Exercise Sheet 5

Discussion: 15. January 2019

Exercise 1: Properties of *st*-Graphs ⋆

Let $D = (V, A)$ be a planar st-graph with a given embedding. For a face f of D denote by V_f and E_f the vertices and edges on f. Let start(f) and target(f) be the source and sink of the graph (V_f, E_f), respectively. Prove or disprove:

(a) D is bimodal.

(b) The boundary of each face f consists of two directed paths from start(f) to target(f).

(c) For every vertex $v \in V$ there is a simple directed st-path that contains v.

Exercise 2: Duals of *st*-Graphs ⋆

Let D be a planar embedded st-graph. For a directed edge $e = (u, v)$, let $\ell(e)$ denote the face left of e, and let $r(e)$ denote the face right of e. Without loss of generality assume that D is embedded such that $r(s, t)$ is the external face. The directed dual graph $D^\star = (V^\star, A^\star)$ of D is defined as follows:

- $V^\star$ is the set of faces of D, where $s^\star = r(s, t)$ and $t^\star = \ell(s, t)$.
- $A^\star = \{(\ell(e), r(e)) \mid e \in A \setminus \{(s, t)\}\} \cup \{(s^\star, t^\star)\}$

(a) Prove that $D^\star$ is a planar st-graph.

(b) Prove that for any two faces f and g of D exactly one of the following properties holds:
  i) D contains a directed path from target(f) to start(g)
  ii) D contains a directed path from target(g) to start(f)
  iii) $D^\star$ contains a directed path from f to g
  iv) $D^\star$ contains a directed path from g to f

Hint: Consider a topological numbering $\sigma : V \to \mathbb{N}$ of the nodes of D, such that for every $(u, v) \in A$ it holds that $\sigma(u) < \sigma(v)$.

Exercise 3: Extended Canonical Ordering for 4-Connected Graphs ⋆⋆

A planar graph $G = (V, E)$ is called *proper triangular planar* (PTP, for short) if every interior face of G is a triangle and the exterior face of G is a quadrangle, and G has no separating triangles.

Let $G = (V, E)$ be a PTP graph with vertices a, b, c, d on the outer face. A labeling $v_1 = a, v_2 = b, v_3, \ldots, v_n = d$ of the vertices of G is called an *extended canonical ordering* of G if for every $4 \leq k \leq n$:

(i) The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and the boundary C_{k-1} of G_{k-1} contains the edge (a, b), and

(ii) the vertex v_k is on the boundary of the exterior face of G_k, and its neighbors in G_{k-1} form a subinterval of the path $C_{k-1} \setminus (a, b)$ with at least two elements. If $k \leq n - 2$, then v_k has at least two neighbors in $G \setminus G_k$.

Let $G = (V, E)$ be a PTP graph with vertices a, b, c, d on the outer face. Prove the following statements. We denote by G_C the graph that is induced by the vertices in the interior and on the boundary of a simple cycle C.

(a) The graph obtained from G by the removal of the vertices c, d and all edges incident to them is biconnected.

(b) Let $C = \{a = u_1, \ldots, u_m = b, a\}$ be a simple cycle of G such that $c, d \notin C$. Let $u_i \in C$, $2 \leq i \leq m - 1$ such that no internal chord of C is incident to u_i. Then the graph $G_C \setminus \{u_i\}$ is biconnected.

(c) Let C be as above and let (v_i, v_j), $1 \leq i < j \leq m$, be an internal chord of C. Then there exists a vertex u_l, $i < l < j$ that is adjacent to at least two vertices of $G \setminus G_C$.

Use the previous statements to prove the following lemma.

Lemma 1 *Every PTP graph G with four vertices a, b, c, d on the outer face has an extended canonical ordering such that $v_1 = a, v_2 = b, v_{n-1} = c, v_n = d$.*