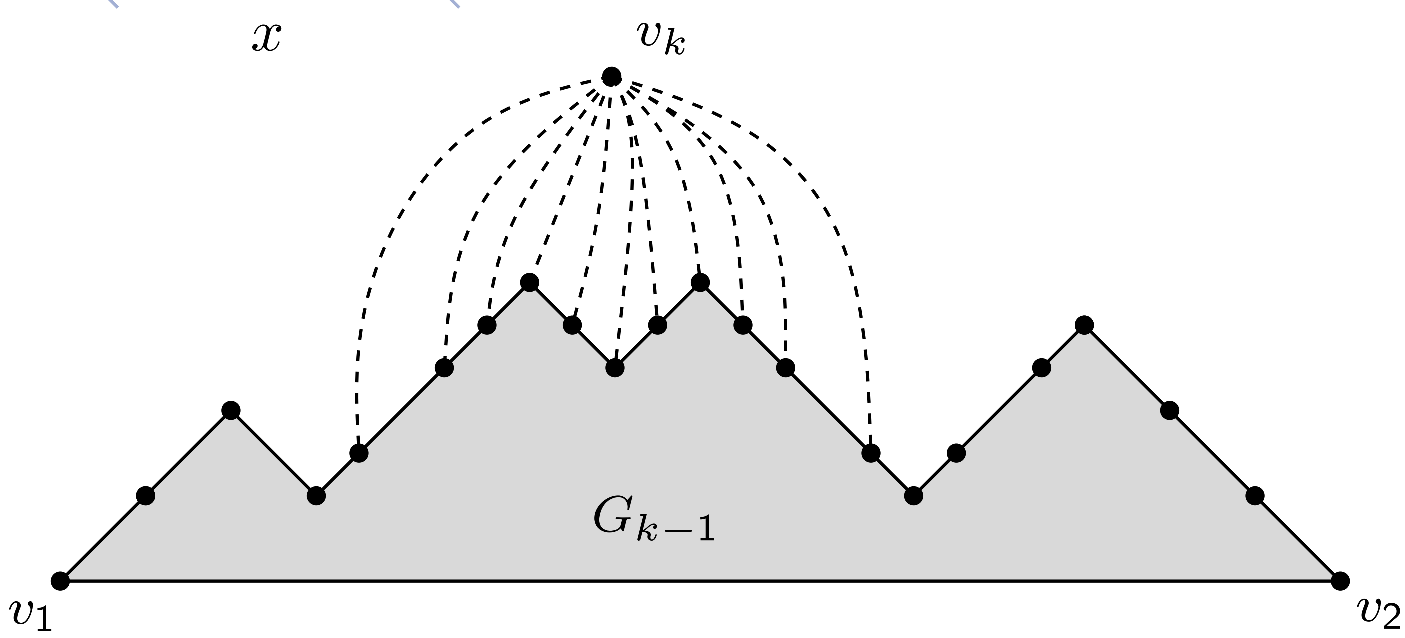
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



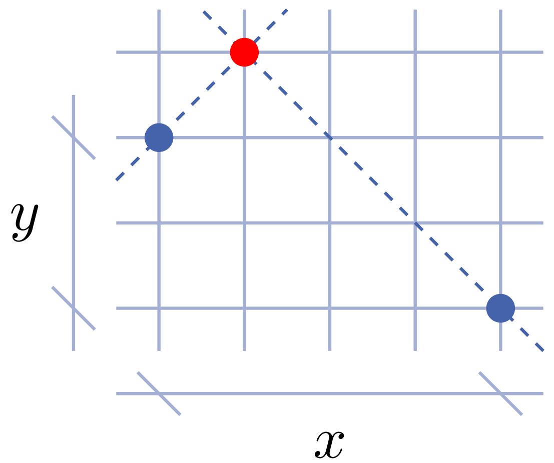
Algorithm invariants: G_{k-1} is drawn such that

- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$
- Boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn with slopes ± 1



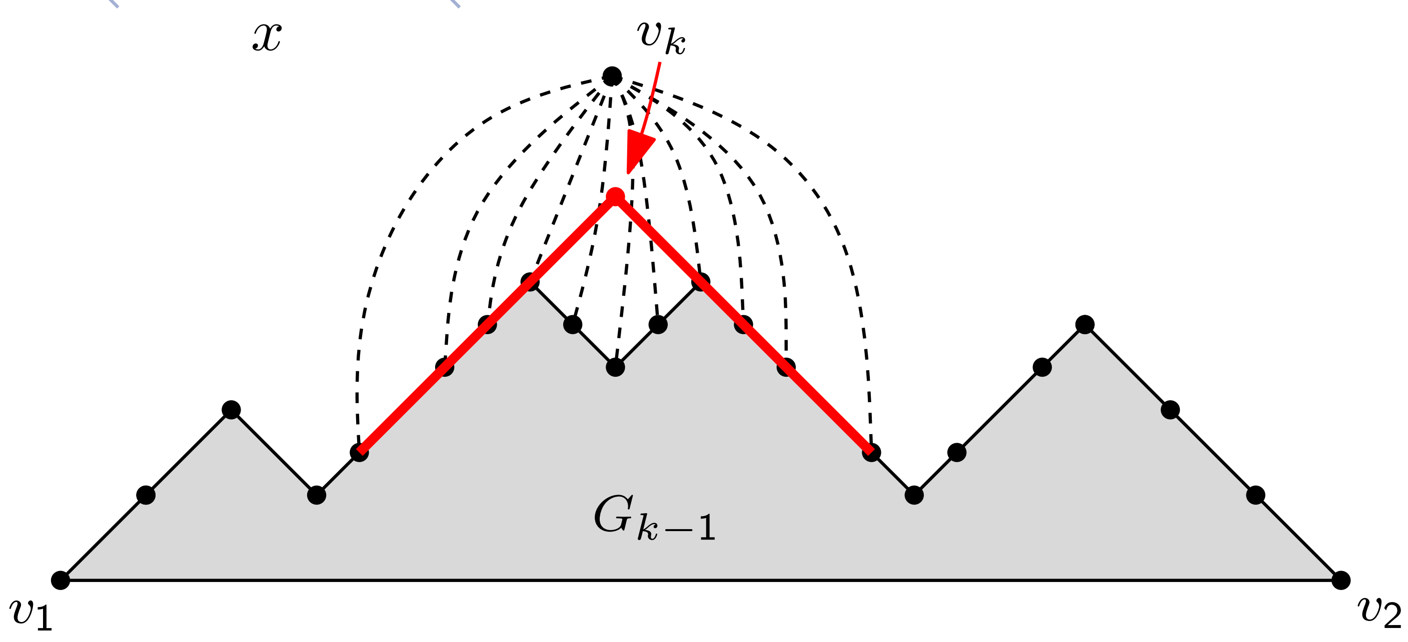
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance

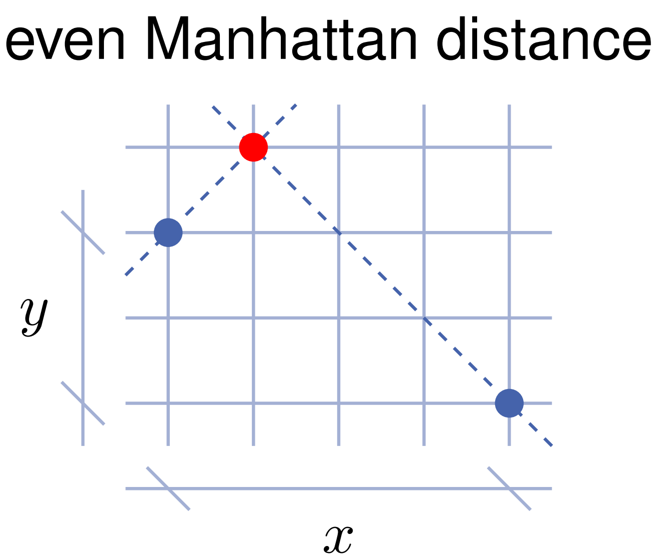


Algorithm invariants: G_{k-1} is drawn such that

- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$
- Boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn with slopes ± 1



De Fraysseix Pach Pollack (Shift) Algorithm



Algorithm invariants: G_{k-1} is drawn such that

- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$
- Boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn x-monotone
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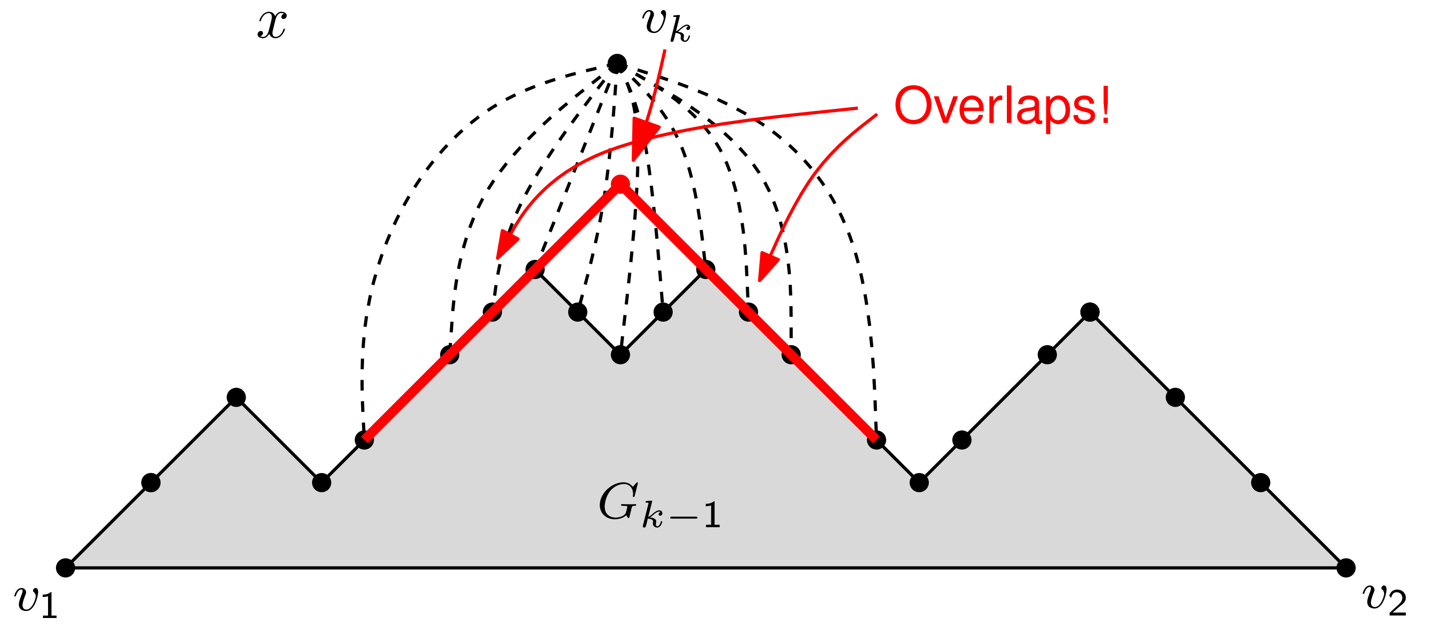
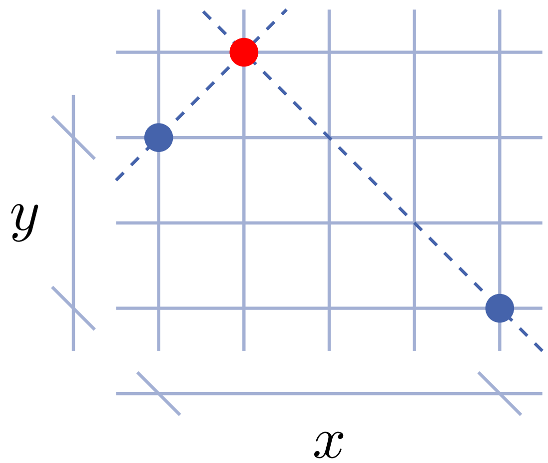


15 - 5

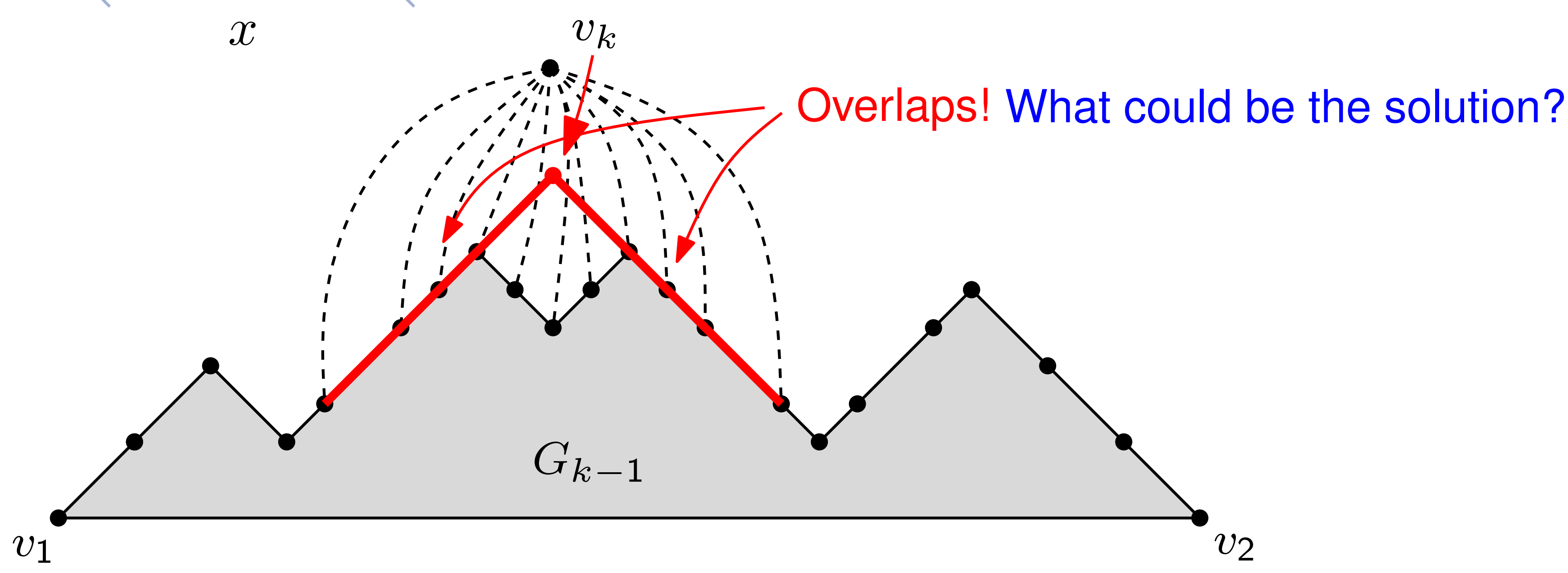
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



Algorithm invariants: G_{k-1} is drawn such that

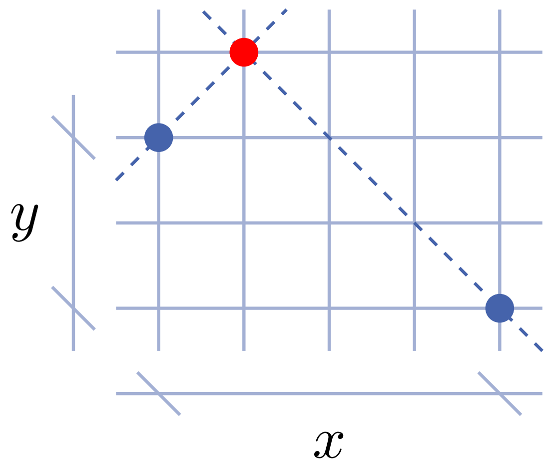
- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$
- Boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn with slopes ± 1



15 - 6

De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



Algorithm invariants: G_{k-1} is drawn such that

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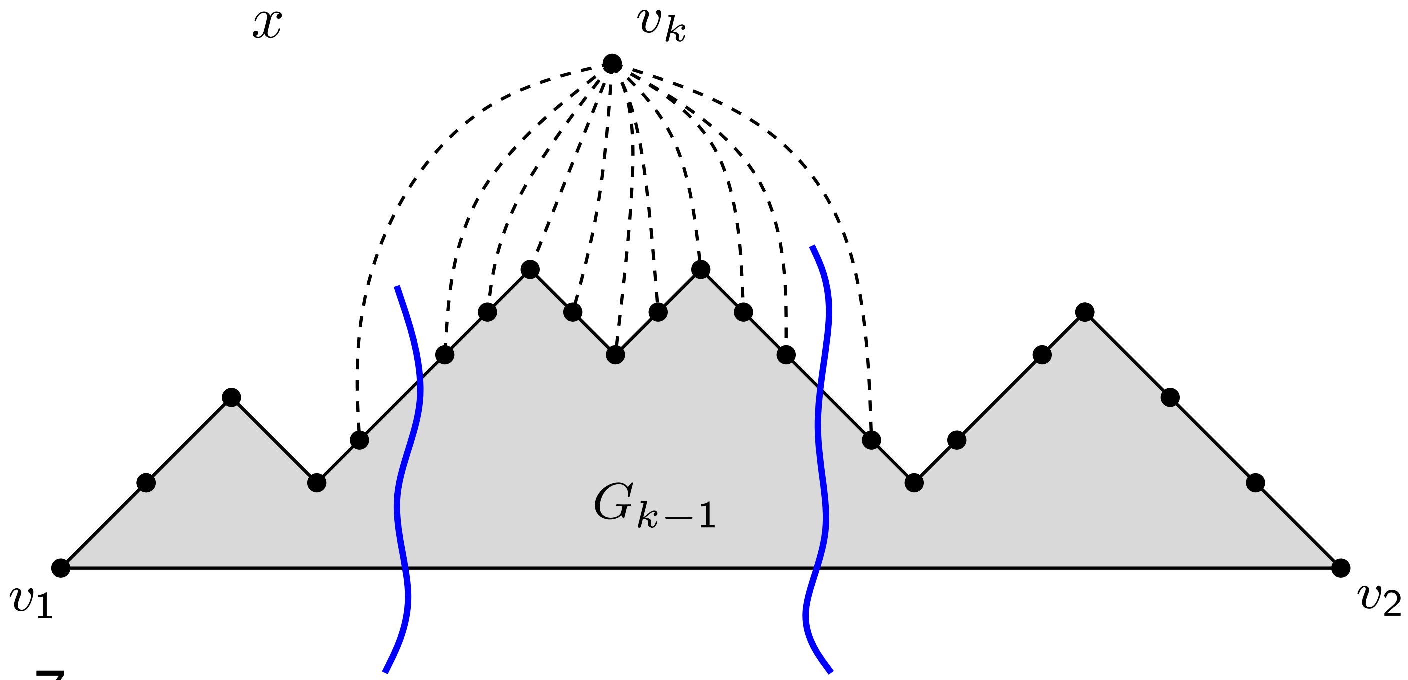
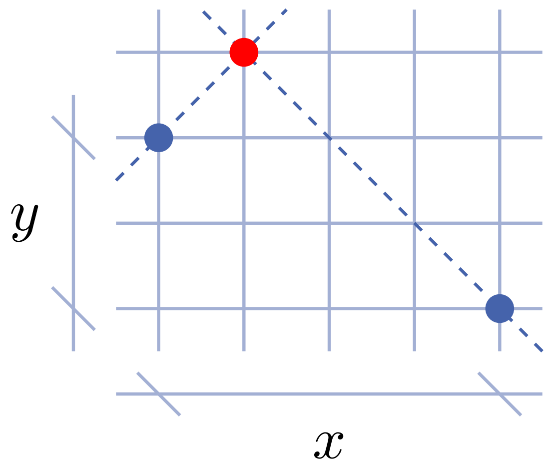
- Boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn with slopes ± 1



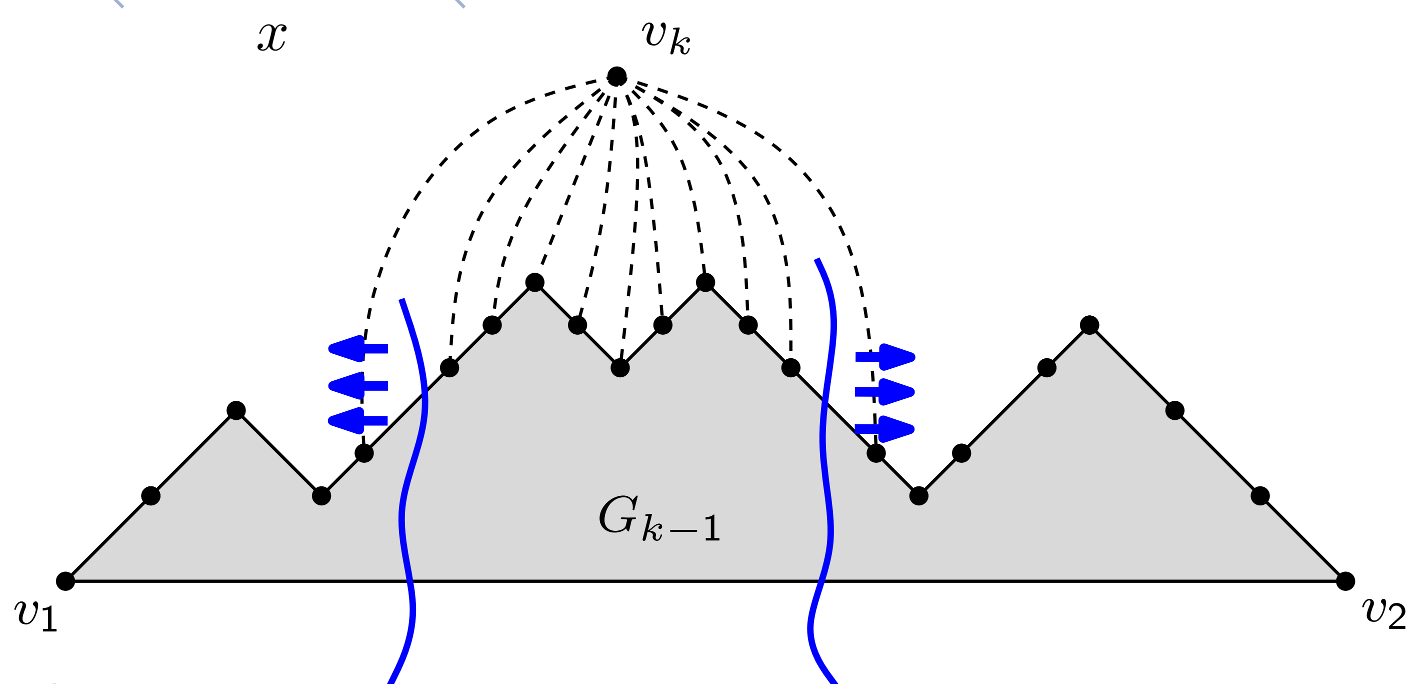
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



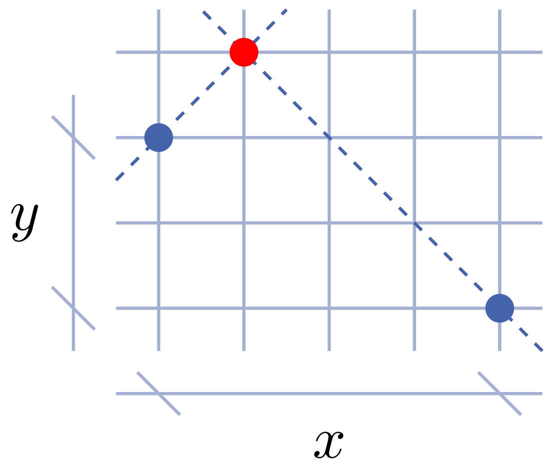
Algorithm invariants: G_{k-1} is drawn such that

- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$
- Boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1,v_2)) is drawn with slopes ± 1



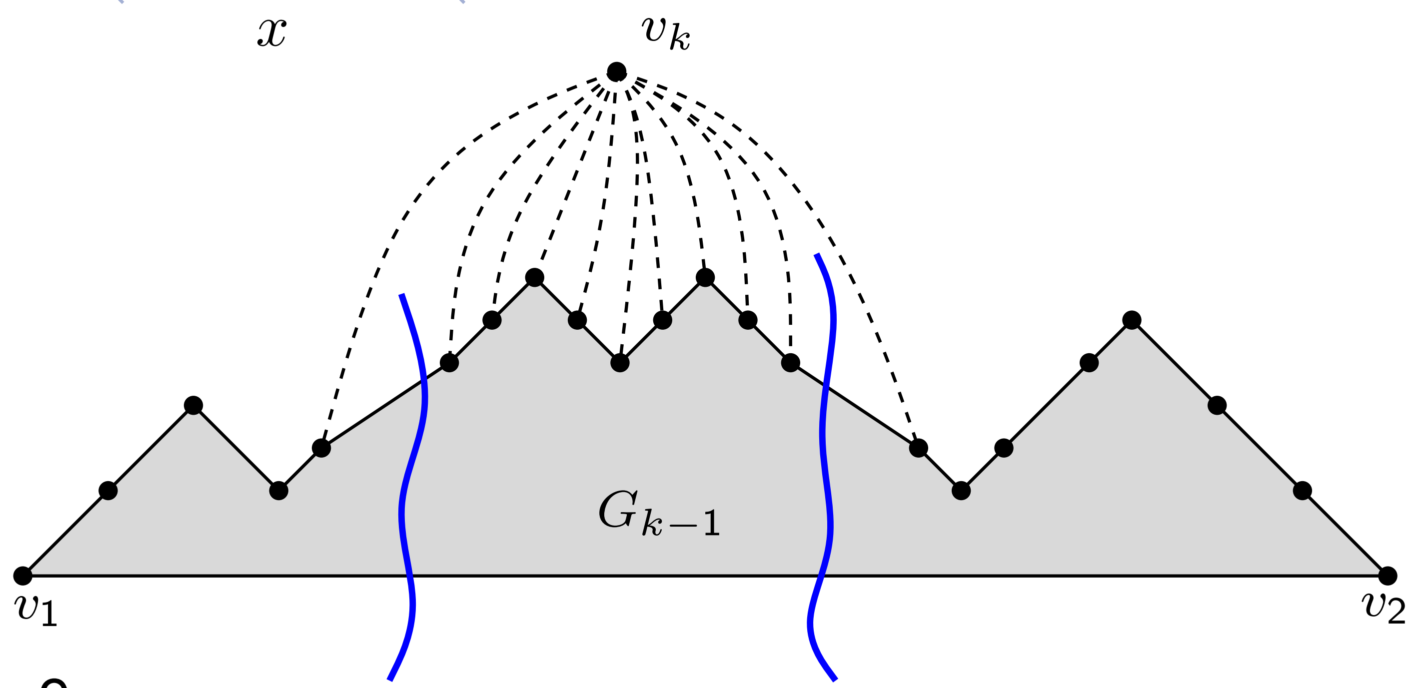
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



