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Algorithms for Graph Visualization

Force-Directed Algorithms

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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20.11.2018

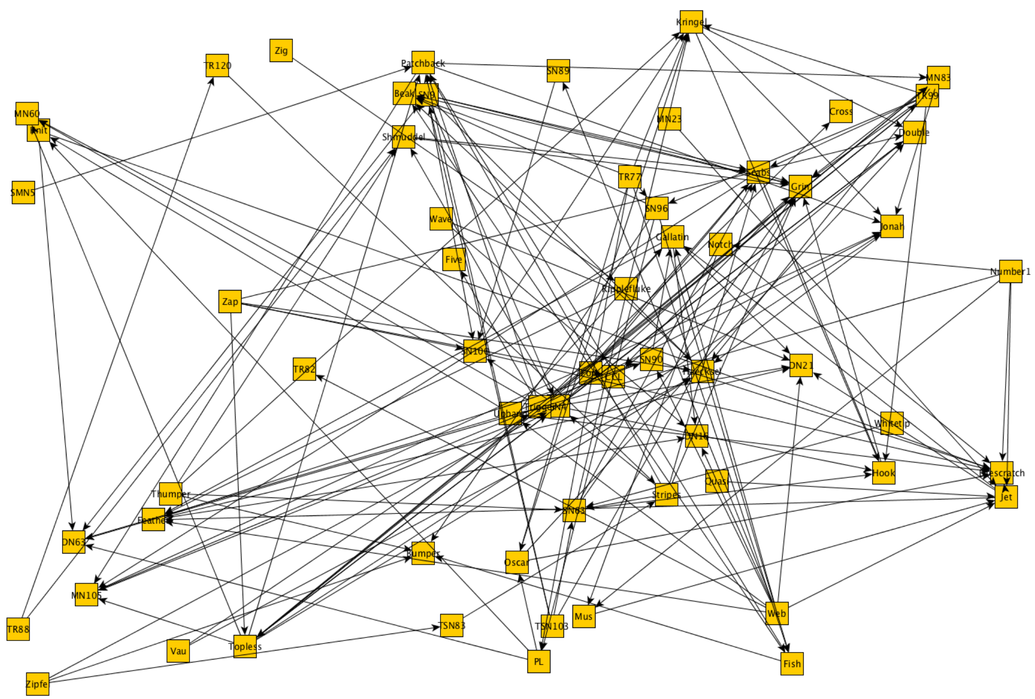
Introduction

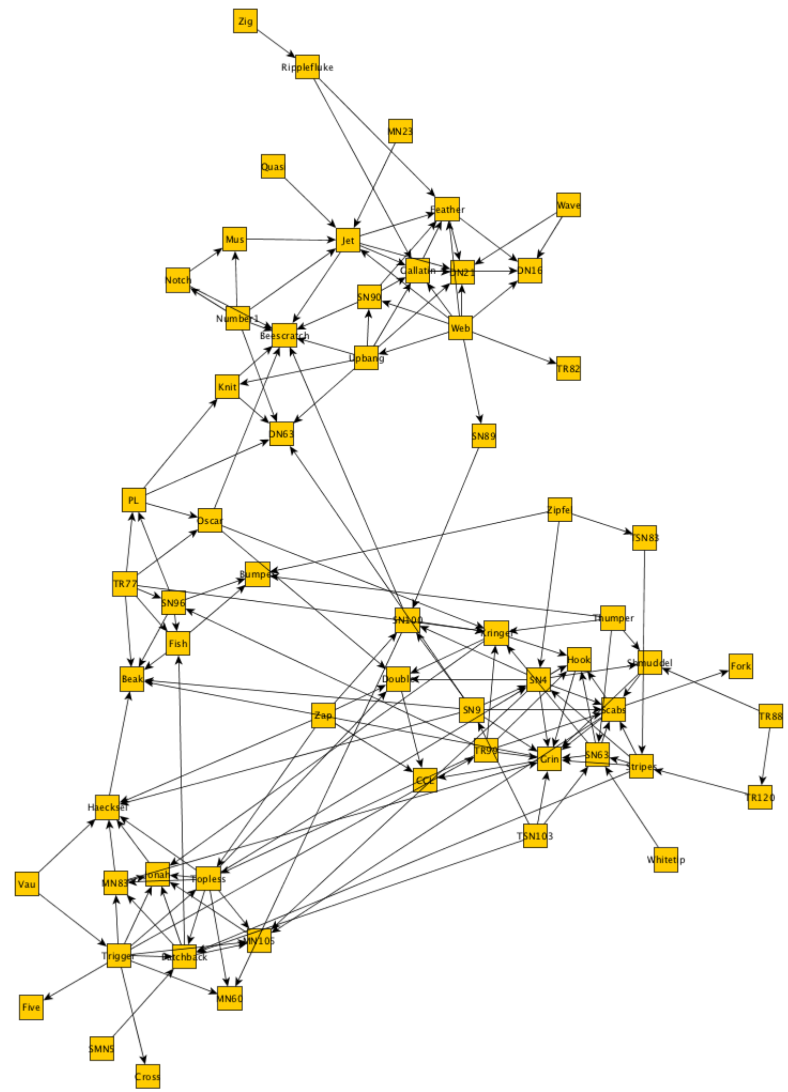
- Before: always based on some properties: tree, series-parallel graph, planar graph
- and on some additional information: ordering of the vertices, decompositions into SP-components
- Today: more direct and intuitive method based on physical analogies
- The methods are very popular: intuitiveness, easy to program, generality, fairly satisfactory results,...

General Layout Problem

Given: Graph $G = (V, E)$

Find: Clear and readable drawing of G





Which aesthetic criteria would you optimize?

General Layout Problem

Given: Graph $G = (V, E)$

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Criteria:

- adjacent nodes are close
- non-adjacent far apart
- edges short, straight-line, similar length
- densly connected parts (clusters) form communities
- as few crossings as possible
- nodes distributed evenly

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⚠ Optimization criteria partially contradict each other

Example: Fixed edge-length

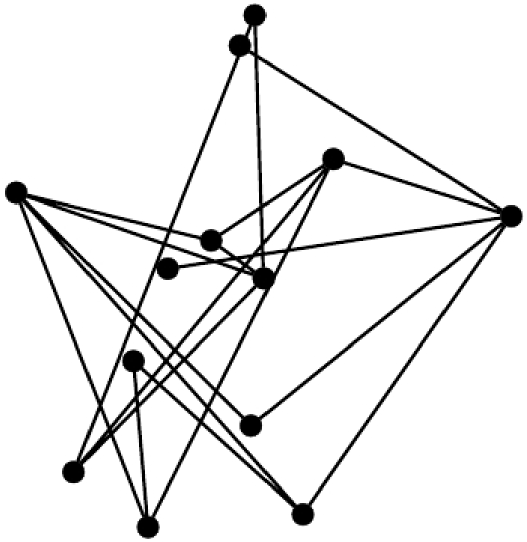
Given: Graph $G = (V, E)$, required edge length $\ell(e)$, $\forall e \in E$

Find: Drawing of G which realizes all the edge lengths

NP-hard for

- edge lengths $\{1, 2\}$ [Saxe, '80]
- planar drawing with unit edge length [Eades, Wormald, '90]

Physical Model



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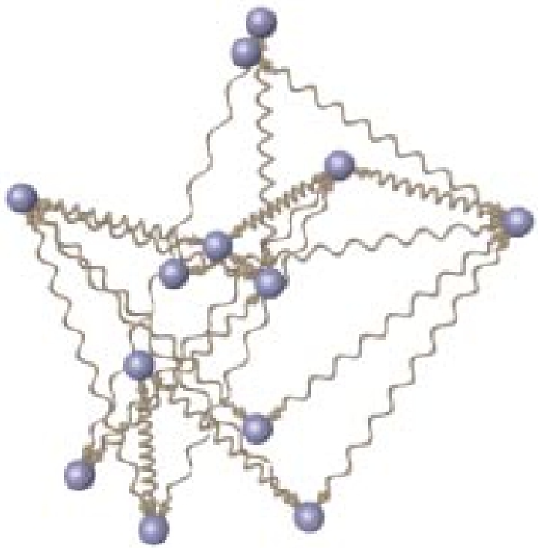

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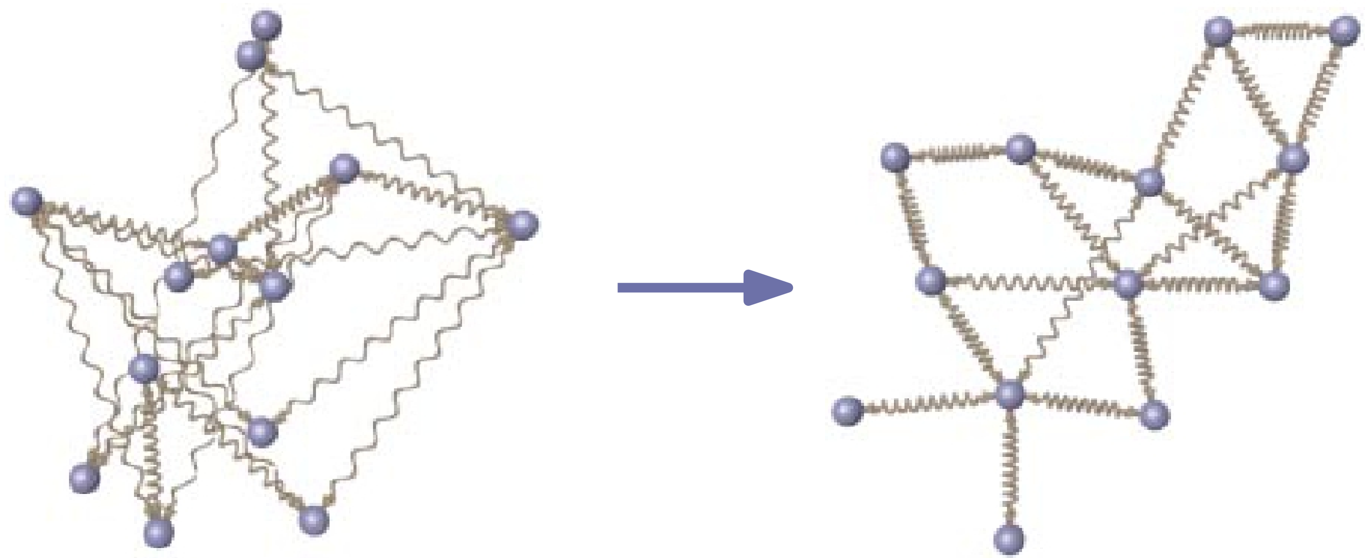

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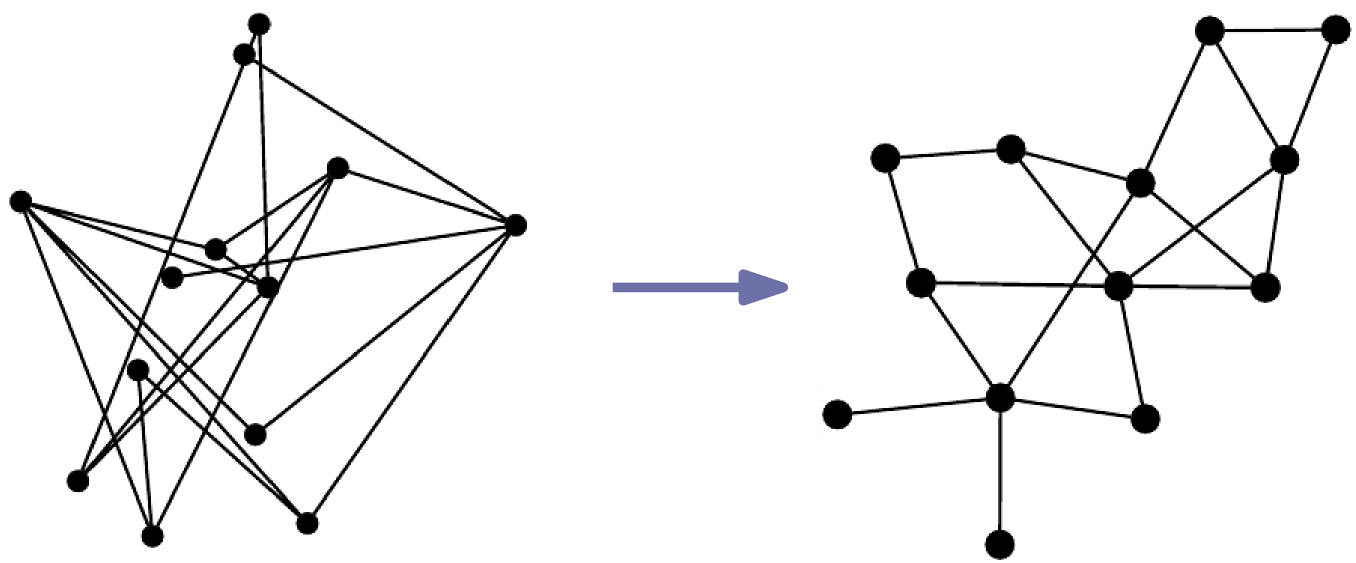

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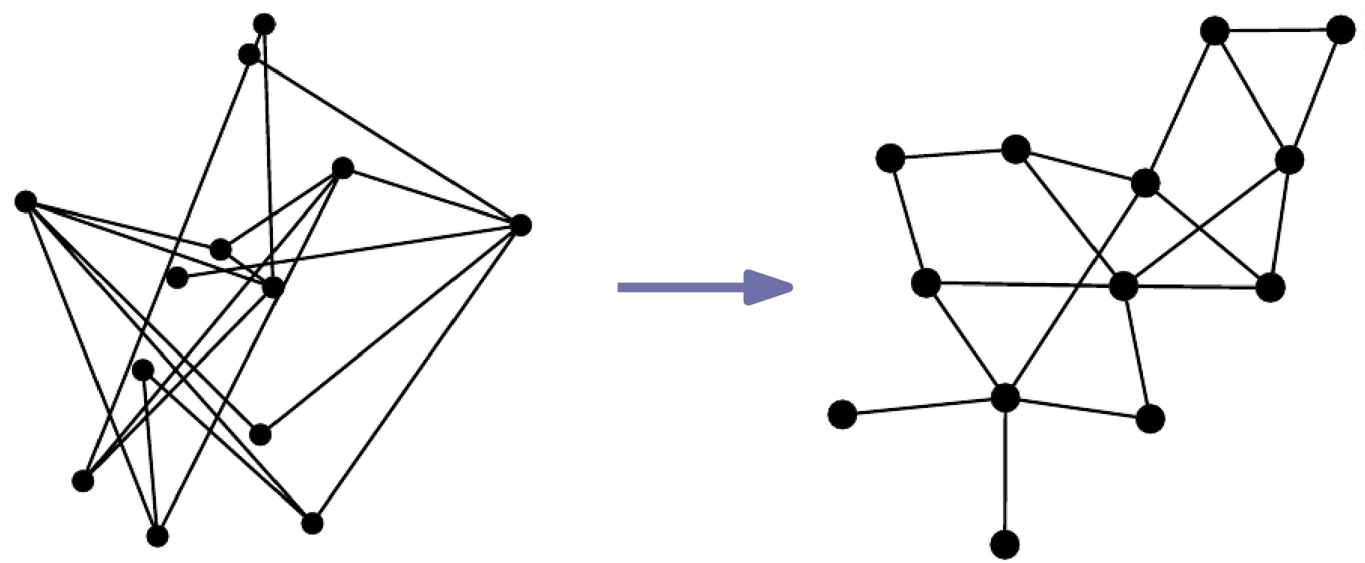

So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

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Notation

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$\ell = \ell(e)$	ideal spring length for edge e
$p_v = (x_v, y_v)$	position of node v
$\|\|p_u - p_v\|\|$	Euclidean distance between u and v
$\overrightarrow{p_u p_v}$	unit vector pointing from u to v

Spring-Embedder (Eades, 1984)

Model:

- repulsive force between two non-adjacent nodes u and v

$$f_{\text{rep}}(p_u, p_v) = \frac{c_{\text{rep}}}{||p_v - p_u||^2} \cdot \overrightarrow{p_u p_v}$$

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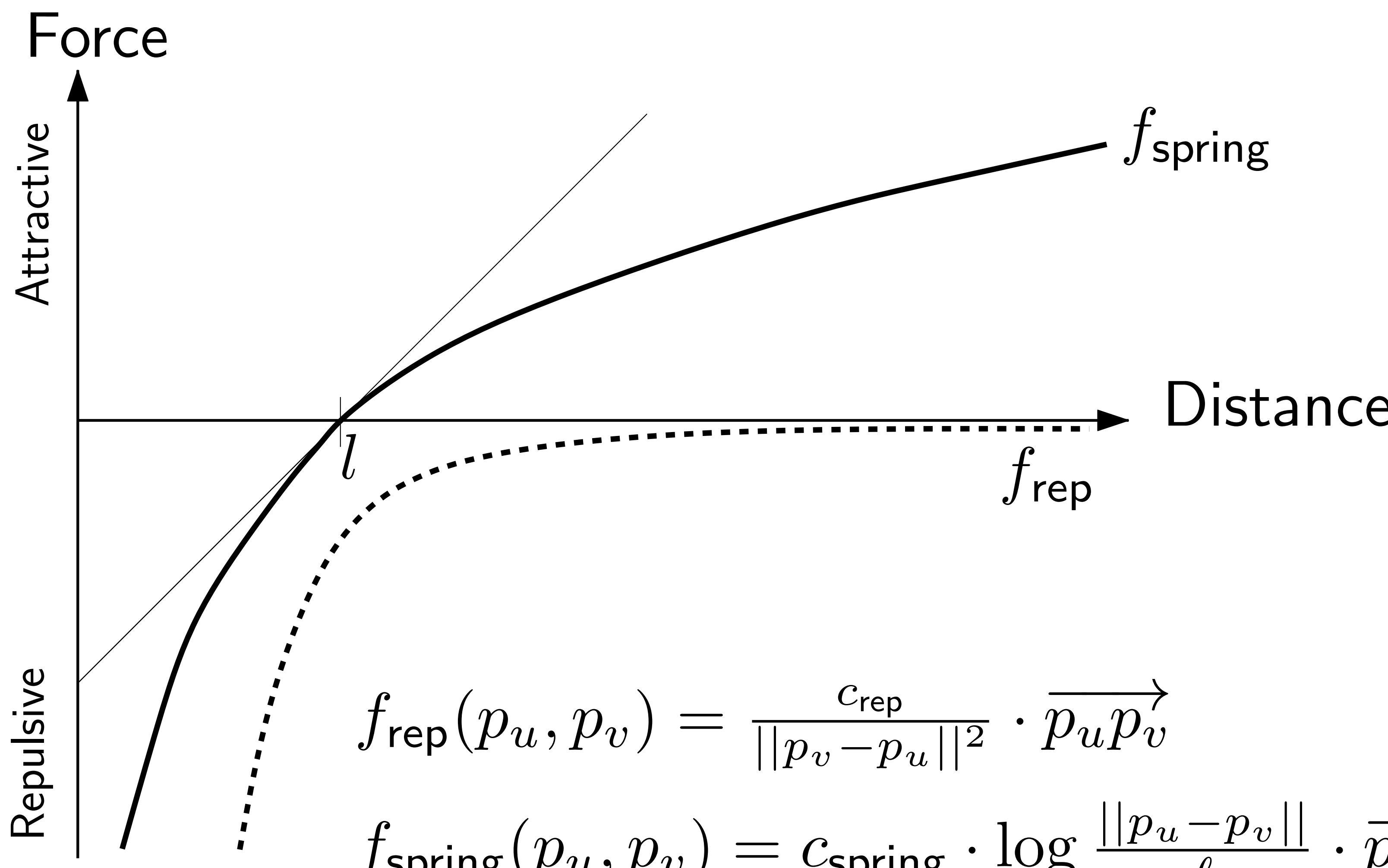
- resulting displacement vector for node v

$$F_v = \sum_{u:\{u,v\} \notin E} f_{\text{rep}}(p_u, p_v) + \sum_{u:\{u,v\} \in E} f_{\text{spring}}(p_u, p_v)$$

