Exercise 1: Properties of \(st\)-Graphs

Let \(D = (V, A)\) be a planar \(st\)-graph with a given embedding. For a face \(f\) of \(D\) denote by \(V_f\) and \(E_f\) the vertices and edges on \(f\). Let \(\text{start}(f)\) and \(\text{target}(f)\) be the source and sink of the graph \((V_f, E_f)\), respectively. Prove or disprove:

(a) \(D\) is bimodal.

(b) The boundary of each face \(f\) consists of two directed paths from \(\text{start}(f)\) to \(\text{target}(f)\).

(c) For every vertex \(v \in V\) there is a simple directed \(st\)-path that contains \(v\).

Exercise 2: Duals of \(st\)-Graphs

Let \(D\) be a planar embedded \(st\)-graph. For a directed edge \(e = (u, v)\), let \(\ell(e)\) denote the face left of \(e\), and let \(r(e)\) denote the face right of \(e\). Without loss of generality assume that \(D\) is embedded such that \(r(s,t)\) is the external face. The directed dual graph \(D^* = (V^*, A^*)\) of \(D\) is defined as follows:

- \(V^*\) is the set of faces of \(D\), where \(s^* = r(s,t)\) and \(t^* = \ell(s,t)\).
- \(A^* = \{(\ell(e), r(e)) \mid e \in A \setminus \{(s,t)\}\} \cup \{(s^*, t^*)\}\)

(a) Prove that \(D^*\) is a planar \(st\)-graph.

(b) Prove that for any two faces \(f\) and \(g\) of \(D\) exactly one of the following properties holds:
   
   i) \(D\) contains a directed path from \(\text{target}(f)\) to \(\text{start}(g)\)
   
   ii) \(D\) contains a directed path from \(\text{target}(g)\) to \(\text{start}(f)\)
   
   iii) \(D^*\) contains a directed path from \(f\) to \(g\)
   
   iv) \(D^*\) contains a directed path from \(g\) to \(f\)

*Hint:* Consider a topological numbering \(\sigma : V \to \mathbb{N}\) of the nodes of \(D\), such that for every \((u, v) \in A\) it holds that \(\sigma(u) < \sigma(v)\).
Exercise 3: Extended Canonical Ordering for 4-Connected Graphs

A planar graph $G = (V, E)$ is called proper triangular planar (PTP, for short) if every interior face of $G$ is a triangle and the exterior face of $G$ is a quadrangle, and $G$ has no separating triangles.

Let $G = (V, E)$ be a PTP graph with vertices $a, b, c, d$ on the outer face. A labeling $v_1 = a, v_2 = b, v_3, \ldots, v_n = d$ of the vertices of $G$ is called an extended canonical ordering of $G$ if for every $4 \leq k \leq n$:

(i) The subgraph $G_{k-1}$ induced by $v_1, \ldots, v_{k-1}$ is biconnected and the boundary $C_{k-1}$ of $G_{k-1}$ contains the edge $(a, b)$, and

(ii) the vertex $v_k$ is on the boundary of the exterior face of $G_k$, and its neighbors in $G_{k-1}$ form a subinterval of the path $C_{k-1} \setminus (a, b)$ with at least two elements. If $k \leq n - 2$, then $v_k$ has at least two neighbors in $G \setminus G_k$.

Let $G = (V, E)$ be a PTP graph with vertices $a, b, c, d$ on the outer face. Prove the following statements. We denote by $G_C$ the graph that is induced by the vertices in the interior and on the boundary of a simple cycle $C$.

(a) The graph obtained from $G$ by the removal of the vertices $c, d$ and all edges incident to them is biconnected.

(b) Let $C = \{a = u_1, \ldots, u_m = b, a\}$ be a simple cycle of $G$ such that $c, d \notin C$. Let $u_i \in C$, $2 \leq i \leq m - 1$ such that no internal chord of $C$ is incident to $u_i$. Then the graph $G_C \setminus \{u_i\}$ is biconnected.

(c) Let $C$ be as above and let $(v_i, v_j), 1 \leq i < j \leq m$, be an internal chord of $C$. Then there exists a vertex $u_l, i < l < j$ that is adjacent to at least two vertices of $G \setminus G_C$.

Use the previous statements to prove the following lemma.

Lemma 1 Every PTP graph $G$ with four vertices $a, b, c, d$ on the outer face has an extended canonical ordering such that $v_1 = a, v_2 = b, v_{n-1} = c, v_n = d$. 

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