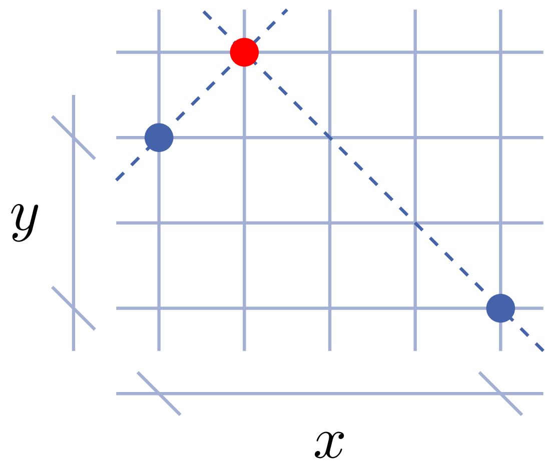
Algorithm invariants: G_{k-1} is drawn such that

- v_1 is on $(0, 0)$, v_2 is on $(2k - 6, 0)$
- Boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn with slopes ± 1



De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



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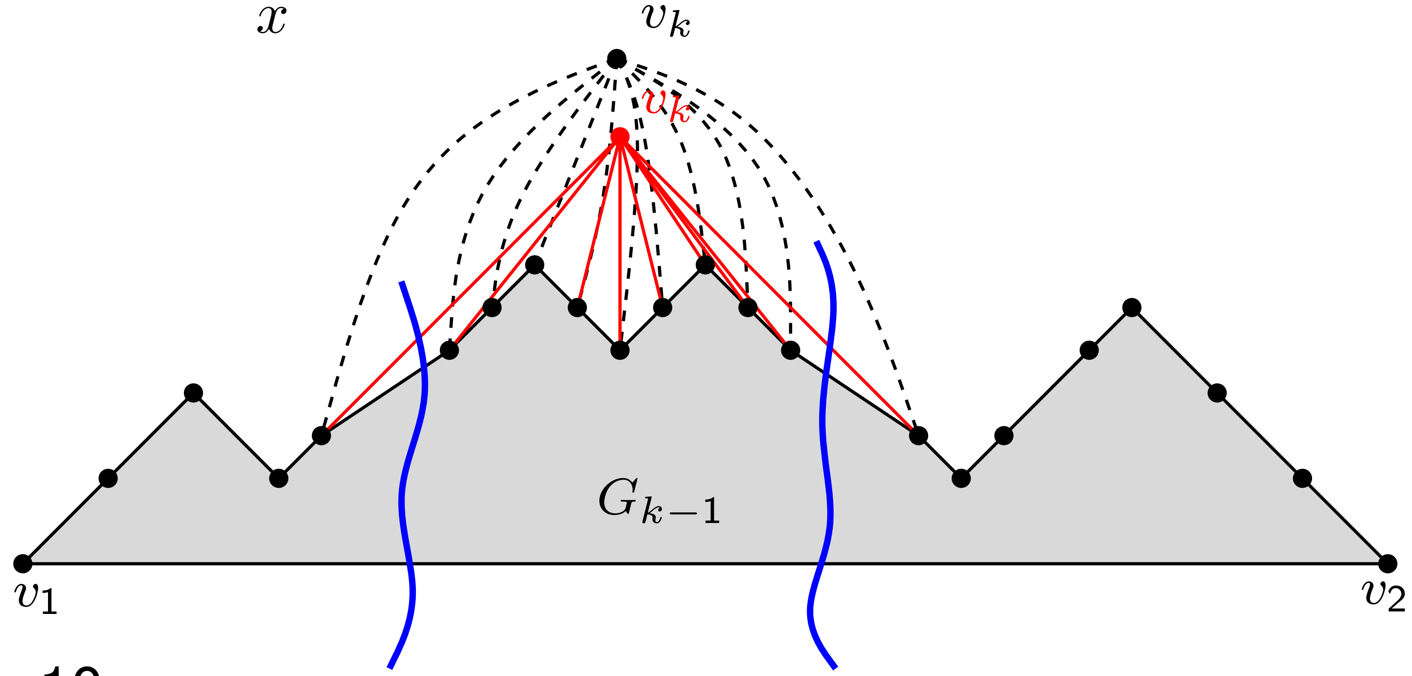
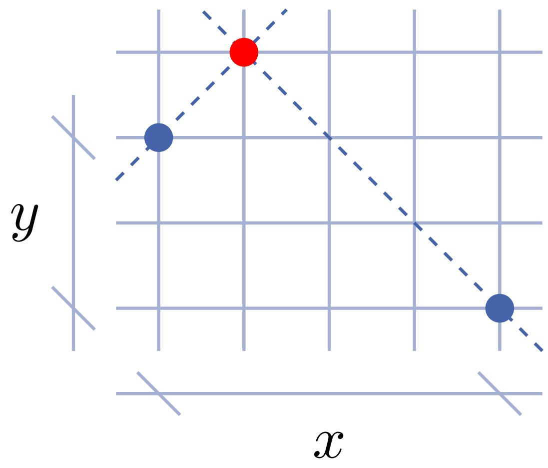


15 - 10

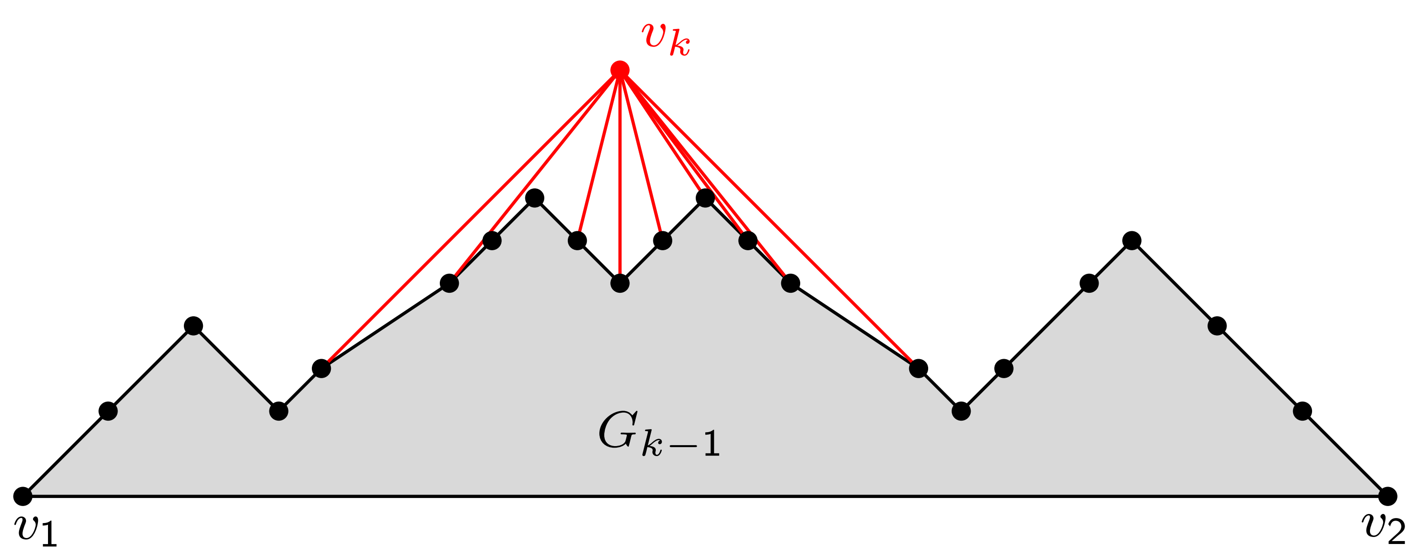
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance



Algorithm invariants: G_{k-1} is drawn such that

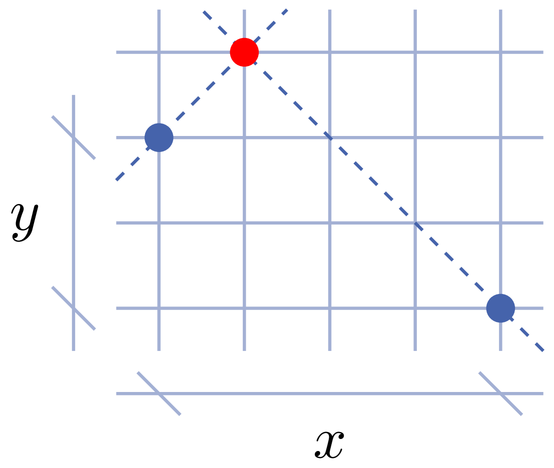
- v_1 is on $(0, 0)$, v_2 is on $(2k - 6, 0)$
- Boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn with slopes ± 1



15 - 11

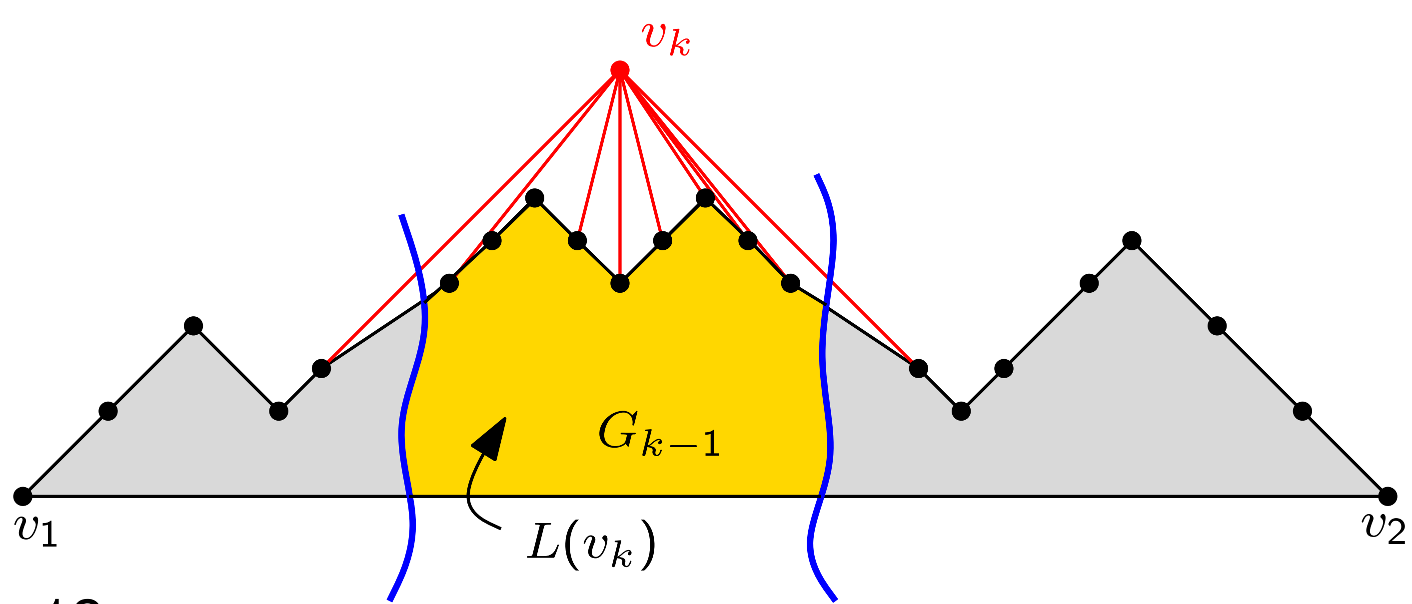
De Fraysseix Pach Pollack (Shift) Algorithm

even Manhattan distance

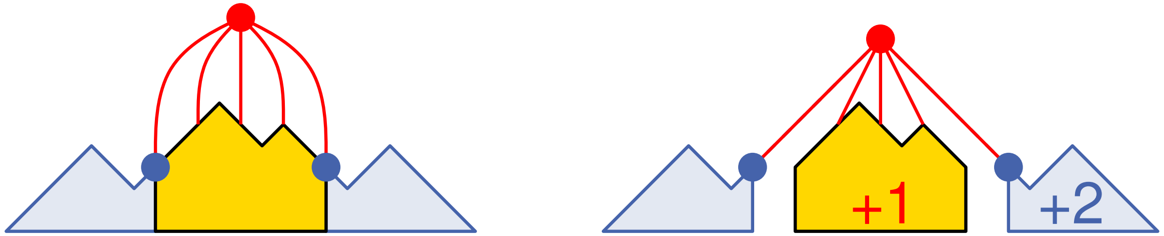


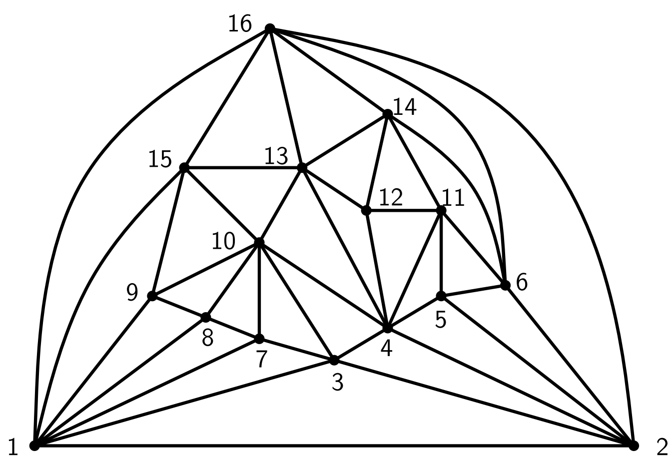
Algorithm invariants: G_{k-1} is drawn such that

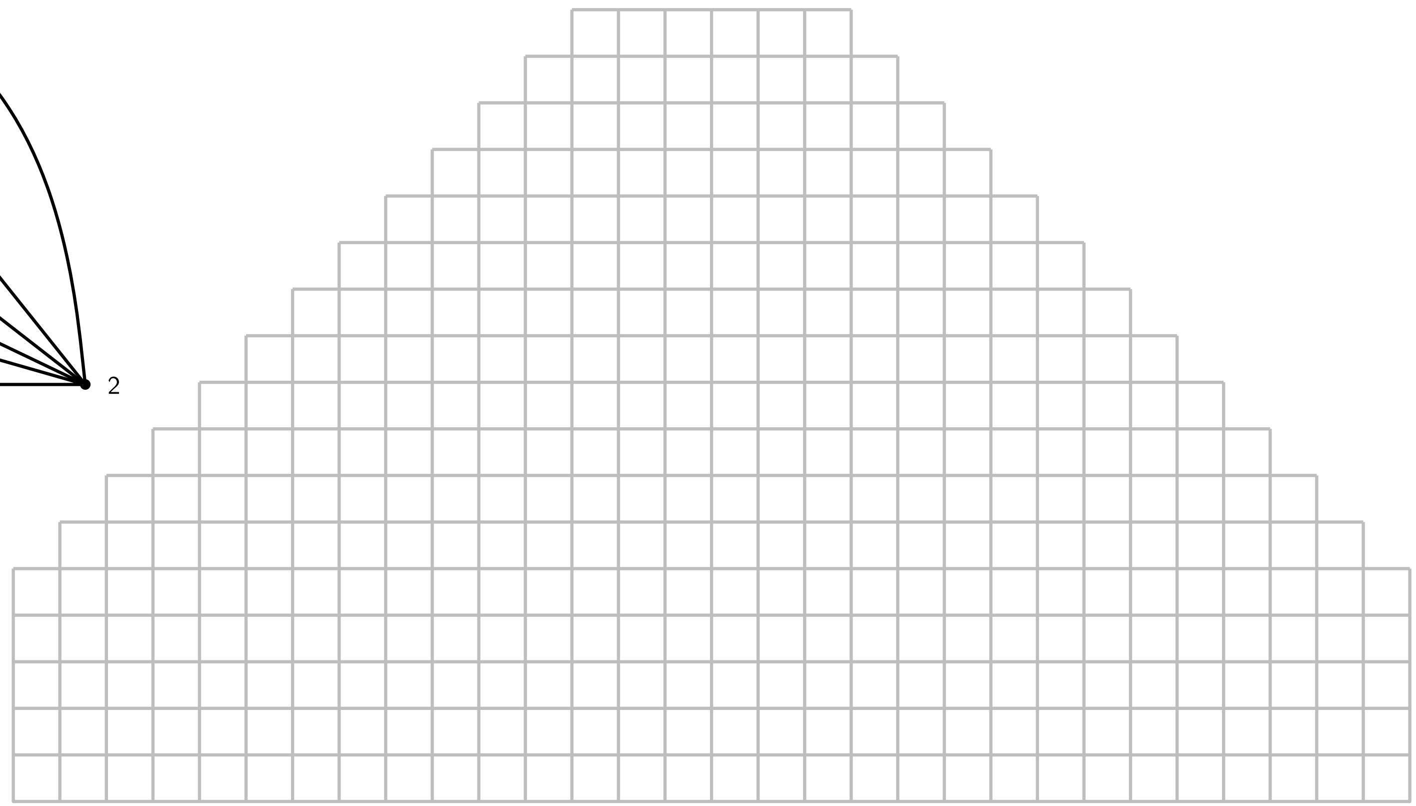
- v_1 is on $(0,0)$, v_2 is on $(2k-6,0)$
- Boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn with slopes ± 1