Diagram of Spring-Embedder Forces (Eades, 1984)



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Algorithm Spring-Embedder (Eades, 1984)

Input: $G = (V, E)$ connected undirected graph with initial placement $p = (p_v)_{v \in V}$, number of interations $K \in \mathbb{N}$, threshold $\varepsilon > 0$, constant $\delta > 0$

Output: Layout p with "low internal stress"

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

  foreach $v \in V$ **do**

    $F_v(t) \leftarrow \sum_{u:\{u,v\} \notin E} f_{\text{rep}}(p_u, p_v) + \sum_{u:\{u,v\} \in E} f_{\text{spring}}(p_u, p_v)$

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    $p_v \leftarrow p_v + \delta \cdot F_v(t)$

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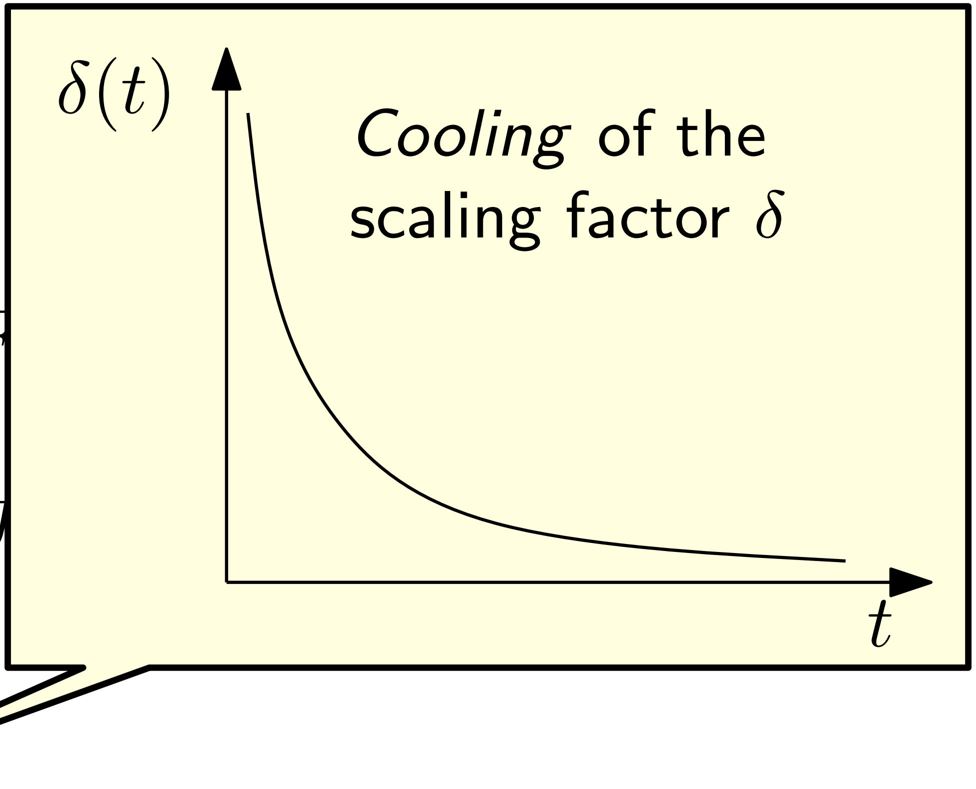

Discussion

Advantages

- very simple Algorithm
- good results for small and medium-sized graphs
- emphirically good representation of symmetry and structure

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Influence

- Original paper by Peter Eades got 1800 citations (400 in the past four years)
- Basis for many further ideas

Variant: Fruchterman & Reingold (1991)

Model:

- repulsive force between **all** node pairs u and v

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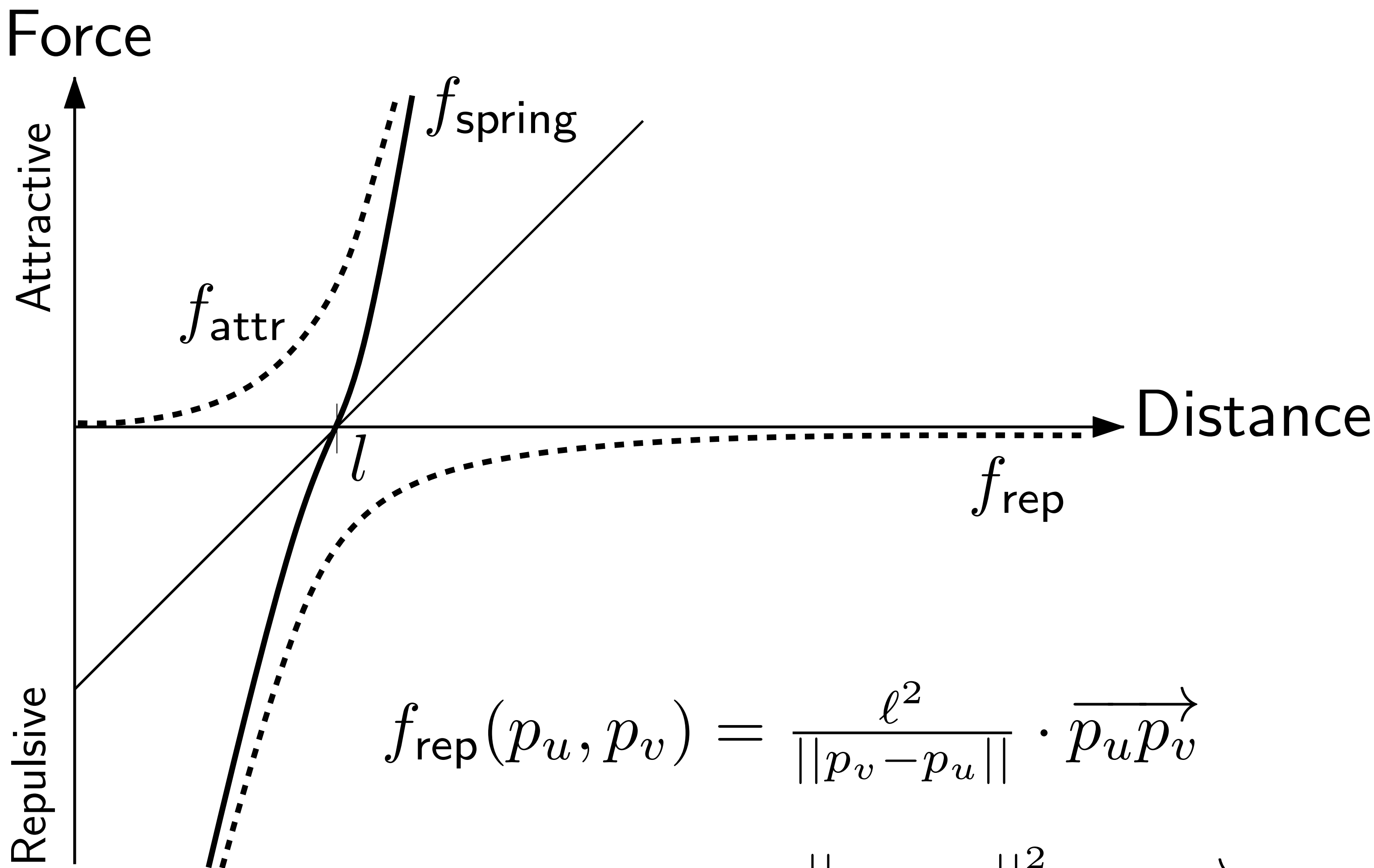

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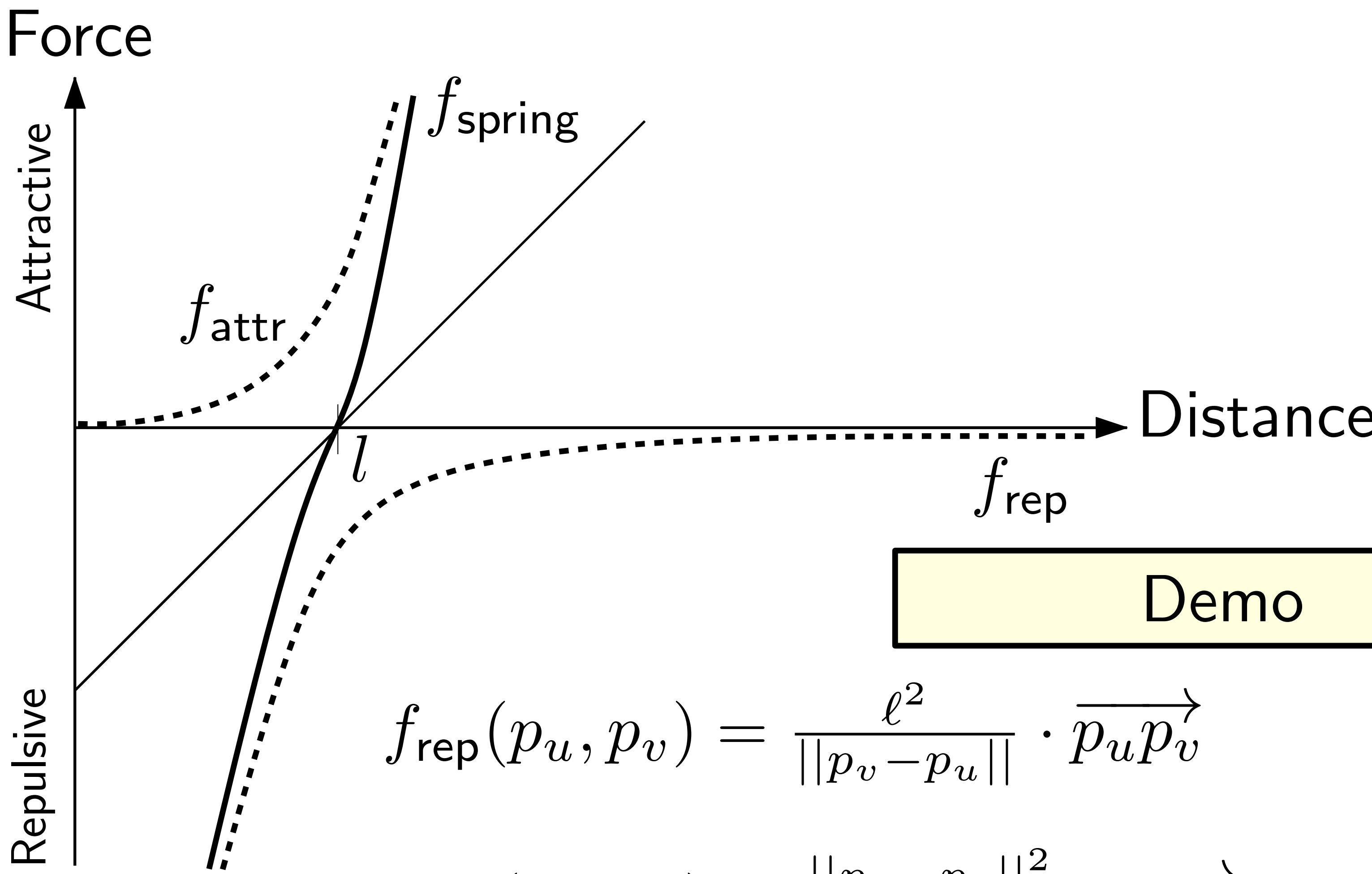

Demo

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- **Gravitation**

- **Magnetic forces**

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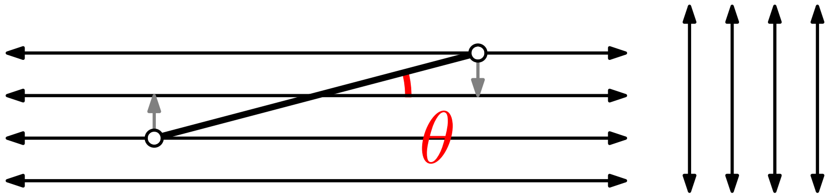
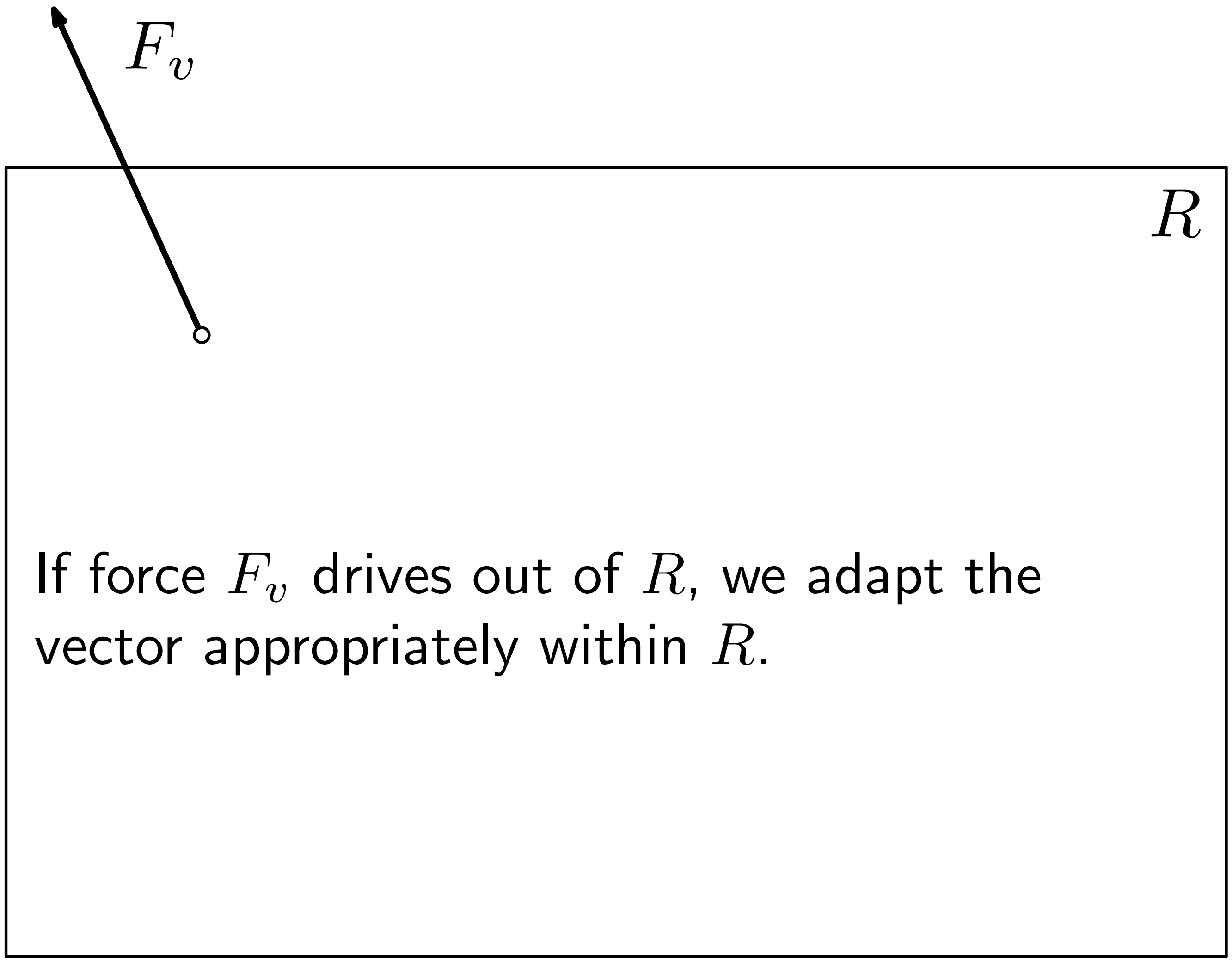
- **Magnetic forces**
  - define magnetic fields (e.g. vertical, horizontal)
  - angle θ between edge and the direction of the field
  - define force that reduces this angle



Bounded Drawing Area



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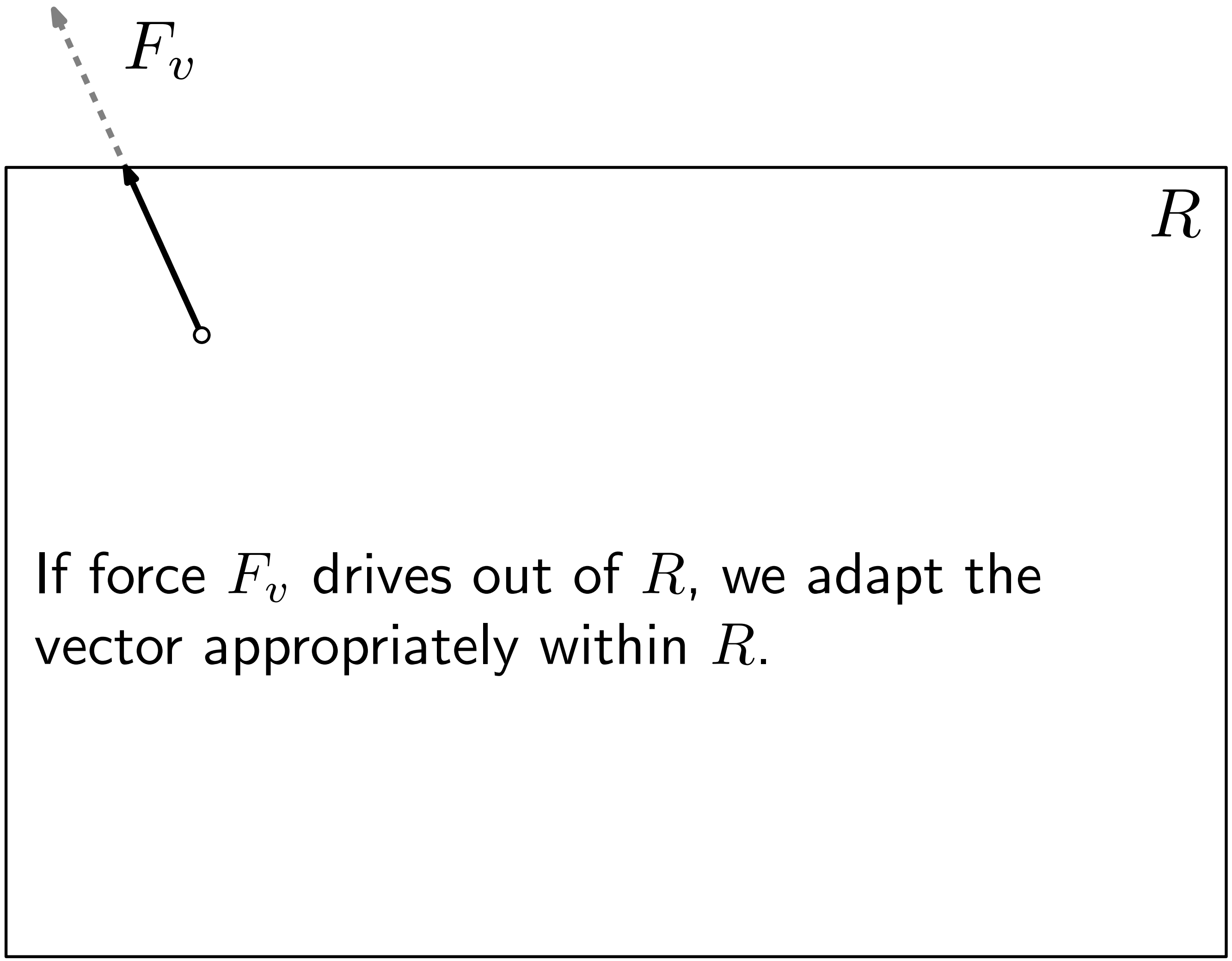

