Exercise 1: Sum of Angles

Let $\Gamma$ be an orthogonal drawing of a planar embedded graph $G$. Denote by $p_f$ the number of vertex-angles and bend-angles inside a face $f$ in $\Gamma$. Prove the following lemma.

**Lemma 1** If $f$ is an interior face, then the sum of all vertex-angles and bend-angles inside a face $f$ in the orthogonal drawing $\Gamma$ is $\pi(p_f - 2)$. If $f$ is an exterior face, then the corresponding sum is $\pi(p_f + 2)$.

Exercise 2: Bends in Octahedrons

A octahedron is a completely triangulated 4-regular graph on 6 vertices. Prove the following.

**Lemma 2** Every orthogonal drawing of a octahedron contains an edge with at least 3 bends.

**Hint:** Count the number of bends of the edges incident to the exterior face.

Exercise 3: Bend Minimization with Additional Constraints

Prove the following.

**Lemma 3** Let $G$ be biconnected planar embedded graph $G$ with maximal vertex degree of 4 and let $k_E : E \to \mathbb{N}_0, k_F : F \to \mathbb{N}_0$ be cost functions on the set of edges $E$ and faces $F$, respectively. There is a polynomial time algorithm that decides whether $G$ has an orthogonal layout that satisfies the following constraints

(a) every edge $e$ has at most $k_E(e)$ bends,
(b) every face $f$ has at most $k_F(f)$ concave corners (interior angle of $3\pi/2$), and
(c) the number of concave corners of all interior faces is minimal.

**Hint:** Adapt the flow network introduced in the lecture.

Given an orthogonal drawing with a minimal number of bends and one edge $e$ with many bends, what is the shape of $e$?

please turn over
Exercise 4: Edges with Many Bends

Define a family of embedded planar graphs with maximum degree 4 and \(O(n)\) vertices, such that for each bend-minimal orthogonal drawing of the given embedding there is an edge that has \(\Omega(n)\) bends.

*Hint:* Consider spirals.

Exercise 5: Area of Grid Drawings

Let \(\Gamma\) be an orthogonal drawing of a planar graph \(G\) and let \([x_{\text{min}}, x_{\text{max}}] \times [y_{\text{min}}, y_{\text{max}}]\) be the bounding box of \(\Gamma\), i.e., the smallest axis-aligned rectangle that contains \(\Gamma\). Rows or columns in \(\Gamma\) that do not contain a bend can be removed without changing the underlying orthogonal representation. Thus, it is natural to require that every column \(x_i \in [x_{\text{min}}, x_{\text{max}}] \cap \mathbb{Z}\) and row \(y_i \in [y_{\text{min}}, y_{\text{max}}] \cap \mathbb{Z}\) contains at least one bend or one vertex. We refer to drawings with this property as *compact*.

Prove the following lemmas.

**Lemma 4** Let \(G\) be a planar graph with minimum vertex degree 2 and let \(H\) be an orthogonal representation of \(G\) with \(b\) bends. Every compact orthogonal drawing of \(G\) that realizes \(H\) has an area of at most \(\lfloor (n + b)/2 \rfloor \cdot \lceil (n + b)/2 \rceil\).

Prove that this bound is tight.

**Lemma 5** There is a family of graphs with orthogonal representations such this family has compact orthogonal drawings with minimal area in \(\Omega((n + b)^2)\).