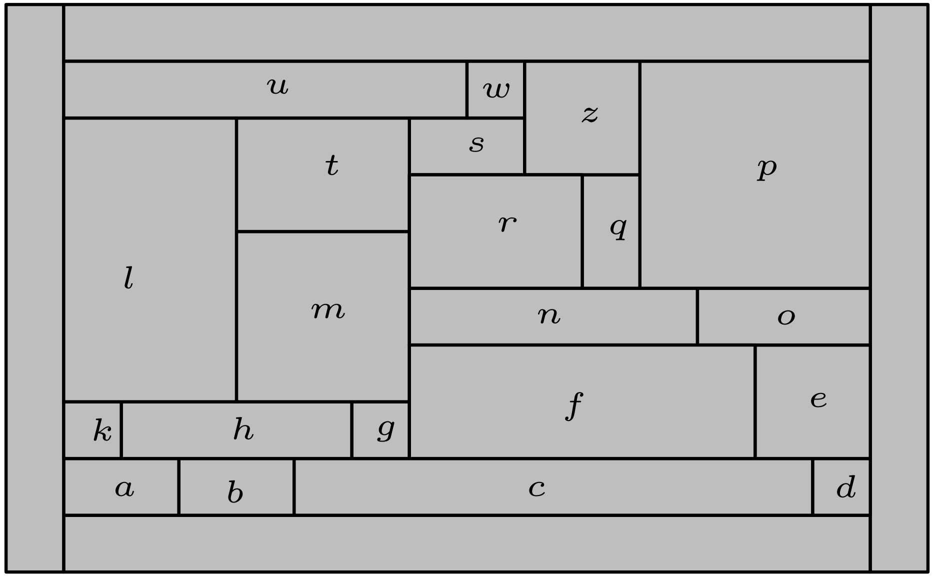
Algorithms for graph visualization

Contact representations of planar graphs.

WINTER SEMESTER 2017/2018

Tamara Mchedlidze



KIT – Universität des Landes Baden-Württemberg und
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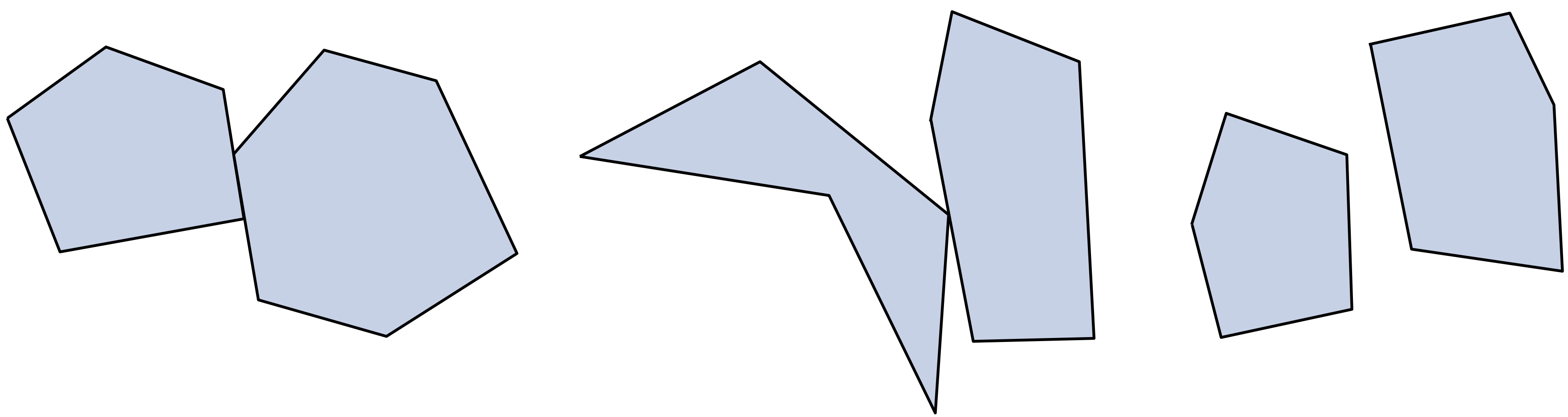
Contact representation

Contact representation

In a contact representation of a planar graph each vertex is represented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.