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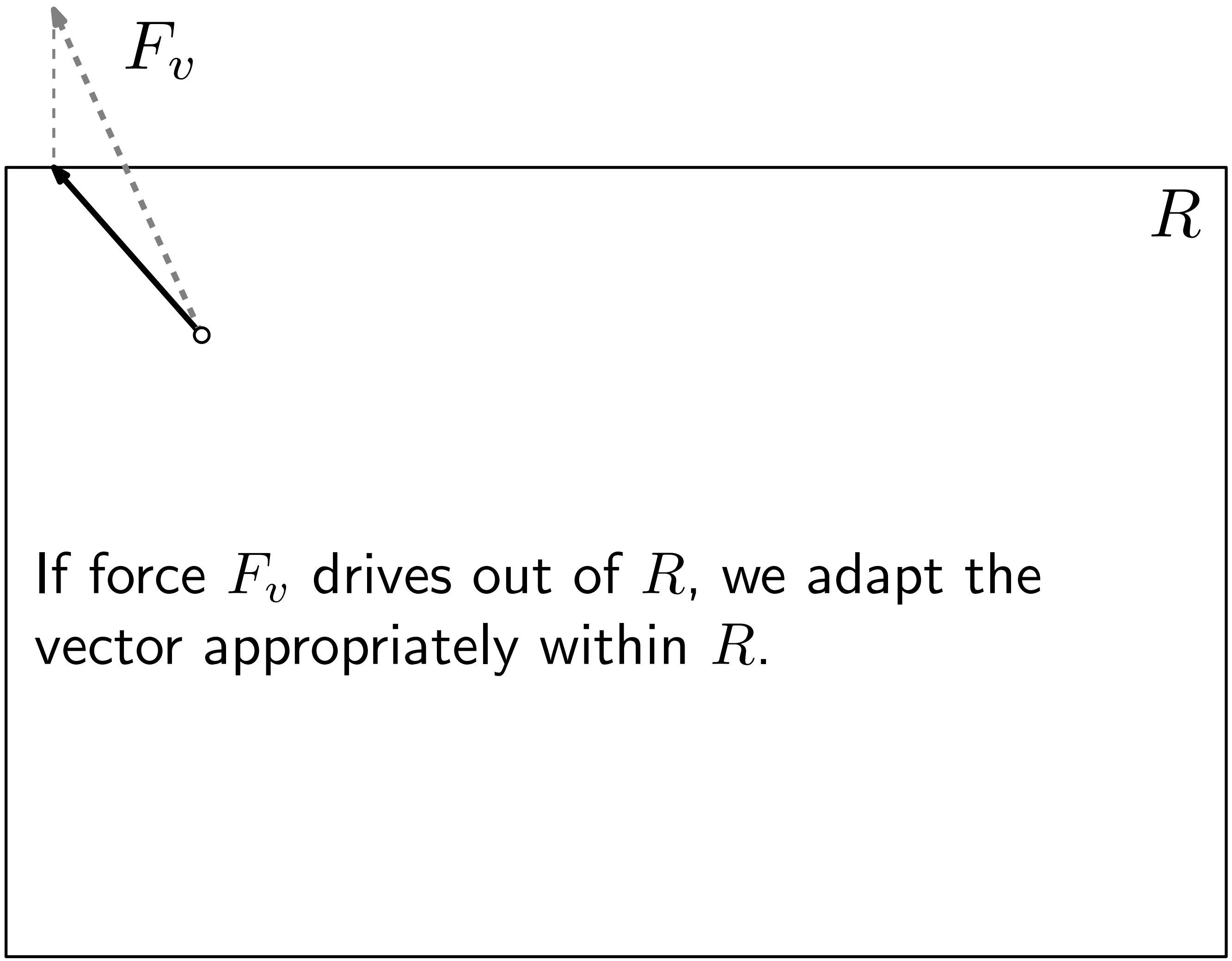

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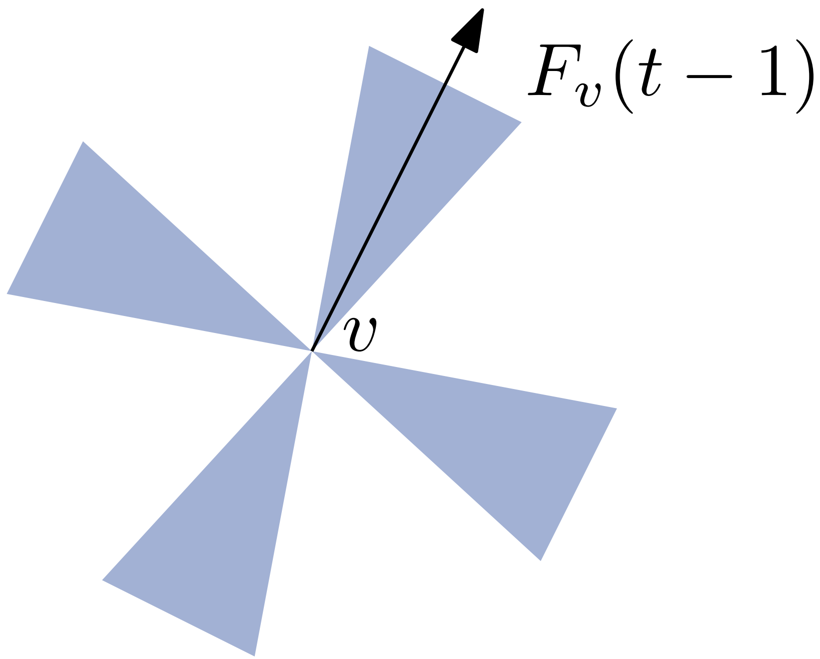

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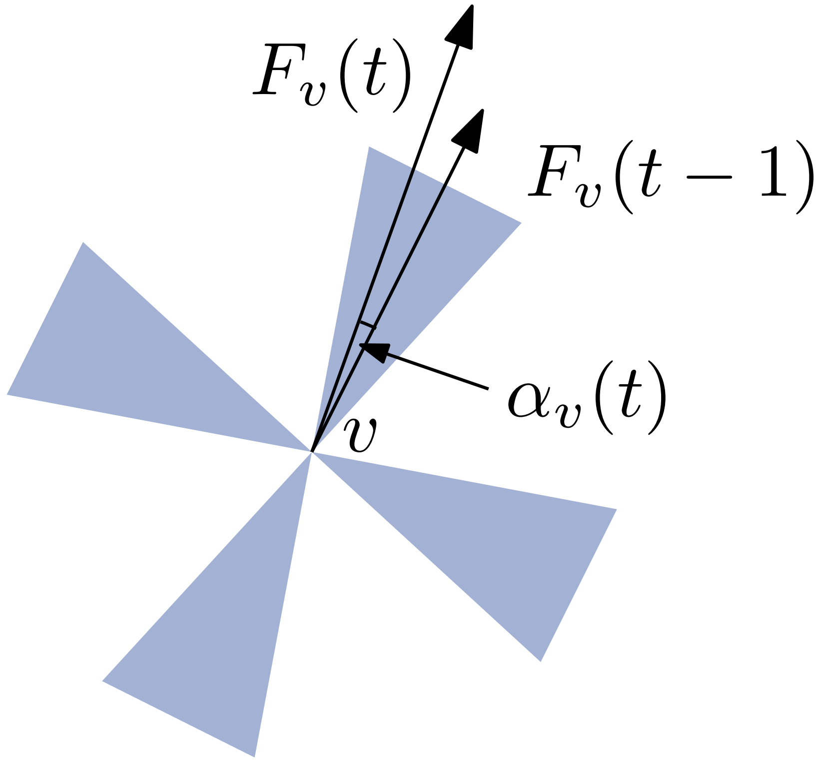

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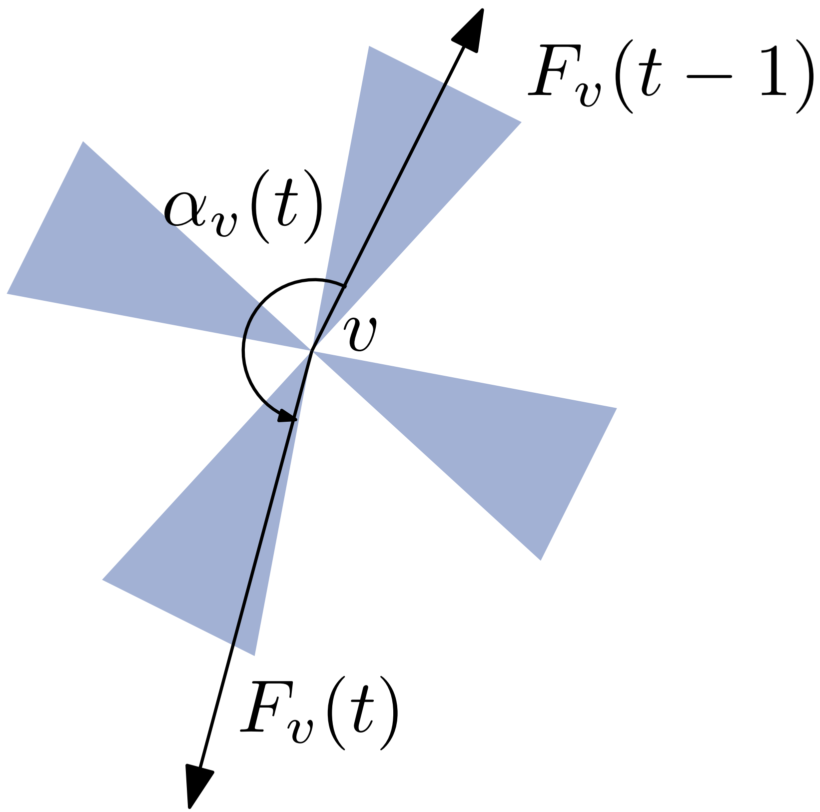

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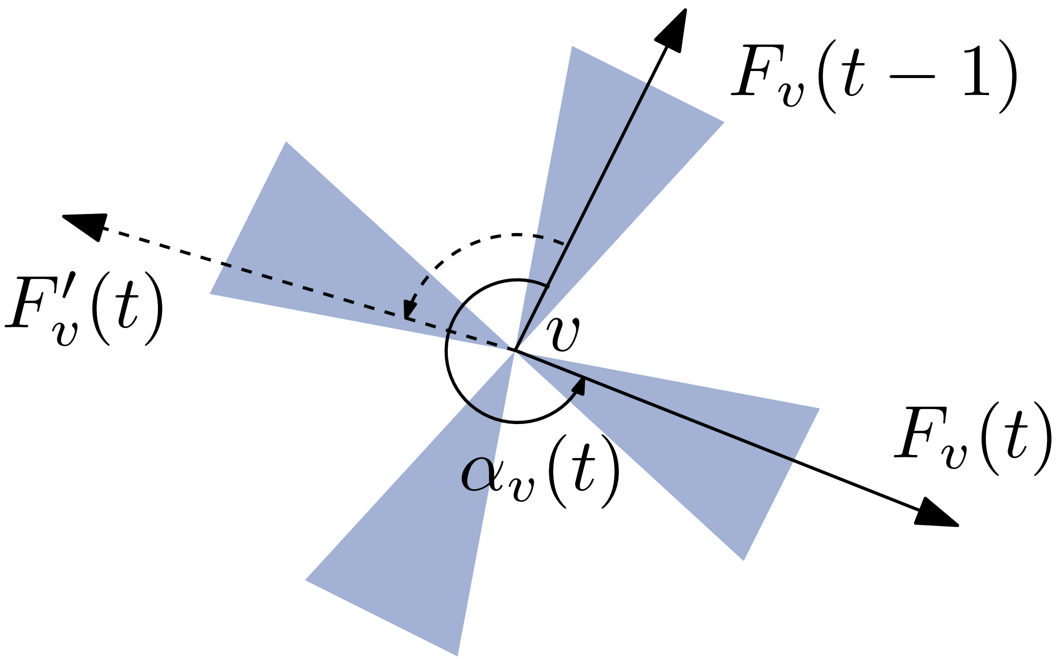

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  $\rightarrow$ reduce the temperature
- $\cos(\alpha_v(t)) \approx 0$:
  Rotation
  $\rightarrow$ update rotation counter and decrease temperature if necessary

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Advantages

- still very simple algorithm
- further modifications improve layout quality and lead to faster convergence

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Could we reduce this?

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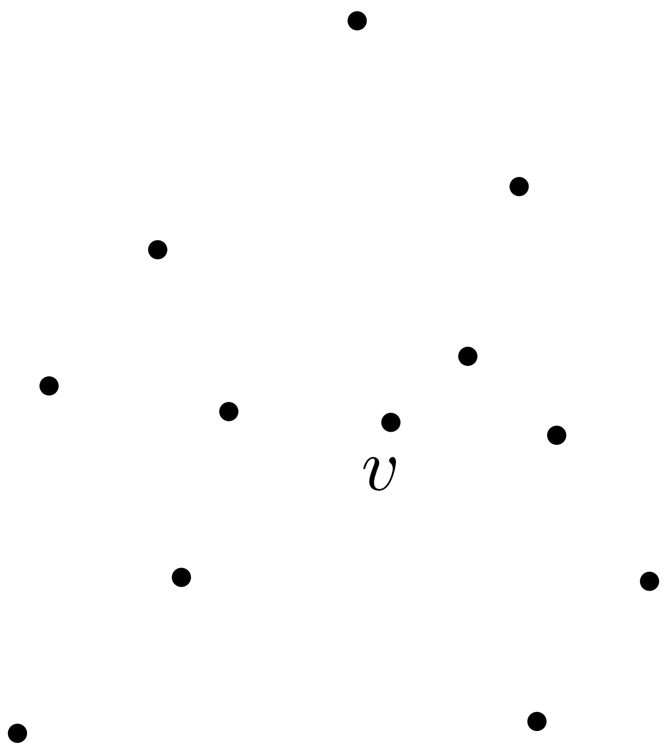

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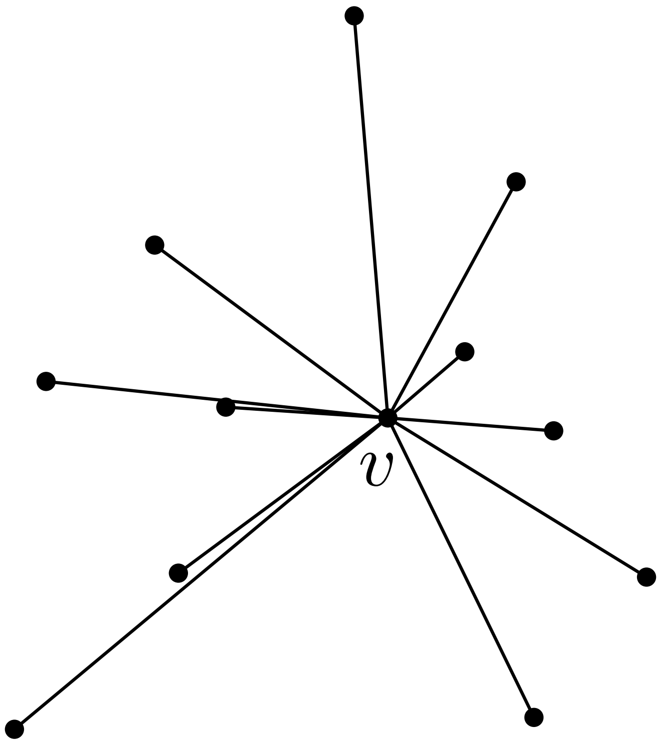

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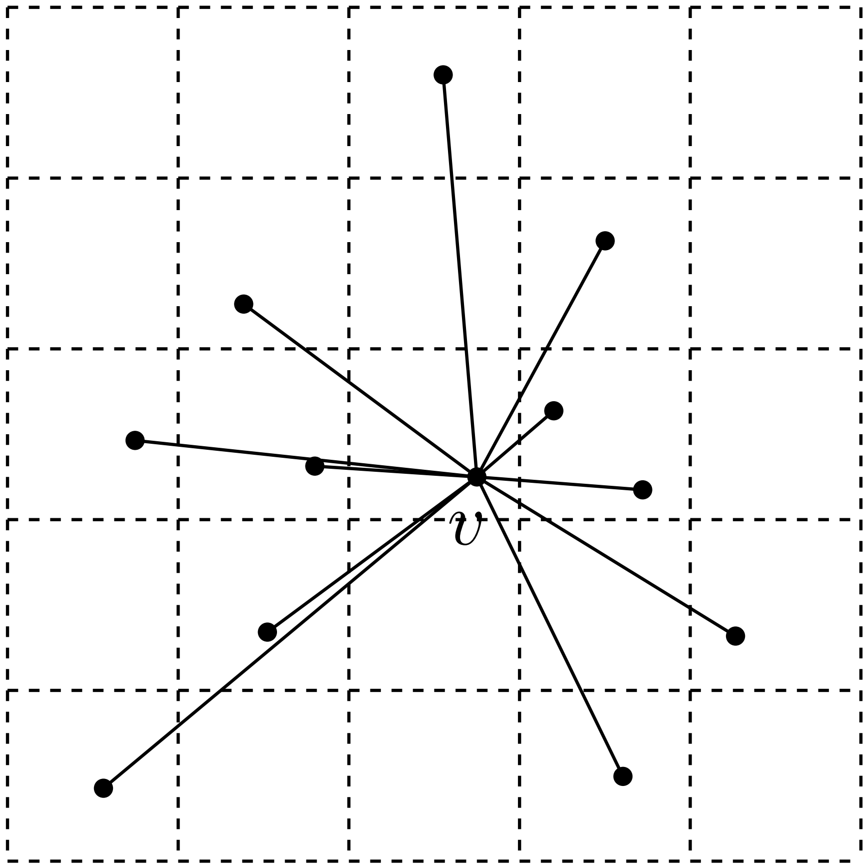

