Algorithms for graph visualization

Incremental algorithms. Orthogonal drawing.
Definition: Orthogonal Drawing

A drawing $\Gamma$ of a graph $G = (V, E)$ is called orthogonal if its vertices are drawn as points and each edge is represented as a sequence of alternating horizontal and vertical segments.
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- Edges lie on the grid, i.e., bends lie on grid points
- Degree of each vertex has to be at most 4
Orthogonal Layout

ER diagram in OGDF

Organigram of HS Limburg

Circuit diagram by Jeff Atwood

UML diagram by Oracle

Algorithmen zur Visualisierung von Graphen
Orthogonal Layout

Aesthetic criteria:
- number of bends
- length of edges
- width, height, area
- monotonicity of edges
- ...
Overview

- Our tool today: $st$-ordering
- Algorithm of Biedl & Kant
- Properties of the drawing, Planarity
- Construction of $st$-ordering through ear decomposition
Definition: \textit{st}-ordering

An \textit{st}-ordering of a graph $G = (V, E)$ is an ordering of the vertices $\{v_1, v_2, \ldots, v_n\}$, such that for each $j$, $2 \leq j \leq n - 1$, vertex $v_j$ has at least one neighbour $v_i$ with $i < j$, and at least one neighbour $v_k$ with $k > j$. 
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Theorem [Lempel, Even, Cederbaum, 66]

Let \( G \) be a biconnected graph \( G \) and let \( s, t \) be vertices of \( G \). \( G \) has an \textit{st}-ordering such that \( s \) appears as the first and \( t \) as the last vertex in this ordering.
Biedl & Kant Orthogonal Drawing Algorithm
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first vertex
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
indegree = 1
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
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Biedl & Kant Orthogonal Drawing Algorithm

first vertex indegree = 1 indegree = 2
Biedl & Kant Orthogonal Drawing Algorithm

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indegree = 2
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
indegree = 1
indegree = 2
indegree = 3
Biedl & Kant Orthogonal Drawing Algorithm

first vertex  indegree = 1  indegree = 2  indegree = 3
Biedl & Kant Orthogonal Drawing Algorithm

first vertex | indegree = 1 | indegree = 2 | indegree = 3 | indegree = 4

1

4

3

5

6

1

2

3

4

5

6
Biedl & Kant Orthogonal Drawing Algorithm

- First vertex
- Indegree = 1
- Indegree = 2
- Indegree = 3
- Indegree = 4

New columns
Biedl & Kant Orthogonal Drawing Algorithm

**Lemma (Area of Biedl & Kant drawing)**

The width is $m - n + 1$ and the height at most $n + 1$. 
Biedl & Kant Orthogonal Drawing Algorithm

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Proof

Width: At each step we increase the number of columns by $\text{outdeg}(v_i) - 1$, if $i > 1$ and $\text{outdeg}(v_1)$ for $v_1$.

Height: Vertices $v_1$ and $v_2$ use two rows, $v_i$, $i = 1, \ldots, n - 1$ is placed in a new row. Vertex $v_n$ uses one more row if $\text{indeg}(v_n) = 4$. 
Biedl & Kant Orthogonal Drawing Algorithm

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There are at most $2m - 2n + 4$ bends.
Biedl & Kant Orthogonal Drawing Algorithm

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Proof

Each vertex \( v_i, i \neq 1, n \), introduces \( \text{indeg}(v_i) - 1 \) and \( \text{outdeg}(v_i) - 1 \) new bends.
Biedl & Kant Orthogonal Drawing Algorithm

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**Proof**

Let \((v_i, v_j), i < j, i, j \neq 1, n.\) Then \(\text{outdeg}(v_i), \text{indeg}(v_j) \leq 3.\) I.e \((v_i, v_j)\) gets at most one bend after placement of \(v_i\) and at most one before placement of \(v_j.\) Edges outgoing from \(v_1\) can me made 2-bend by using the column below \(v_1\) for the edge \((v_1, v_2).\)
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For planar embedded graphs, with \(v_1\) and \(v_n\) on the outer face, the algorithm produces a planar drawing.
## Biedl & Kant Orthogonal Drawing Algorithm

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For planar embedded graphs, with \(v_1\) and \(v_n\) on the outer face, the algorithm produces a planar drawing.