Touch

Do not touch

Contact representation

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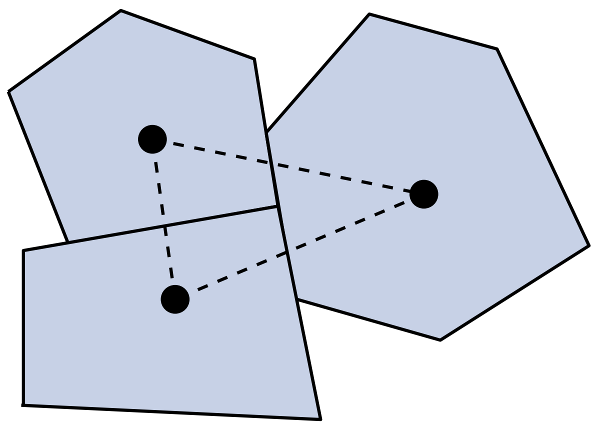

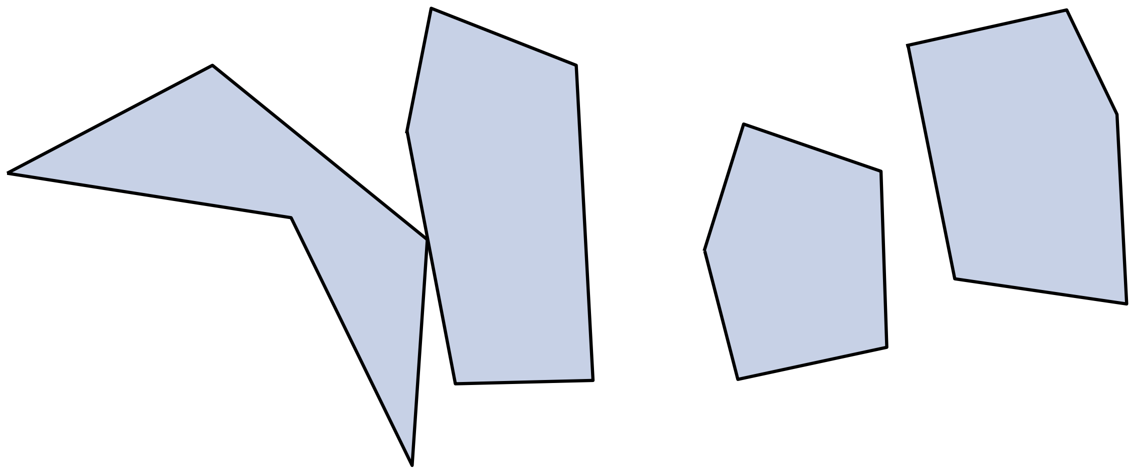

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Contact representation

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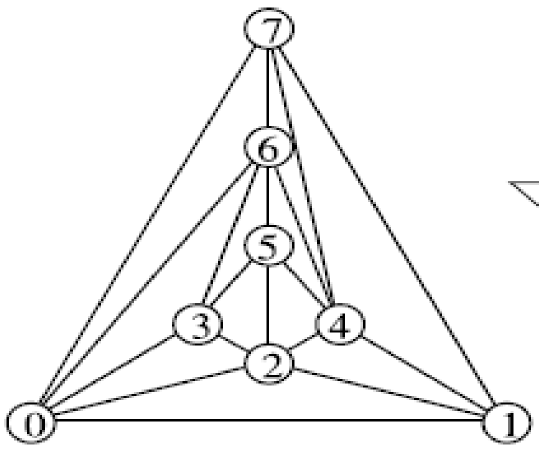

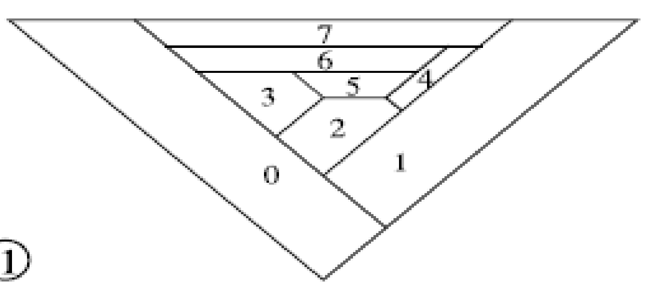

- 6-gons are necessary and suffisient for planar graphs! (Gansner et. al. 2010)