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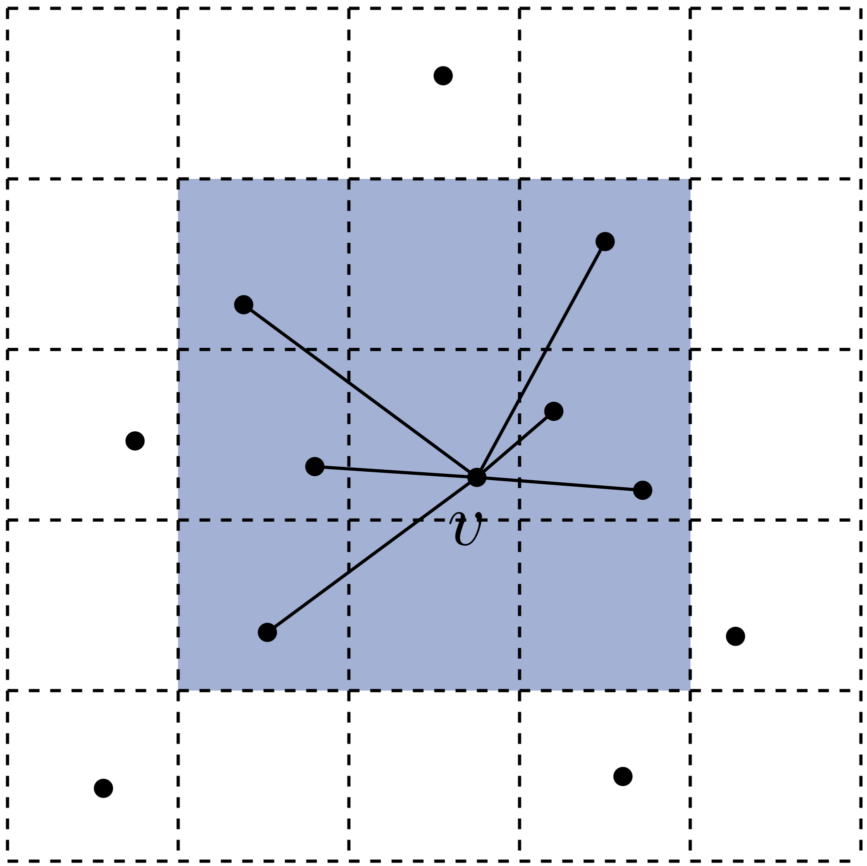

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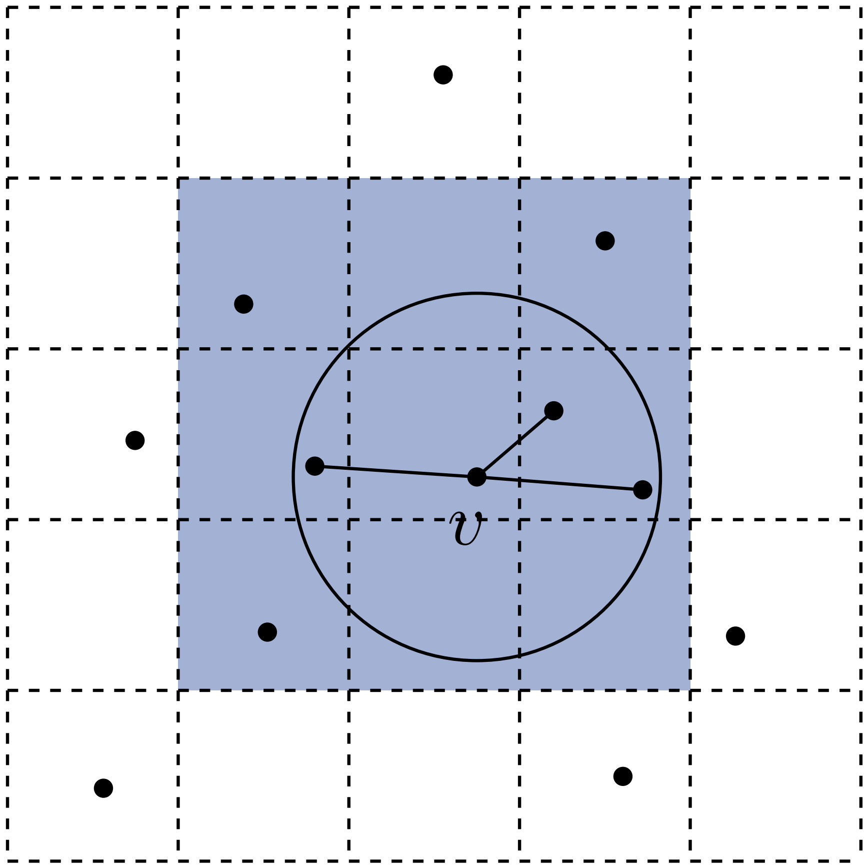

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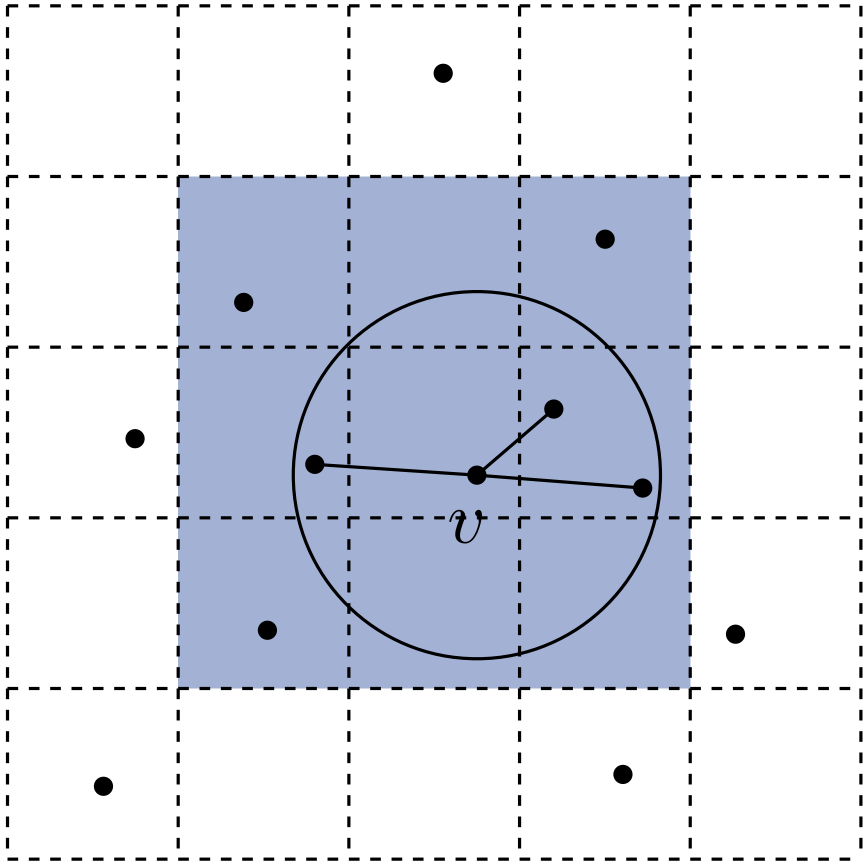

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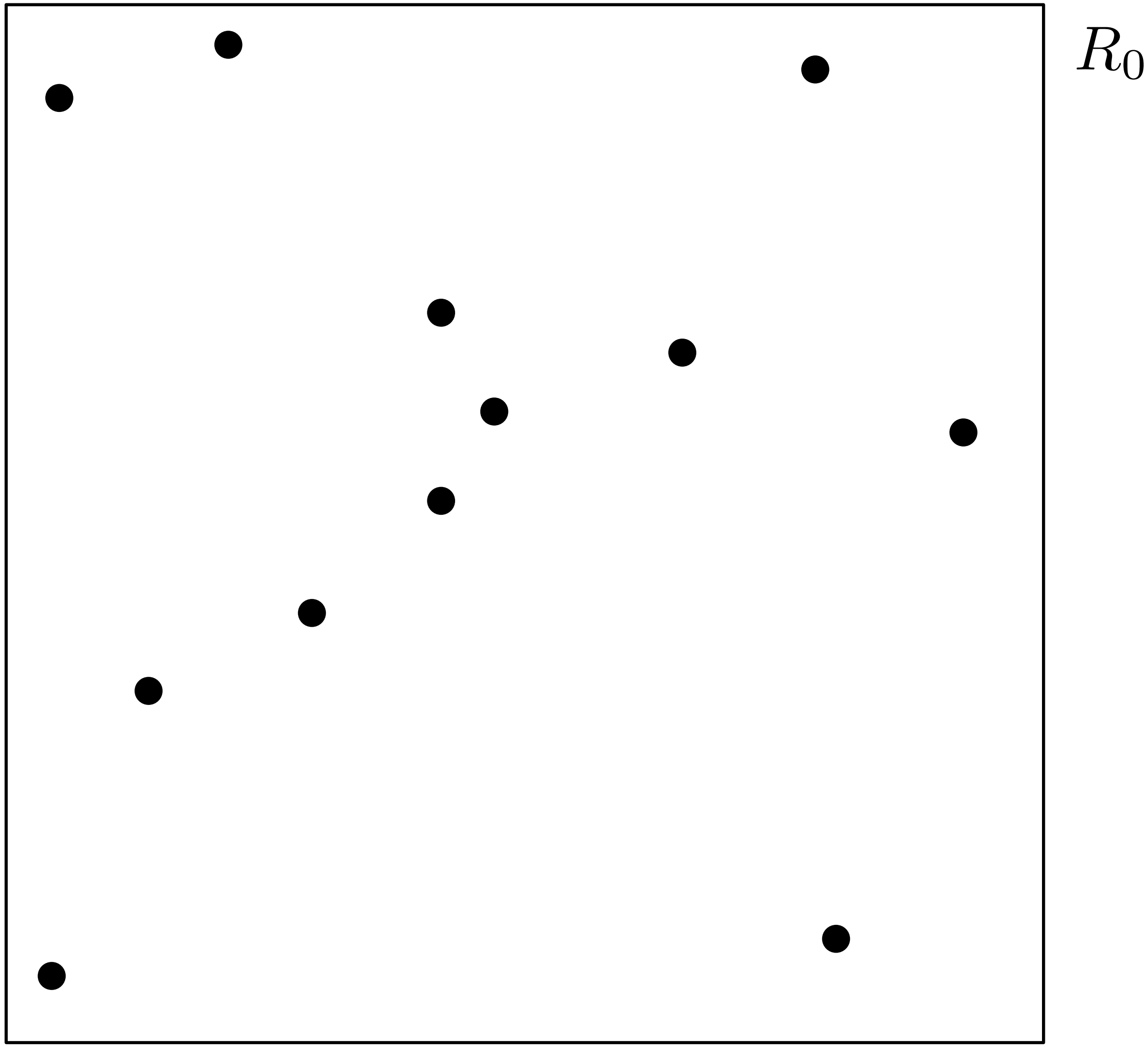
Discussion

- meaningful idea to improve runtime
- worst-case no advantage
- Quality loss

Quad-Tree



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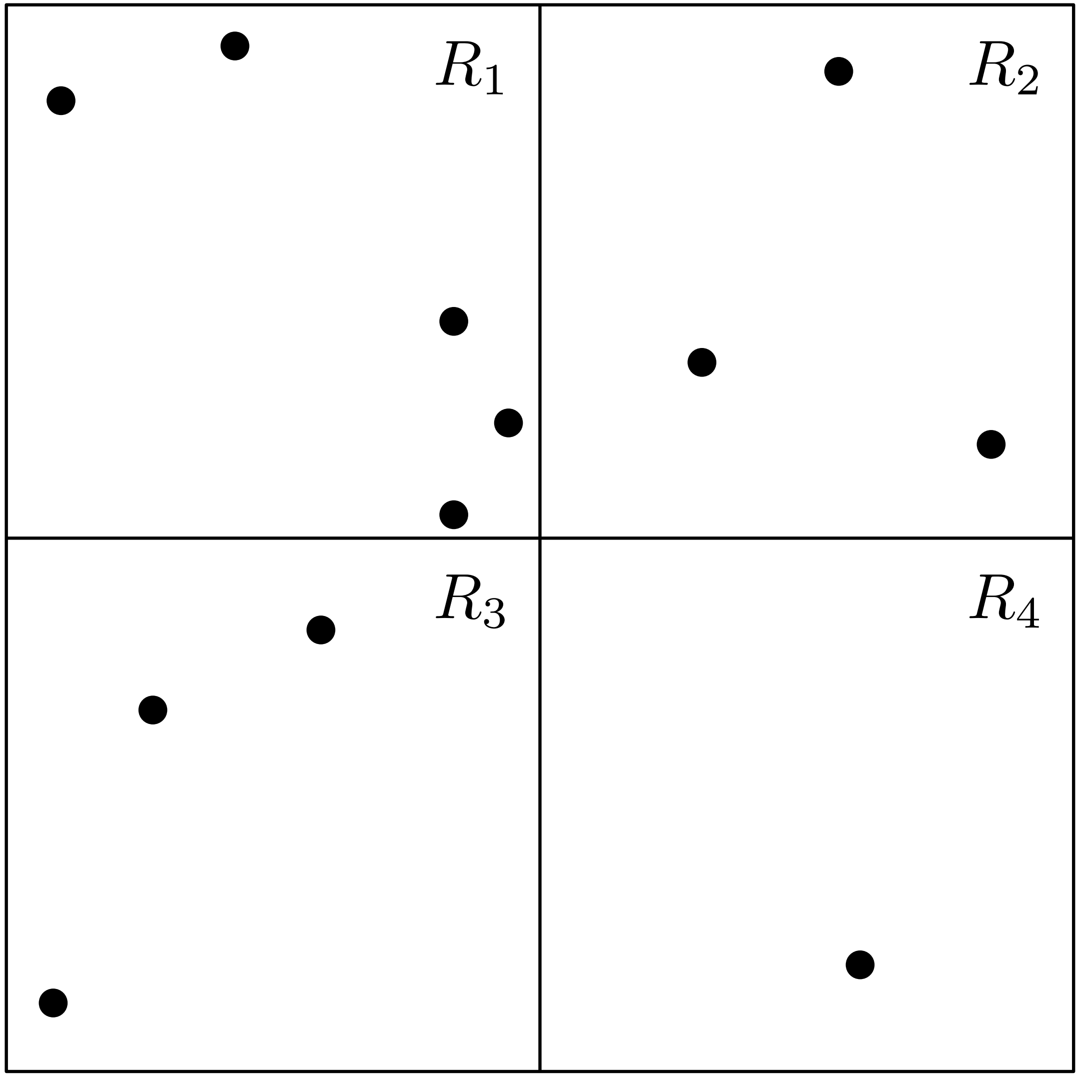

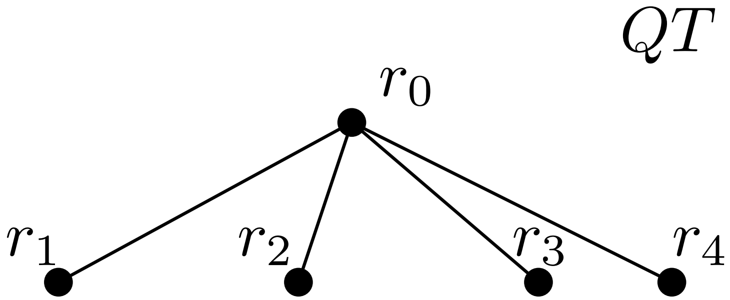

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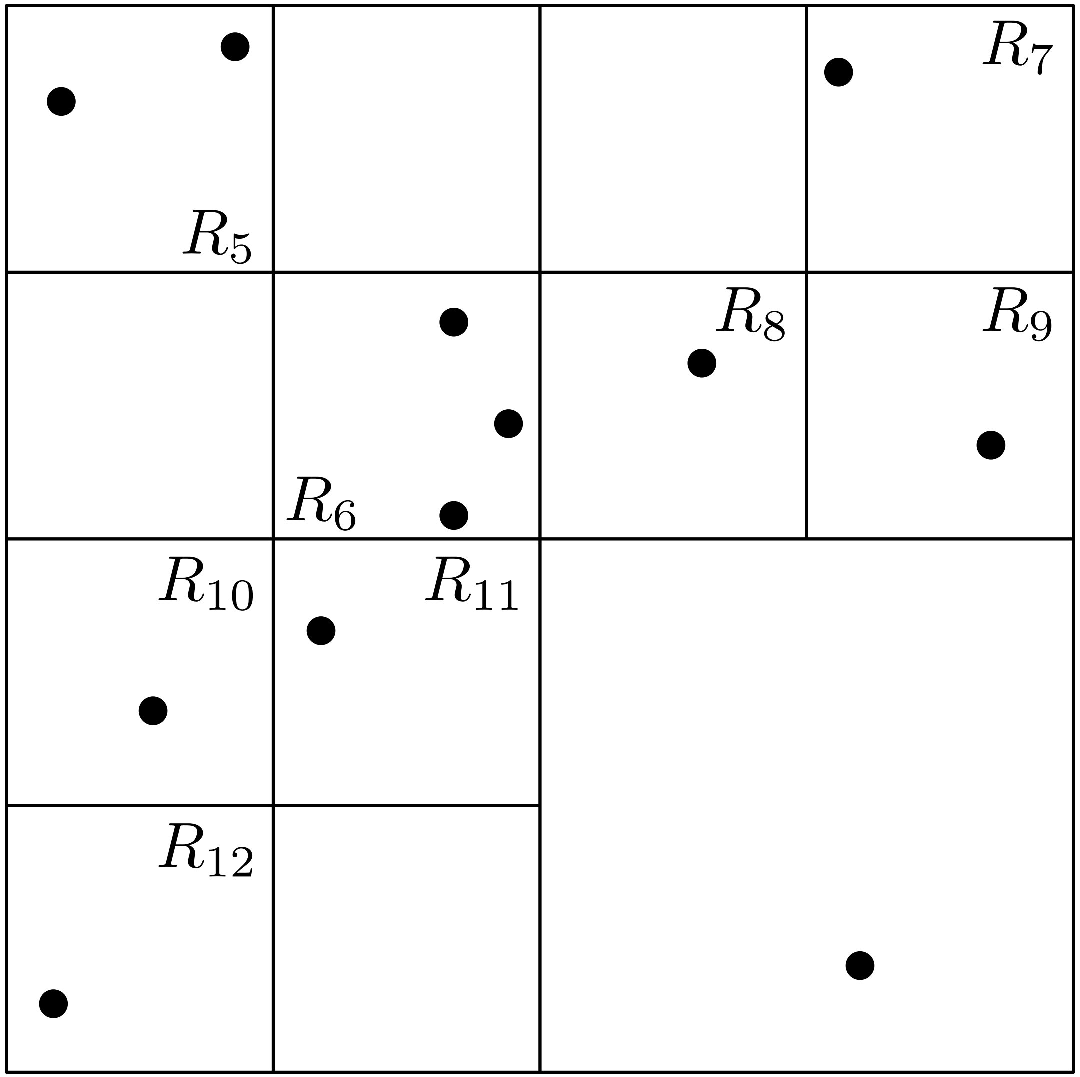