De Fraysseix Pach Pollack (Shift) Algorithm

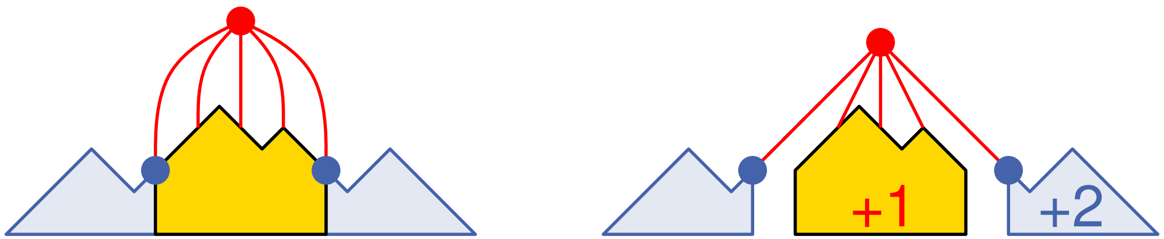


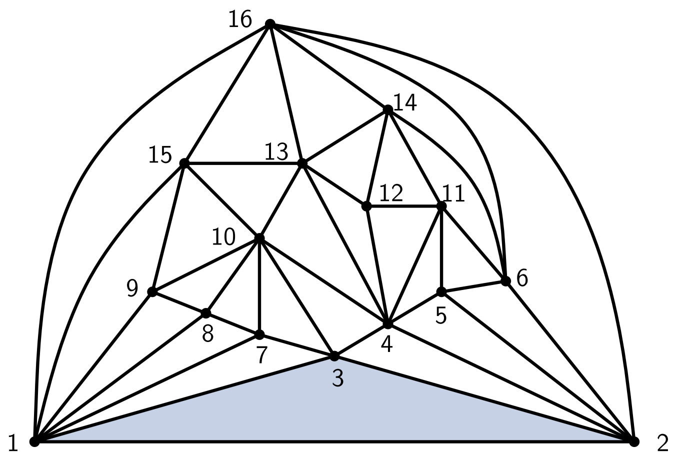


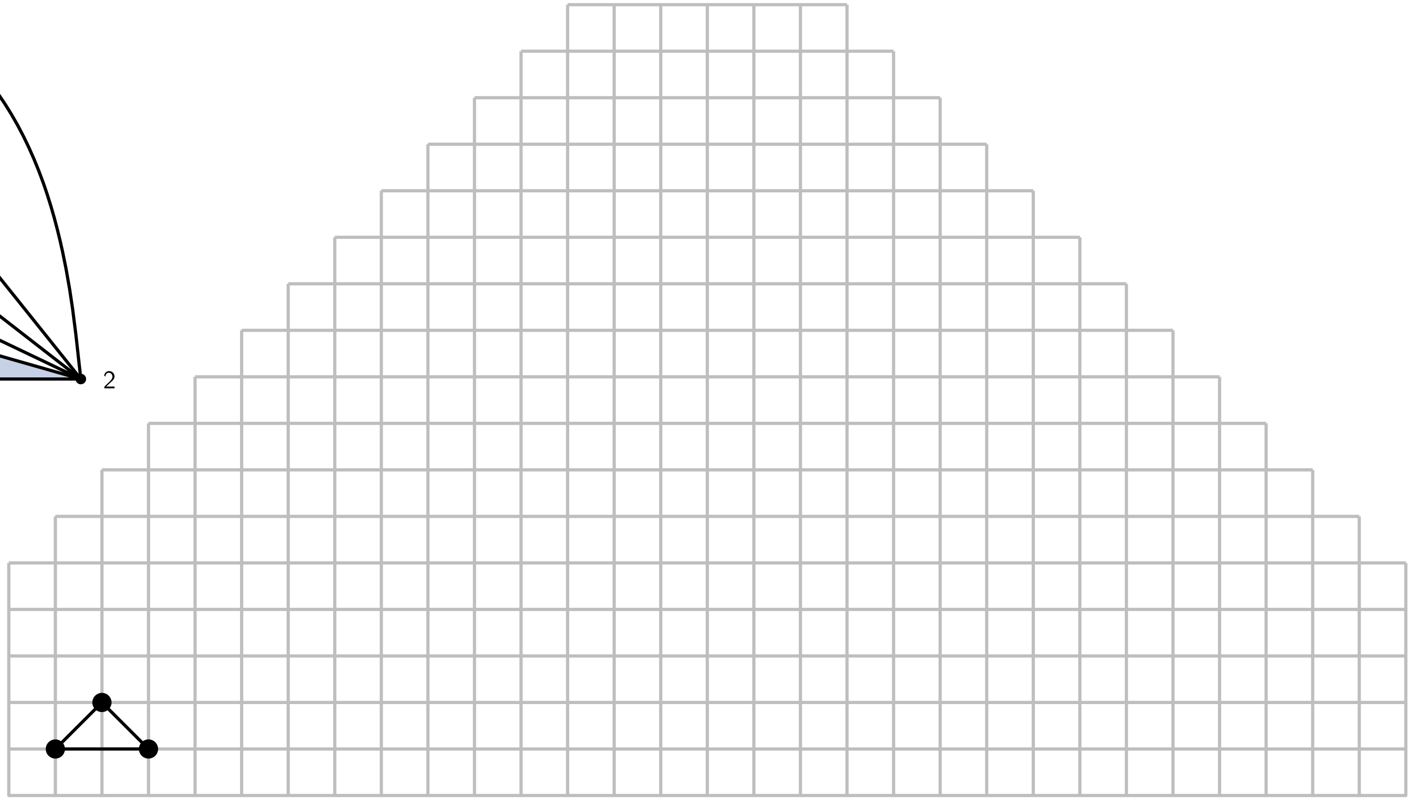


16 - 1

De Fraysseix Pach Pollack (Shift) Algorithm

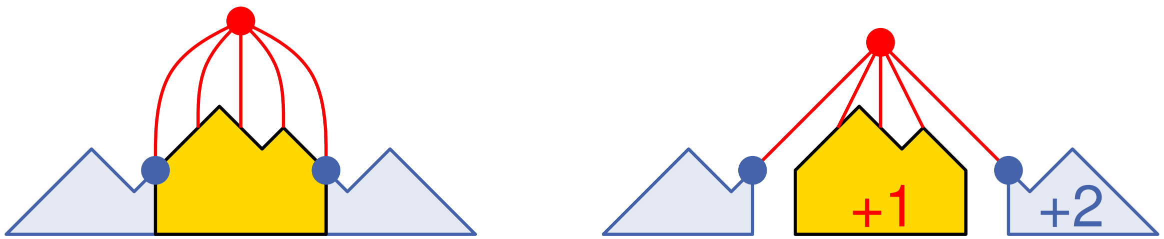


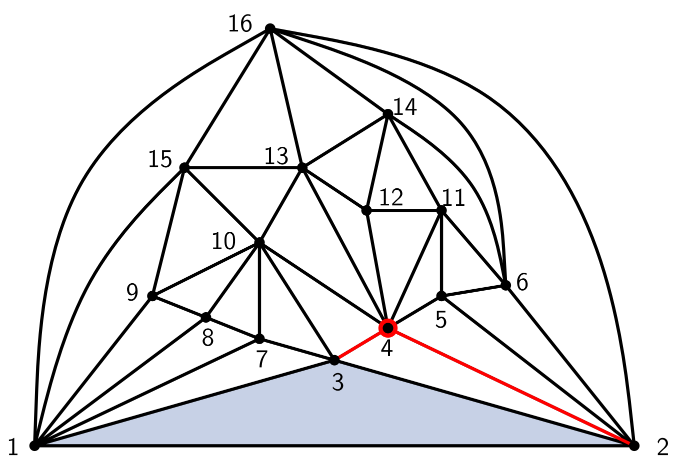


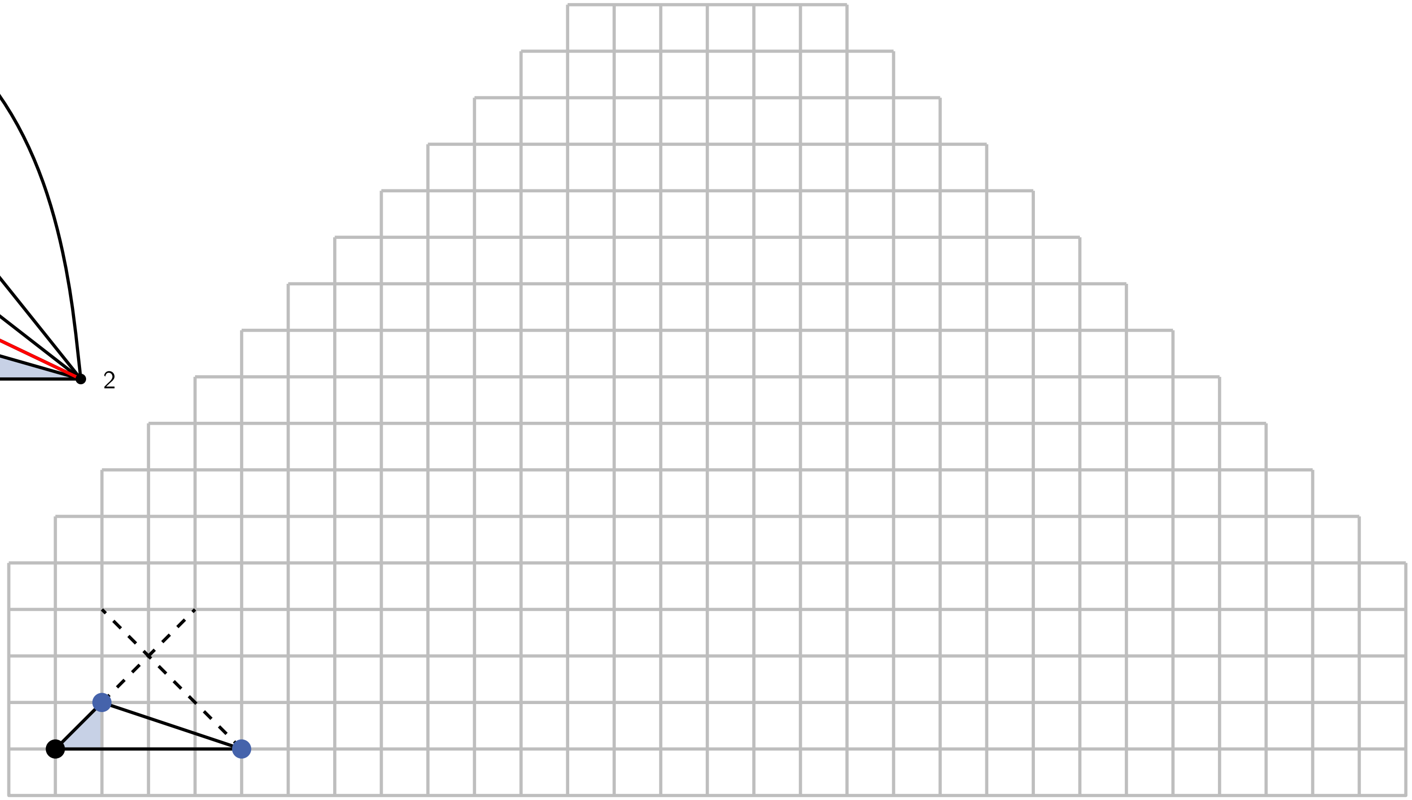


16 - 2

De Fraysseix Pach Pollack (Shift) Algorithm

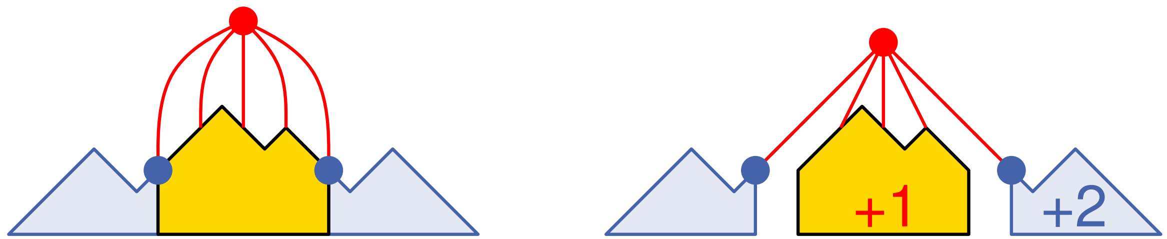


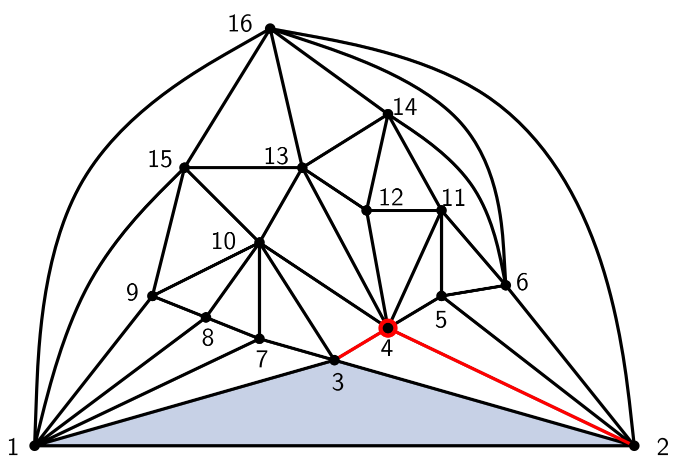


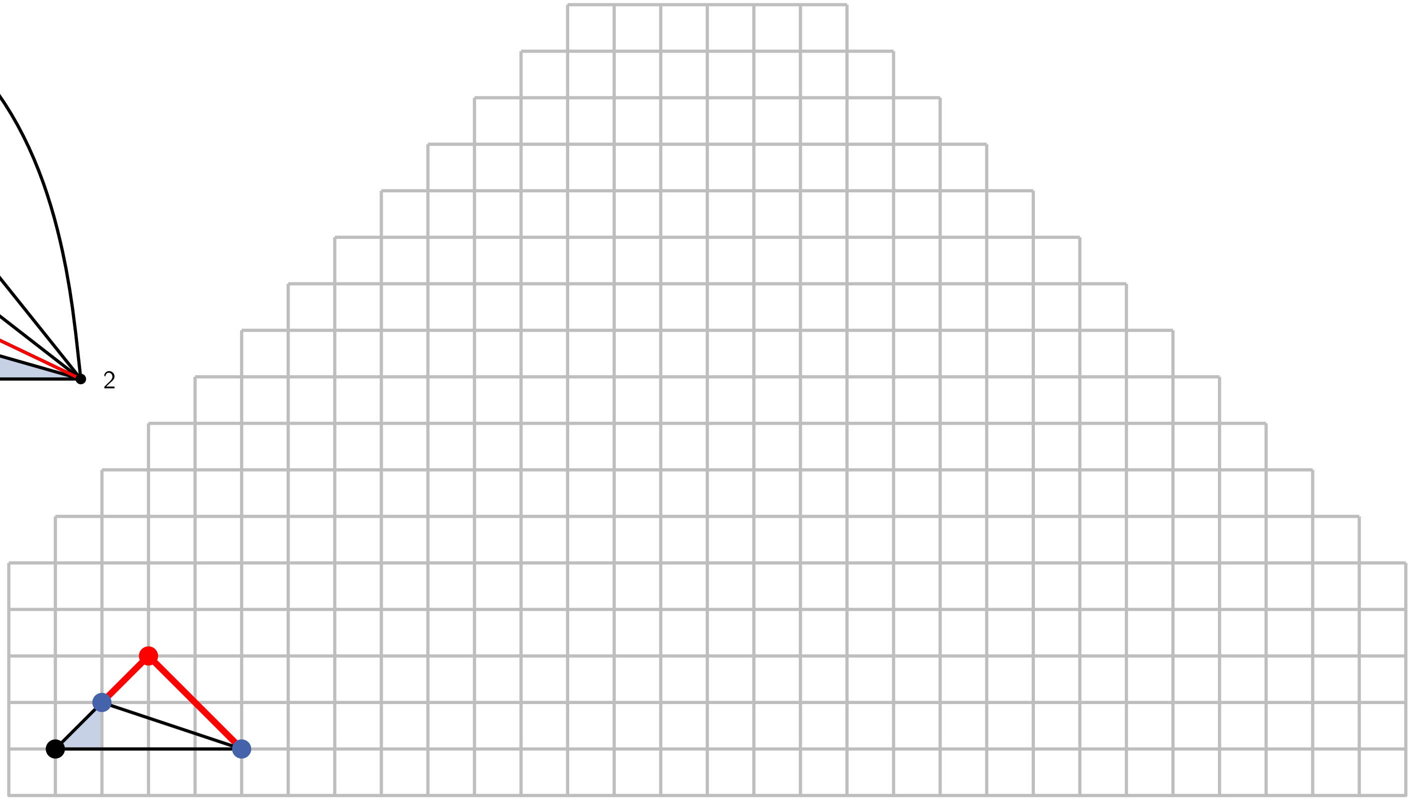


16 - 3

De Fraysseix Pach Pollack (Shift) Algorithm

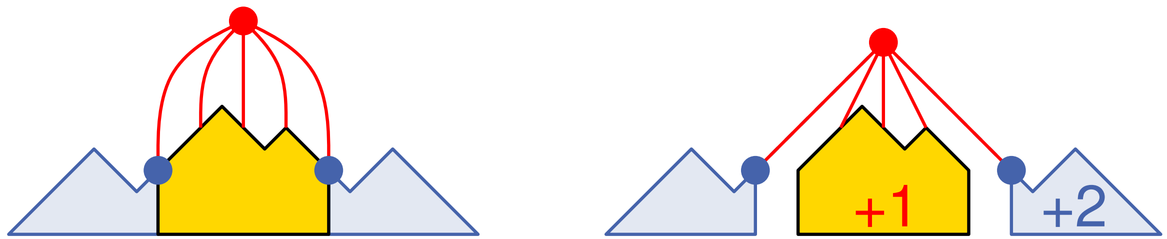


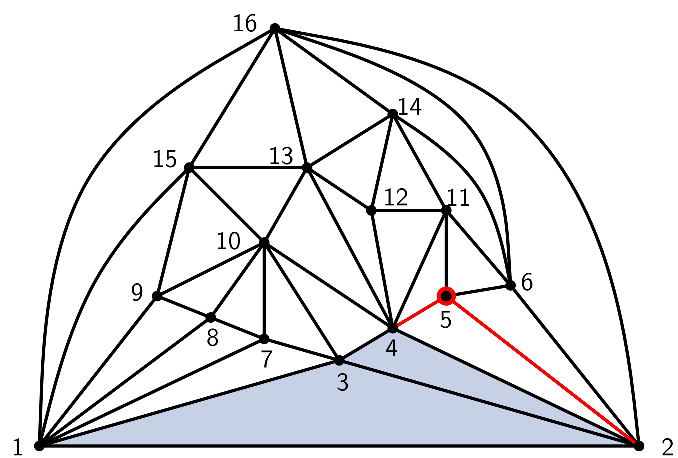


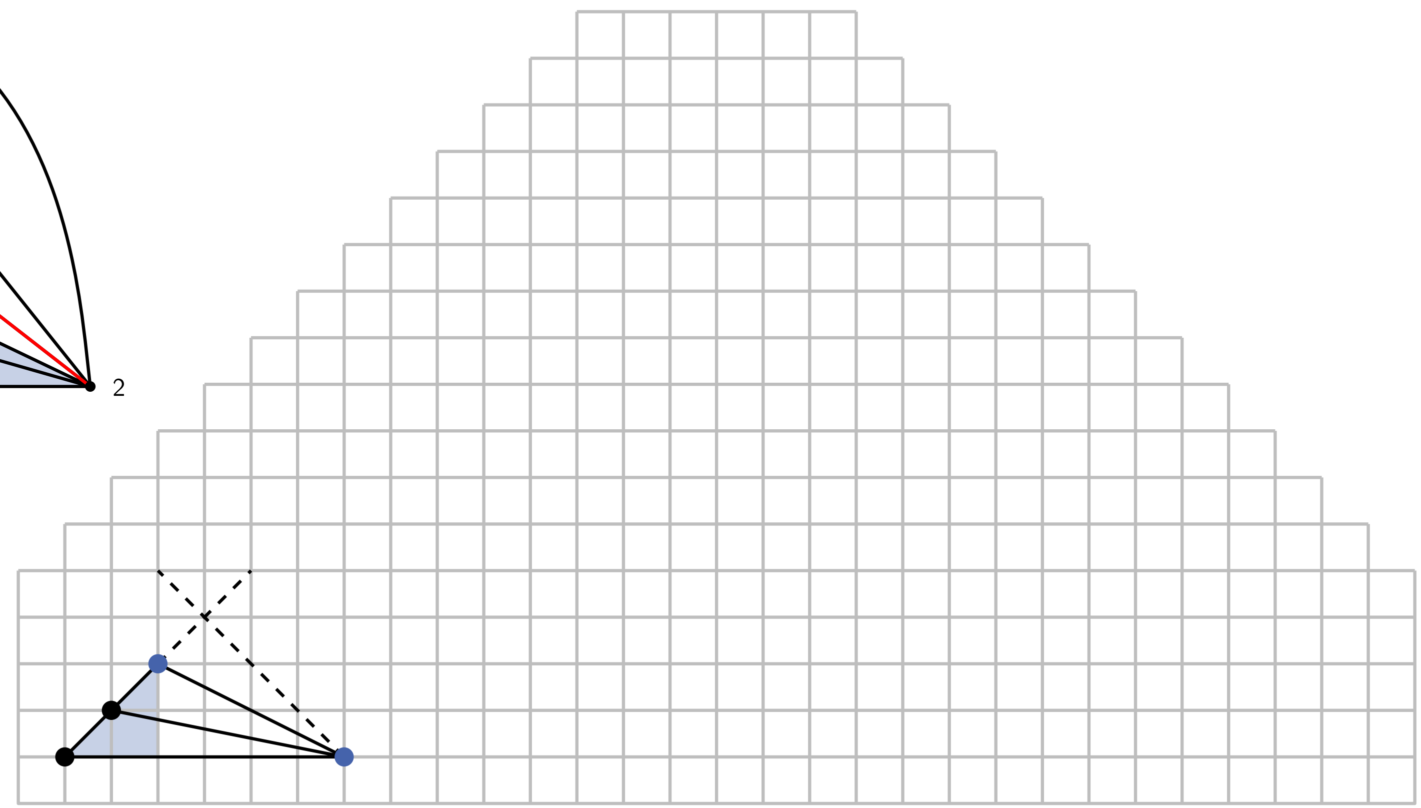


16 - 4

De Fraysseix Pach Pollack (Shift) Algorithm

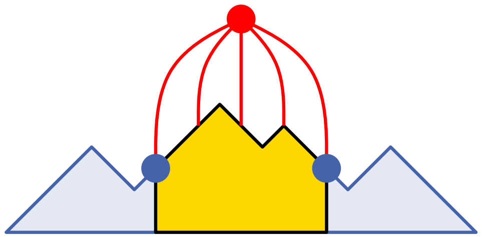


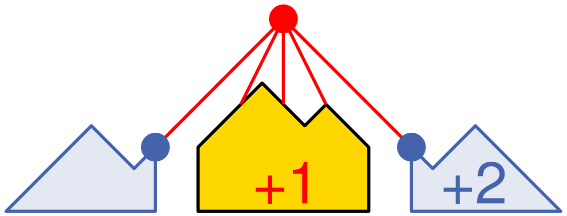


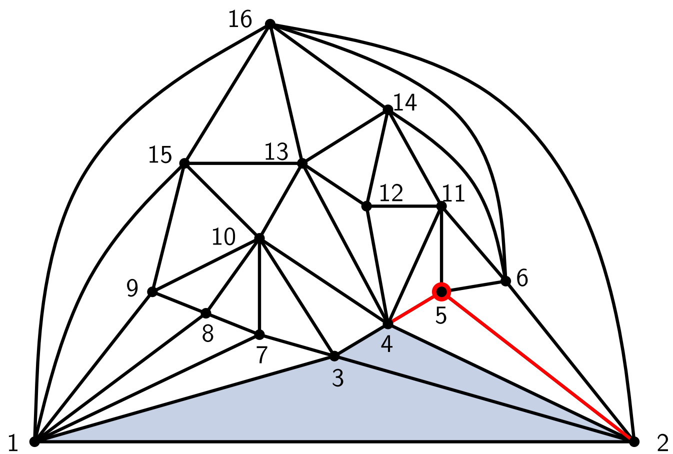