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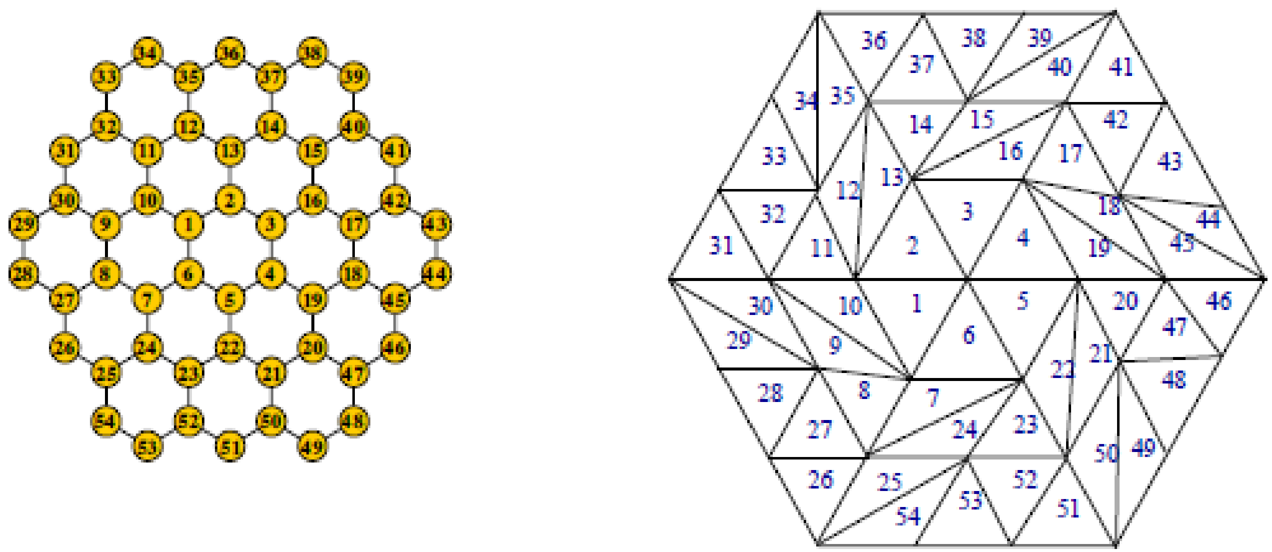
Touching triangle representation



- Every 3-connected cubic planar graph admits a contact representation with triangles (Kobourov et. al. 2012)

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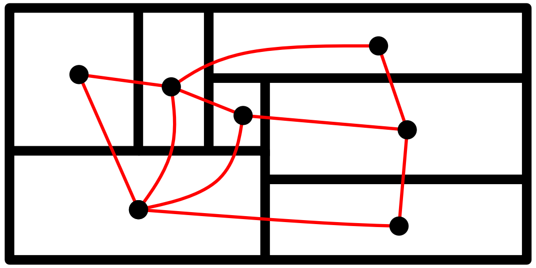
Contact representation with rectangles



- Every 4-connected planar graph* admits a contact representation with rectangles (Xin He 1993*)