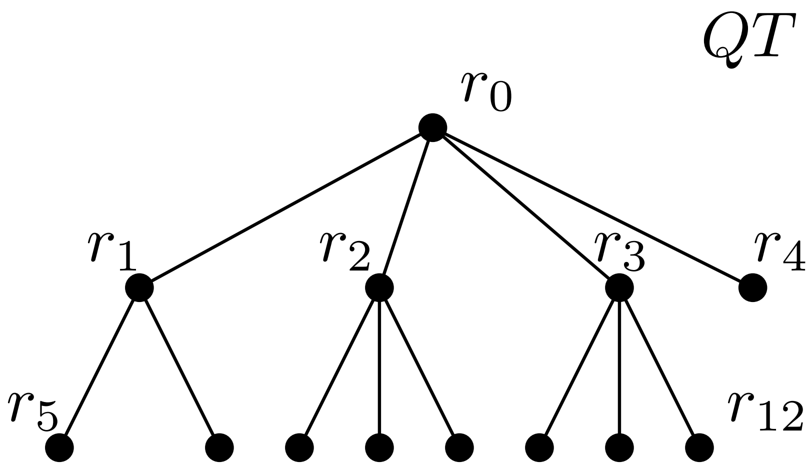

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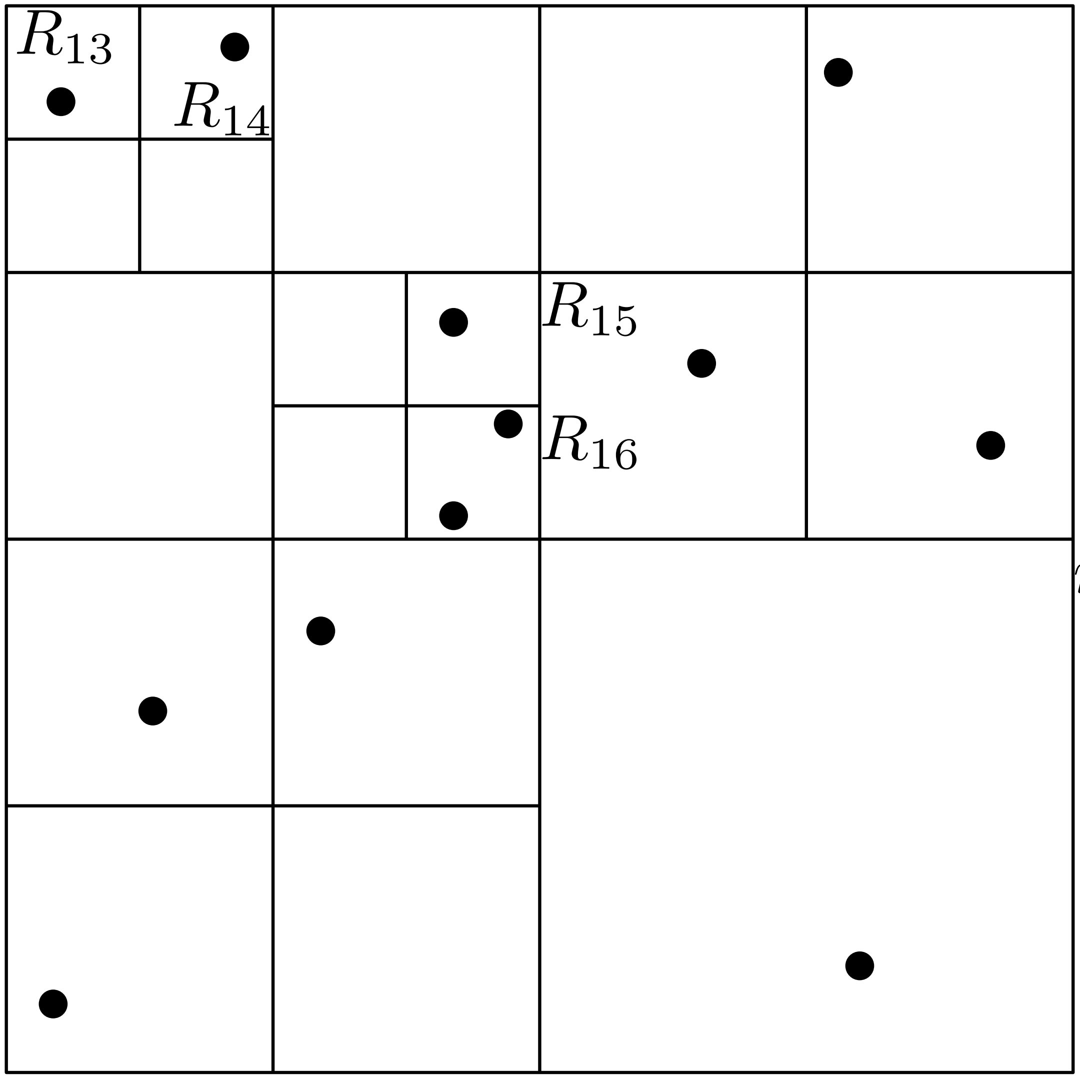

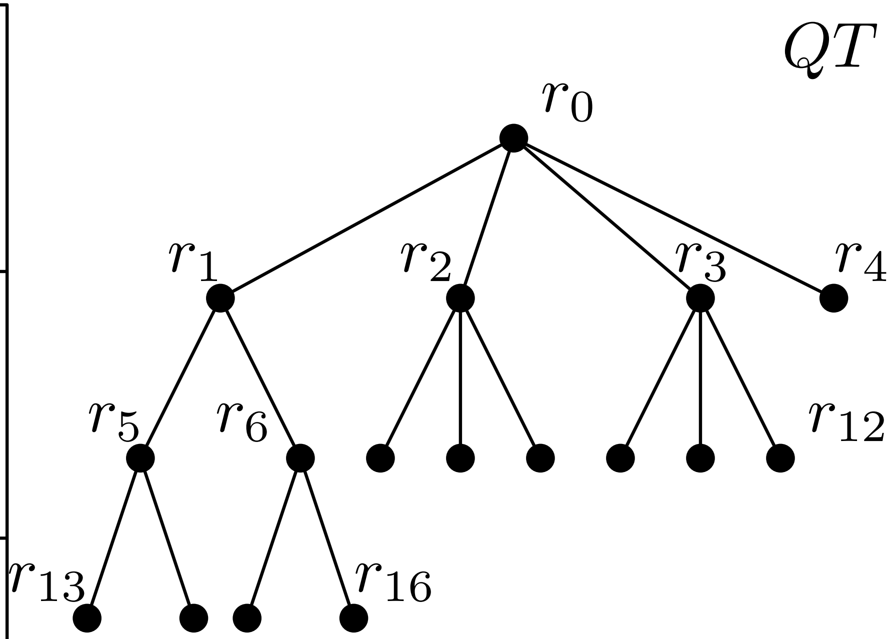

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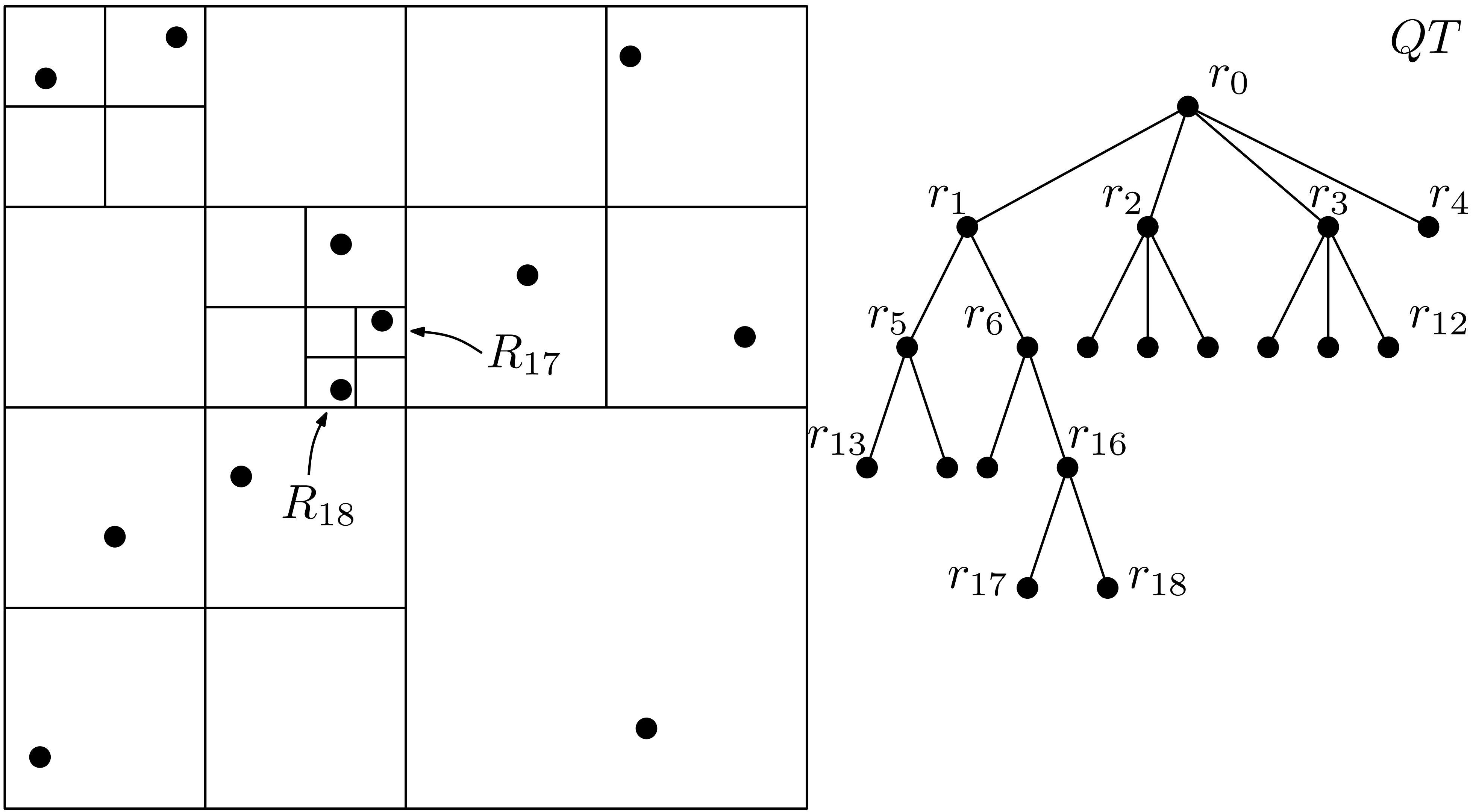
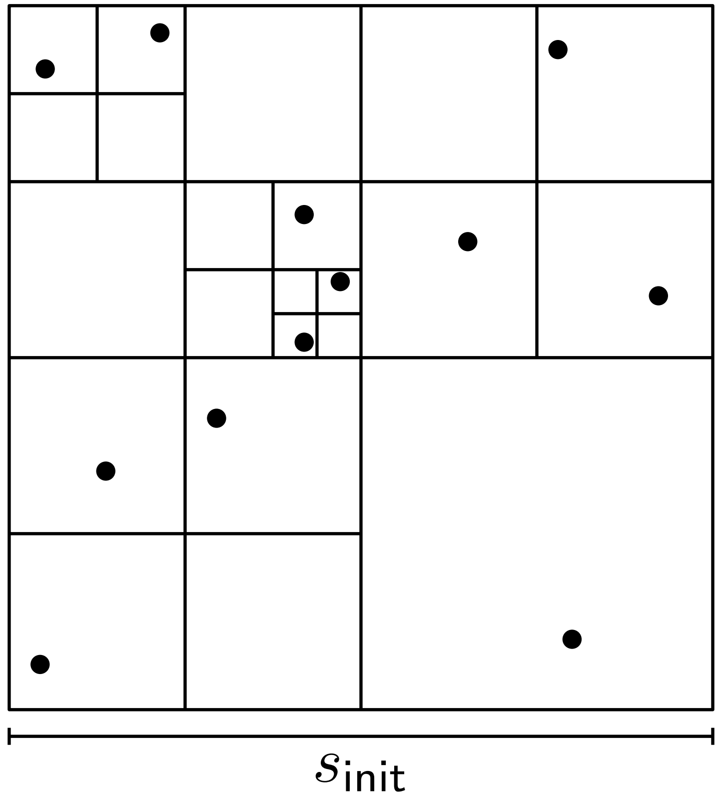
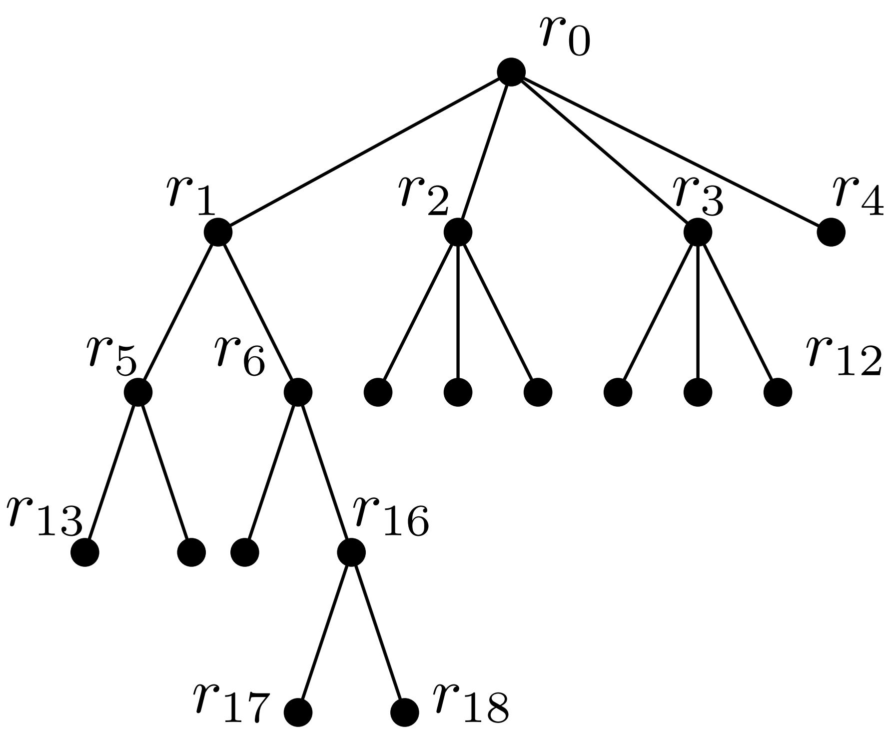


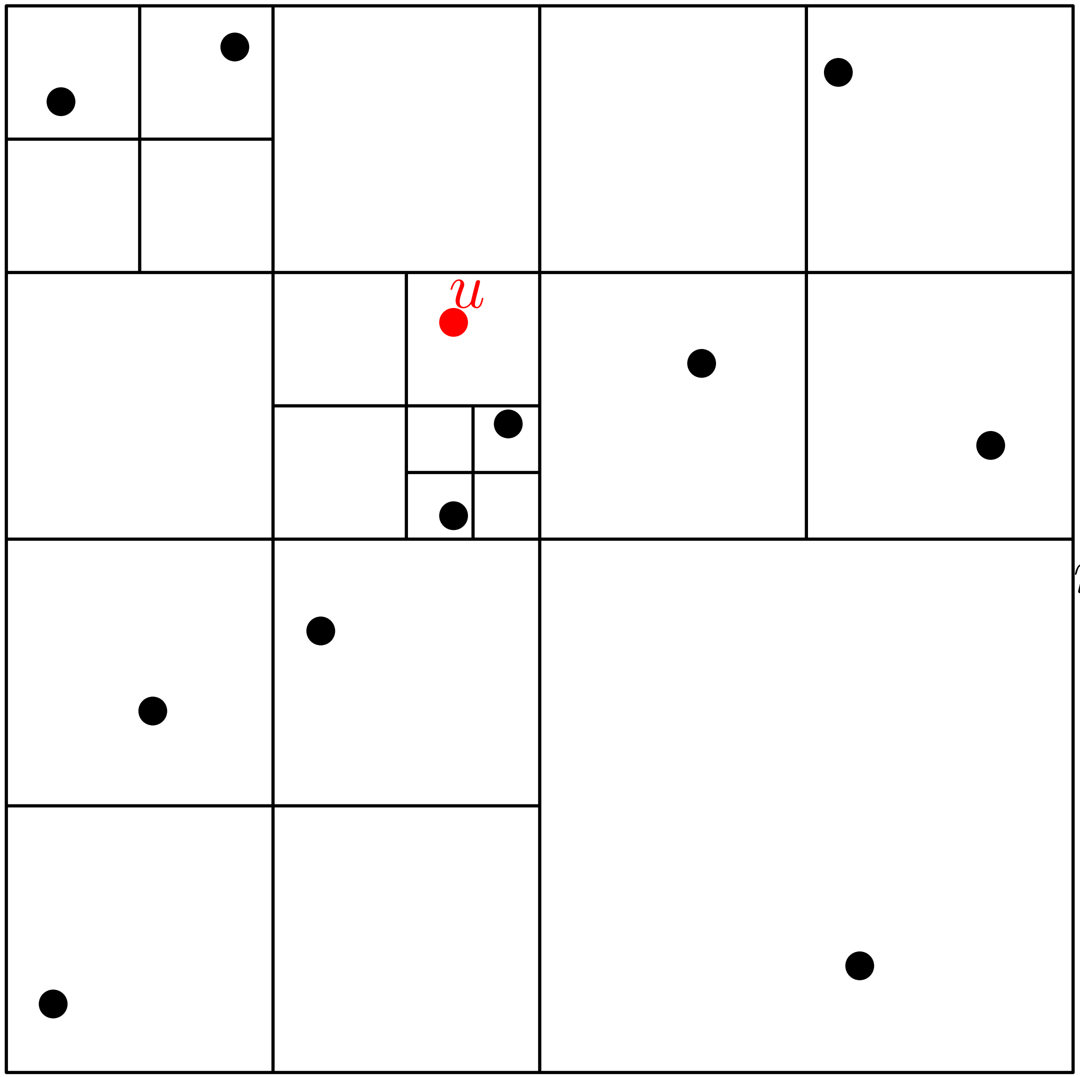
Properties of Quad-Tree

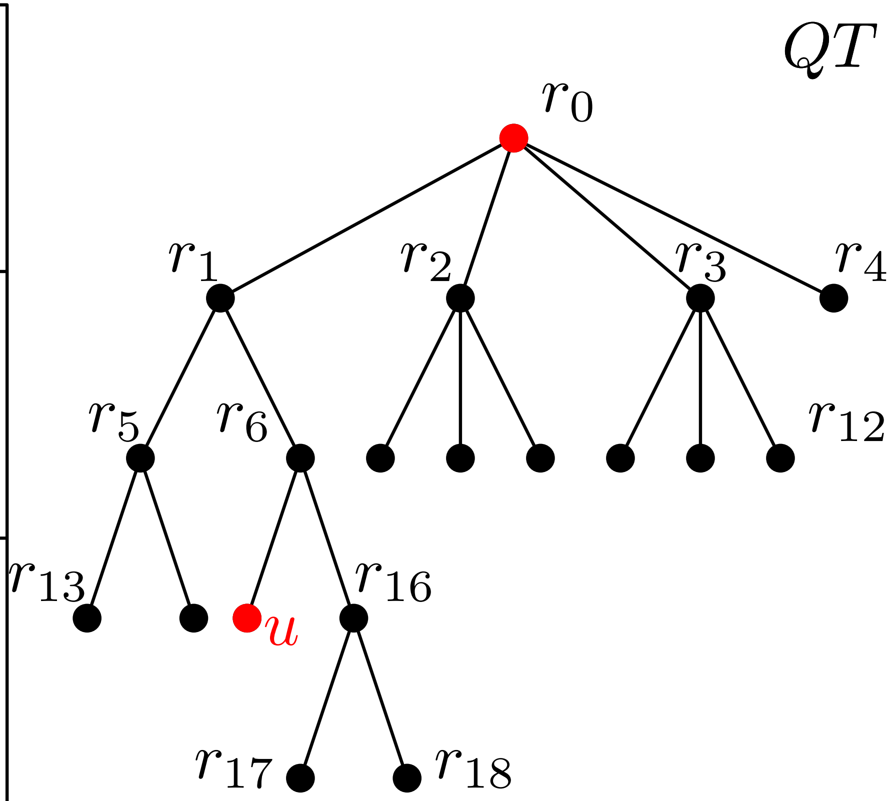
- height $h \leq \log \frac{s_{\text{init}}}{d_{\text{min}}} + \frac{3}{2}$, here d_{min}-smallest distance
- time and space $O(hn)$
- *compressed* quad-tree in $O(n \log n)$ time





