Algorithms for graph visualization

Layouts for planar graphs. Realizer method.
Today

Last lecture:

**Theorem [De Fraysseix, Pach, Pollack ’90]**

Every $n$-vertex planar graph has a planar straight-line drawing of a size $(2n - 4) \times (n - 2)$.

This lecture:

**Theorem [Schnyder ’90]**

Every $n$-vertex planar graph has a planar straight-line drawing of a size $(n - 2) \times (n - 2)$.
Overview

- Barycentric Coordinates and Barycentric Graph Representation.
- Schnyder Labelling. Existence.
- Schnyder Realizer.
- From Schnyder Realizer to Barycentric Representation.
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Barycentric Coordinates

Let $A, B, C, P \in \mathbb{R}^2$. A triple $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ such that:

- $\alpha + \beta + \gamma = 1$
- $P = \alpha A + \beta B + \gamma C$

is called barycentric coordinates of $P$ with respect to $\triangle ABC$. 
Barycentric Representation

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Barycentric Representation

A **Barycentric Representation** of a graph $G = (V, E)$ is an assignment of barycentric coordinates to the vertices of $G$, i.e. it is an **injective** function $v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$, such that:

- $v_a + v_b + v_c = 1$ for all $v \in V$
- for each $(x, y) \in E$ and each $z \in V \setminus \{x, y\}$, $\exists k \in \{a, b, c\}$ with $x_k < z_k$ and $y_k < z_k$. 
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What does this condition mean?
Barycentric Representation

Discuss with your neighbour(s) and then share

- Consider the a-coordinates of points $x$ and $y$, which is bigger?
- Interpret the condition below.

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Barycentric Representation

Lemma [Schnyder ’90]

Let $v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$ be a barycentric representation of a graph $G = (V, E)$ and let $A, B, C \in \mathbb{R}^2$. The function

$$f: v \in V \mapsto v_a A + v_b B + v_c C$$

gives a planar drawing of $G$ inside triangle $\triangle ABC$. 
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Schnyder Labeling

**Definition: Schnyder-Labeling**

A **Schnyder-Labeling** of a planar triangulated graph $G$ is a labeling of all internal angles with labels 1, 2 and 3 such that:

- Each internal face contains vertices with all three labels $1$, $2$, and $3$, appearing in a counterclockwise order.
- The counterclockwise ordering of the labels around each vertex consists of a nonempty interval of $1$'s followed by a nonempty interval of $2$'s followed by a nonempty interval of $3$'s.
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Face  Each internal face contain vertices with all three labels 1, 2 and 3, appearing in a counterclockwise order.

Vertex  The counterclockwise ordering of the labels around each vertex consists of a nonempty interval of 1’s followed by a nonempty interval of 2’s followed by a nonempty interval of 3’s.
**Theorem [Schnyder '90]**

Every triangulated plane graph has a Schnyder labeling.
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- Edge contraction. Contractible edge. Notation: $G \setminus (u, v)$. 
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**Lemma**

Let $G$ be a triangulated plane graph with vertices $a, b, c$ on the outer face. There exists a contractible edge $(a, x)$ in $G$, $x \neq b, c$. 

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Proof

- By induction on the number of vertices in a graph.
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- Assume that every graph with less or equal than \( k - 1 \) vertices has a Schnyder labeling in which all labels at \( a \) are 1.
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### Definition: Schnyder Forest

A Schnyder Forest or a Realizer of a planar triangulated graph $G = (V, E)$ is a partition of the inner edges of $E$ into three sets of oriented edges $T_1, T_2, T_3$ such that for each inner vertex $v \in V$ hold:

- $v$ has an outgoing edge in each of $T_1, T_2, T_3$
- The counterclockwise order of the edges around $v$ is as follows: edges leaving in $T_1$, entering in $T_3$, leaving in $T_2$, entering in $T_1$, leaving in $T_3$, entering in $T_2$. 
Schnyder Realizer

Recall that:

**Theorem [Schnyder ’90]**

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Every triangulated plane graph has a Schnyder labeling.