**Proof**

Consider a planar embedding of \(G\). Let \(v_1, \ldots, v_n\) be an \(st\)-ordering of \(G\). Let \(G_i\) be the graph induced by \(v_1, \ldots, v_i\). It holds that

- **if** \(G\) is planar, vertex \(v_{i+1}\) lies on the outer face of \(G_i\).
Biedl & Kant Orthogonal Drawing Algorithm

Proof (Continuation)

- The proof is by induction on $G_i$, $i = 1, \ldots, n$, with $G_n = G$.
- Let $E_i$ be the edges outgoing from the vertices of $G_i$ in the order they appear in the embedded $G$.
- We use as an invariant that edges $E_i$ appear in the same order in the orthogonal drawing of $G_i$.
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\[ \begin{align*}
  v_{i1} & \quad e_1 \quad v_{ik} \\
  e_2 & \quad v_{i1} \quad v_{ik} \\
  e_3 & \\
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\end{align*} \]
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Biedl & Kant Orthogonal Drawing Algorithm

Theorem (Biedl & Kant 98)

A biconnected graph $G$ with vertex-degree at most 4 admits an orthogonal drawing such that:
- Area is $(m - n + 1) \times n + 1$
- Each edge (except maybe for one) has at most 2 bends
- The exceptional edge has at most 3 bends
- The total number if bends is at most $2m - 2n + 4$
- If $G$ is plane, the orthogonal drawing is planar
- Finally, provided an $st$-ordering such a drawing can be constructed in $O(n)$ time.
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- For the construction we have used an $st$-ordering of $G$!
### Definition: st-digraph

Let $G$ be a directed graph. A vertex $s$ (resp. $t$) is called **source** (resp. **sink**) of $G$ if it has only outgoing (resp. incoming) edges. A directed acyclic graph with one source and one sink is called **$st$-digraph**.

![Diagram of an $st$-digraph](image-url)
st-digraph, topological ordering

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A **topological ordering** of a directed graph $G$ (with $n$ vertices) is an assignment of numbers $\{1, \ldots, n\}$ to the vertices of $G$, such that for every edge $(u, v)$, $\text{number}(v) > \text{number}(u)$. 
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How to construct a topological ordering?
$st$-ordering

Construction of an $st$-ordering:

$G$ is undirected biconnected graph
$st$-ordering

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Orient edges of $G$
$st$-ordering

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$G'$ is an $st$-digraph
**st-ordering**

**Construction of an st-ordering:**

- \( G \) is undirected biconnected graph
- Orient edges of \( G \)
- \( G' \) is an st-digraph
- Let \( v_1, \ldots, v_n \) be a topological ordering of \( G' \)
Construction of an $st$-ordering:

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Orient edges of $G$ to form $G'$.

$G'$ is an $st$-digraph

Let $v_1, \ldots, v_n$ be a topological ordering of $G'$.

Since $G'$ is an $st$-digraph, for $v_i$ ($i \neq 1, n$) there exist $(v_j, v_i)$ and $(v_i, v_k)$. By the property of topological ordering, $j < i$ and $i < k$. 
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_**HOW?**_

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**EXAMPLE**
**Definition: Ear decomposition**

An ear decomposition $D = (P_0, \ldots, P_r)$ of an undirected graph $G = (V, E)$ is a partition of $E$ into an ordered collection of edge disjoint paths $P_0, \ldots, P_r$, such that:

- $P_0$ is an edge
- $P_0 \cup P_1$ is a simple cycle
- both end-vertices of $P_i$ belong to $P_0 \cup \cdots \cup P_{i-1}$
- no internal vertex of $P_i$ belong to $P_0 \cup \cdots \cup P_{i-1}$

An ear decomposition is open if $P_0, \ldots, P_r$ are simple paths.
Lemma (Ear decomposition)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. $G$ has an open ear decomposition $(P_0, \ldots, P_r)$, where $P_0 = (s, t)$. 

*st*-ordering
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**Proof**

- Let $P_0 = (s, t)$ and $P_1$ be path between $s$ and $t$, it exists since $G$ is biconnected.
**st-ordering**

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- Let $w$ be the first vertex of $P$ that is contained in $P_0 \cup \cdots \cup P_i$. Set $P_{i+1} = (u, v) \cup P(v - \cdots - w)$.
**st-ordering**

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Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$. 
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Let $G = (V, E)$ be a biconnected graph and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an st-digraph. $G'$ is called st-orientation of $G$.

**Proof**

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

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![Diagram showing an ear decomposition with an st-path from s to t through P0 and P1.]
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**Diagram**

![Diagram of st-ordering](image)
*st*-ordering

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- Distinguish two cases based on whether $u$ and $v$ are connected by a directed path or not.
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Let \(G = (V, E)\) be a biconnected graph and let \((s, t) \in E\). There is an orientation \(G'\) of \(G\) which represents an \(st\)-digraph. \(G'\) is called \(st\)-orientation of \(G\).

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Construction of an $st$-ordering:

$G$ is undirected biconnected graph

HOW?

Orient edges of $G$  

$G'$ is an $st$-digraph

Let $v_1, \ldots, v_n$ be a topological ordering of $G'$

Since $G'$ is an $st$-digraph, for $v_i$

$(i \neq 1, n) \exists (v_j, v_i)$

and $(v_i, v_k)$. By the property of topological ordering $j < i$ and $i < k$.