Forces with Quad-Trees (Barnes, Hut, 1986)





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, where p_{R_i} - the barycentre of the points in R_i

Compute repulsive force between u and the barycentre of the vertices in the corresponding square

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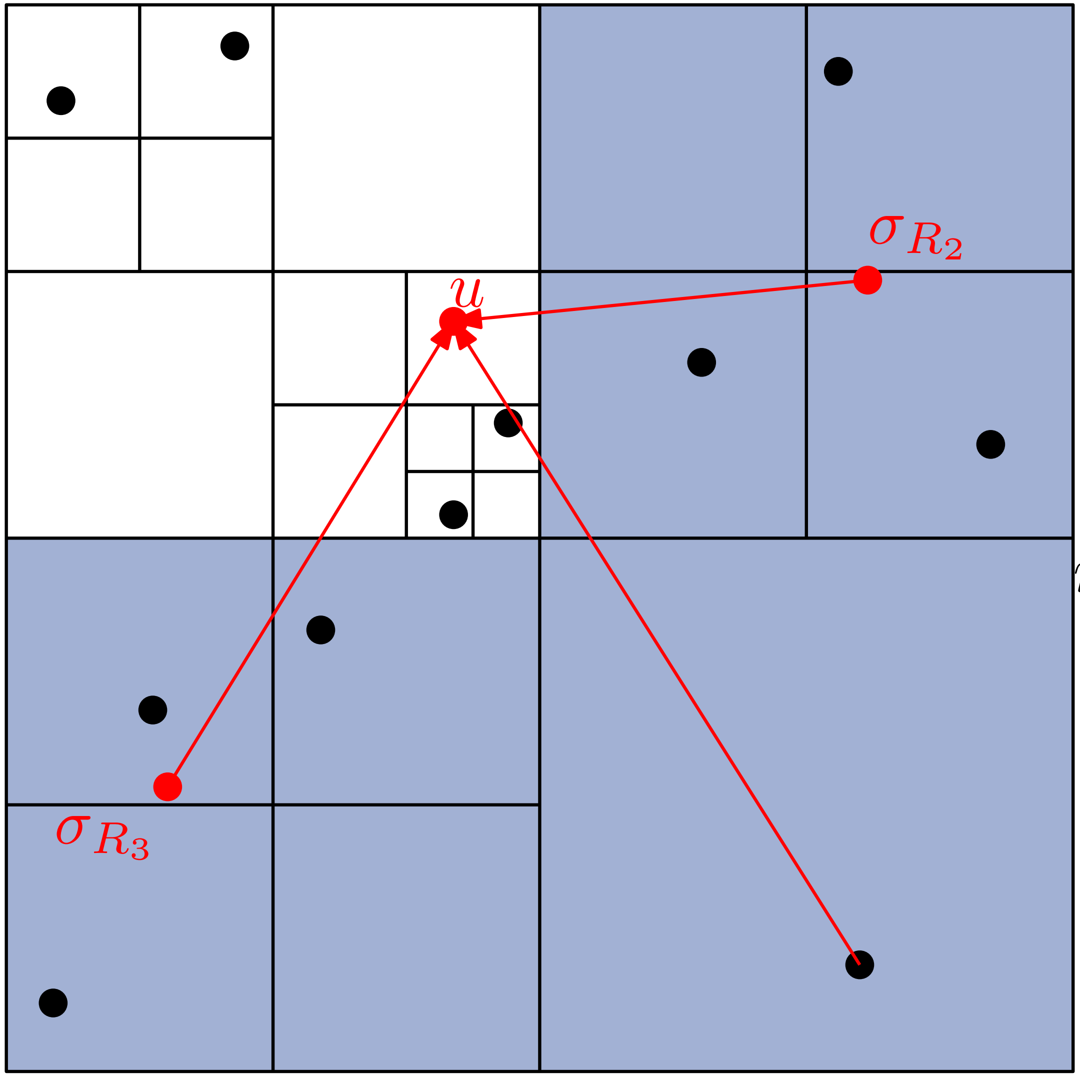

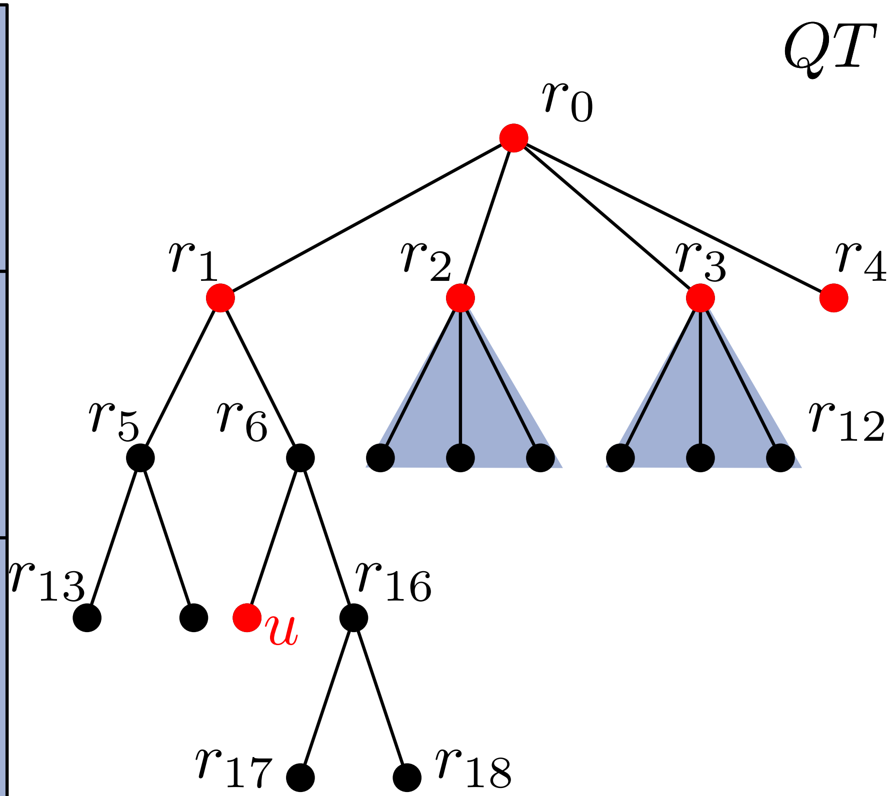

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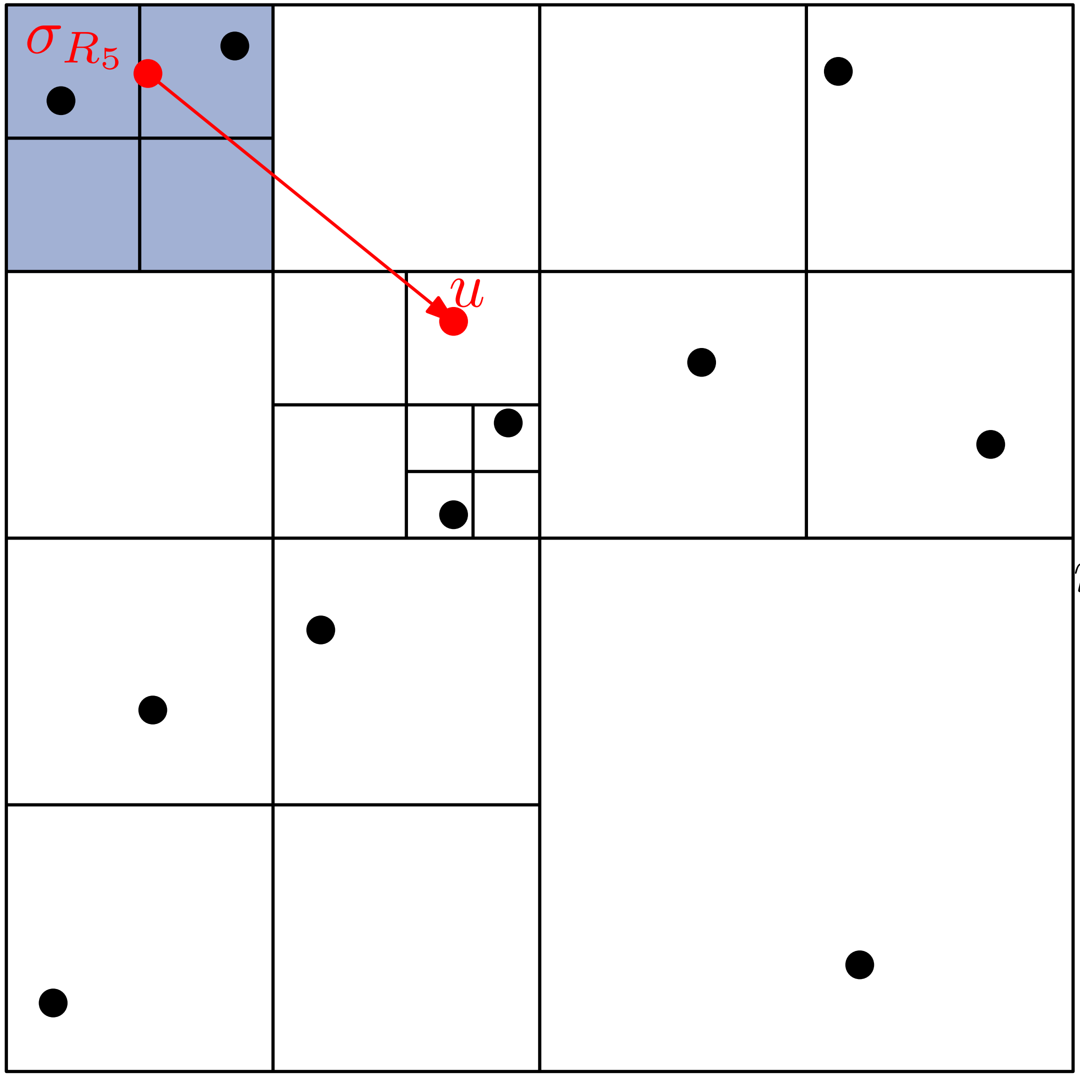

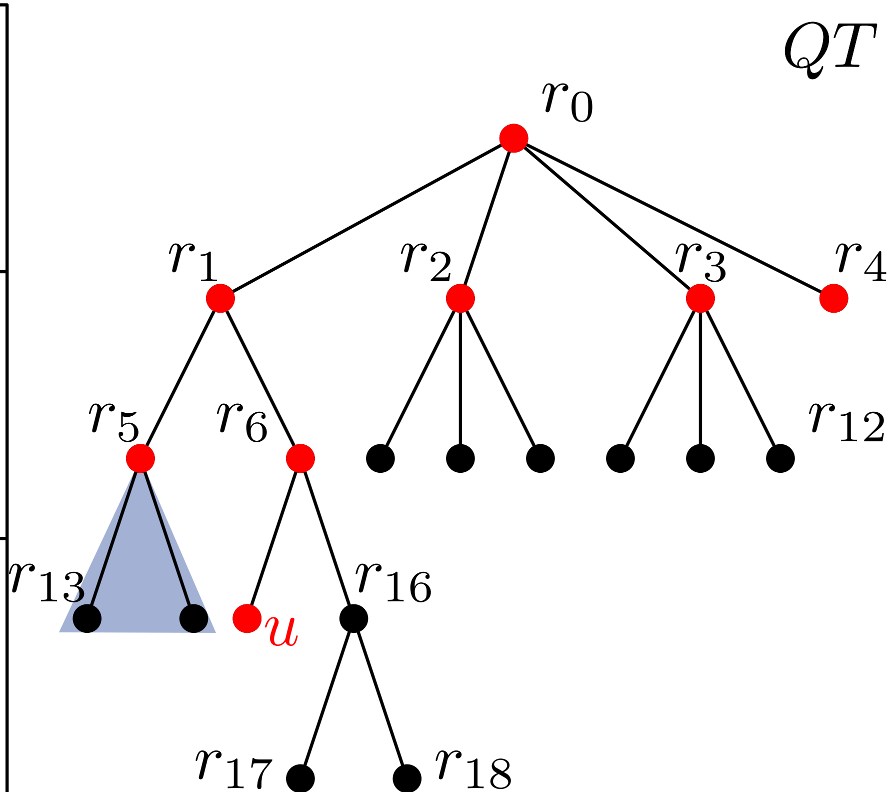

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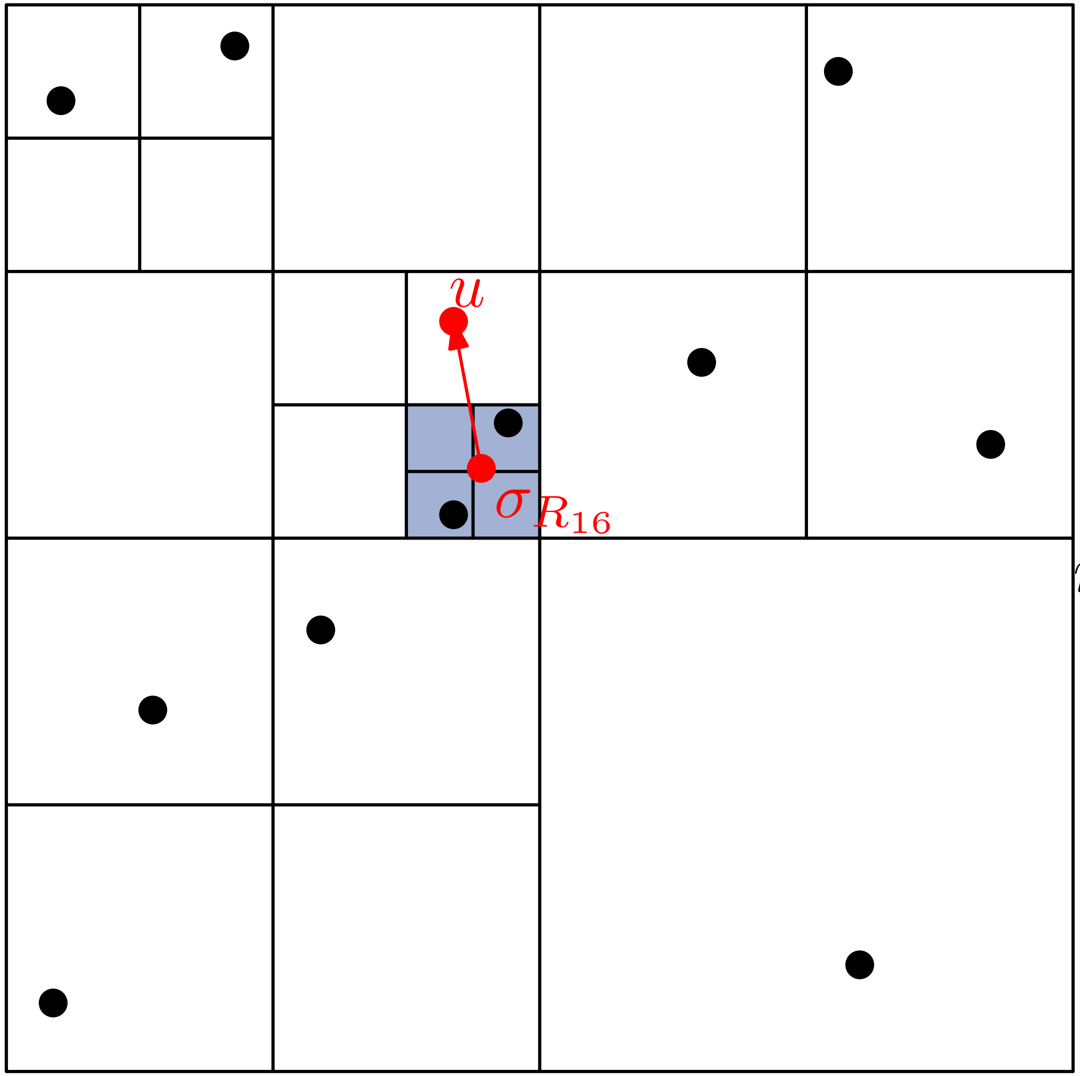