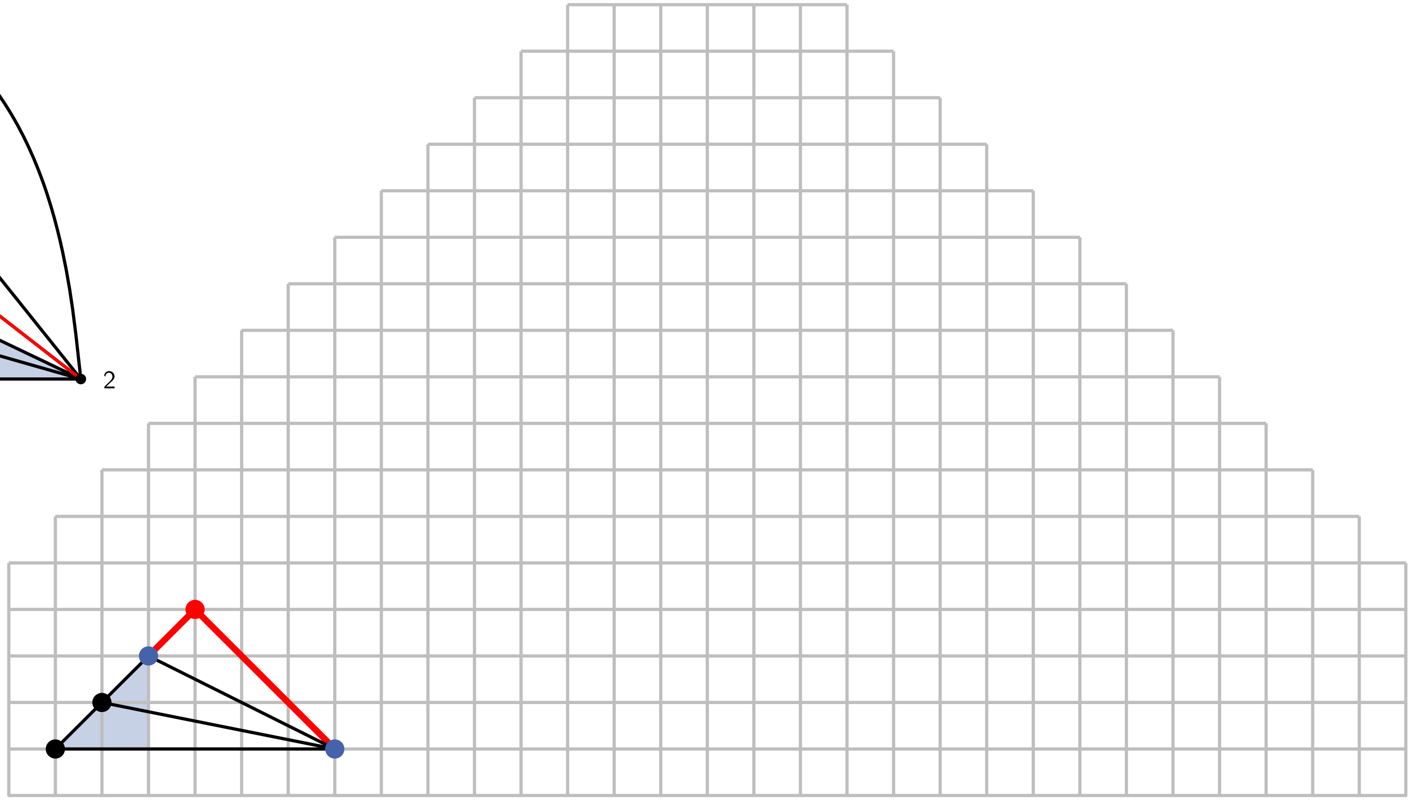
16 - 5

De Fraysseix Pach Pollack (Shift) Algorithm





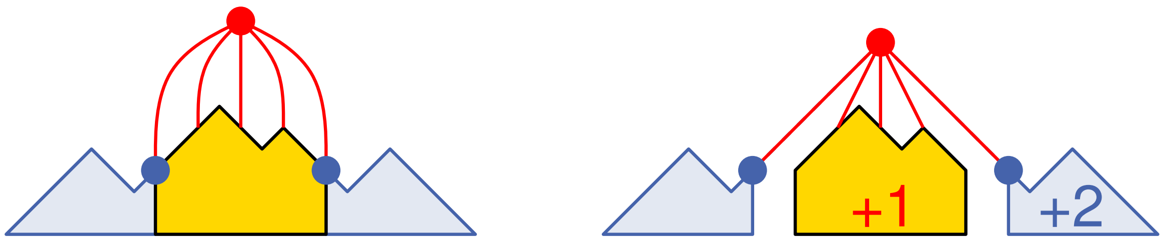


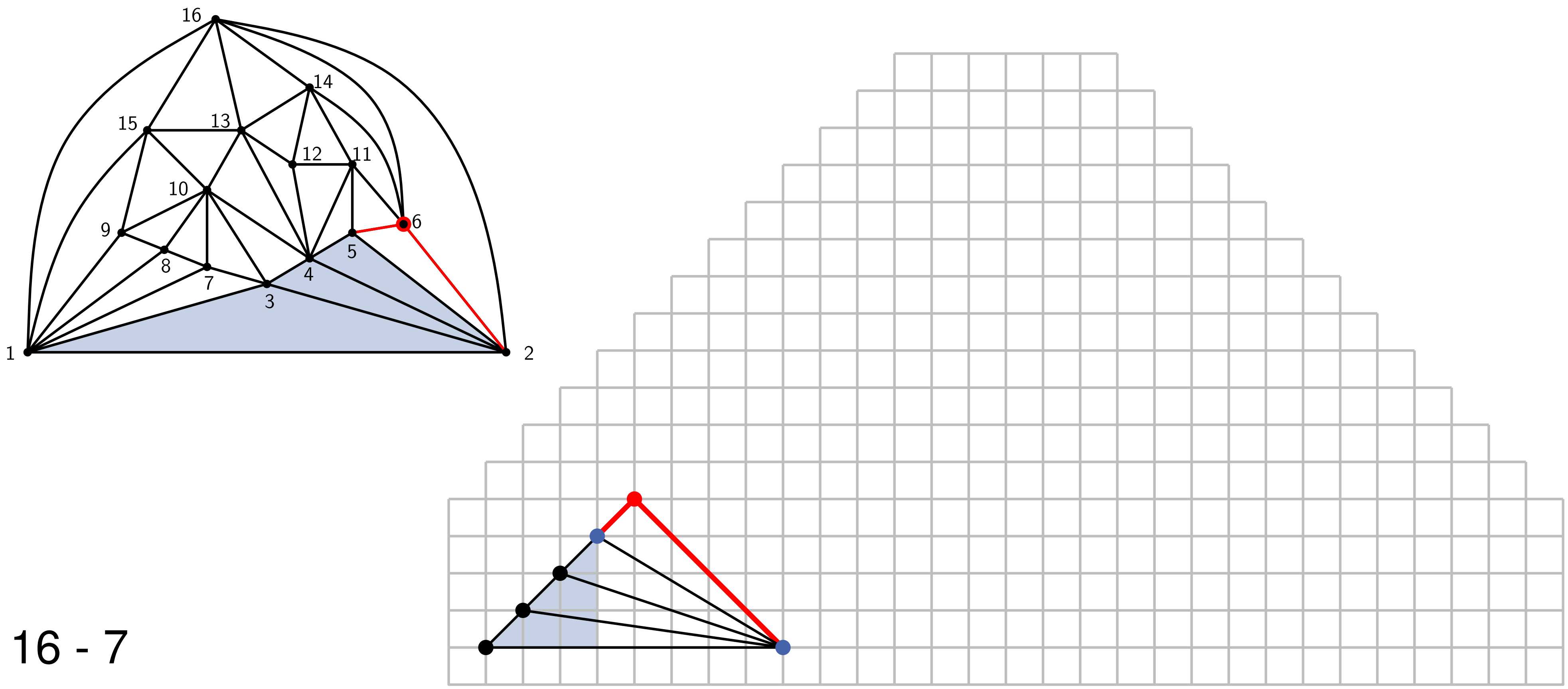


16 - 6

Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

De Fraysseix Pach Pollack (Shift) Algorithm

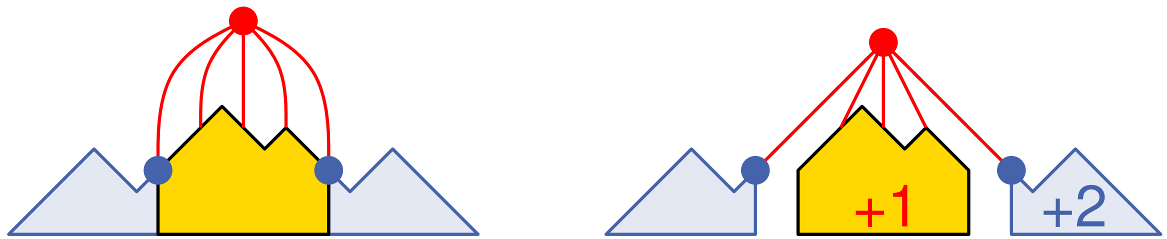


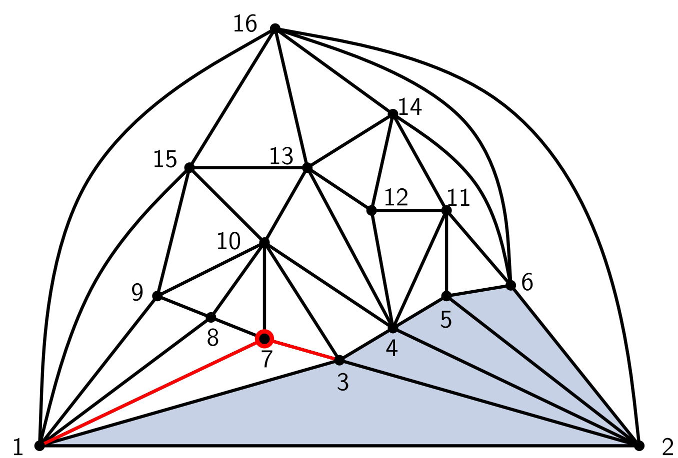


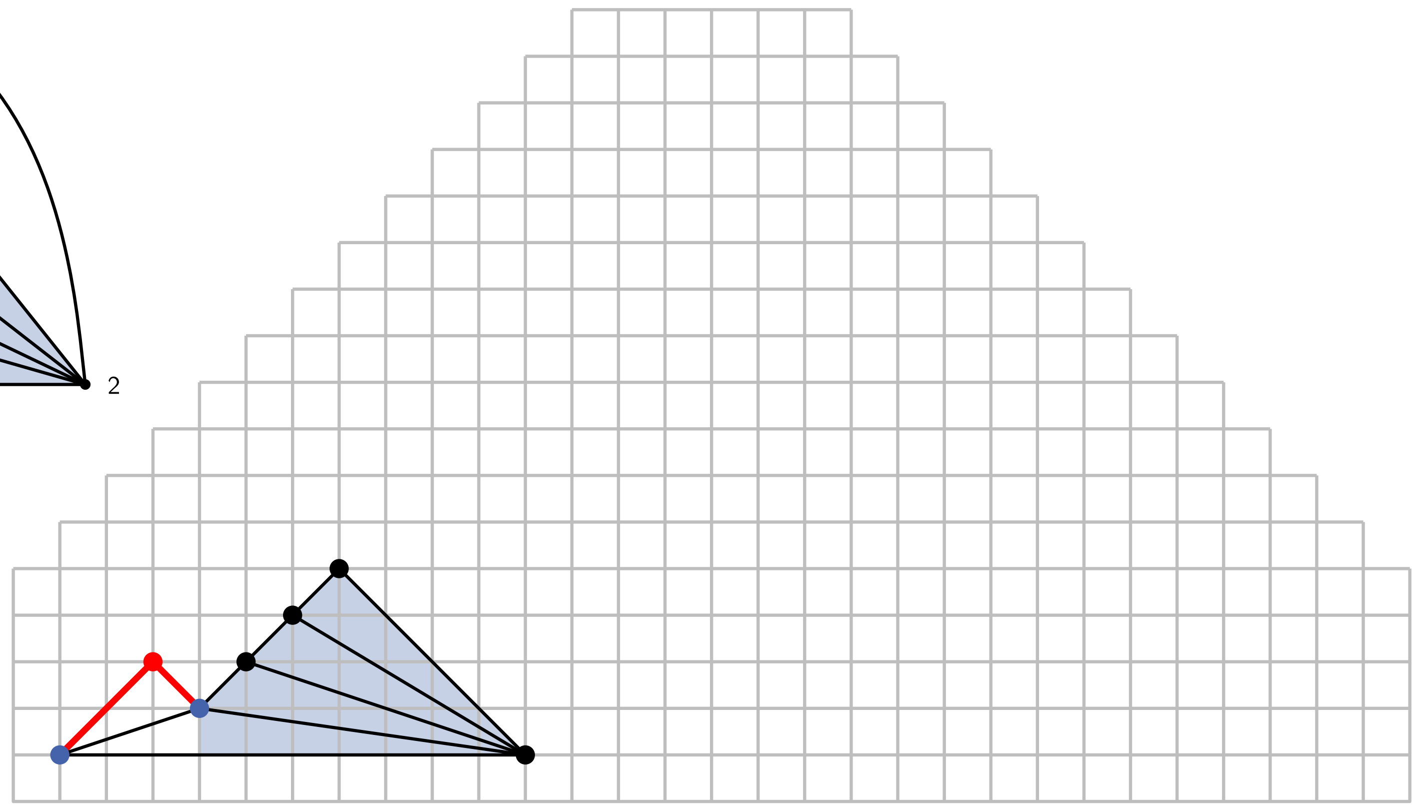
16 - 7

Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

De Fraysseix Pach Pollack (Shift) Algorithm

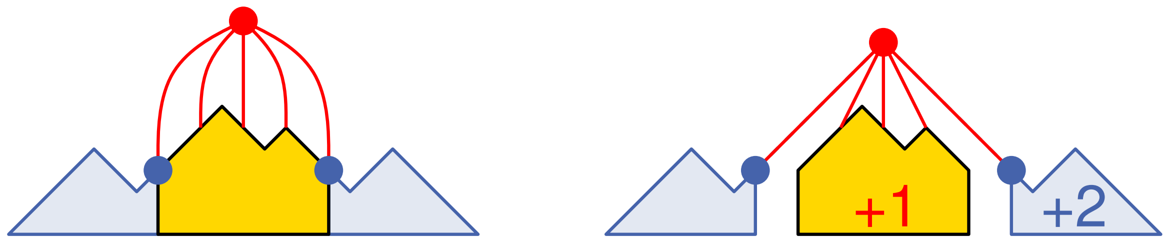


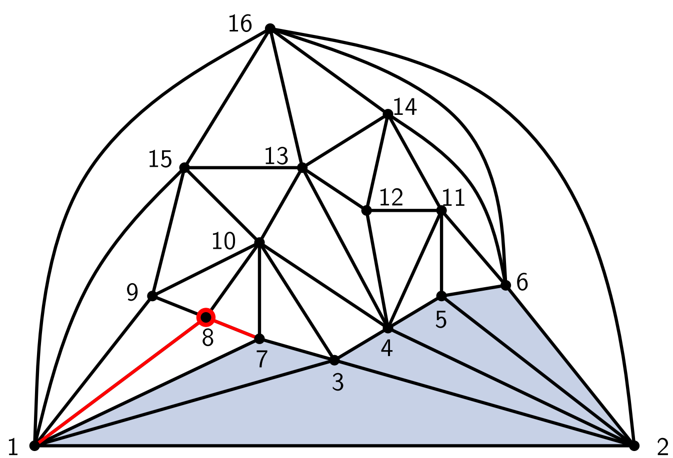


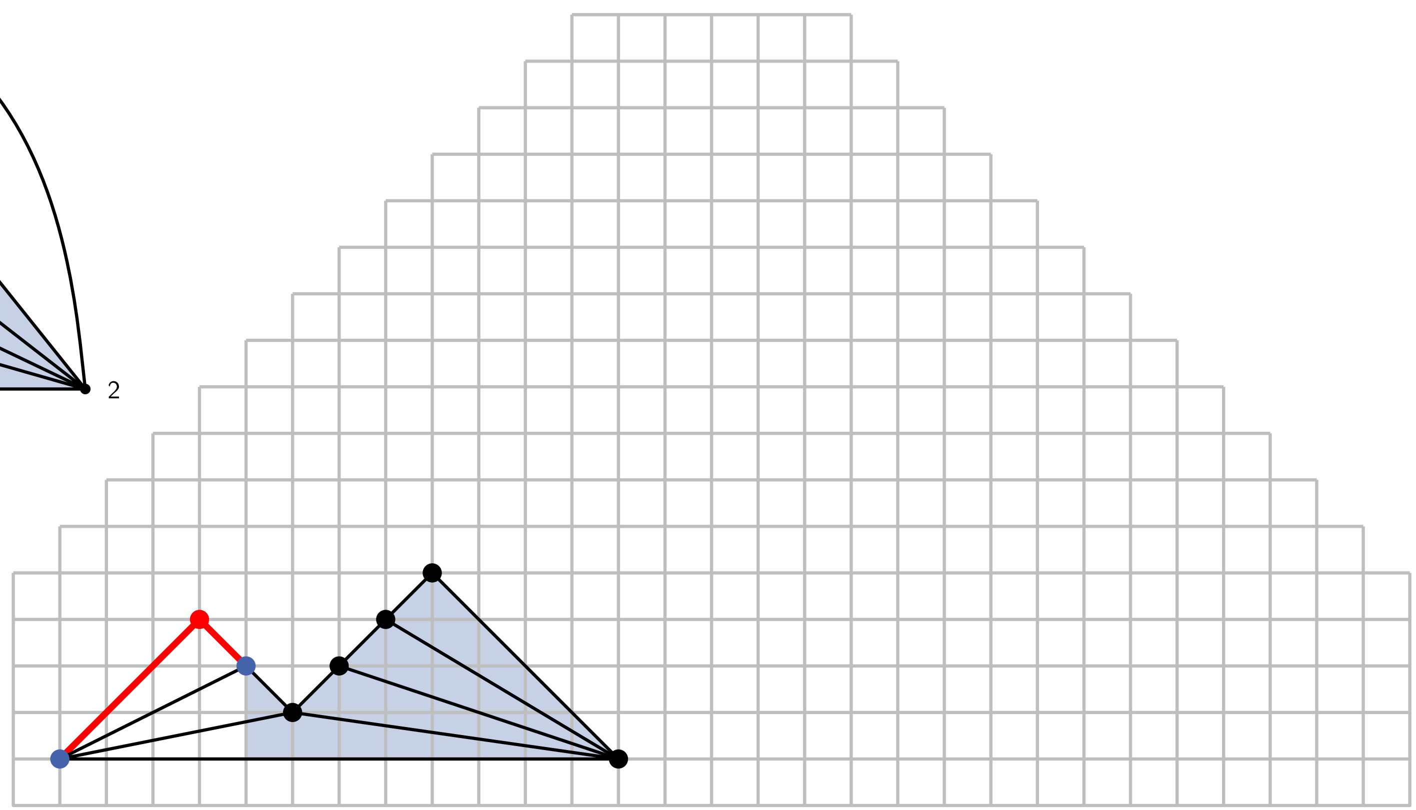


16 - 8

De Fraysseix Pach Pollack (Shift) Algorithm

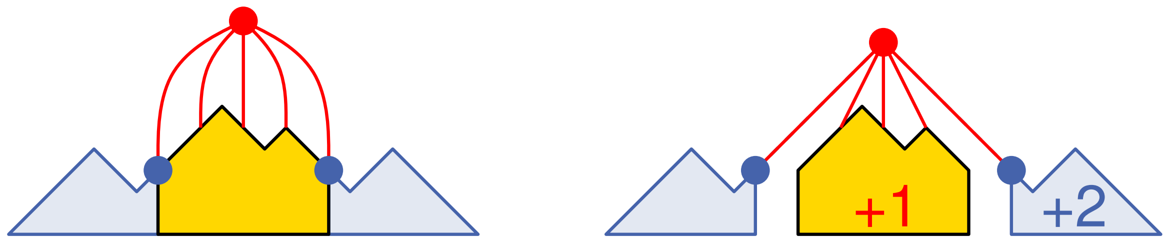


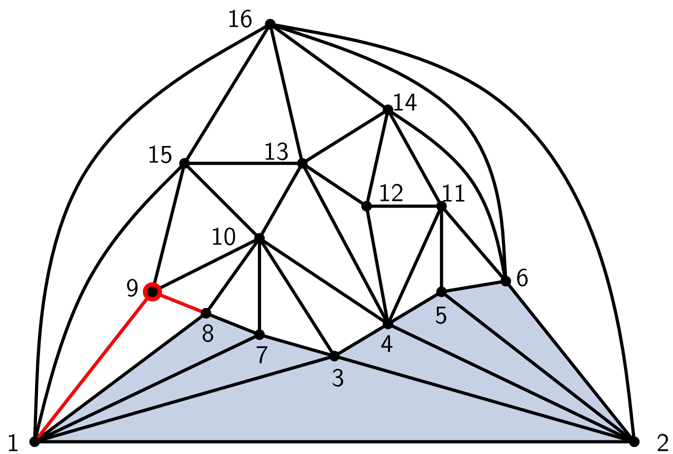


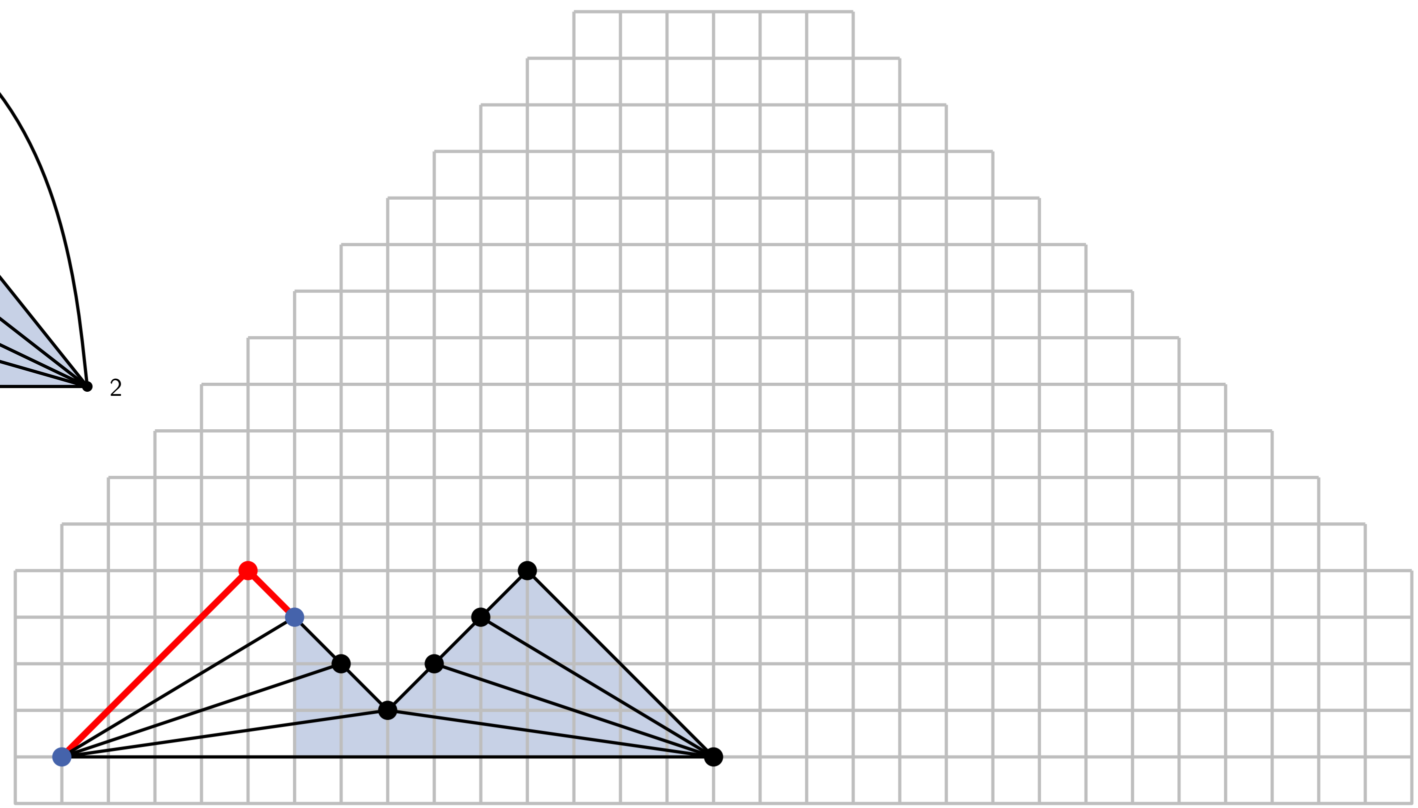


16 - 9