Contact representation

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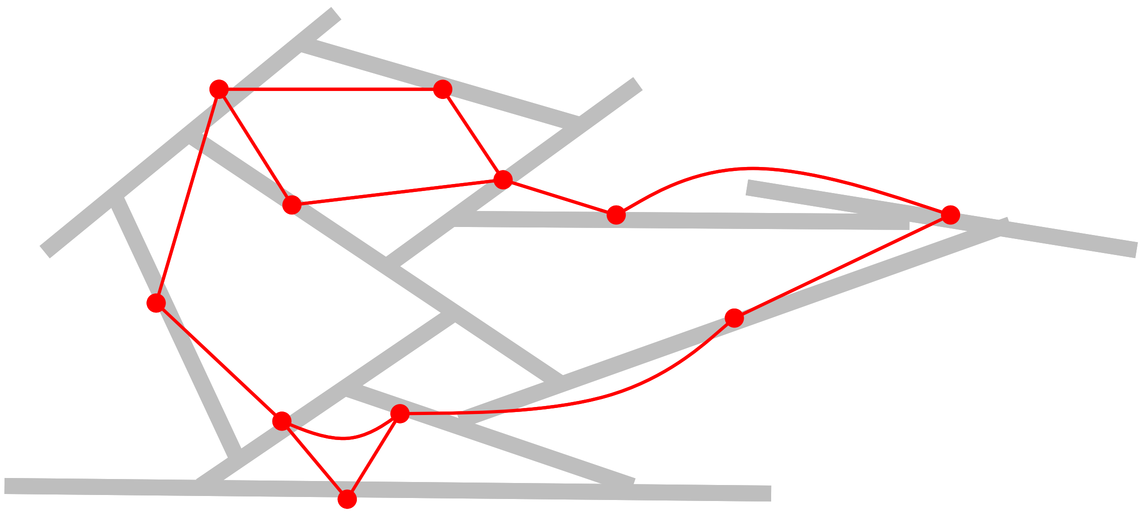
Touching segment representation



- Every triangle-free planar graph has a contact representation with line segments in just three directions (de Castro et. al. 1999)

Contact representation

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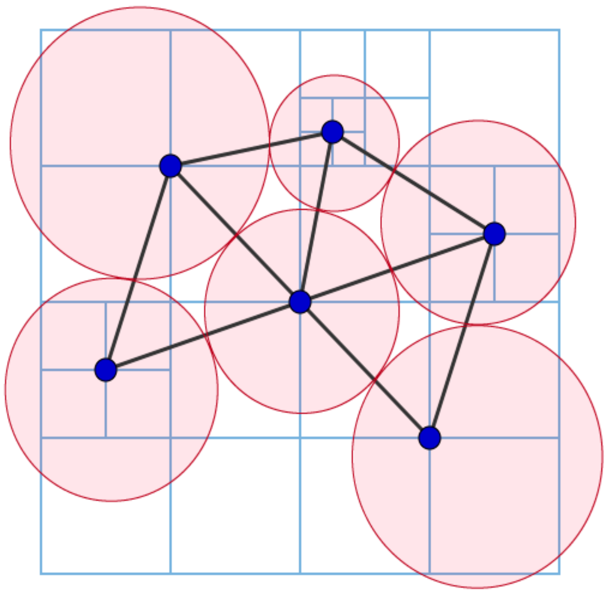
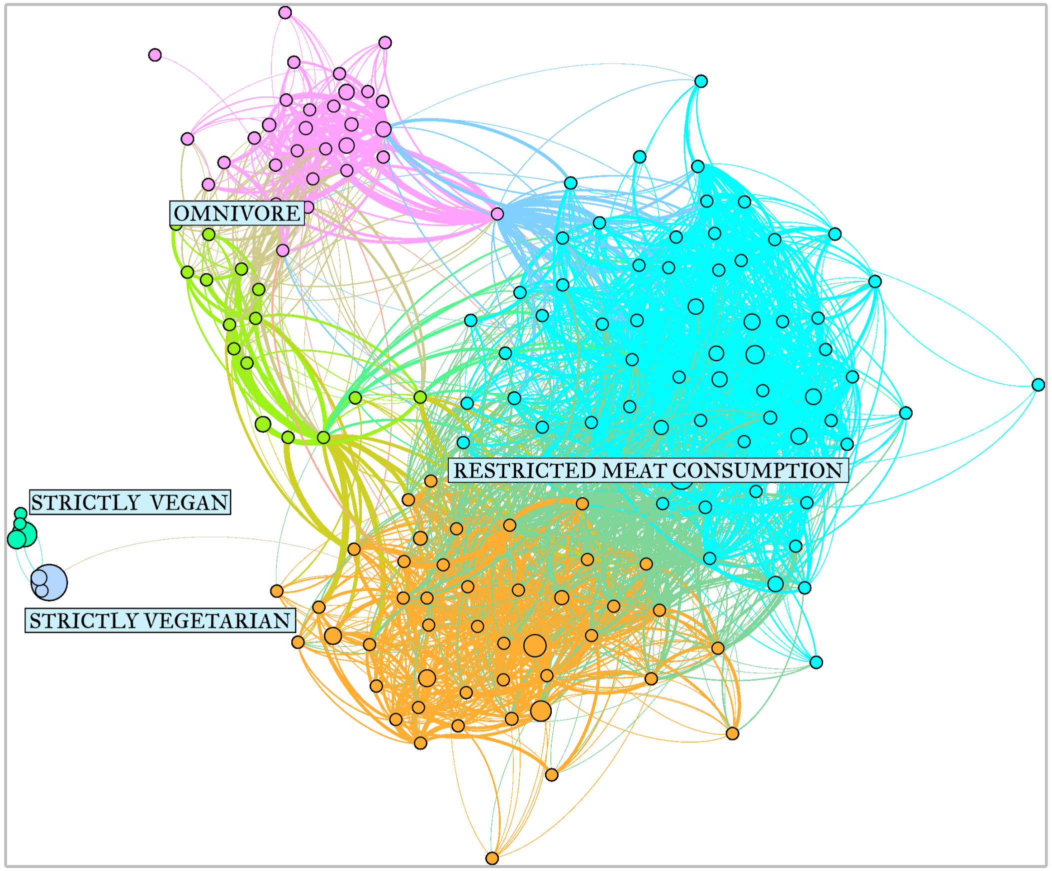
Touching disk representation