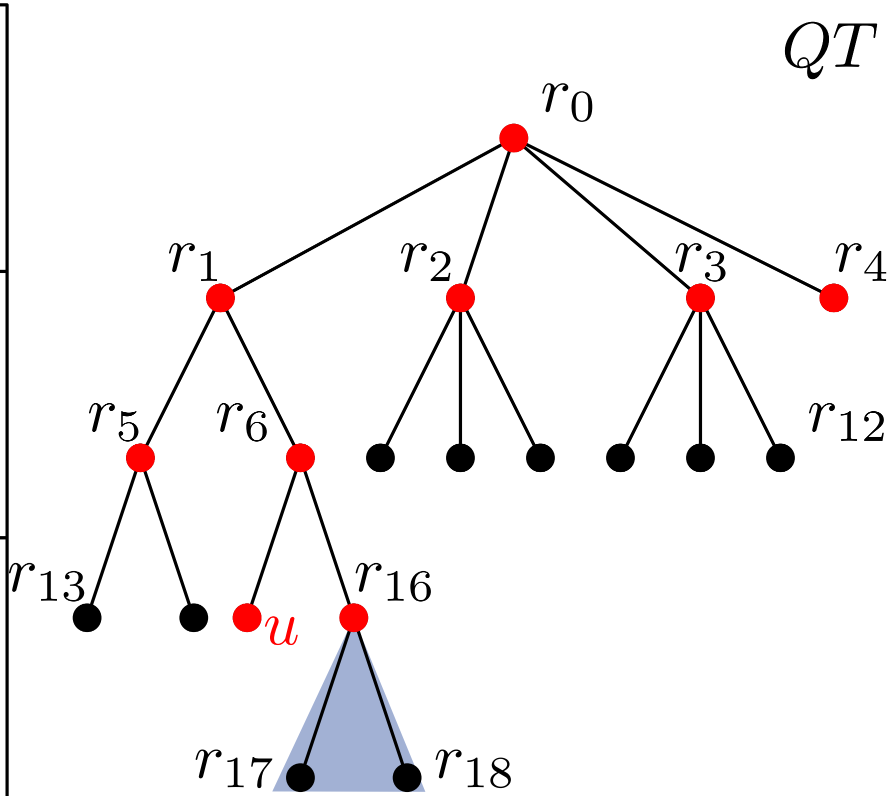

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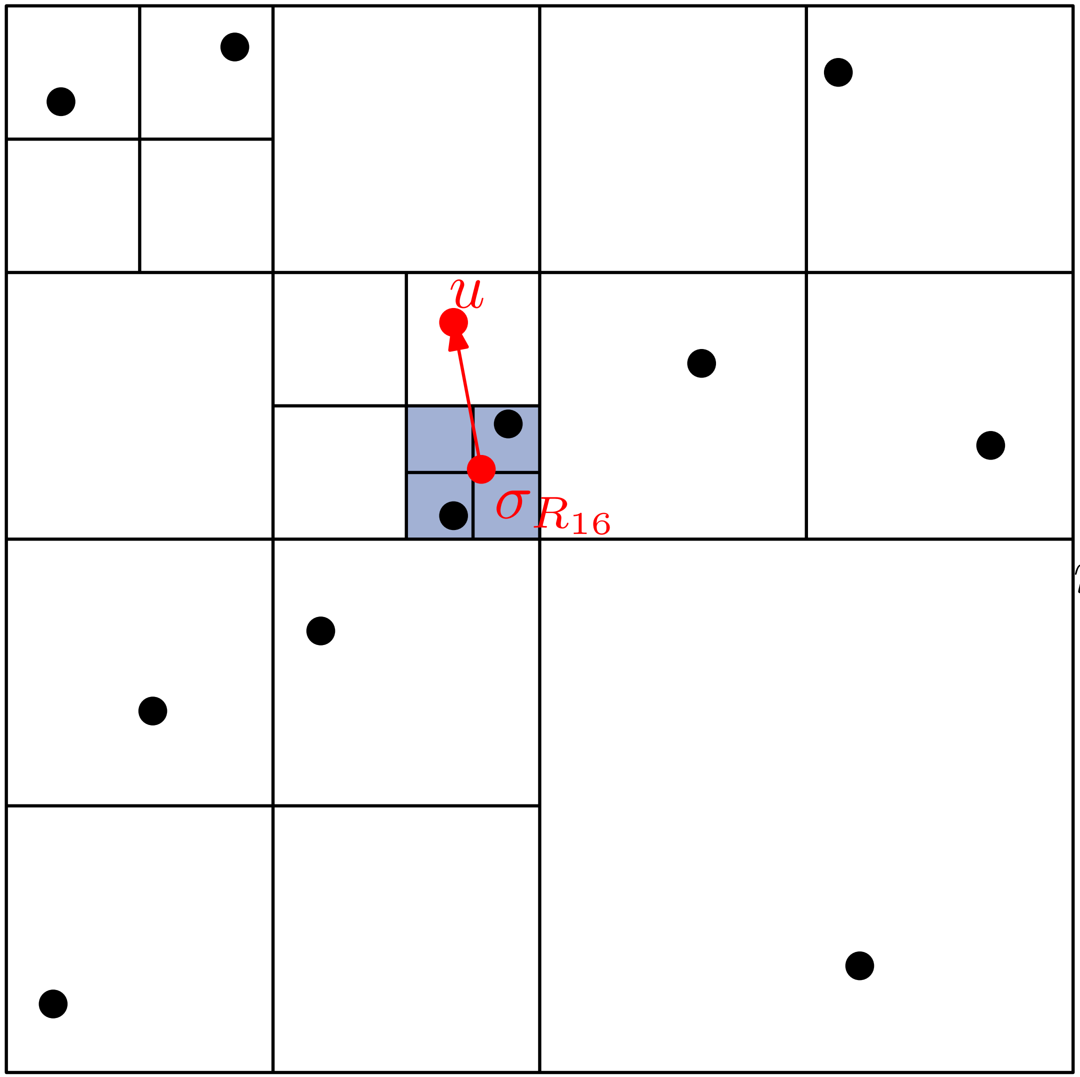

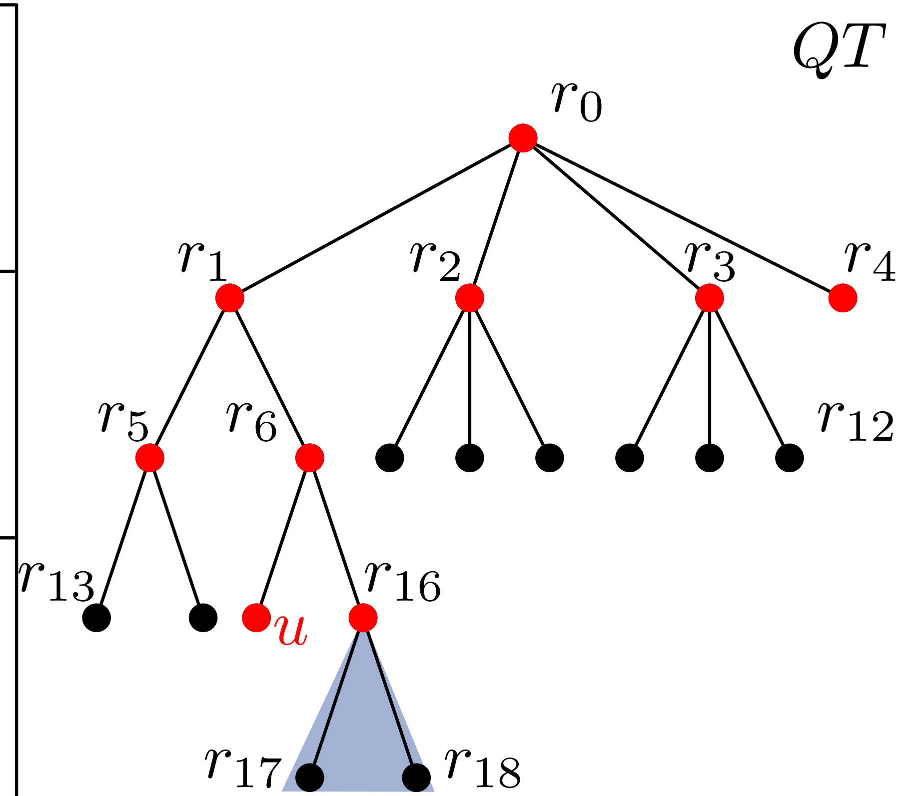

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Approximation control: if $\frac{\ell(R_i)}{dist(R_i,u)} > \gamma \rightarrow$ recurse in R_i

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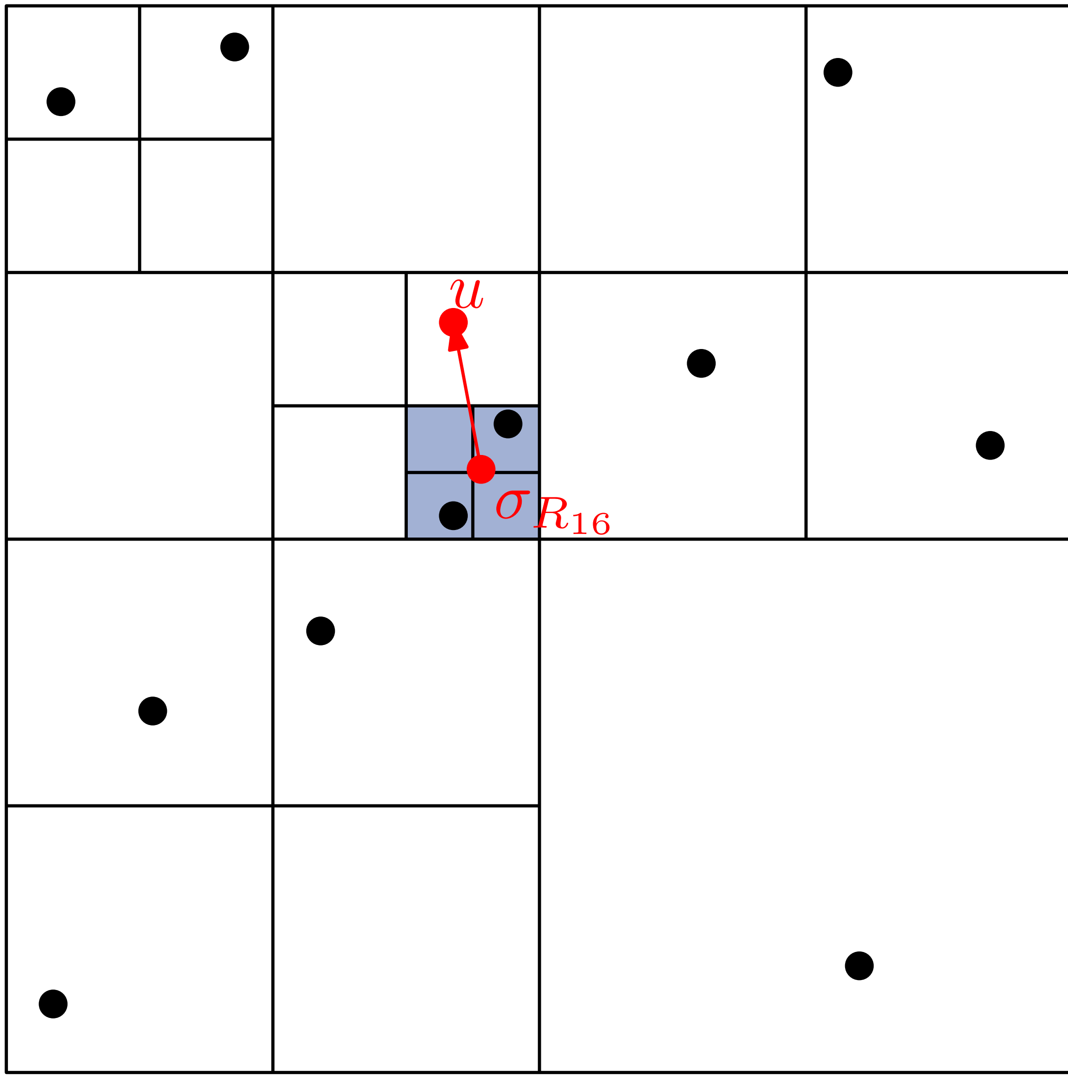

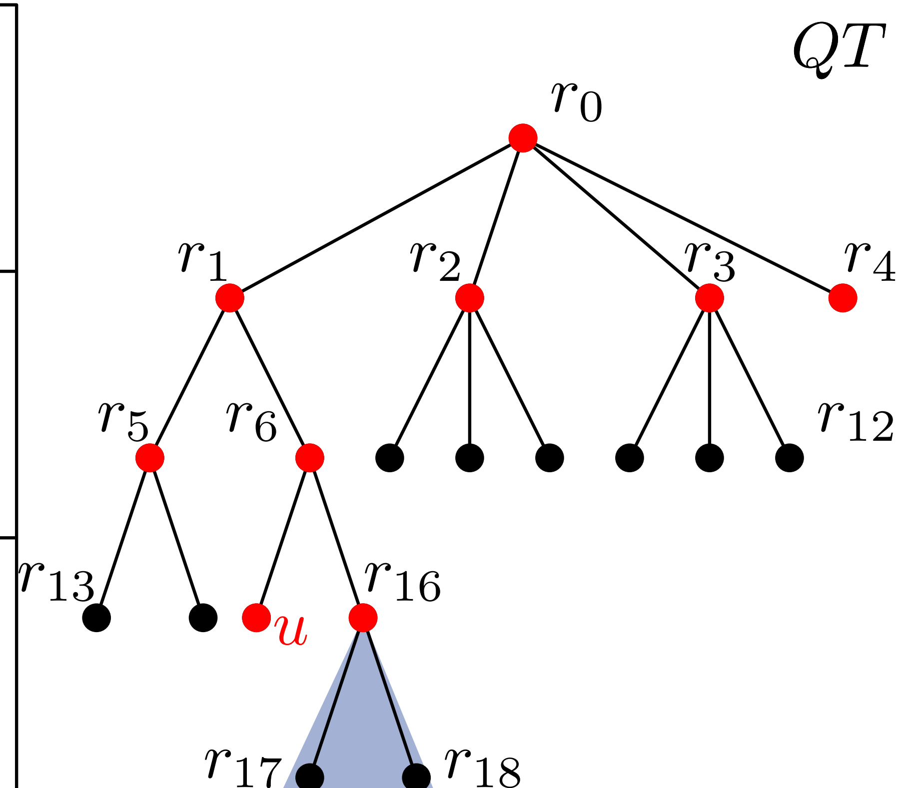

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- If $\frac{\ell(R_i)}{dist(R_i,u)} < \gamma$ always hold then we do $O(h)$ computations for u
- Computation time per iteration is $O(nh)$

Multilevel-Algorithms for Large Graphs

Motivation

- classical Spring-Embedder for large graphs are too slow
- scensitivity to random initialization of node positions

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GRIP – Graph dRawing with Intelligent Placement

(Gajer, Kobourov, 2004)

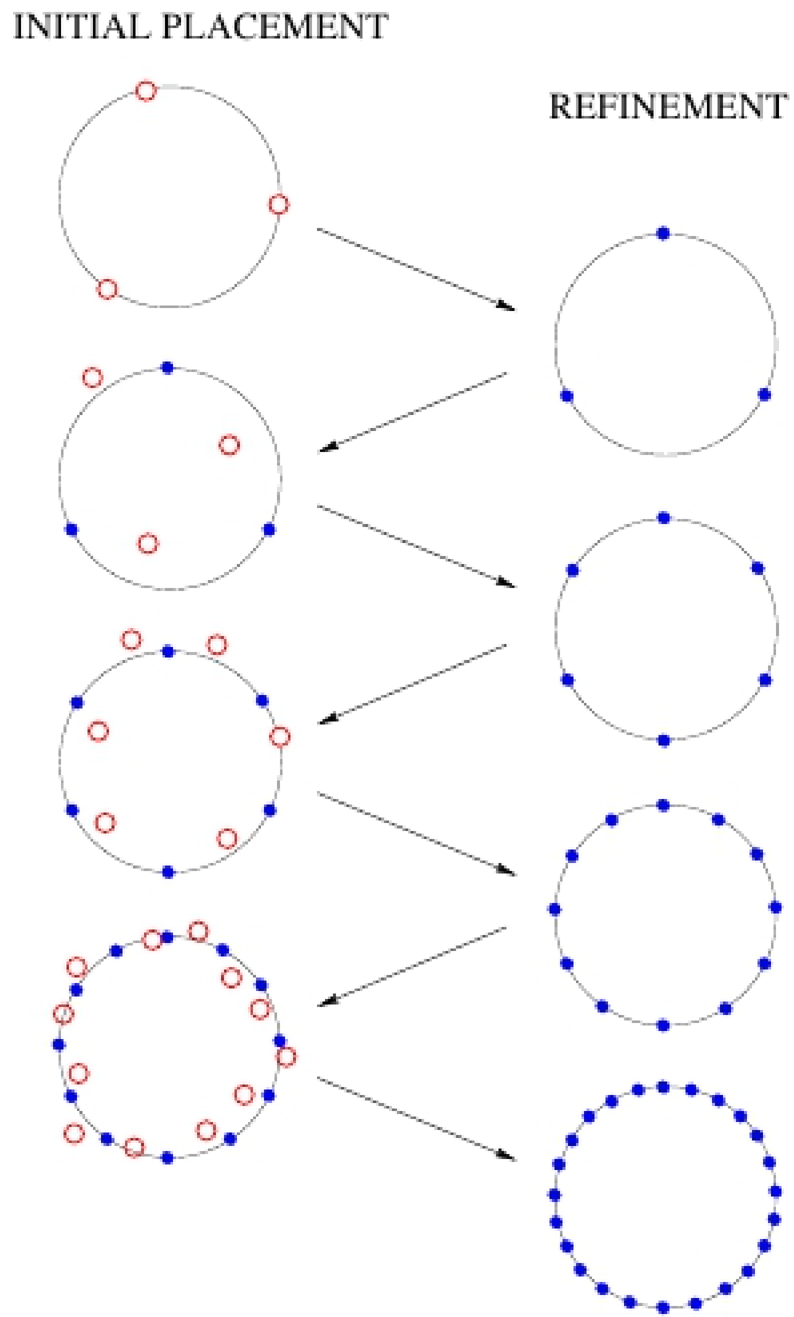
Approach

- top-down graph coarsening/filtration
- bottom-up calculation of the layout:
- clever placement of new nodes
- force-based refinement of their positions

GRIP Algorithm

Input: Graph $G = (V, E)$

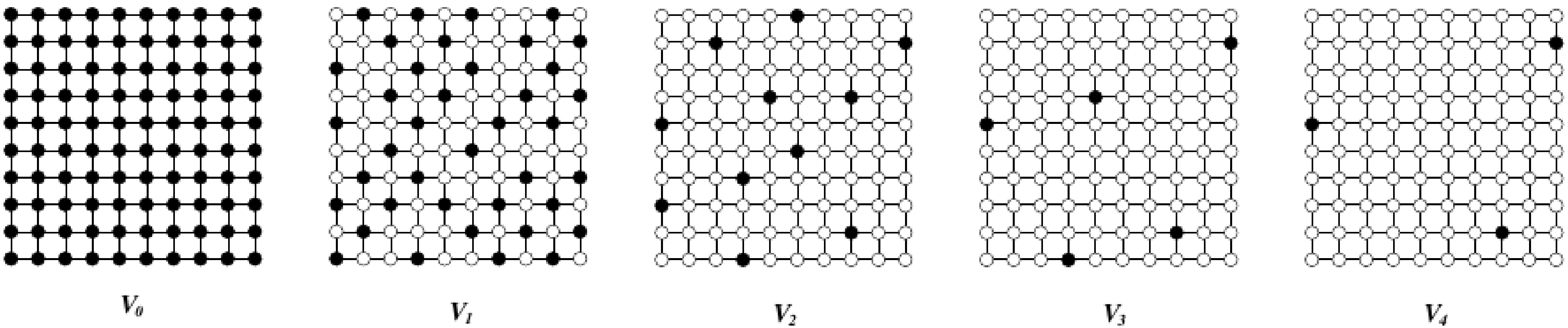
$\mathcal{V} \leftarrow$ Filtering $V = V_0 \supset V_1 \supset \cdots \supset V_k$
for $i = k$ **to** 0 **do**
  foreach $v \in V_i \setminus V_{i+1}$ **do**
    compute neighbours of v
    compute initial position of v
  for $j = 1$ **to** `rounds` **do**
    foreach $v \in V_i$ **do**
      force-based relaxation