De Fraysseix Pach Pollack (Shift) Algorithm

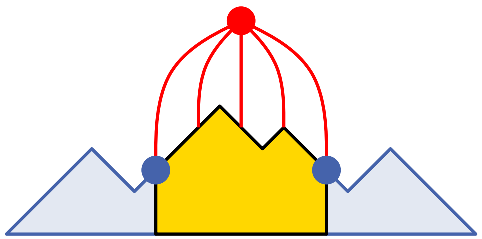


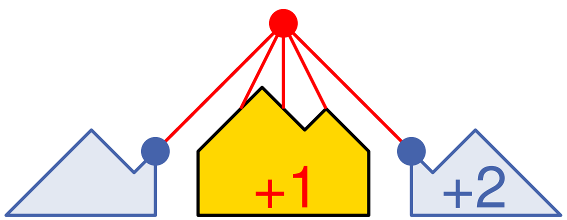


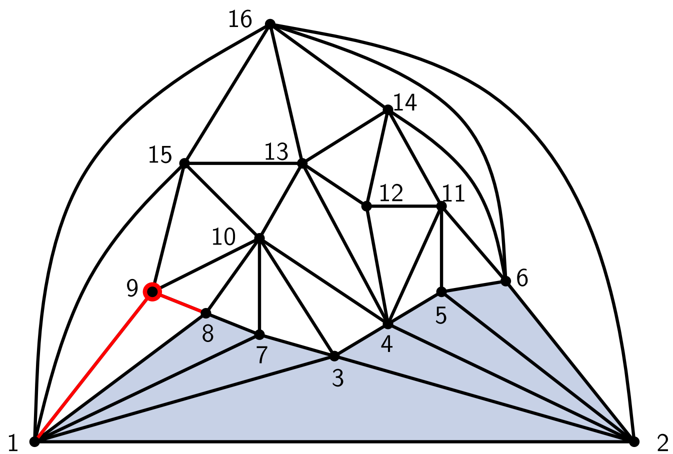


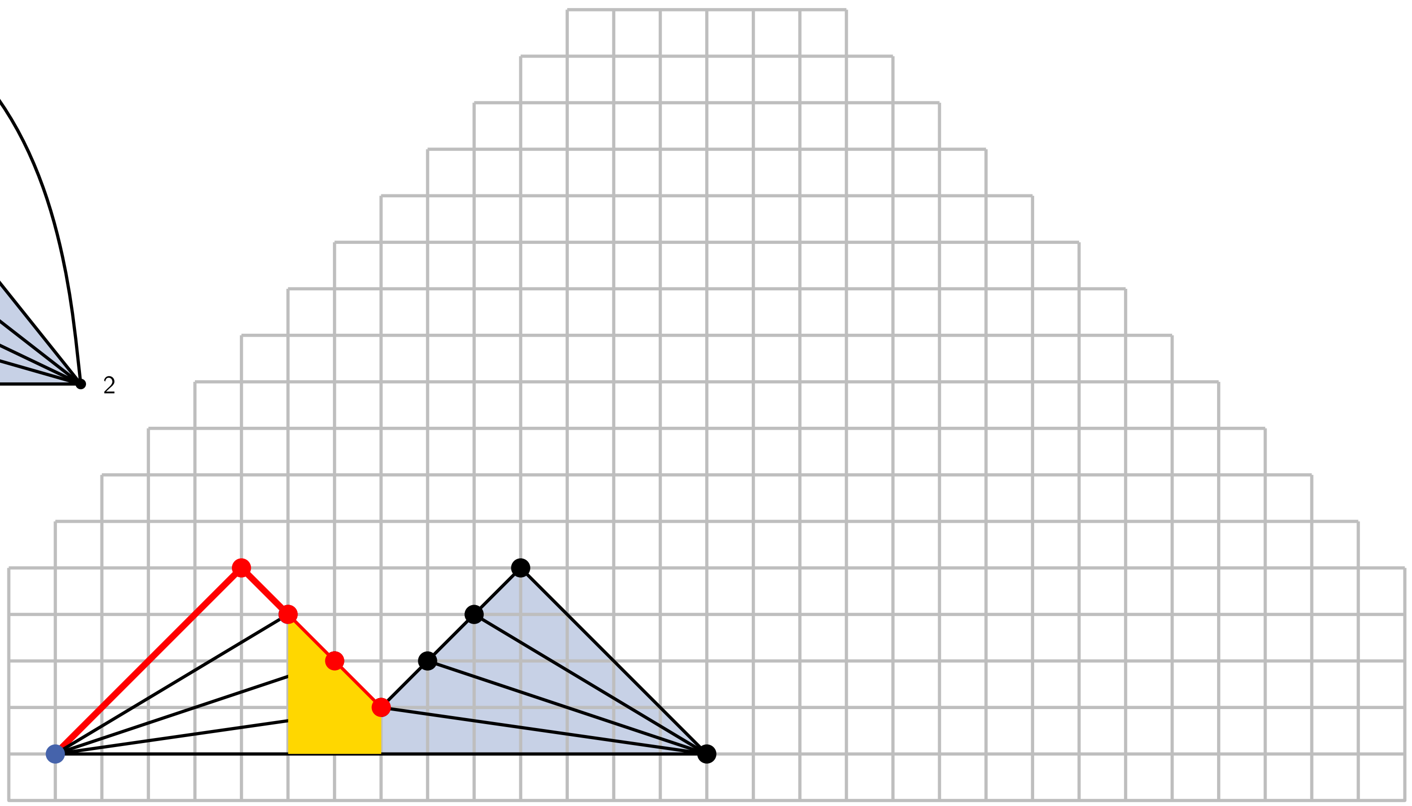
16 - 10

De Fraysseix Pach Pollack (Shift) Algorithm



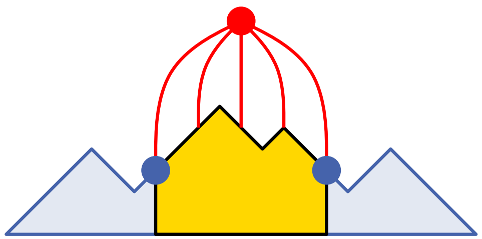


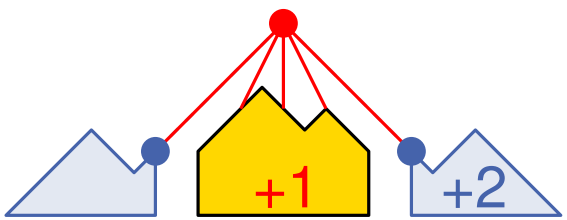


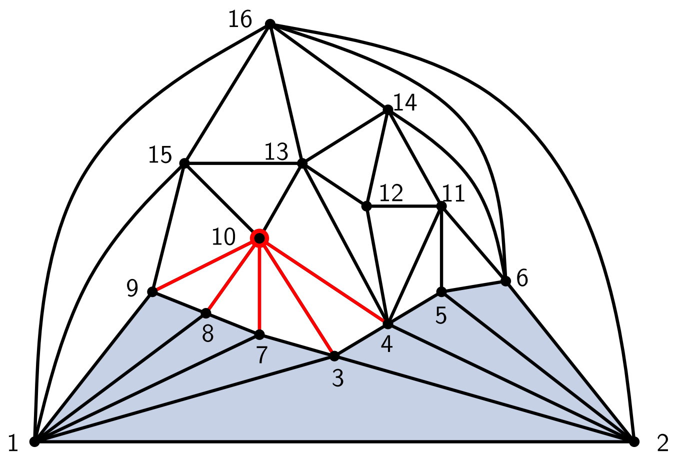


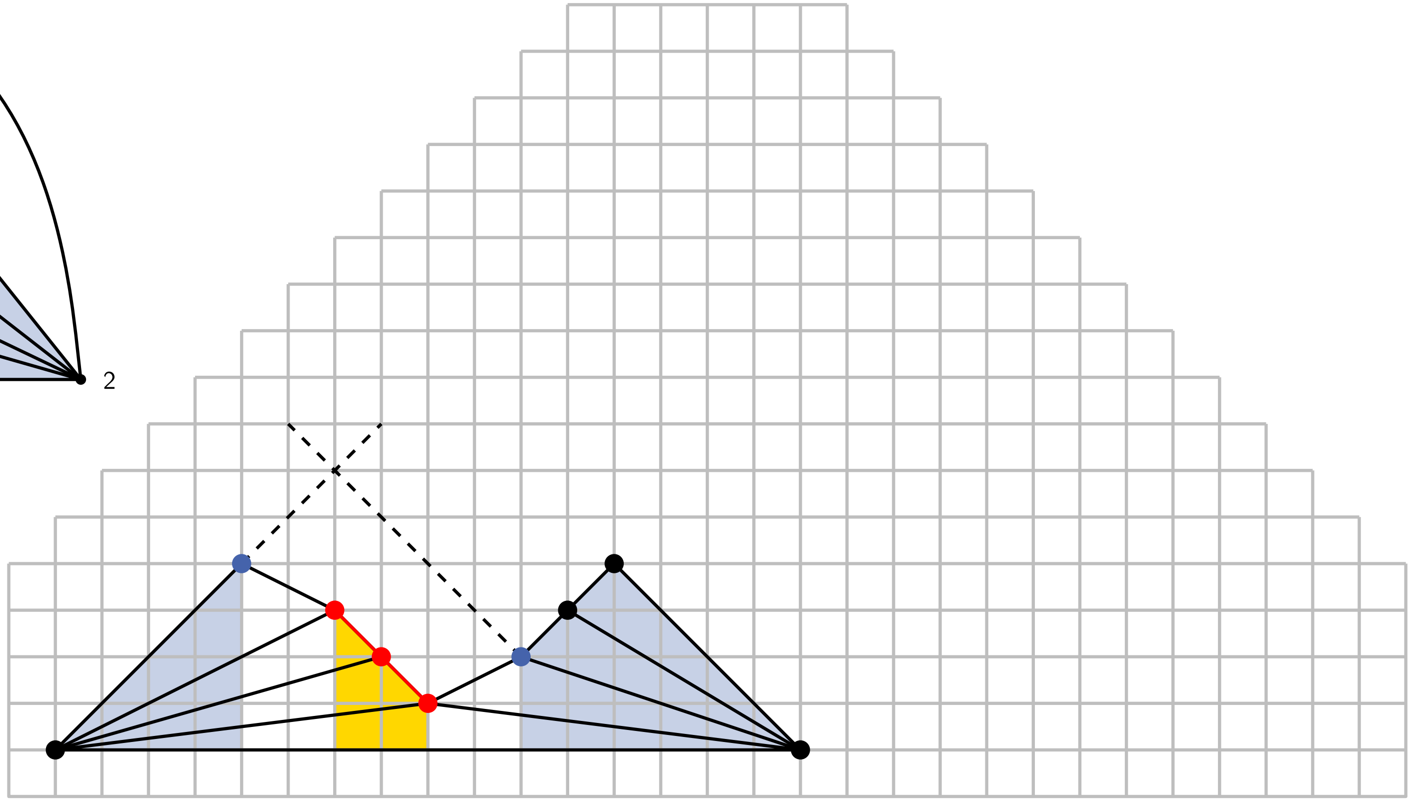
16 - 11

De Fraysseix Pach Pollack (Shift) Algorithm



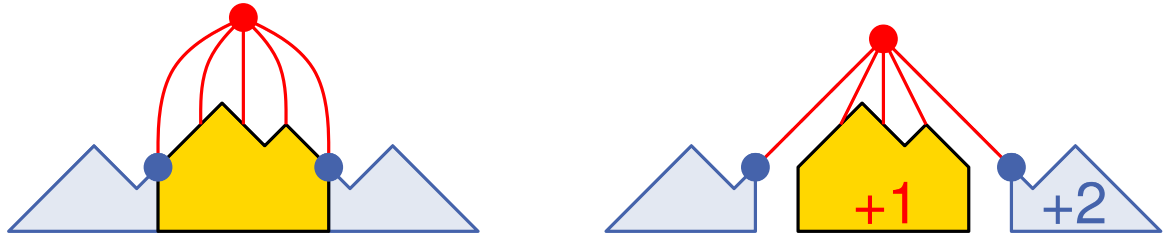


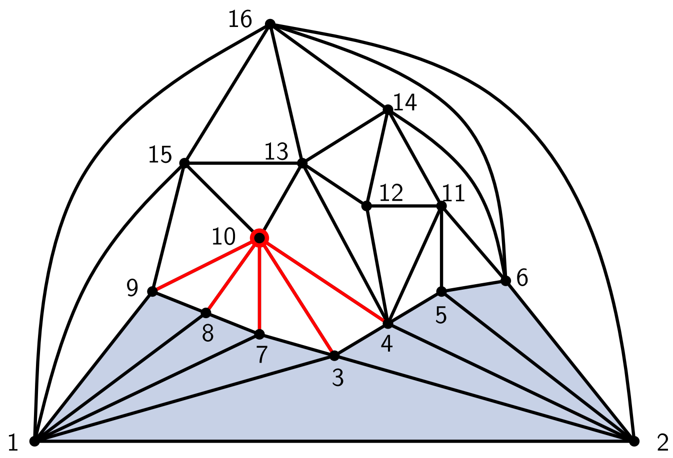


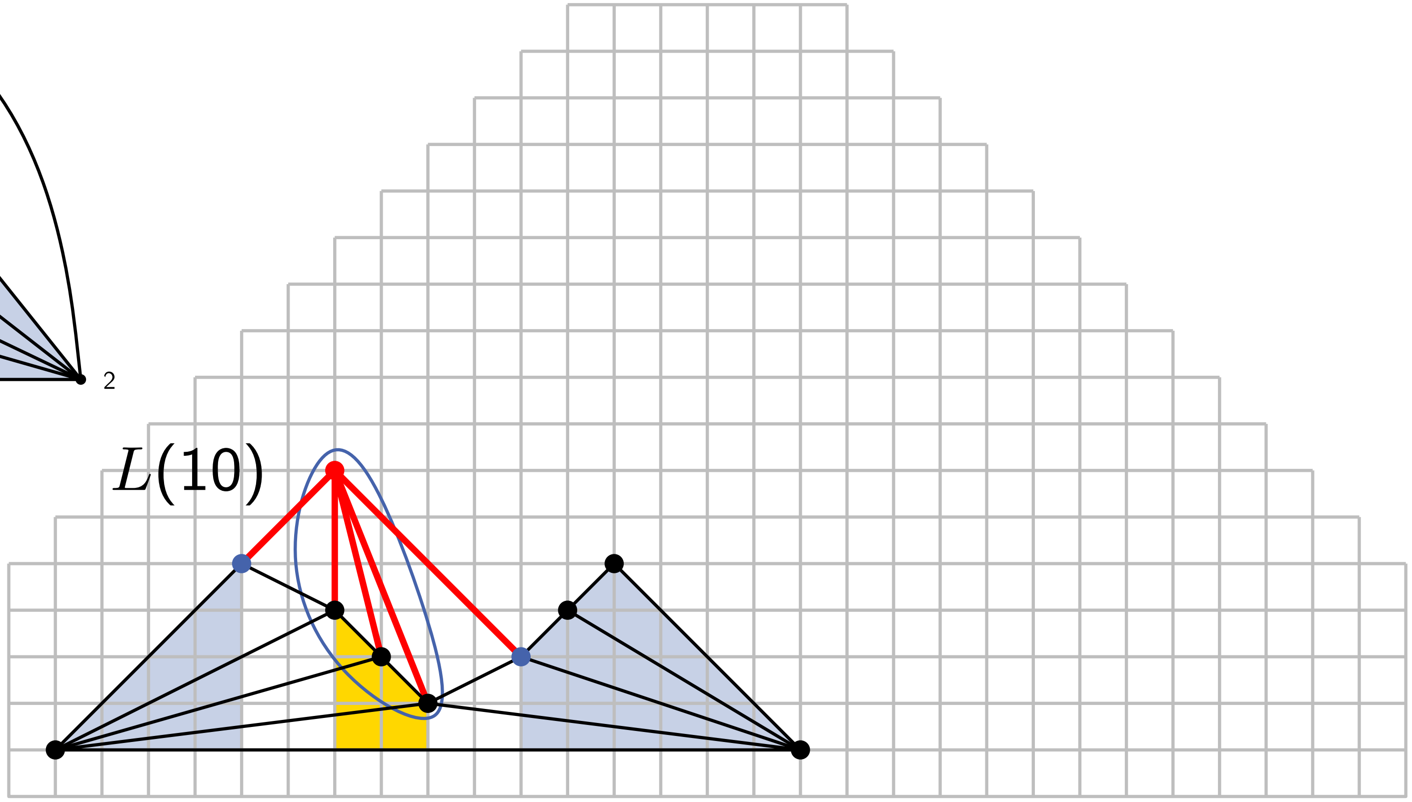


16 - 12

De Fraysseix Pach Pollack (Shift) Algorithm

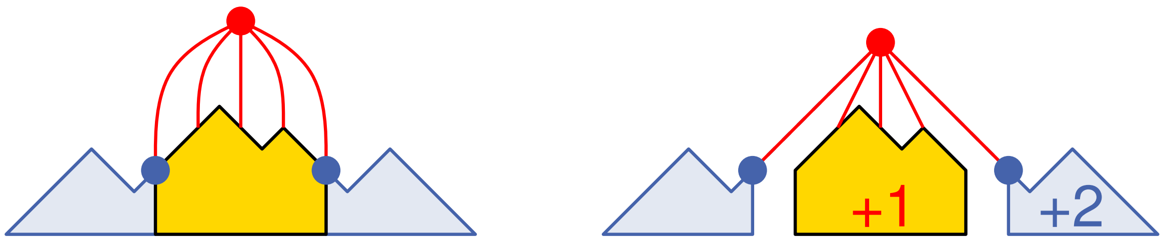


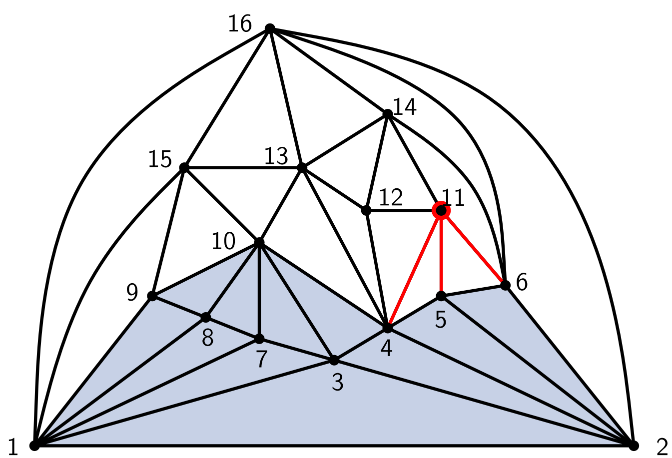


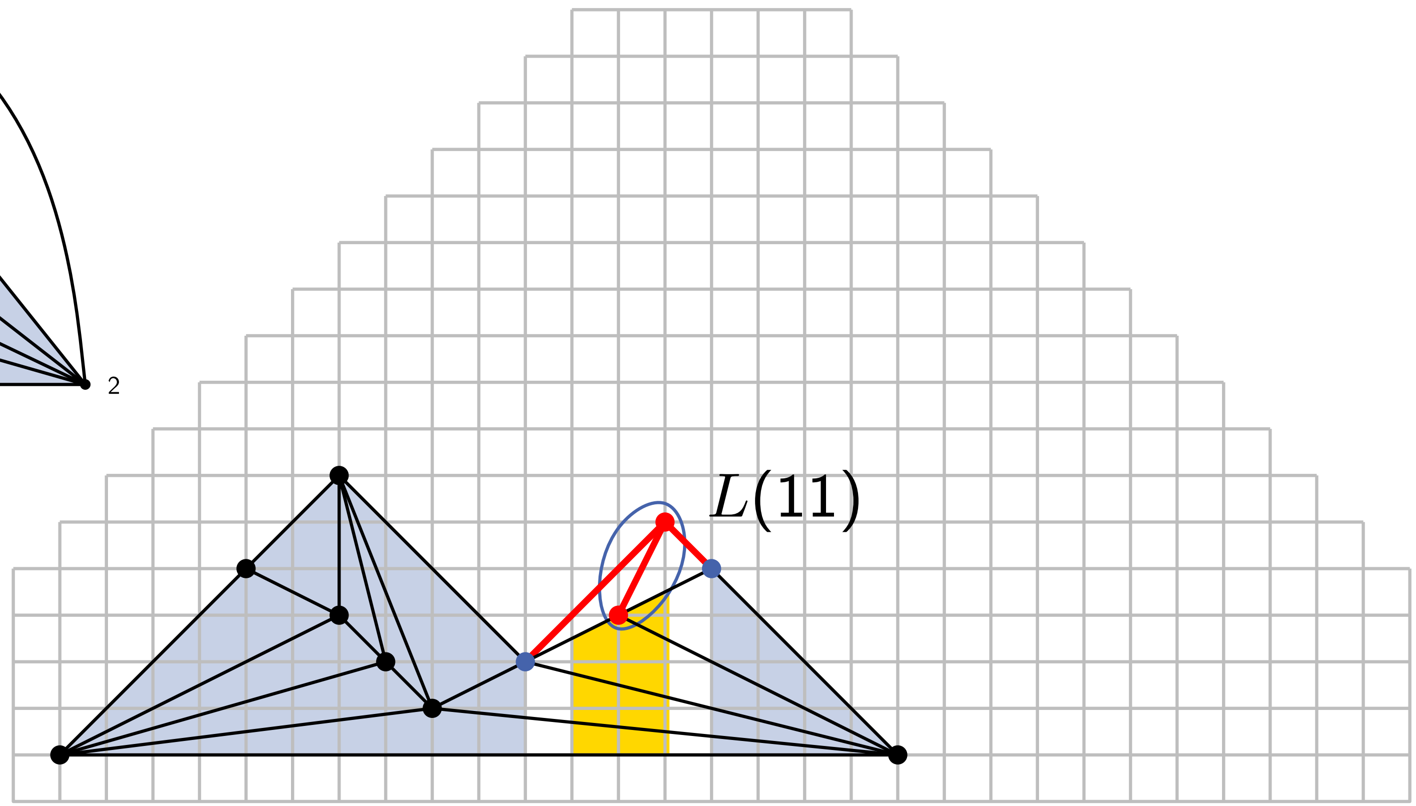


16 - 13

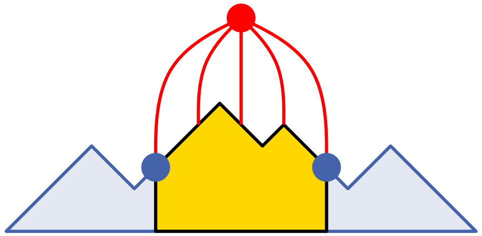
De Fraysseix Pach Pollack (Shift) Algorithm

