Exercise Sheet 2
Discussion: 12. November 2019

Exercise 1: Outerplanar and Series-Parallel Graphs

A graph $G$ is called **outerplanar** if it has a planar drawing where all vertices lie on the boundary of the outer face. Prove the following lemma.

**Lemma 1**

1. There is an outerplanar graph that is not series-parallel.
2. Every biconnected outerplanar graph is series-parallel.

Exercise 2: Visibility Representation

In a **visibility representation** of a graph $G = (V,E)$ the vertices are represented by horizontal segments (**vertex-segments**). We say that two vertices $u$ and $v$ see each other, if their vertex-segments can be connected by a vertical rectangle of non-zero width that does not cross any other vertex-segment. Thus, in a visibility representation of $G$, two vertices $u, v$ see each other if and only if $(u,v) \in E$; see Figure 1. Prove the following lemma.

**Lemma 2** Every series-parallel graph has a visibility representation.
Exercise 3: Canonical Orderings for Triconnected Planar Graphs

Let $G = (V, E)$ be a triconnected plane graph with a vertex $v_1$ on the outer face. Further, let $\pi = (V_1, \ldots, V_K)$ be an ordered partition of $V$, i.e., $V_1 \cup \cdots \cup V_K = V$ and $V_i \cap V_j = \emptyset$ for $i \neq j$. We define $G_k$ to be the subgraph of $G$ induced by $V_1 \cup \cdots \cup V_K$ and denote by $C_k$ the outer face of $G_k$.

The sequence $\pi$ is a canonical ordering of $G$, if

- $V_1$ consists of $\{v_1, v_2\}$, where $v_2$ lies on the outer face and $(v_1, v_2) \in E$.
- $V_K = \{v_n\}$ is a singleton, where $v_n$ lies on the outer face, $\{v_1, v_n\} \in E$, and $v_n \neq v_2$.
- Each $C_k$ $(k > 1)$ is a cycle containing $\{v_1, v_2\}$.
- Each $G_k$ is biconnected and internally triconnected, that is, removing two interior vertices of $G_k$ does not disconnect it.
- For each $k$ with $2 \leq k \leq K - 1$, one of the following conditions holds:
  1. $V_k = \{z\}$, where $z$ belongs to $C_k$ and has at least one neighbor in $G - G_k$.
  2. $V_k = \{z_1, \ldots, z_\ell\}$ is a chain, where each $z_i$ has at least one neighbor in $G - G_k$ and where $z_1$ and $z_\ell$ each have one neighbor on $C_k - 1$, and these are the only two neighbors of $V_k$ in $G_{k-1}$.

Prove the following lemma.

**Lemma 3** Every triconnected planar graph admits a canonical ordering.

**Hint:** Use reverse induction. For the induction step, consider the two cases that $G_k$ is triconnected and $G_k$ is not triconnected.
Exercise 4: Barycentric Coordinates

Let \( \Delta_{a,b,c} \) be a triangle on the plane on vertices \( a, b \) and \( c \). For each point \( x \) laying inside triangle \( \Delta_{a,b,c} \) there exists a triple \( (x_a, x_b, x_c) \) such that \( x_a \cdot a + x_b \cdot b + x_c \cdot c = x \) and \( x_a + x_b + x_c = 1 \). The triple \( (x_a, x_b, x_c) \) is called barycentric coordinates of \( x \) with respect to \( \Delta_{a,b,c} \).

Prove that:

(a) If \( A(\Delta) \) denotes the area of the triangle \( \Delta \), then

\[
x_a = \frac{A(\Delta_{b,c,x})}{A(\Delta_{a,b,c})}, \quad x_b = \frac{A(\Delta_{a,c,x})}{A(\Delta_{a,b,c})}, \quad x_c = \frac{A(\Delta_{a,b,x})}{A(\Delta_{a,b,c})}
\]

(b) Equations \( x_a = 0, x_b = 0, x_c = 0 \) represent lines through \( bc, ab \) and \( ab \), respectively.

(c) Let \( (x_a, x_b, x_c) \) be barycentric coordinates of point \( x \) in triangle \( \Delta_{abc} \). The set of points \( \{(x_a, x_b, x_c') : x_b', x_c' \in \mathbb{R}\} \) represents a line parallel to edge \( bc \) passing through point \( x \). Similarly, sets of points \( \{(x_a', x_b, x_c) : x_a', x_c \in \mathbb{R}\}, \{(x_a', x_b', x_c) : x_a', x_b' \in \mathbb{R}\} \) represent lines parallel to edges \( ac, ab \), respectively, passing through point \( x \).