MIS-filtering

Maximal Independent Set (MIS) filtering

- sequence of node sets $V = V_0 \supset V_1 \supset \cdots \supset V_k \supset \emptyset$
- V_i is (inclusion-) maximal set of nodes, so that
- distance in G between nodes in V_i is $\geq 2^{i-1} + 1$
- good balance betwen size of a level and depth of decompositon



Computing MIS-Filtering

Algorithm

- incremental procedure: given V_i compute V_{i+1}
- select random element v in V_i
- remove all elements from V_i with distance $\leq 2^i$ to v
- use BFS from v with search depth 2^i
- until no further node remains in V_i

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Depth of Filtration

- for the last level k it holds that $2^k > \text{diam}(G)$
- therefore the depth is $O(\log \text{diam}(G))$

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Alternative

- coarsening with contracting matchings: end vertices of each matching edge are contracted (Walshaw, JGAA 2003)

Level-based Node Placement

Step 1

- for each node $v \in V_i \setminus V_{i+1}$ find optimal position with respect to three adjacent nodes V_{i+1}

Step 2

- perform force-based refinement of V_i, where forces are computed locally only to a constant number of nearest neighbors in V_i

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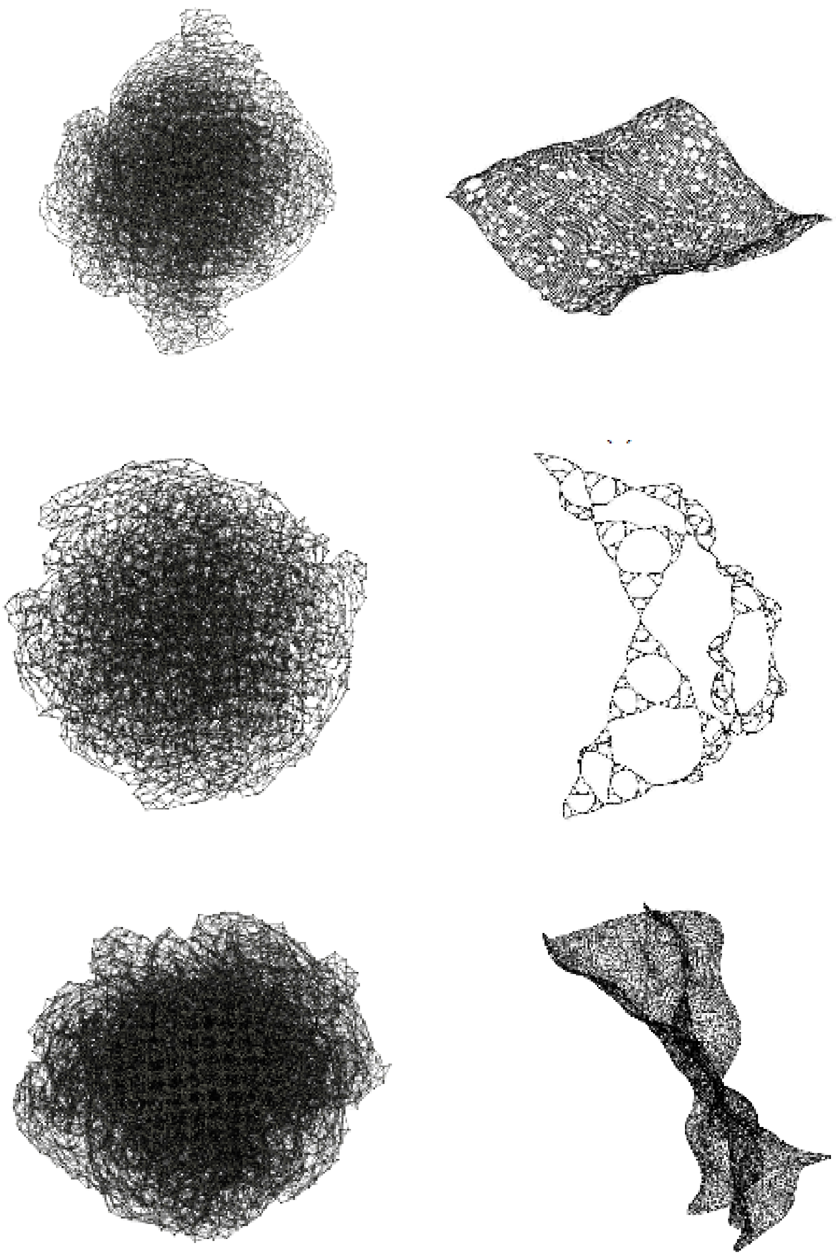
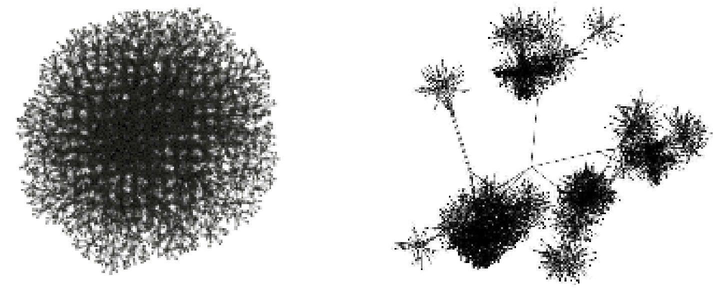
Properties of GRIP

- intelligent step-by-step calculation starting from a good layout and using MIS-filtering
- significantly faster convergence
- graphs with $> 10{,}000$ nodes in seconds (2004)

Experimens (Hachul, Jünger 2007)



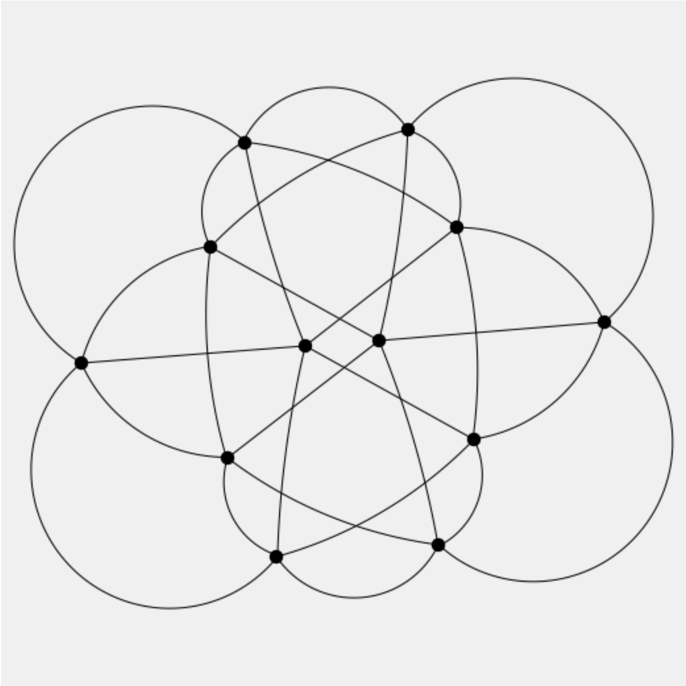


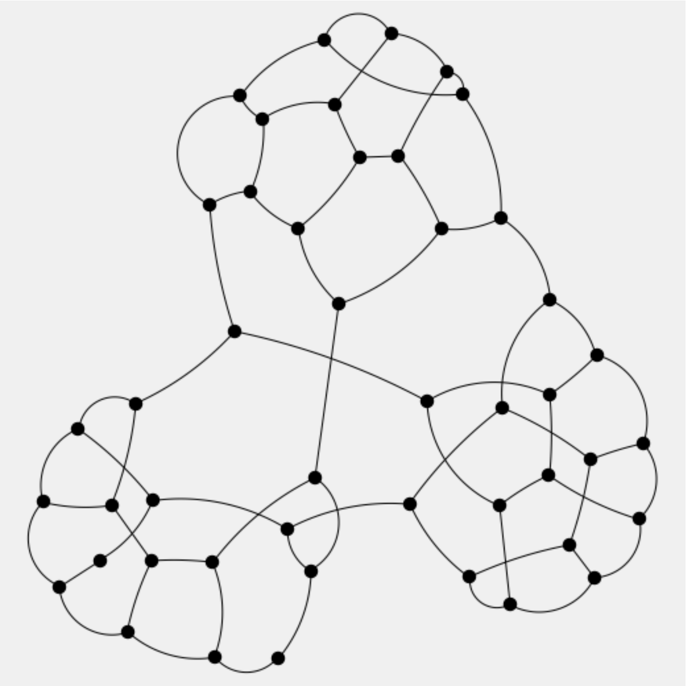
Left: Grid version of Fruchterman Reingold.
Right: GRIP

Other Examples

Lombardi-Spring-Embedder (Chernobelskiy et al. 2012)

- edges are circular arcs
- goal: optimal angular resolution $2\pi/\deg(v)$ at each node v
- additional rotational forces





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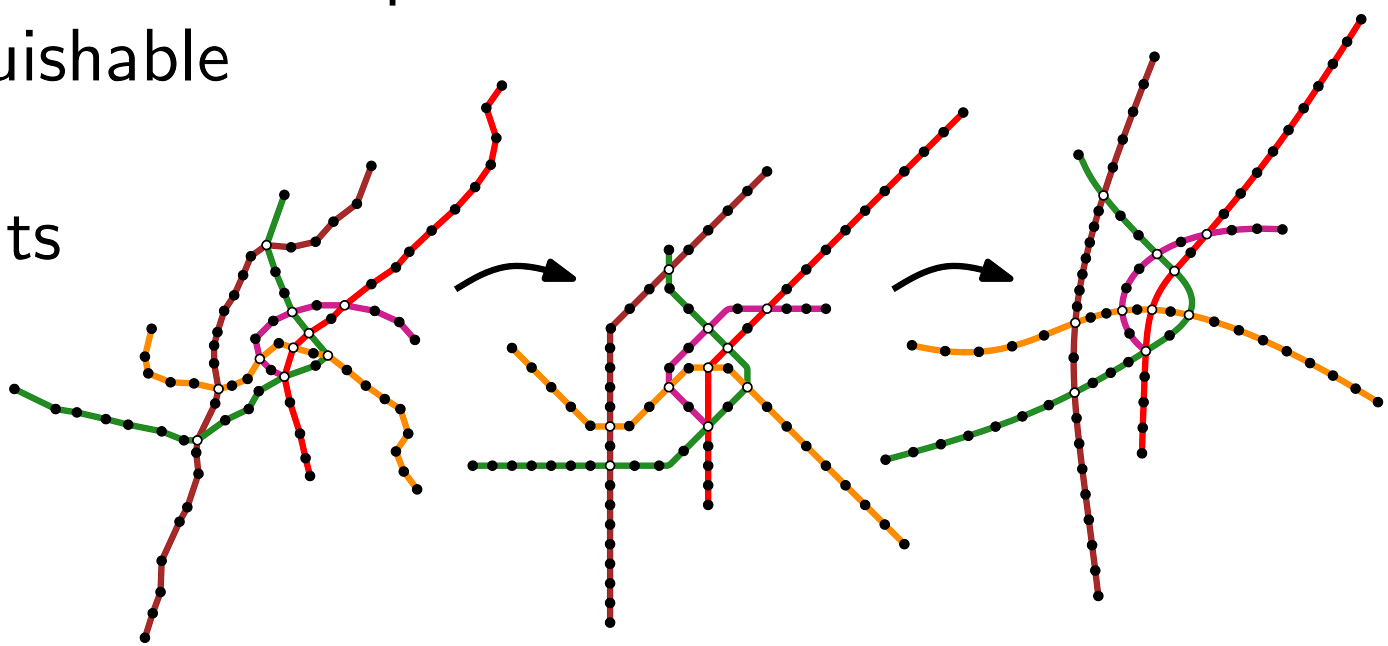
Metro Maps with Bézier curves (Fink et al. 2013)

- model paths as Bézier curves
- forces on nodes and control points:
- lines are distinguishable
- few bend points
- few control points



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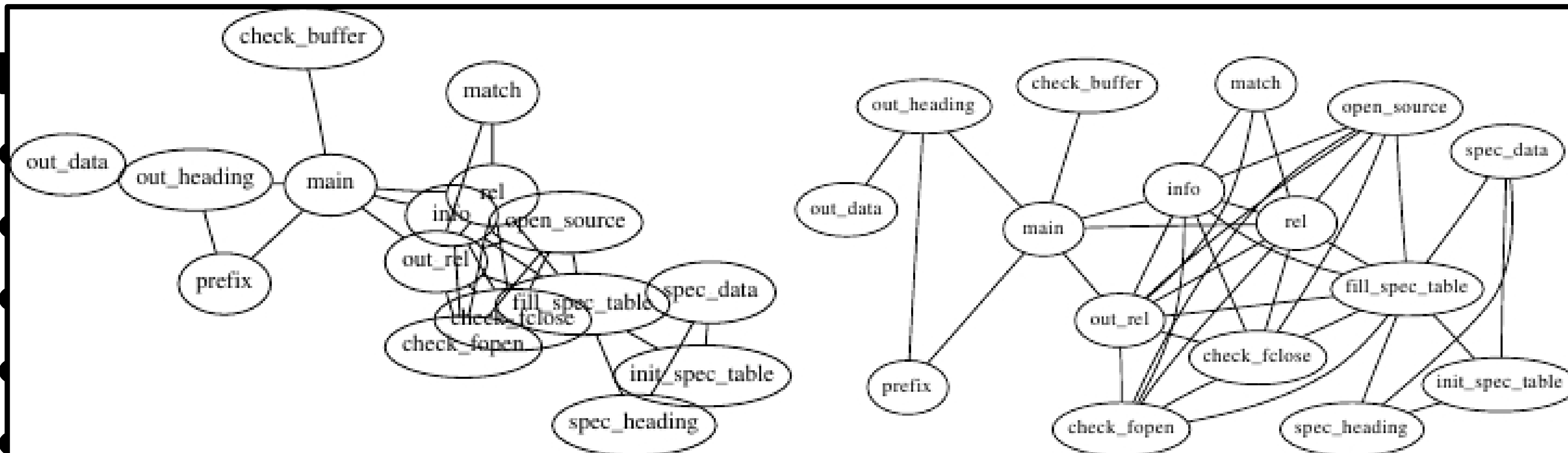

Realistic Node Sizes (Gansner, North 1998)

- node positions are adjusted to avoid overlaps

Summary

Force-based Approaches are

- easily understandable and implementable
- no special requirements on the input graph
- depending on the graphs (small and sparce) amazingly good layouts (Symmetries, Clustering, ...)
- easily adaptable and configurable
- robust
- scalable

Summary

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- easily understandable and implementable
- no special requirements on the input graph
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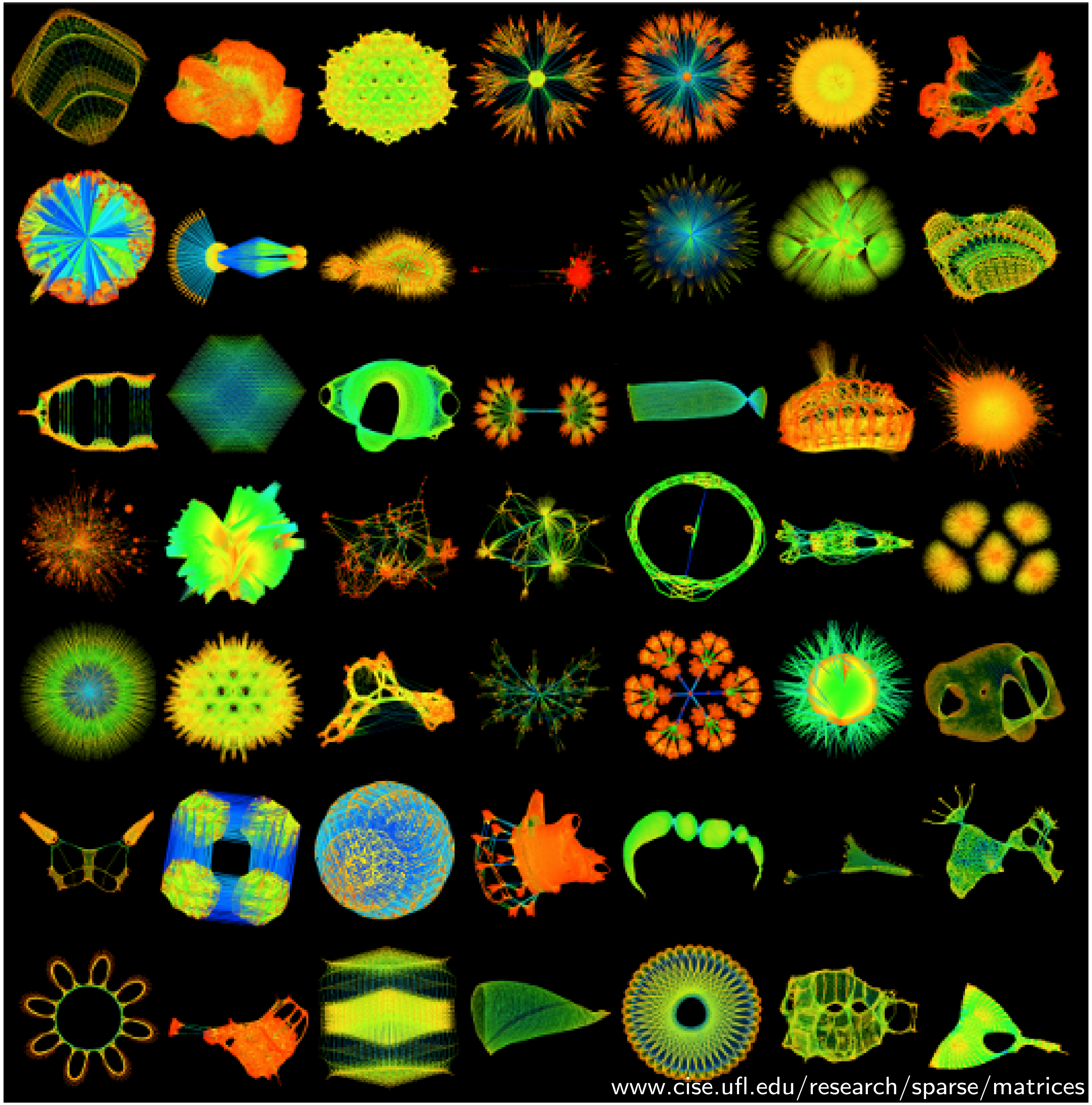
- easily adaptable and configurable
- robust
- scalable

But...

- usually no quality and running time guaranees
- bad choice of starting layout $\rightarrow$ slow convergence
- possibly slow for large graphs
- fine-turning can be done by experts
- no explicit optimization of aesthetic functions

Gallerie



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