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Direct construction of st-ordering from ear decomposition
$st$-ordering

Direct construction of $st$-ordering from ear decomposition

We construct it incrementally, considering $G_i = P_0 \cup \cdots \cup P_i$, $i = 0, \ldots, r$. 
Direct construction of $st$-ordering from ear decomposition

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- Assume that $L$ contains an $st$-ordering of $G_i$ and let ear $P_{i+1} = \{v_1, \ldots, v_q\}$. We insert vertices $v_1, \ldots, v_q$ to $L$ after vertex $v_1$ (or before $v_q$).
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### Direct construction of \(st\)-ordering from ear decomposition

- We construct it incrementally, considering \(G_i = P_0 \cup \cdots \cup P_i, i = 0, \ldots, r\).

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- **Why this is an \(st\)-ordering?** Let \(G'_{i+1}\) be an \(st\)-orientation of \(G_i\) as constructed in the previous proof. \(L\) is a topological ordering of \(G'_{i+1}\) and therefore an \(st\)-ordering of \(G_i\).
**Algorithm: st-ordering (example)**

(Implementation details - Based on DFS)
\textit{st-ordering}

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![Graph Example](image-url)
Algorithm: \textit{st}-ordering (example)
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**st-ordering**

Algorithm: *st*-ordering (example)
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```
s, b, f, g, t
```
Algorithm: $st$-ordering (example)
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Algorithm: \textit{st-ordering} (example)
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\begin{itemize}
\item \texttt{s, b, f, g, h, t}
\end{itemize}
Algorithm: \textit{st}-ordering (example)
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\[ s, e, b, f, g, h, t \]
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\begin{itemize}
\item $s$
\item $e$
\item $b$
\item $a$
\item $f$
\item $g$
\item $h$
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![Diagram of a directed graph with nodes labeled s, e, b, a, f, c, d, g, h, t and edges connecting them.]

\[ s, e, b, a, f, c, d, g, h, t \]
Algorithm \textit{st}-ordering

\textbf{Data:} Undirected biconnected graph $G = (V, E)$, edge $\{s, t\} \in E$

\textbf{Result:} List $L$ of nodes representing an \textit{st}-ordering of $G$

\begin{algorithmic}
\Function{dfs}{vertex $v$}
\State $i \leftarrow i + 1$; $DFS[v] \leftarrow i$
\While{there exists non-enumerated $e = \{v, w\}$}
\State $DFS[e] \leftarrow DFS[v]$
\If{$w$ not enumerated}
\State $CHILDEDGE[v] \leftarrow e$; $PARENT[w] \leftarrow v$
\State $dfs(w)$
\Else
\State $\{w, x\} \leftarrow CHILDEDGE[w]$; $D[\{w, x\}] \leftarrow D[\{w, x\}] \cup \{e\}$
\If{$x \in L$}
\State process\_ears$(w \rightarrow x)$
\EndIf
\EndIf
\EndWhile
\EndFunction

\begin{algorithmic}
\State initialize $L$ as $\{s, t\}$
\State $DFS[s] \leftarrow 1$; $i \leftarrow 1$; $DFS[\{s, t\}] \leftarrow 1$; $CHILDEDGE[s] \leftarrow \{s, t\}$
\State $dfs(t)$
\end{algorithmic}
**st-ordering**

**Function process_ears**

```plaintext
process_ears(tree edge \( w \rightarrow x \)) begin
    foreach \( v \leftrightarrow w \in D[w \rightarrow x] \) do
        \( u \leftarrow v; \)
        while \( u \not\in L \) do \( u \leftarrow PARENT[u]; \)
        \( P \leftarrow (u \ast \rightarrow v \leftrightarrow w); \)
        if \( w \rightarrow x \) is oriented from \( w \) to \( x \) (resp. from \( x \) to \( w \)) then
            orient \( P \) from \( w \) to \( u \) (resp. from \( u \) to \( w \));
            paste the inner nodes of \( P \) to \( L \)
            before (resp. after) \( u \);
        foreach tree edge \( w' \rightarrow x' \) of \( P \) do process_ears(\( w' \rightarrow x' \));
    D[{\( w, x \}]} \leftarrow \emptyset;
```

Function process_ears
The described algorithm produces an \(st\)-ordering of a given biconnected graph \(G = (V, E)\) in \(O(E)\) time.
**Theorem**

The described algorithm produces an \(st\)-ordering of a given biconnected graph \(G = (V, E)\) in \(O(E)\) time.

**Theorem (Biedl & Kant 98)**

A biconnected graph \(G\) with vertex-degree at most 4 admits an orthogonal drawing such that:
- Area is \((m - n + 1) \times n + 1\)
- Each edge (except maybe for one) has at most 2 bends
- The exceptional edge has at most 3 bends
- The total number of bends is at most \(2m - 2n + 4\)
- If \(G\) is plane, the orthogonal drawing is planar
- Finally, provided an \(st\)-ordering such a drawing can be constructed in \(O(n)\) time.
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### Theorem

The described algorithm produces an *st*-ordering of a given biconnected graph $G = (V, E)$ in $O(E)$ time.

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Together imply an $O(n)$ algorithm for constructing an orthogonal drawing.