Exercise Sheet 6
Discussion: 30. January 2019

Exercise 1: Properties of \textit{st}-Graphs
Let $D=(V,A)$ be a planar \textit{st}-graph with a given embedding. For a face $f$ of $D$ denote by $V_f$ and $E_f$ the vertices and edges on $f$. Let \text{start}(f)$ and \text{target}(f)$ be the source and sink of the graph $(V_f,E_f)$, respectively. Prove or disprove:

(a) $D$ is bimodal.
(b) The boundary of each face $f$ consists of two directed paths from \text{start}(f) to \text{target}(f).
(c) For every vertex $v \in V$ there is a simple directed \textit{st}-path that contains $v$.

Exercise 2: Duals of \textit{st}-Graphs
Let $D$ be a planar embedded \textit{st}-graph. For a directed edge $e=(u,v)$, let $\ell(e)$ denote the face left of $e$, and let $r(e)$ denote the face right of $e$. Without loss of generality assume that $D$ is embedded such that $r(s,t)$ is the external face. The directed dual graph $D^*=(V^*,A^*)$ of $D$ is defined as follows:

• $V^*$ is the set of faces of $D$, where $s^*=r(s,t)$ and $t^* = \ell(s,t)$.
• $A^* = \{(\ell(e),r(e)) \mid e \in A \setminus \{(s,t)\}\} \cup \{(s^*,t^*)\}$

(a) Prove that $D^*$ is a planar \textit{st}-graph.
(b) Prove that for any two faces $f$ and $g$ of $D$ exactly one of the following properties holds:
   i) $D$ contains a directed path from \text{target}(f) to \text{start}(g)
   ii) $D$ contains a directed path from \text{target}(g) to \text{start}(f)
   iii) $D^*$ contains a directed path from $f$ to $g$
   iv) $D^*$ contains a directed path from $g$ to $f$

\textit{Hint:} Consider a topological numbering $\sigma : V \to \mathbb{N}$ of the nodes of $D$, such that for every $(u,v) \in A$ it holds that $\sigma(u) < \sigma(v)$.
Exercise 3: Extended Canonical Ordering for 4-Connected Graphs

A planar graph \( G = (V,E) \) is called proper triangular planar (PTP, for short) if every interior face of \( G \) is a triangle and the exterior face of \( G \) is a quadrangle, and \( G \) has no separating triangles.

Let \( G = (V,E) \) be a PTP graph with vertices \( a,b,c,d \) on the outer face. A labeling \( v_1 = a, v_2 = b, v_3, \ldots, v_n = d \) of the vertices of \( G \) is called an extended canonical ordering of \( G \) if for every \( 4 \leq k \leq n \):

(i) The subgraph \( G_{k-1} \) induced by \( v_1, \ldots, v_{k-1} \) is biconnected and the boundary \( C_{k-1} \) of \( G_{k-1} \) contains the edge \((a,b)\), and

(ii) the vertex \( v_k \) is on the boundary of exterior face of \( G_{k-1} \), and its neighbors in \( G_{k-1} \) form a subinterval of the path \( C_{k-1} \setminus (a,b) \) with at least two elements. If \( k \leq n - 2 \), then \( v_k \) has at least two neighbors in \( G \setminus G_{k-1} \).

Let \( G = (V,E) \) be a PTP graph with vertices \( a,b,c,d \) on the outer face. Prove the following statements. We denote by \( G_C \) the graph that is induced by the vertices in the interior and on the boundary of a simple cycle \( C \).

(a) The graph obtained from \( G \) by the removal of the vertices \( c,d \) and all edges incident to them is biconnected.

(b) Let \( C = \{a = u_1, \ldots, u_k = b, a\} \) be a simple cycle of \( G \) such that \( c,d \notin C \). Let \( u_i \in C \), \( 2 \leq i \leq k - 1 \) such that no internal chord of \( C \) is incident to \( u_i \). Then the graph \( G_C \setminus \{u_i\} \) is biconnected.

(c) Let \( C \) be as above and let \((v_i, v_j)\), \( 1 \leq i < j \leq k \), be an internal chord of \( C \). Then there exists a vertex \( v_l \), \( i < l < j \) that is adjacent to at least two vertices of \( G \setminus G_C \).

Use the previous statements to prove the following lemma.

**Lemma 1** Every PTP graph \( G \) with four vertices \( a,b,c,d \) on the outer face has an extended canonical ordering such that \( v_1 = a, v_2 = b, v_{n-1} = c, v_n = d \).
Exercise 4: Contact Representation of Maximal Planar Graphs

The figure below gives an example of contact representation of a planar graph with T-shapes. Prove the following Lemma.

**Lemma 2** Every maximal planar graph admits a contact representation with T-shapes.

**Hint**: Use canonical ordering in the way similar to the construction of a visibility representation (Exercise Sheet 3).

Exercise 5: Construction of Rectangular Dual

Consider the graph $G$ of the figure below. Check whether $G$ satisfies the necessary conditions to have a rectangular dual. In affirmative, construct a rectangular dual of $G$. 