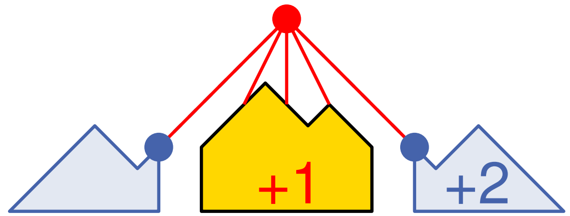


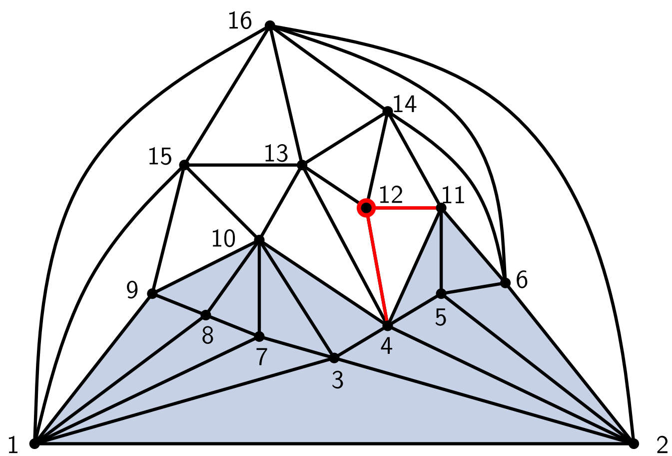


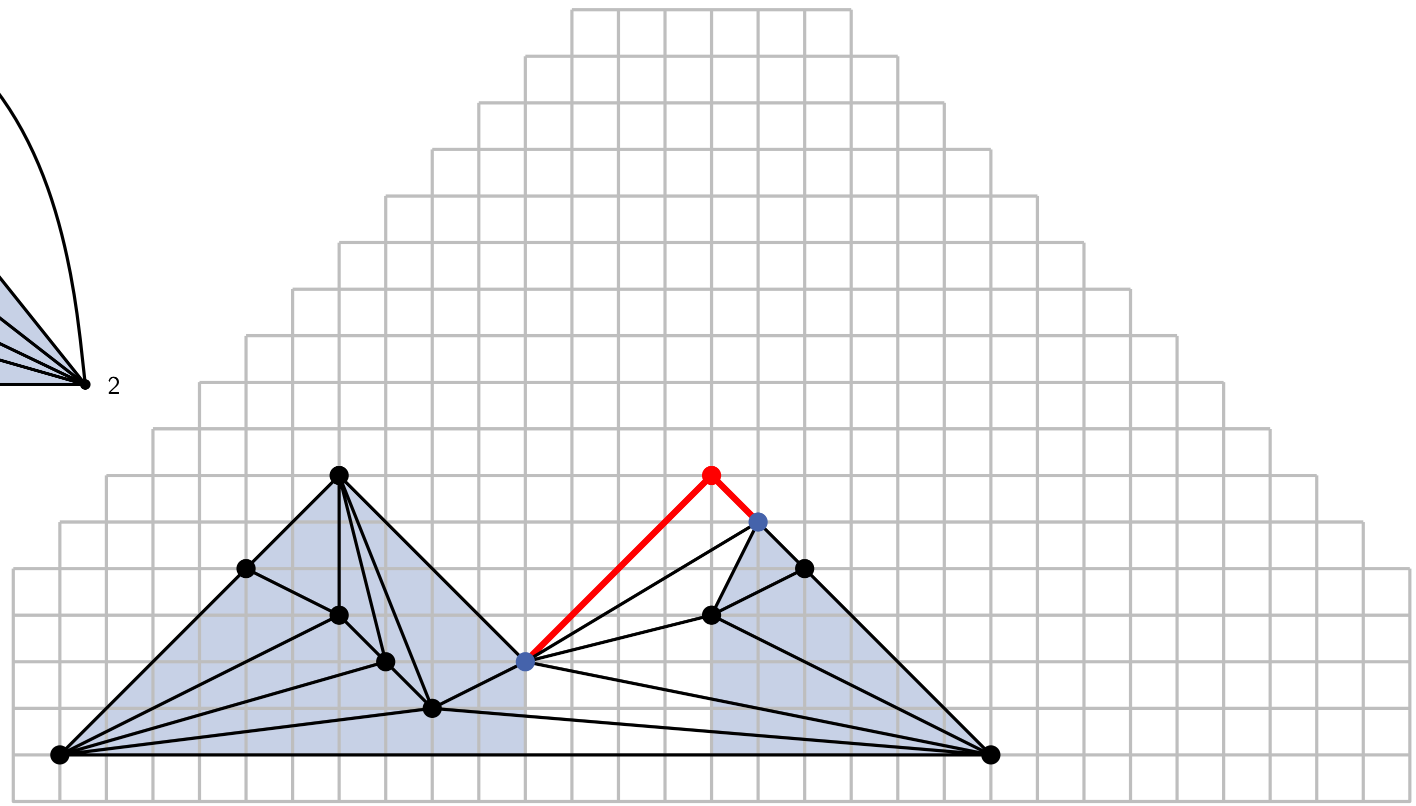


De Fraysseix Pach Pollack (Shift) Algorithm



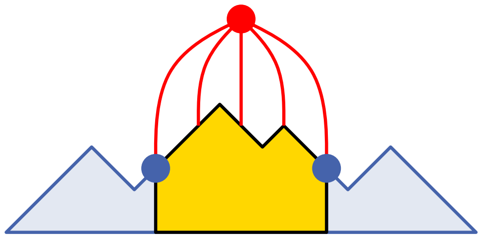


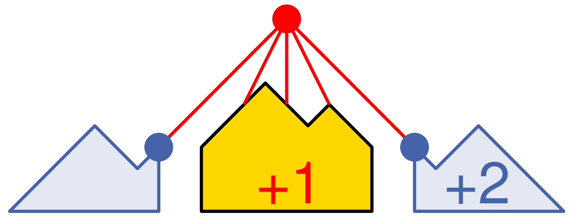


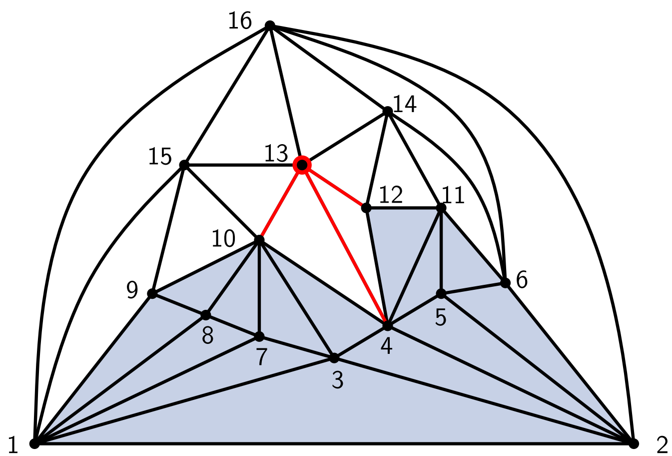


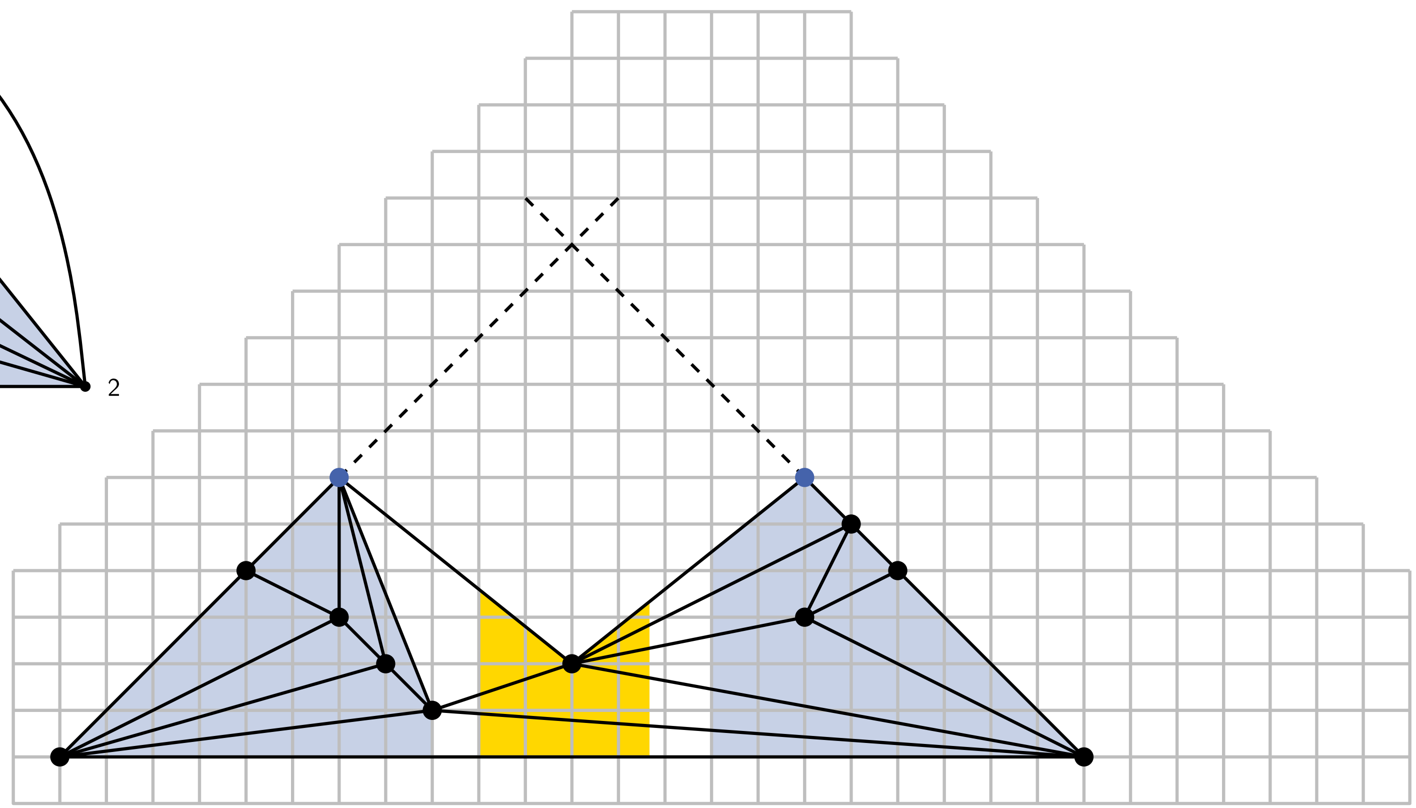
16 - 15

De Fraysseix Pach Pollack (Shift) Algorithm



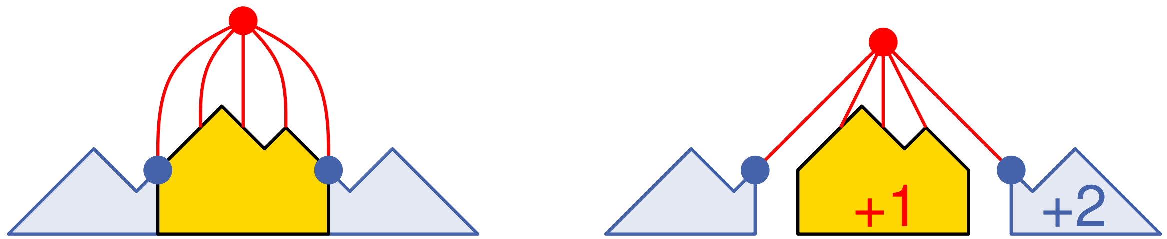


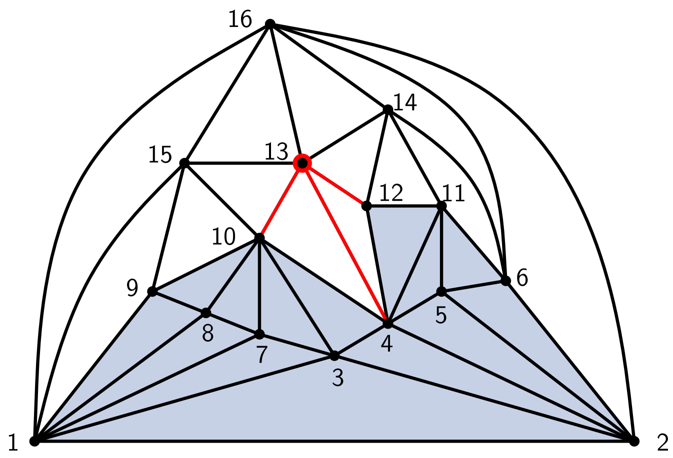


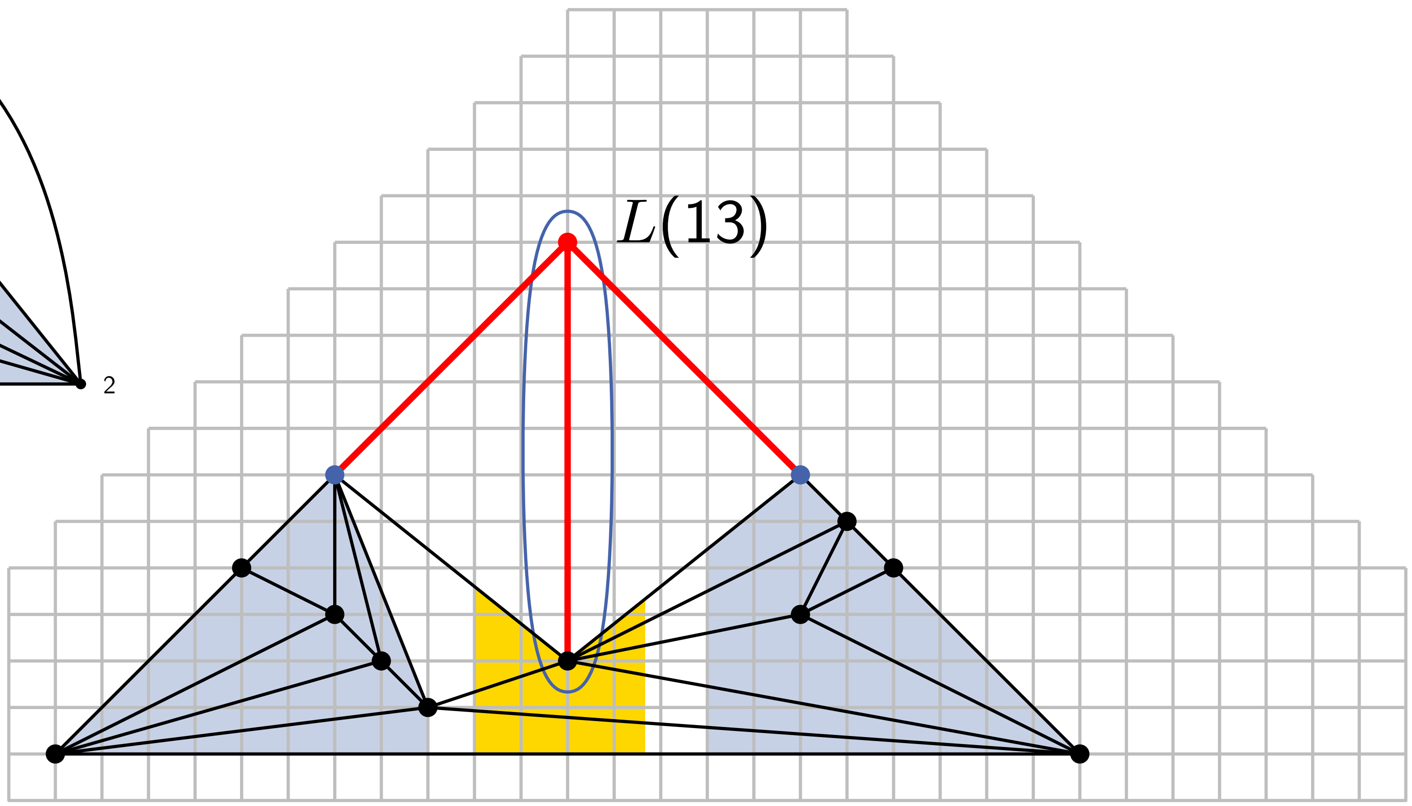


16 - 16

De Fraysseix Pach Pollack (Shift) Algorithm

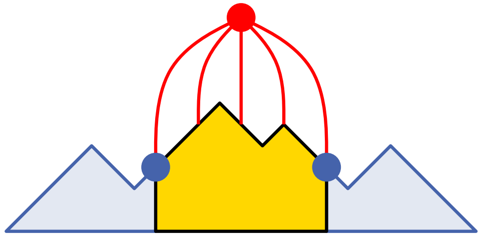


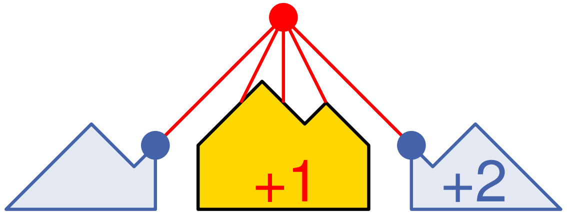


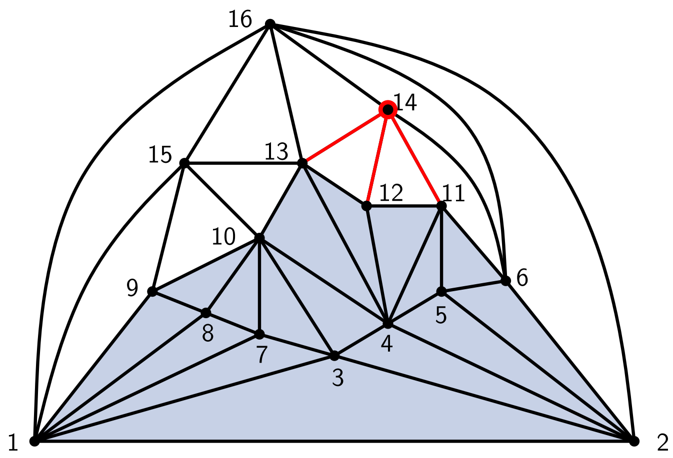


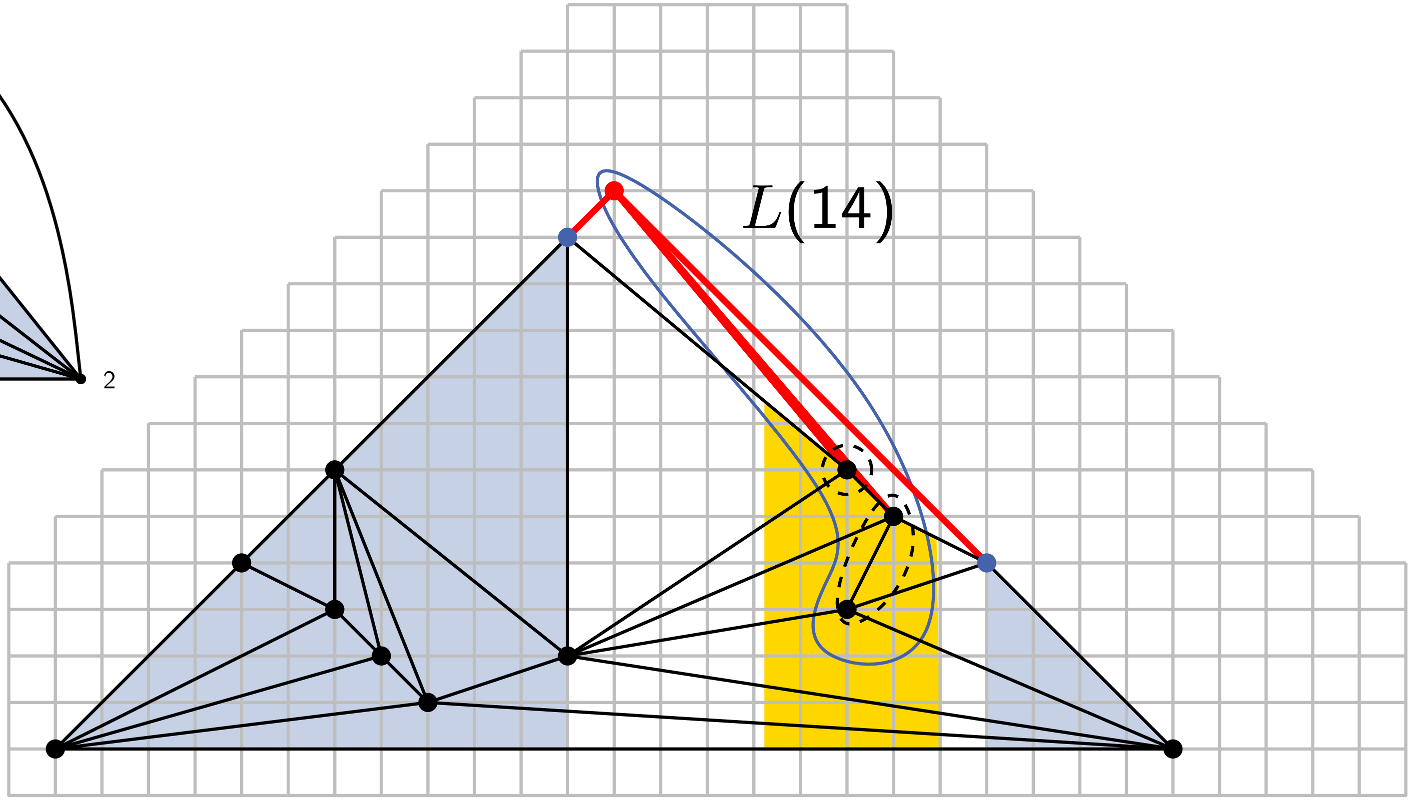
16 - 17

De Fraysseix Pach Pollack (Shift) Algorithm



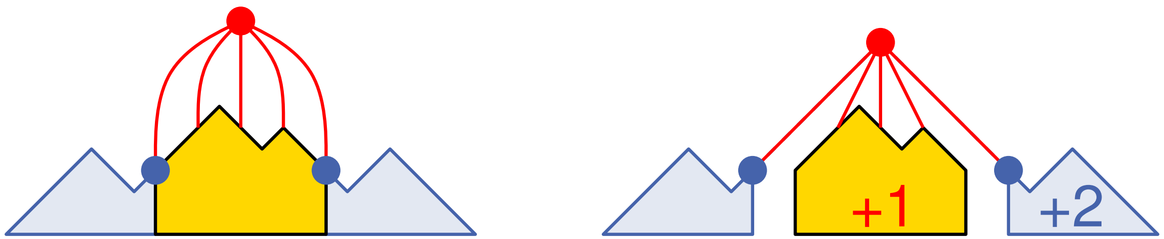


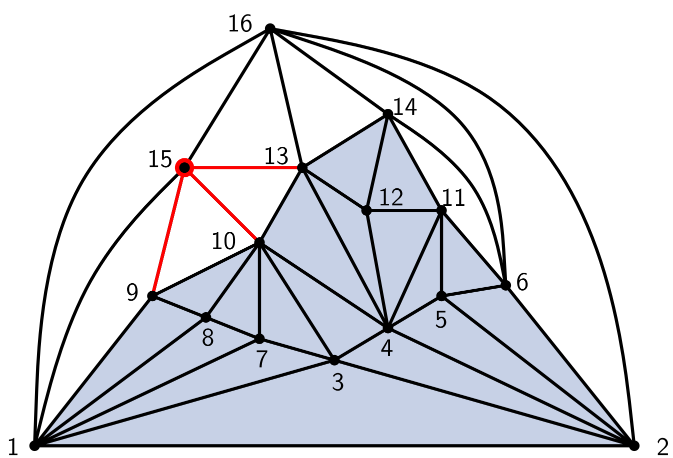


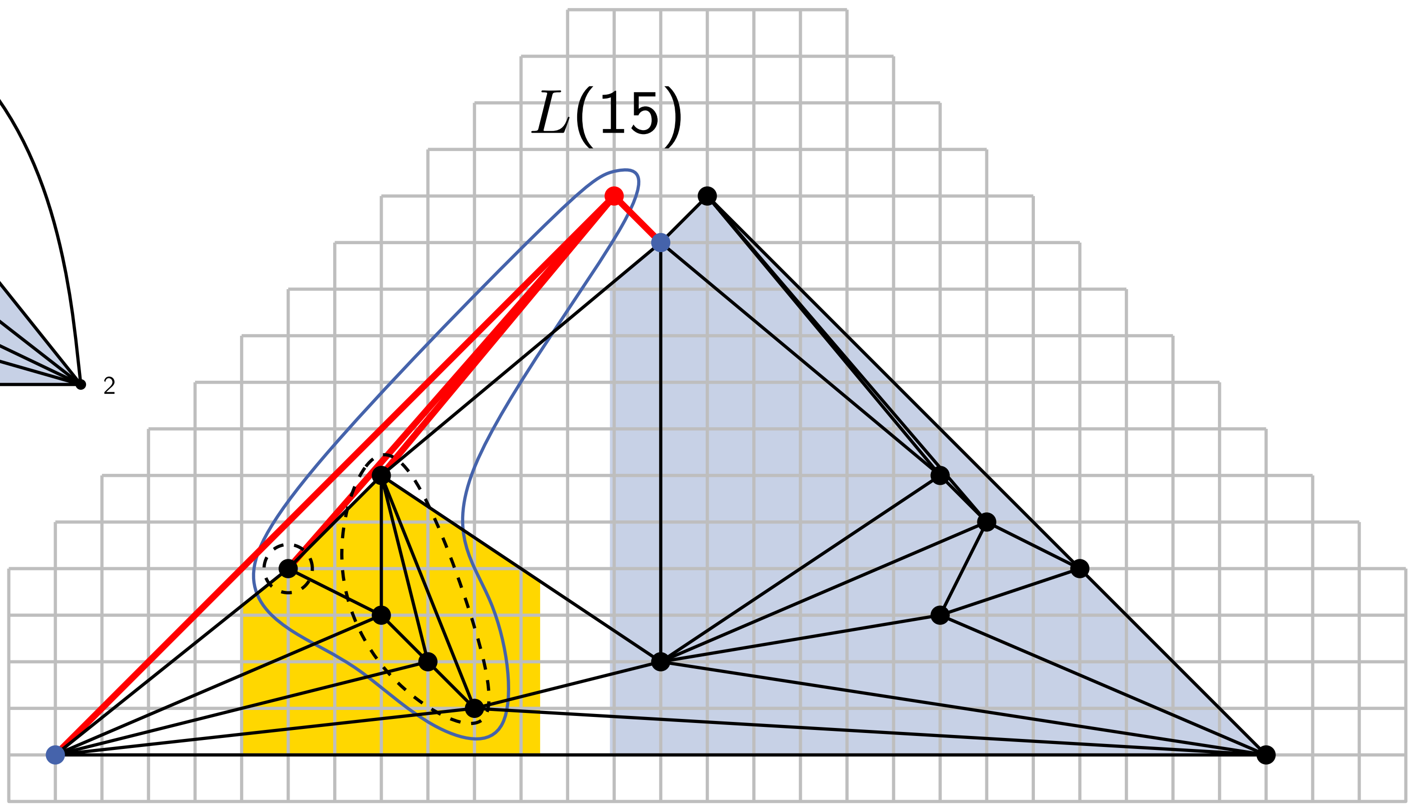


16 - 18

De Fraysseix Pach Pollack (Shift) Algorithm







16 - 19

De Fraysseix Pach Pollack (Shift) Algorithm

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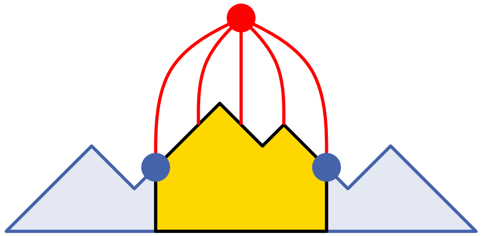
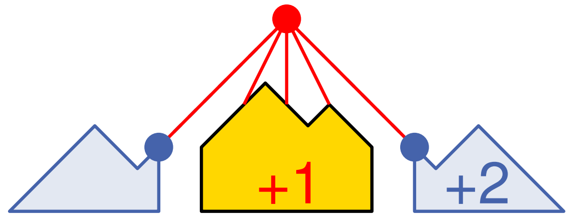
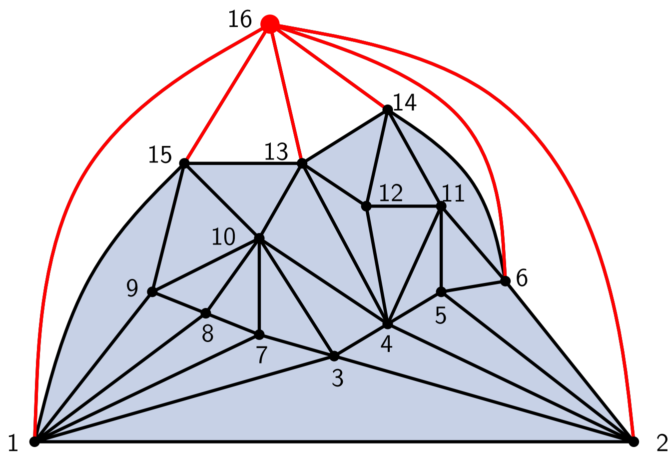
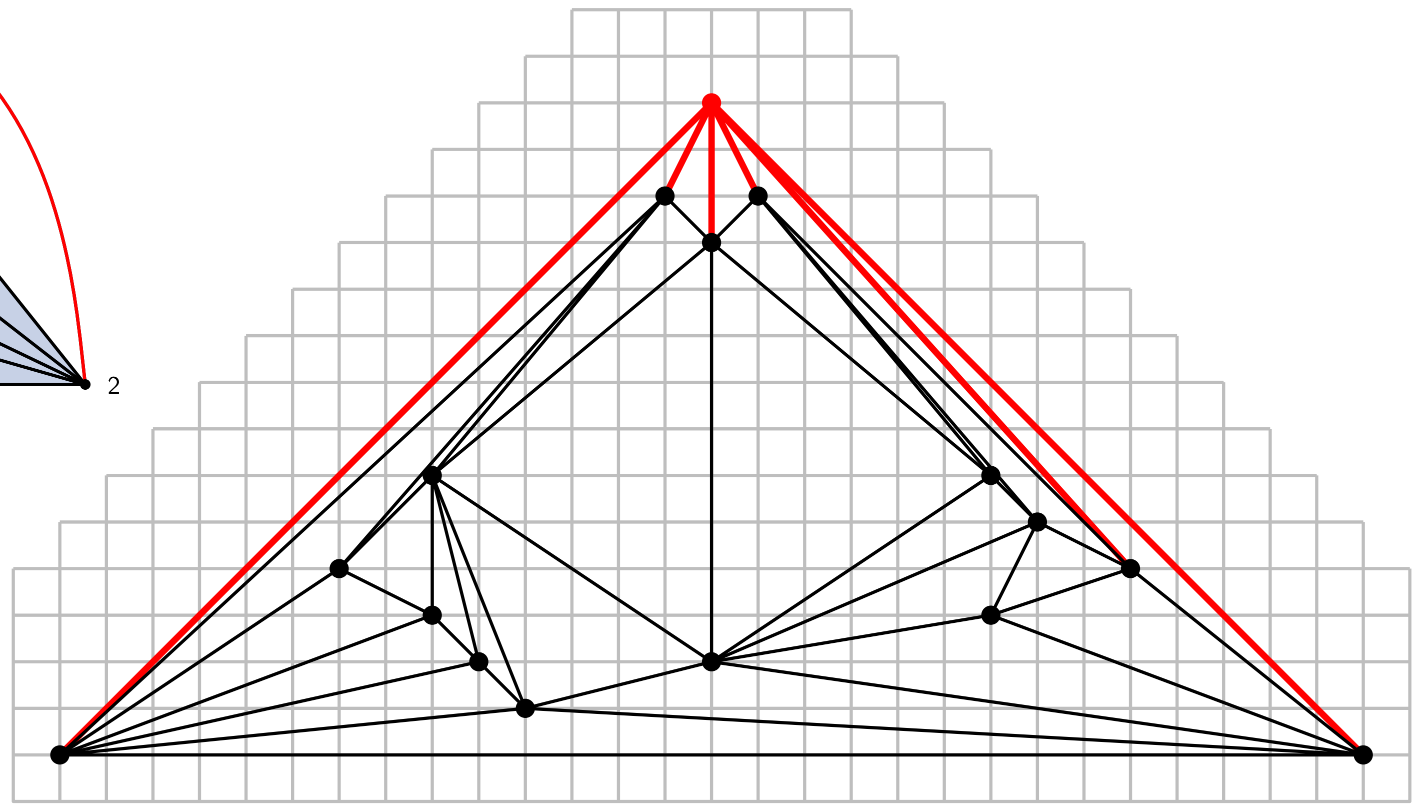




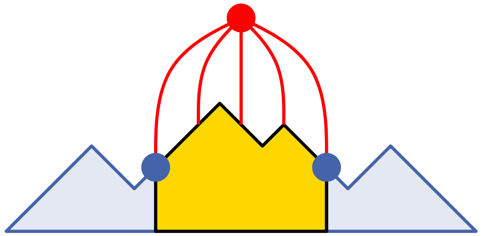


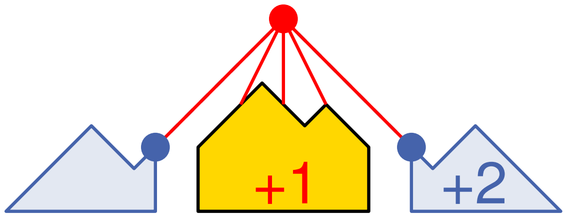


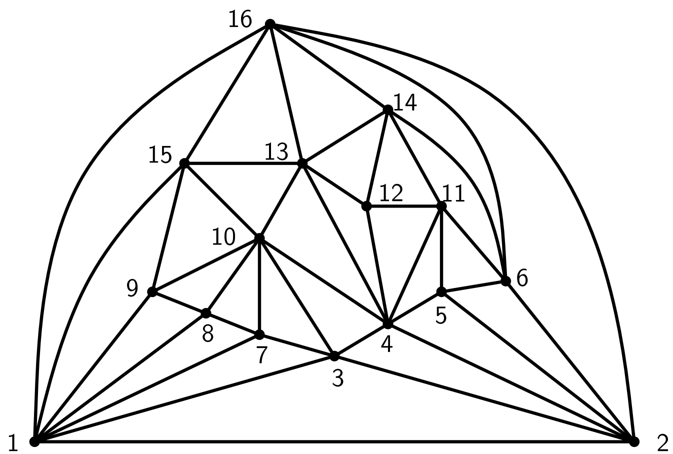
16 - 20

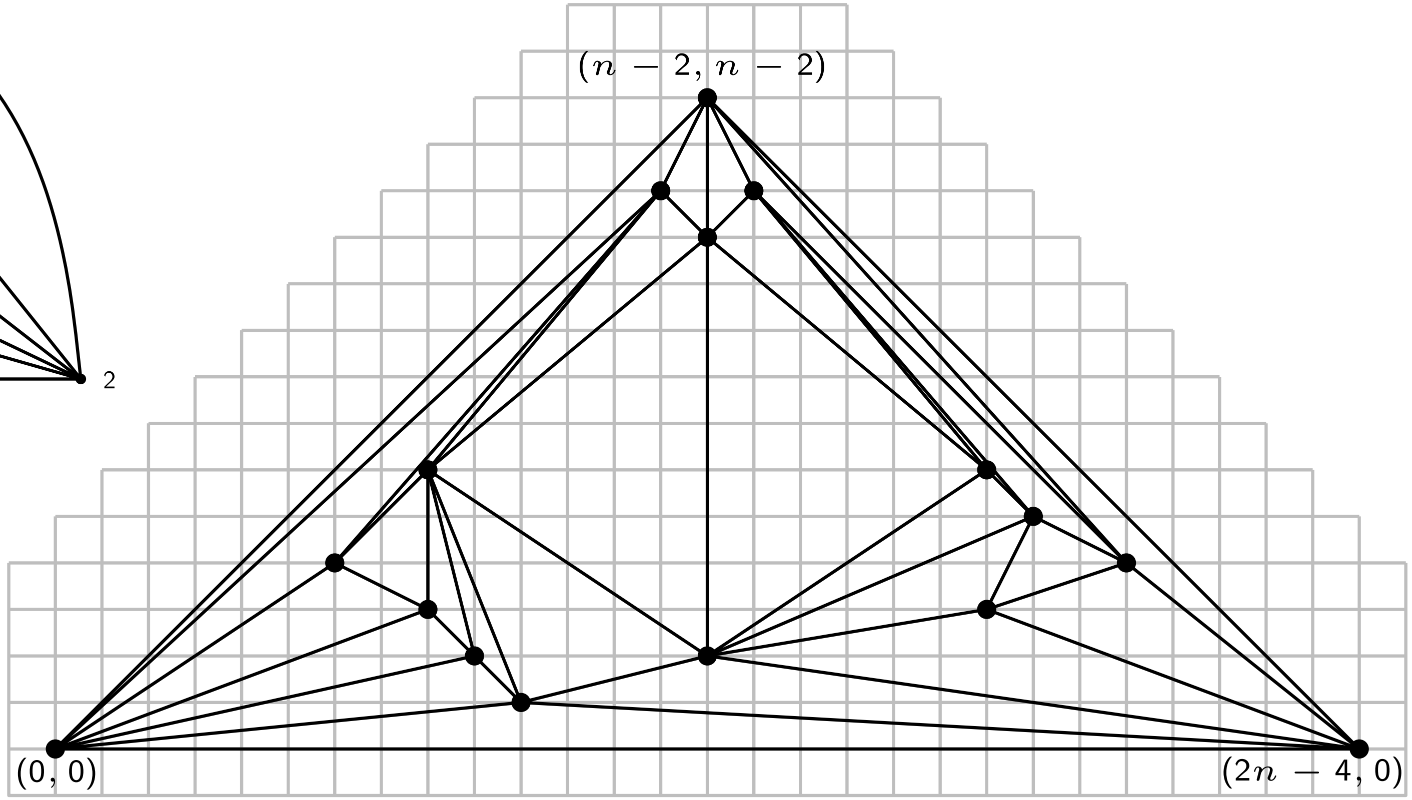
Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

De Fraysseix Pach Pollack (Shift) Algorithm



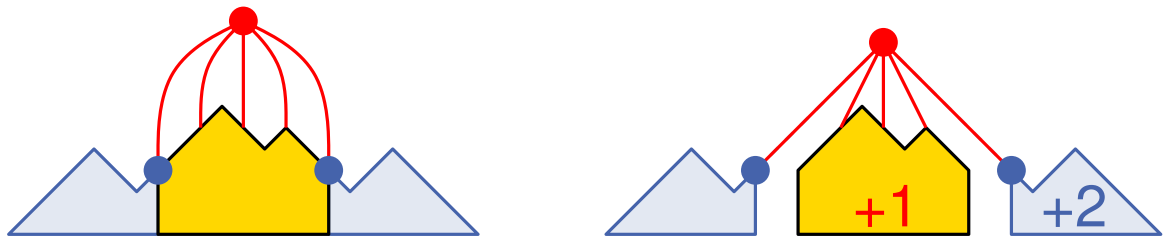






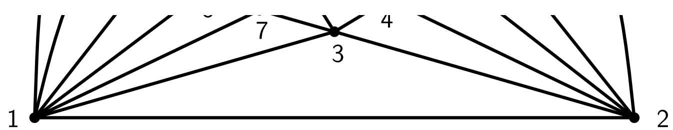
16 - 21

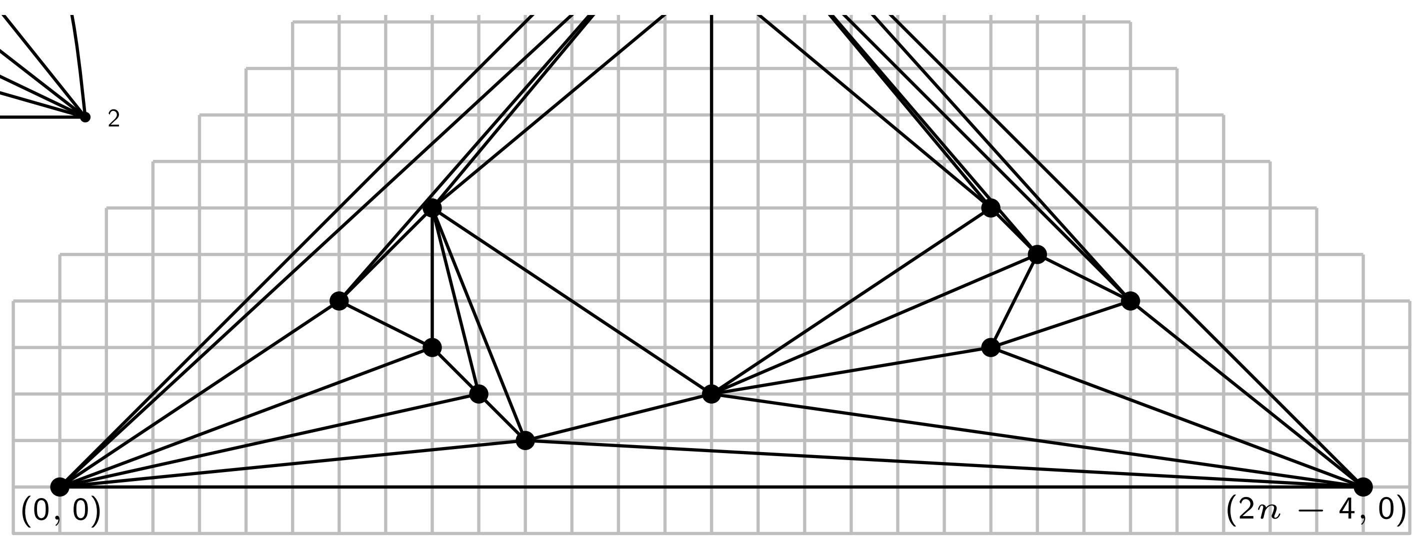
De Fraysseix Pach Pollack (Shift) Algorithm



Take a minute to think about the algorithm. Any questions?