- Each planar graph has a touching disks representation (Koebe 1936)

Application: visualization of a clustering



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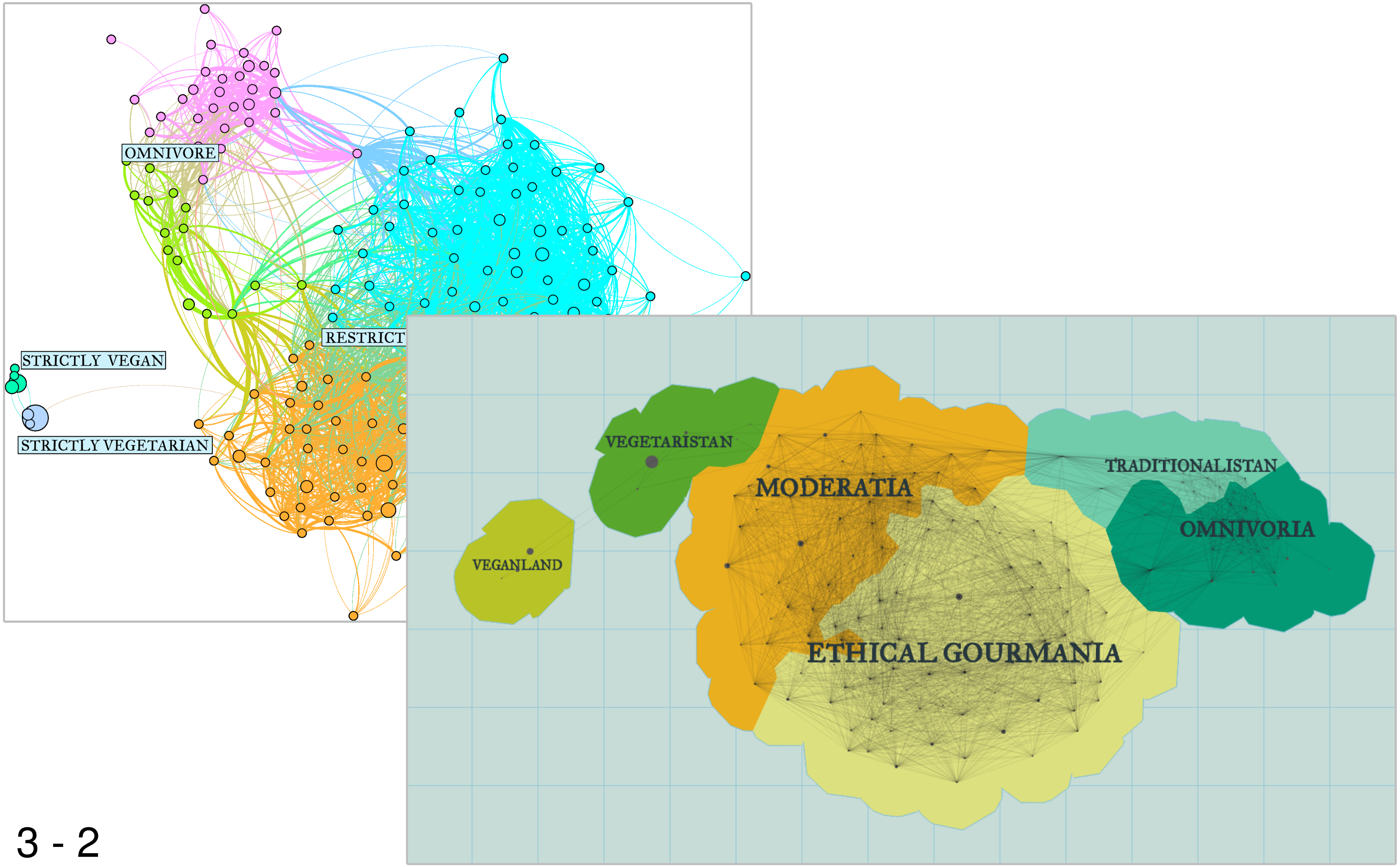