By this theorem and by previous construction:

**Theorem [Schnyder ’90]**
Every triangulated plane graph has a Schnyder realizer.
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- For each \( v \) there exists a directed red, blue, green paths from \( v \) to \( a, b, c \), respectively.
- No monochromatix cycle exists

**Diagram**:

- For each \( v \) there exists a directed red, blue, green paths from \( v \) to \( a, b, c \), respectively.
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Schnyder Realizer

- For each $v$ there exists a directed red, blue, green paths from $v$ to $a, b, c$, respectively.
- No monochromatic cycle exists
- Each monochromatic subgraph is a tree!
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- Each monochromatic subgraph is a tree!
- The sinks of red/blue/green trees are the vertices $a$, $b$, $c$.
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No monochromatix cycle exists.

Each monochromatic subgraph is a tree!

The sinks of red/blue/green trees are the vertices $a, b, c$.

Work with your neighbour(s) and then share

Prove that each monochromatic graph is a tree.

5 min
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Face Regions

- Paths $P_a(v)$, $P_b(v)$, $P_c(v)$ cross only at vertex $v$.
- $R_a(v)$, $R_b(v)$, $R_c(v)$ are sets of faces.
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For inner vertices $u \neq v$ it holds that $u \in R_i(v) \Rightarrow R_i(u) \subsetneq R_i(v)$.

**Proof ...**
Barycentric Representation

- Let barycentric coordinates of \( v \in G \setminus \{a, b, c\} \) be \((v_a, v_b, v_c)\), where 
  \[ v_a = \frac{|R_a(v)|}{2n - 5}, \quad v_b = \frac{|R_b(v)|}{2n - 5} \quad \text{and} \quad v_c = \frac{|R_c(v)|}{2n - 5}. \]
- We set: \( A = (2n - 5, 0), \quad B = (0, 2n - 5), \quad C = (0, 0). \)
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**Theorem [Schnyder ’90]**

The function

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 f : v \mapsto (v_a, v_b, v_c) = \frac{1}{2n - 5} (|R_a(v)|, |R_b(v)|, |R_c(v)|)
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is a barycentric representation of \( G \).
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**Proof**

- **Condition 1**: \( v_a + v_b + v_c = 1 \).
Barycentric Representation

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Proof

- Condition 1: \( v_a + v_b + v_c = 1 \).

- Condition 2: For each edge \((u, v)\) and vertex \( w \neq u, v\) at least one of three is true: \( w_a > u_a, v_a \), \( w_b > u_b, v_b \), \( w_c > u_c, v_c \).
Final Remarks

- The resulting drawing is a grid drawing.
- It is bounded by the triangle $\triangle ABC$ with $A = (2n - 5, 0)$, $B = (0, 2n - 5)$, $C = (0, 0)$.
- It has area $2n - 5 \times 2n - 5$. 
Final Remarks

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How to obtain area \( n - 2 \times n - 2 \)?
- Use weak barycentric coordinates \( \frac{1}{n-1} (n_1(v), n_2(v), n_3(v)) \),
  \( n_i(v) = |\text{vertices in } R_i(v)| - |P_{i-1}(v)| \) with respect to \( A = (n - 1, 0), B = (0, n - 1), C = (0, 0) \).
The resulting drawing is a grid drawing.

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Weak barycentric coordinates: Triple $(v_a, v_b, v_c)$ such that

- $v_a + v_b + v_c = 1$

- For each edge $(u, v)$ and vertex $w \neq u, v$, $\exists k \in \{a, b, c\}$, such that $(u_k, u_{k+1}) <_{lex} (w_k, w_{k+1})$, and $(v_k, v_{k+1}) <_{lex} (w_k, w_{k+1})$.

- Here we say that $(a, b) <_{lex} (c, d)$ iff $a < c$ or $a = c$ and $b < d$. 
Schnyder Realizer Method