1 min



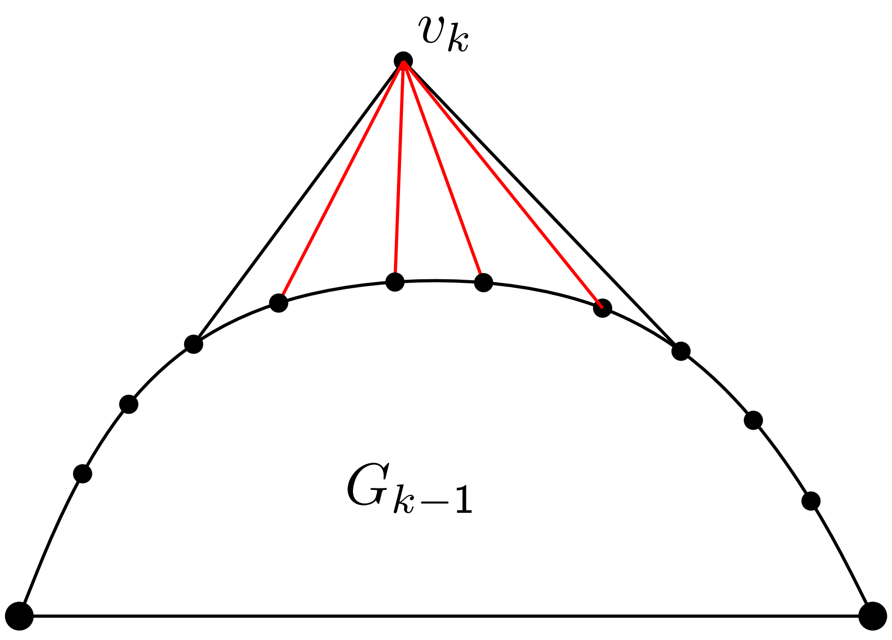


Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

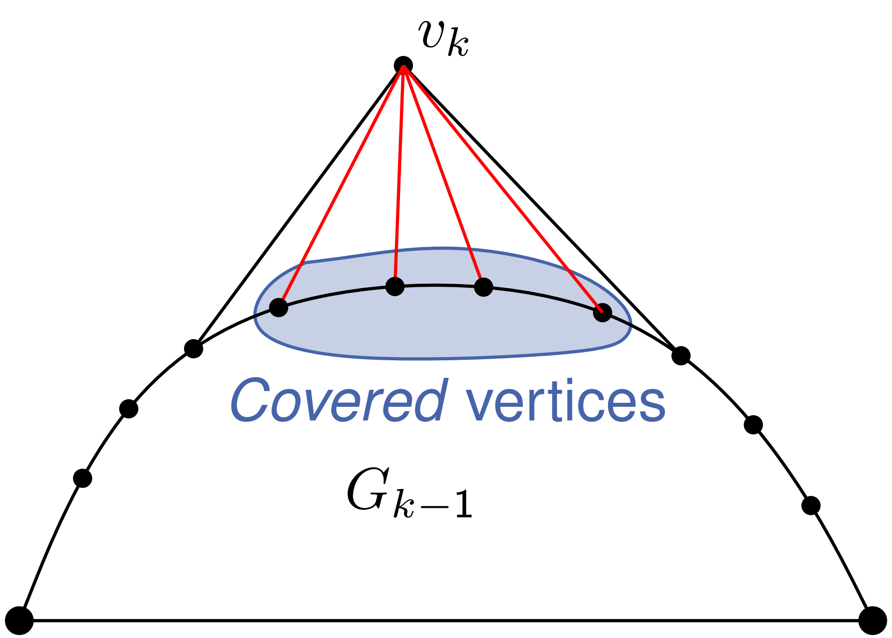
Outline

- Canonical ordering. Existence.
- Canonical ordering. Computation.
- Shift algorithm.
- Proof of planarity.
- Implentational details.

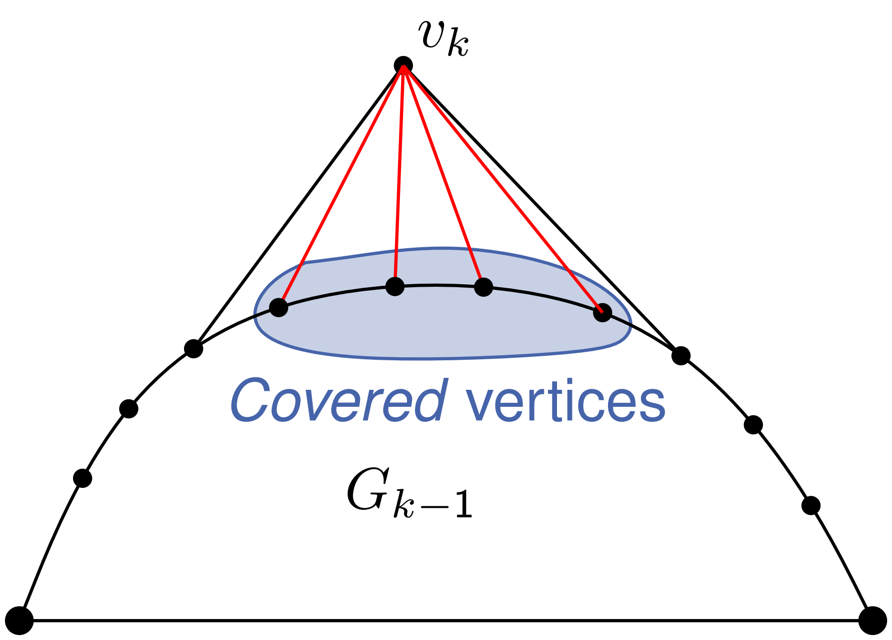
Proof of Planarity



Proof of Planarity

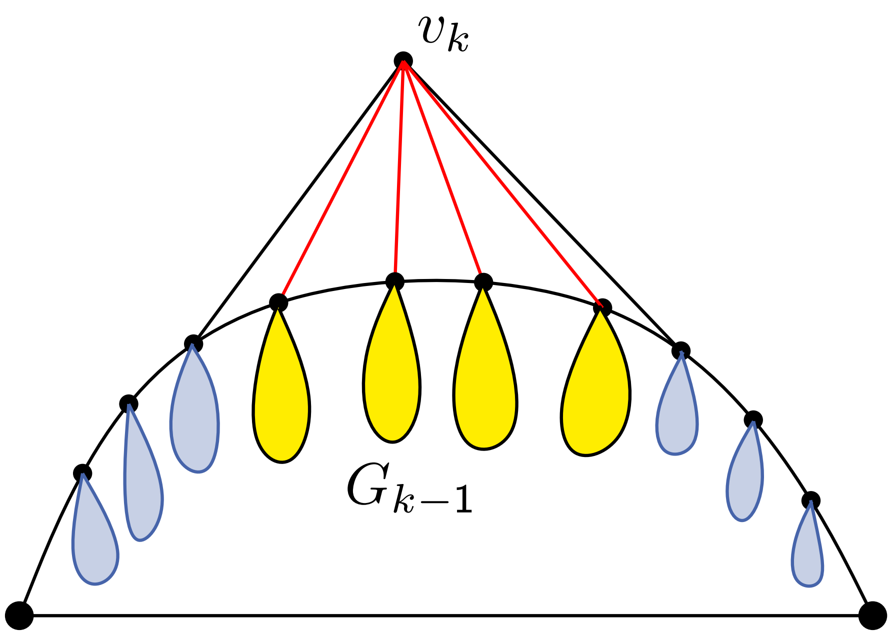


Proof of Planarity



- Each internal vertex is covered exactly once
- Coverence relation defines a tree in G
- But a forest in G_i, $1 \leq i \leq n-1$

Proof of Planarity



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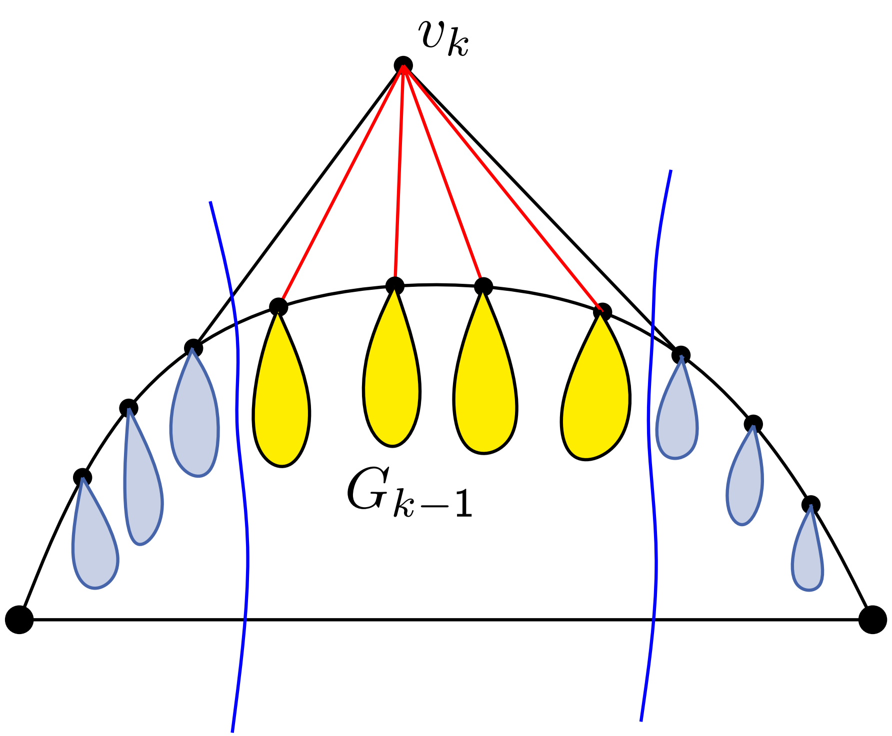
Proof of Planarity



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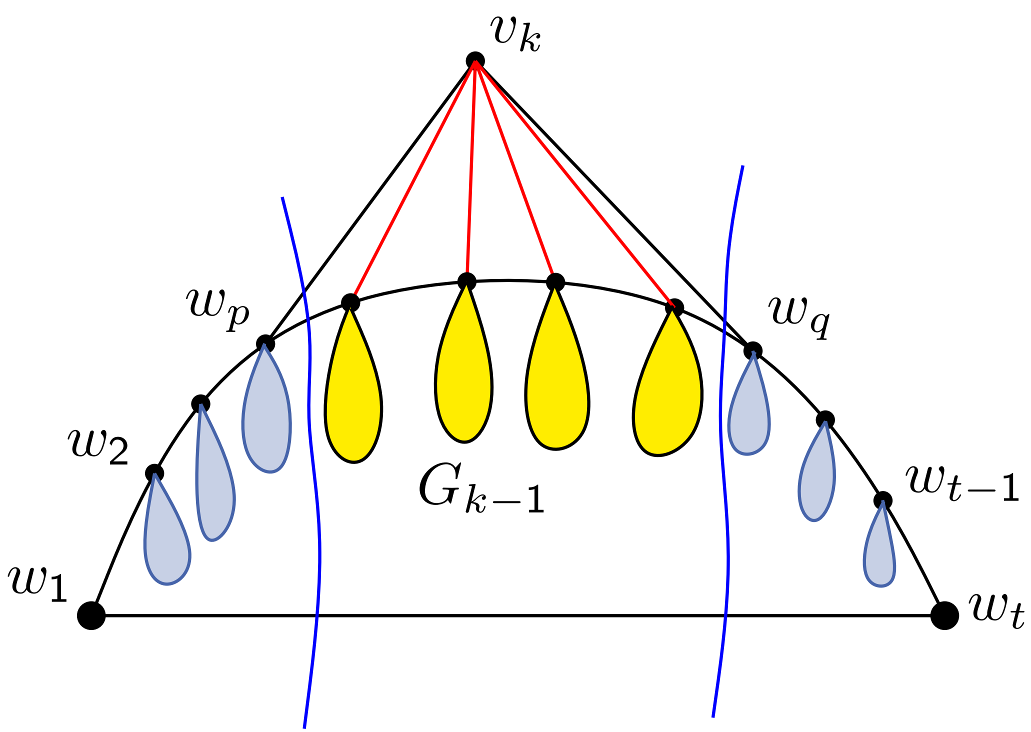
Proof of Planarity



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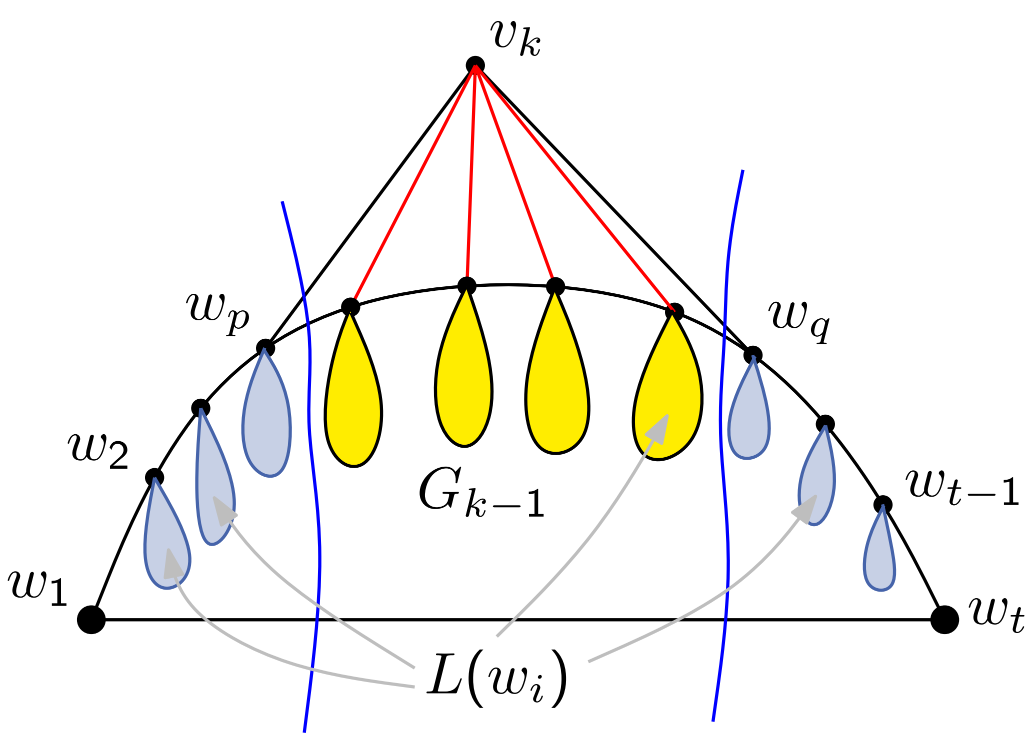
Proof of Planarity



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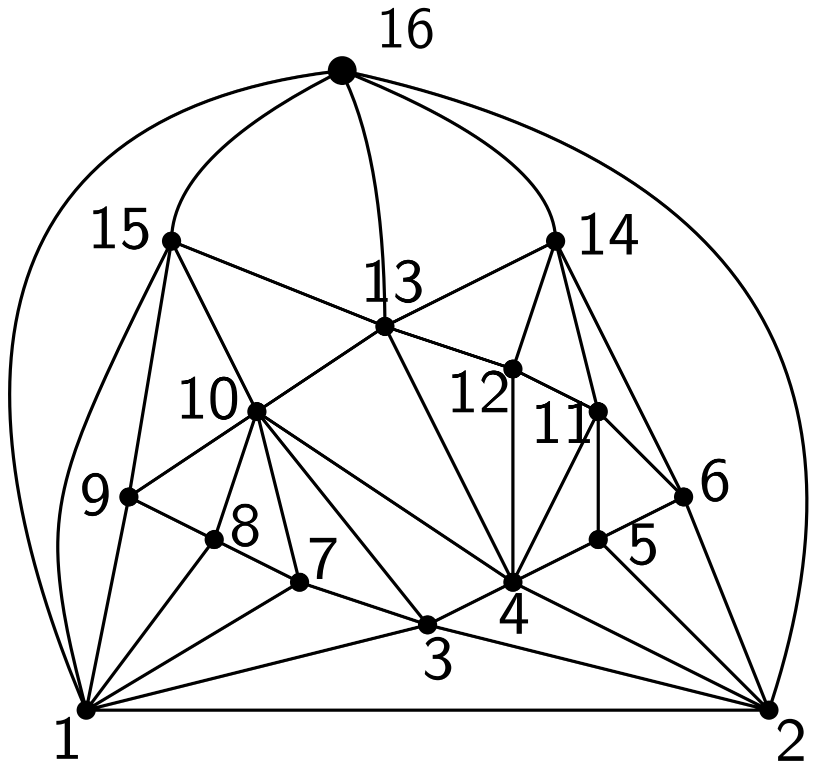
- But a forest in G_i, $1 \leq i \leq n-1$

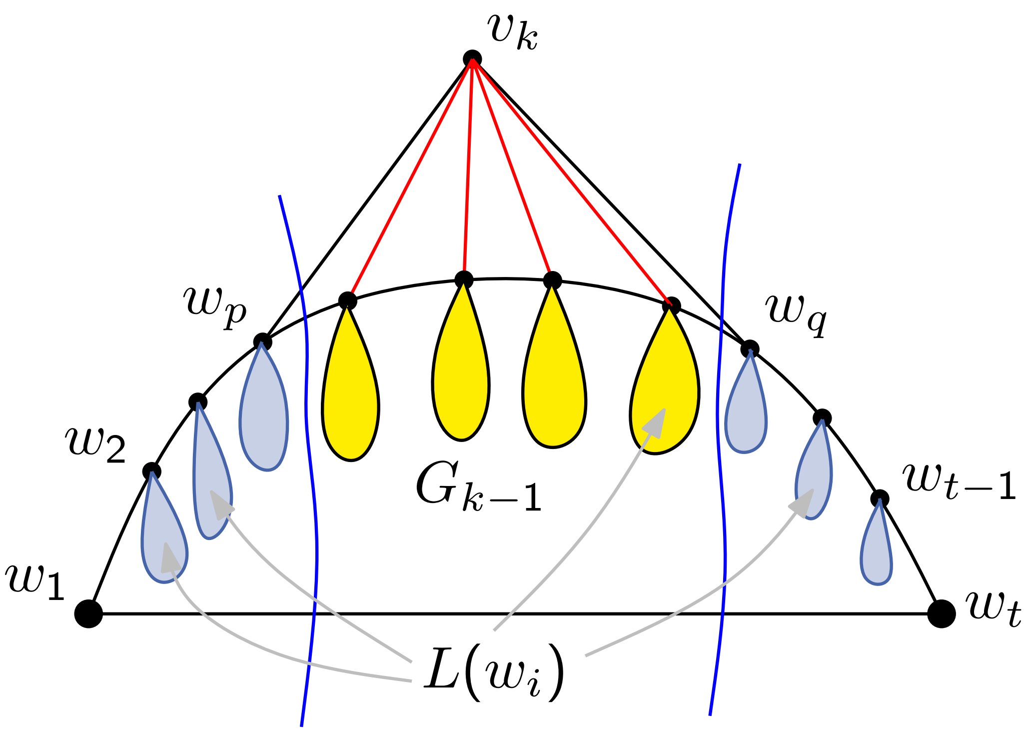
Proof of Planarity



Work with your neighbour(s) and then share **5 min**

- Compute the cover trees of vertices 15, 13, and 14.

Proof of Planarity



- Each internal vertex is covered exactly once
- Coverence relation defines a tree in G
- But a forest in G_i, $1 \leq i \leq n-1$

Lemma

Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and is even. If we shift $L(w_i)$ by δ_i to the right, we get a planar straight line grid drawing.

Proof of Planarity

Lemma

Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and is even. If we shift $L(w_i)$ by δ_i to the right, we get a planar straight line grid drawing.

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Proof

- The proof is by induction on i, i.e. we consider $G_3, \ldots, G_n$.

Proof of Planarity

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Proof

- The proof is by induction on i, i.e. we consider $G_3, \ldots, G_n$.
- Assume that this is true for G_{k-1}.

Proof of Planarity

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Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and is even. If we shift $L(w_i)$ by δ_i to the right, we get a planar straight line grid drawing.

Proof

- The proof is by induction on i, i.e. we consider $G_3, \ldots, G_n$.
- Assume that this is true for G_{k-1}.
- Let $w_1, \ldots, w_p, v_k, w_q, \ldots, w_t$ be the boundary of G_k.

Proof of Planarity

Lemma

Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and is even. If we shift $L(w_i)$ by δ_i to the right, we get a planar straight line grid drawing.

Proof

- The proof is by induction on i, i.e. we consider $G_3, \ldots, G_n$.
- Assume that this is true for G_{k-1}.
- Let $w_1, \ldots, w_p, v_k, w_q, \ldots, w_t$ be the boundary of G_k.
- Let $\delta_1 \leq \cdots \leq \delta_p \leq \delta \leq \delta_q \leq \cdots \leq \delta_t$.

Proof of Planarity

Lemma

Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and is even. If we shift $L(w_i)$ by δ_i to the right, we get a planar straight line grid drawing.

Proof

- The proof is by induction on i, i.e. we consider $G_3, \ldots, G_n$.
- Assume that this is true for G_{k-1}.
- Let $w_1, \ldots, w_p, v_k, w_q, \ldots, w_t$ be the boundary of G_k.
- Let $\delta_1 \leq \cdots \leq \delta_p \leq \delta \leq \delta_q \leq \cdots \leq \delta_t$.
- We set $\delta'_i = \delta_i$ for $1 \leq i \leq p$,
- $\delta'_i = \delta$ for $p+1 \leq i \leq q-1$ (for the neighbors of v_k)
- $\delta'_i = \delta_i$ for $q \leq i \leq t$.

Proof of Planarity

Lemma

Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and is even. If we shift $L(w_i)$ by δ_i to the right, we get a planar straight line grid drawing.

Proof

- The proof is by induction on i, i.e. we consider $G_3, \ldots, G_n$.
- Assume that this is true for G_{k-1}.
- Let $w_1, \ldots, w_p, v_k, w_q, \ldots, w_t$ be the boundary of G_k.
- Let $\delta_1 \leq \cdots \leq \delta_p \leq \delta \leq \delta_q \leq \cdots \leq \delta_t$.
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- $\delta'_i = \delta$ for $p + 1 \leq i \leq q - 1$ (for the neighbors of v_k)
- $\delta'_i = \delta_i$ for $q \leq i \leq t$.
- By induction hypothesis we can move $w_1 \ldots, w_t$ by $\delta'_1 \ldots \delta'_t$, respectively.

Proof of Planarity

Lemma

Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and is even. If we shift $L(w_i)$ by δ_i to the right, we get a planar straight line grid drawing.

Proof

- The proof is by induction on i, i.e. we consider $G_3, \ldots, G_n$.
- Assume that this is true for G_{k-1}.
- Let $w_1, \ldots, w_p, v_k, w_q, \ldots, w_t$ be the boundary of G_k.
- Let $\delta_1 \leq \cdots \leq \delta_p \leq \delta \leq \delta_q \leq \cdots \leq \delta_t$.
- We set $\delta'_i = \delta_i$ for $1 \leq i \leq p$,
- $\delta'_i = \delta$ for $p+1 \leq i \leq q-1$ (for the neighbors of v_k)
- $\delta'_i = \delta_i$ for $q \leq i \leq t$.
- By induction hypothesis we can move $w_1 \ldots, w_t$ by $\delta'_1 \ldots \delta'_t$, respectively.
- We can complete the drawing by placing v_k, v_k is moved with $L(w_{p+1}), \ldots, L(w_{q-1})$ by δ.

Outline

- Canonical ordering. Existence.
- Canonical ordering. Computation.
- Shift algorithm.
- Proof of planarity.
- Implentational details.

Implementation Details

Algorithm Shift

Let $v_1, \ldots, v_n$ be a canonical ordering of G
for $i = 1$ **to** n **do**
  $L(v_i) \leftarrow \{v_i\}$;

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$; $P(v_3) \leftarrow (1,1)$;
for $i = 4$ **to** n **do**
  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1}; and let $w_p, \ldots, w_q$ be the neighbors v_i;
  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**
    $x(v) \leftarrow x(v) + 1$;
  for $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do**
    $x(v) \leftarrow x(v) + 2$;
  $P(v_i) \leftarrow$ intersection of $+1$ and -1 edges from $P(w_p)$ and $P(w_q)$;
  $L(v_i) = \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$;

Implementation Details

Karlsruhe Institute of Technology

Algorithm Shift

Let $v_1, \ldots, v_n$ be a canonical ordering of G

for $i = 1$ **to** n **do**

  $L(v_i) \leftarrow \{v_i\}$;

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$; $P(v_3) \leftarrow (1,1)$;

for $i = 4$ **to** n **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1};
  and let $w_p, \ldots, w_q$ be the neighbors v_i;

  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 1$;

  for $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 2$;

  $P(v_i) \leftarrow$ intersection of $+1$ and -1 edges from $P(w_p)$ and $P(w_q)$;

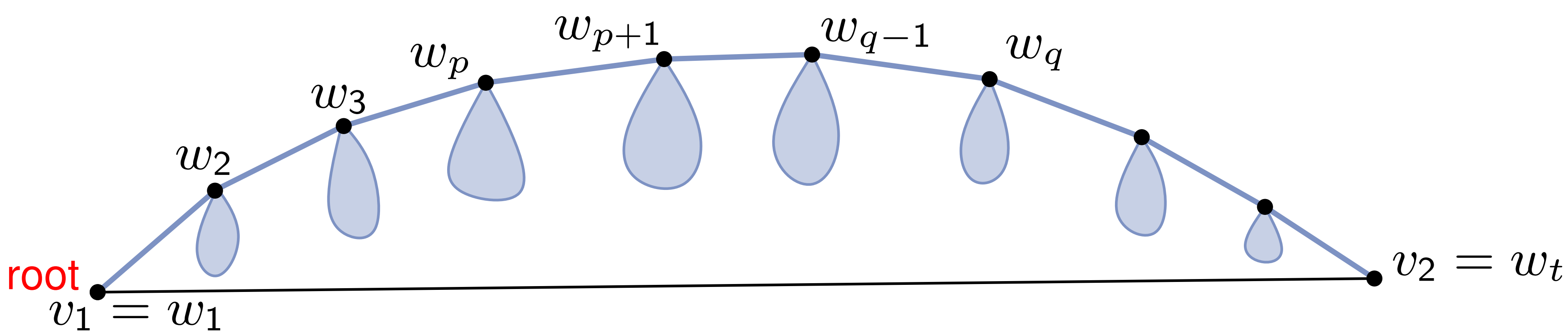
  $L(v_i) = \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$;

Take a minute to think about the time complexity of the algorithm.
Can we do better? **2 min**

Implementation Details

relative x-distance tree

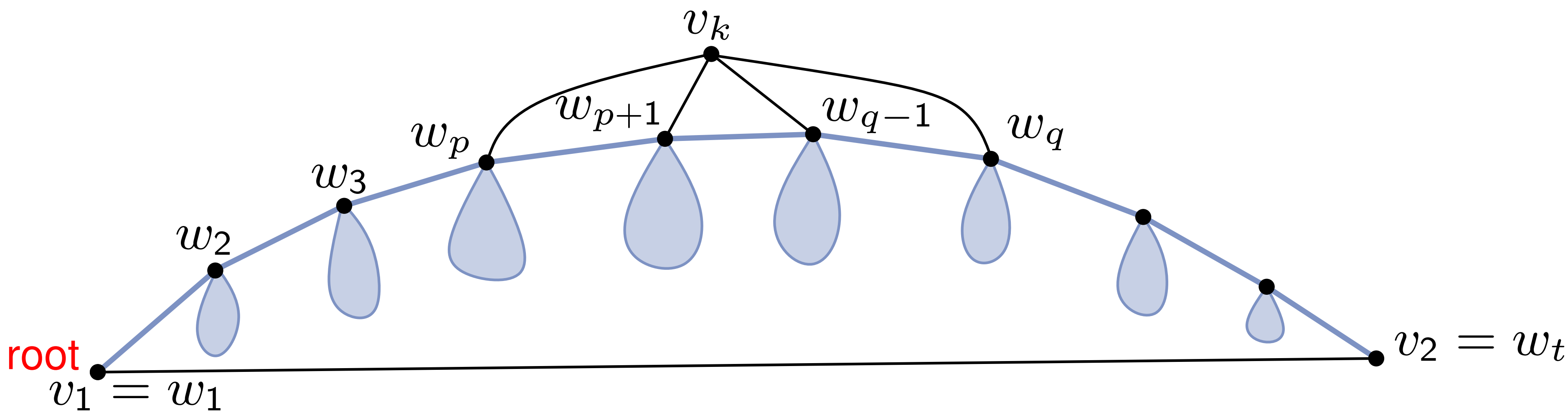
For each vertex store the x-offset from its parent and the y-coordinate



Implementation Details

relative x-distance tree

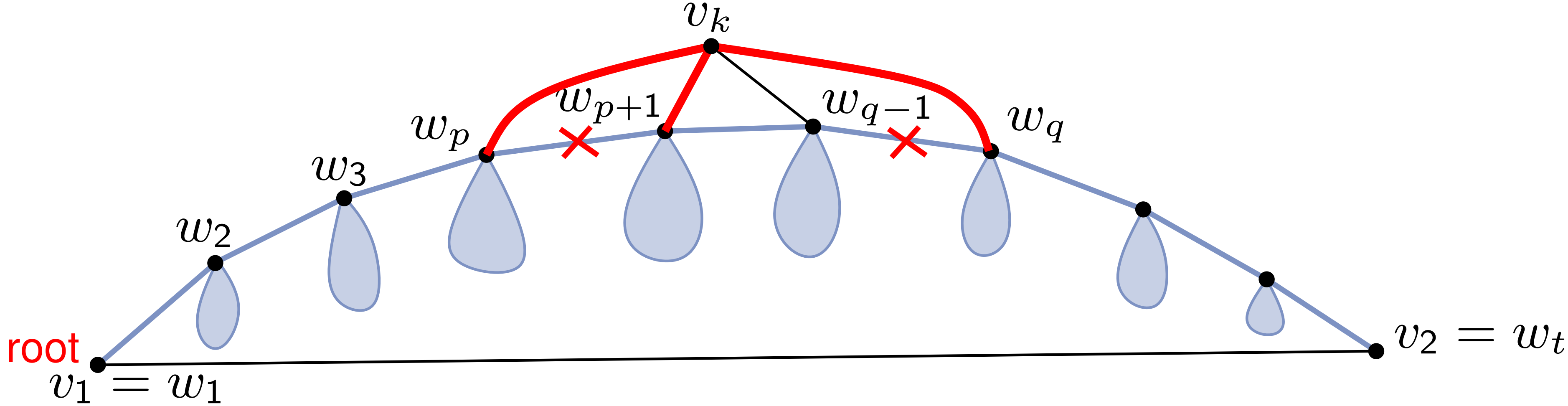
For each vertex store the x-offset from its parent and the y-coordinate



Implementation Details

relative x-distance tree

For each vertex store the x-offset from its parent and the y-coordinate



Implementation Details

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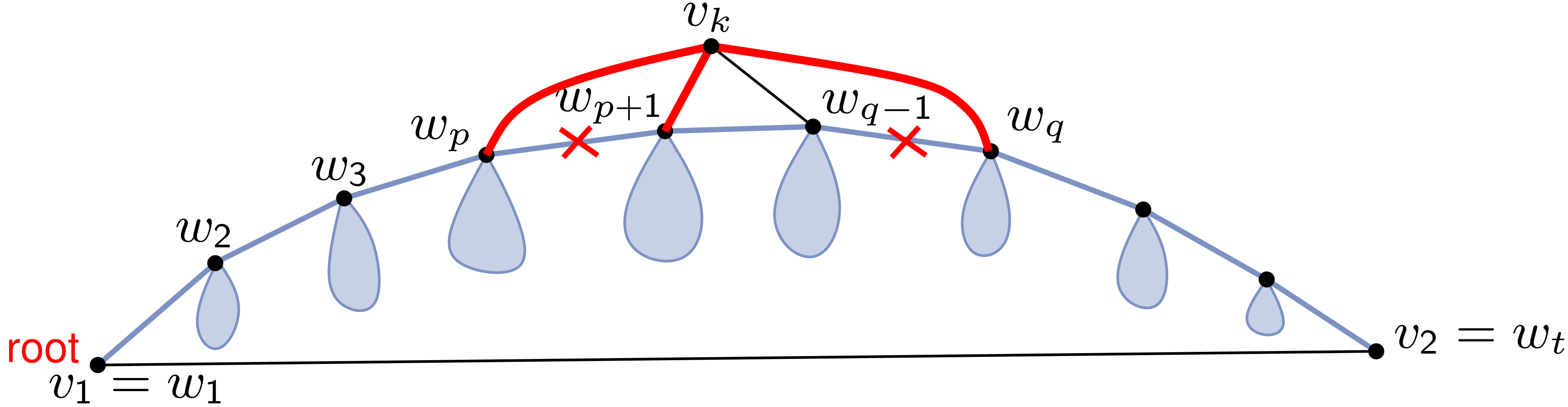

- $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$ (1)
- $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$ (2)
- $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$ (3)

Implementation Details

- In the binary tree at each vertex we keep its relative x-distance from its parent and its y-coordinate
- If we know the y-coordinates of w_p and w_q and the difference $x(w_p) - x(w_q)$, we can compute the difference $x(v_k) - x(w_p)$

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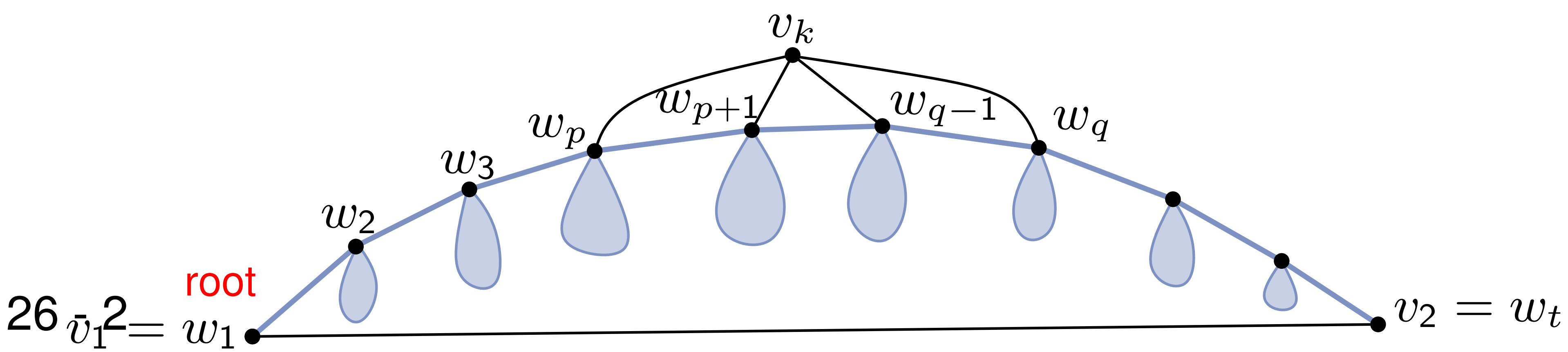

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- Calculate $\Delta_x(v_k)$ by eq. (3)
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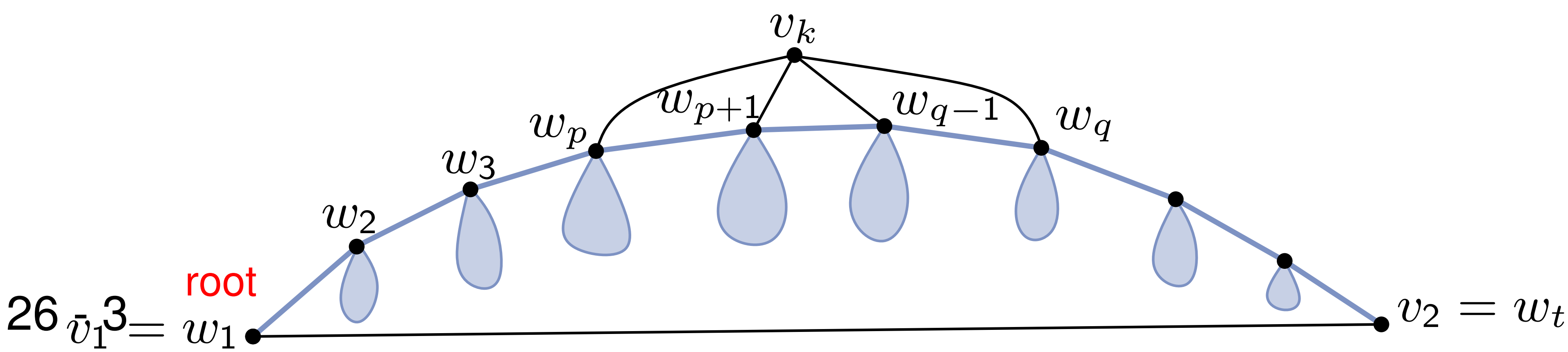

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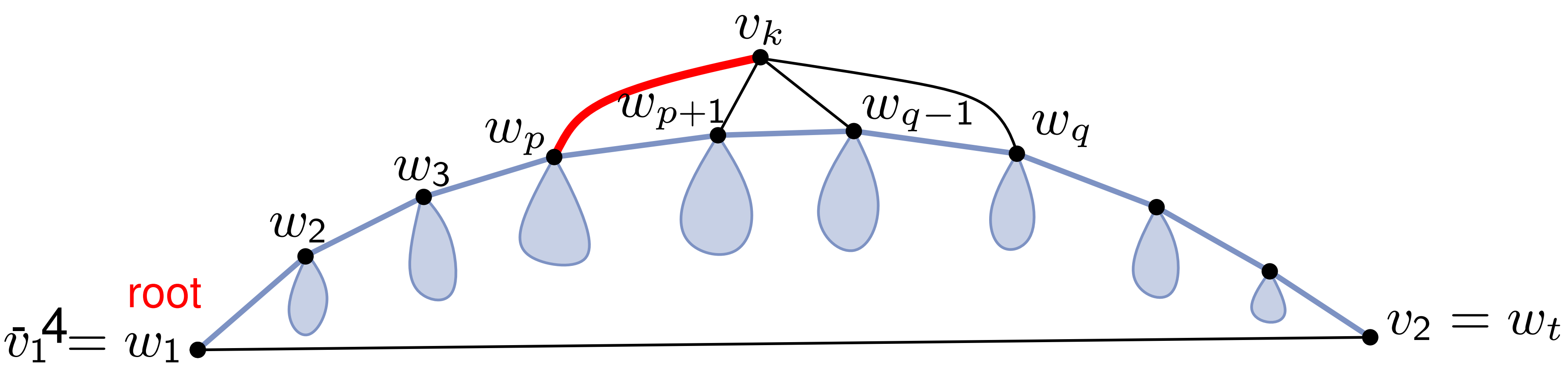

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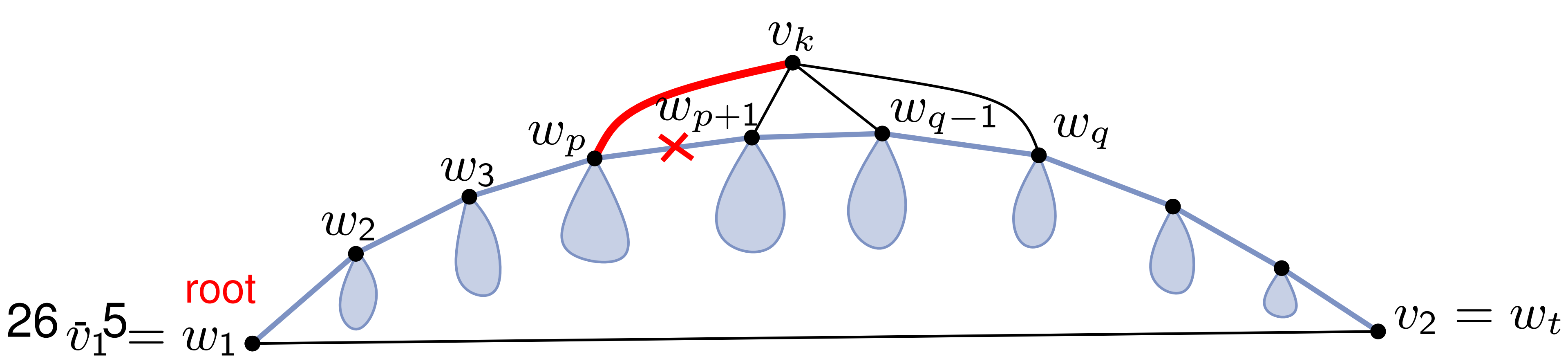

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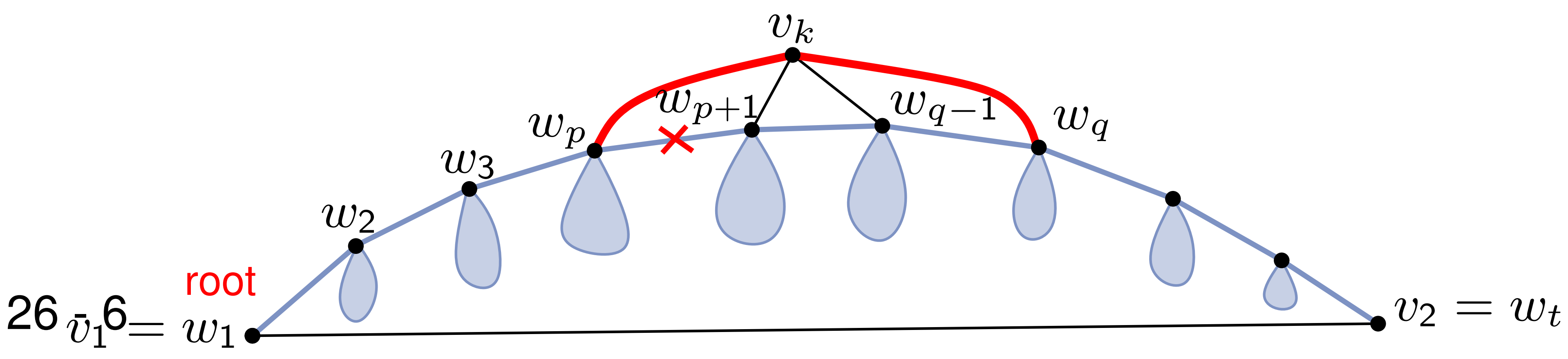

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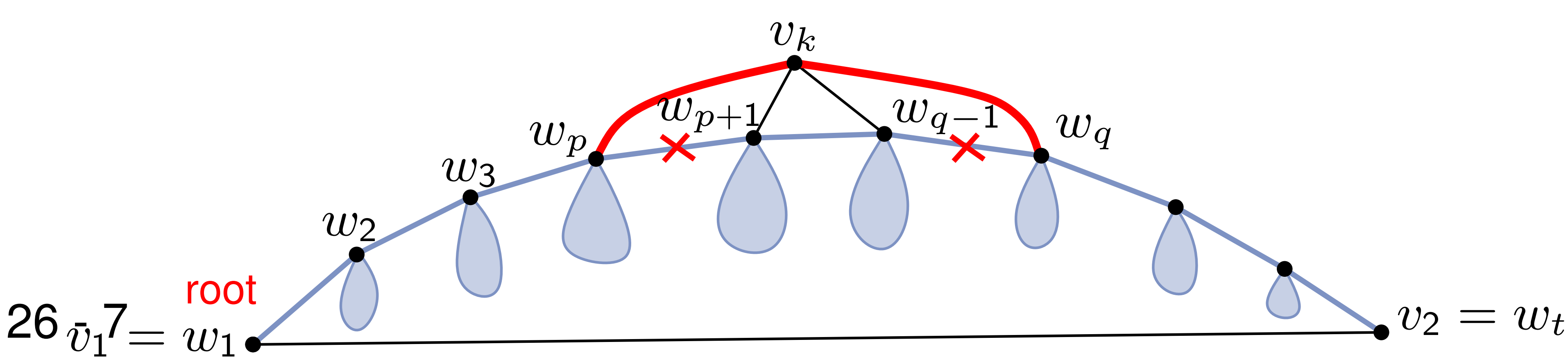

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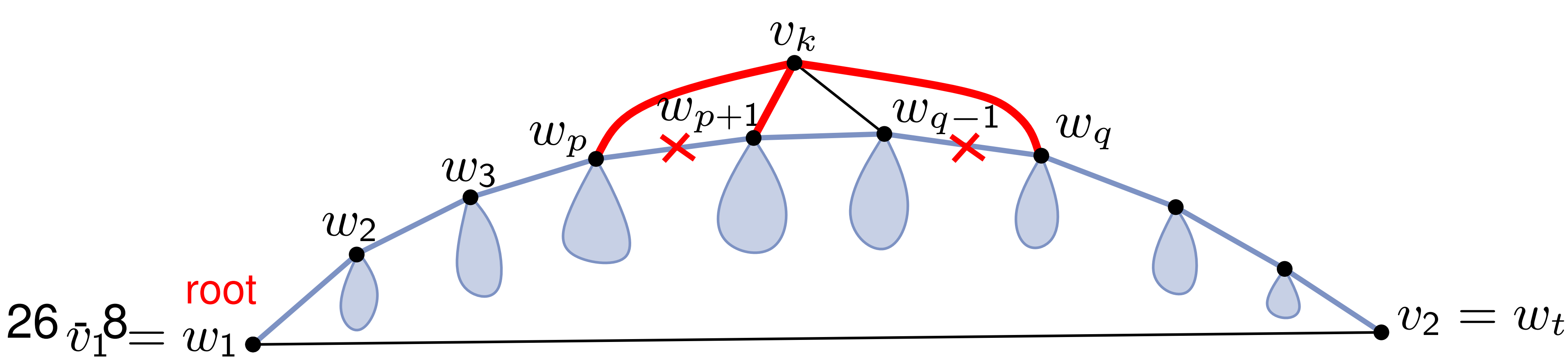

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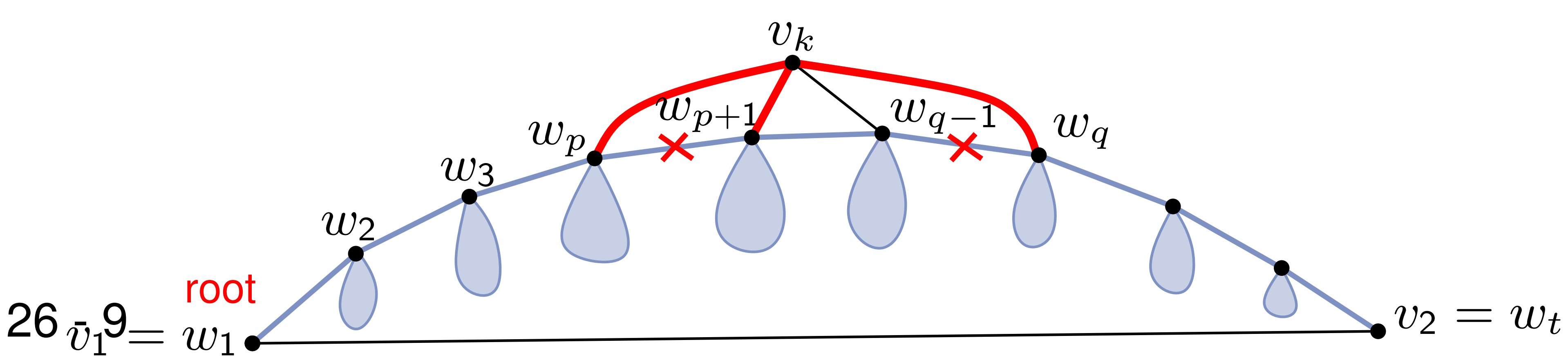

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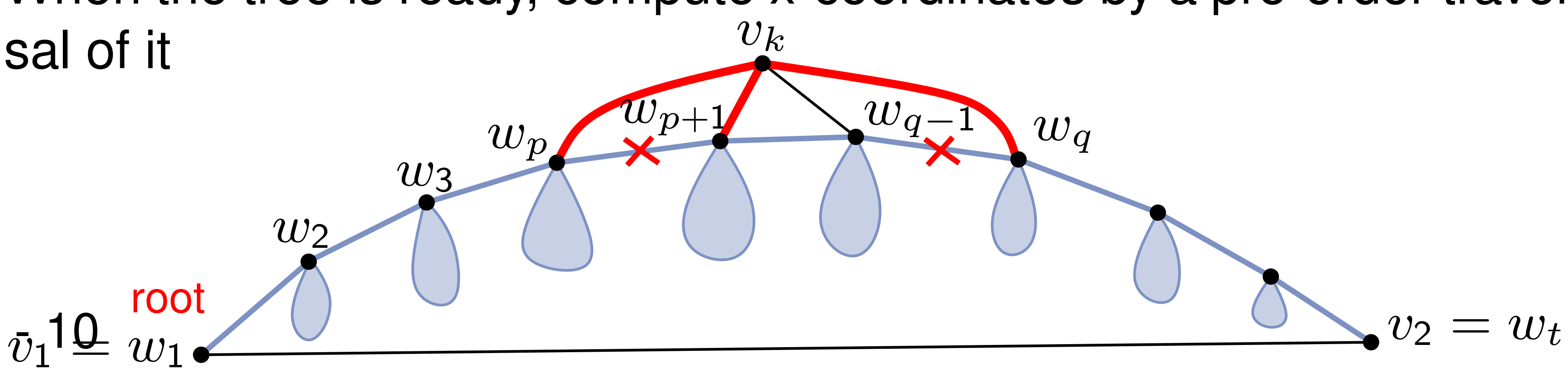
  - $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$
- When the tree is ready, compute x-coordinates by a pre-order traversal of it



Summary

This lecture:

Theorem [De Fraysseix, Pach, Pollack '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(2n-4) \times (n-2)$.

Shift Algorithm

- [NR04] Book T. Nishizeki, Md. S. Rahman "'Planar Graph Drawing. Chapter 4.2.

Summary

This lecture:

Theorem [De Fraysseix, Pach, Pollack '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(2n-4) \times (n-2)$.

Next lecture:

Theorem [Schnyder '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(n-2) \times (n-2)$.