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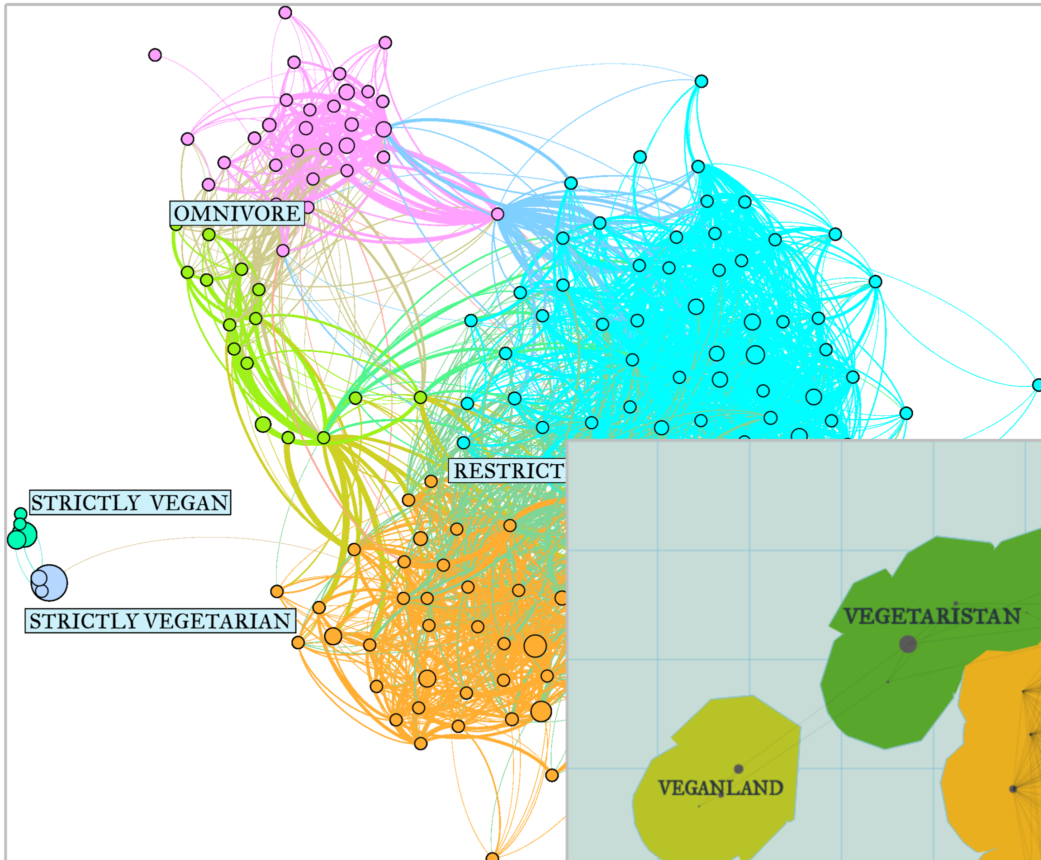

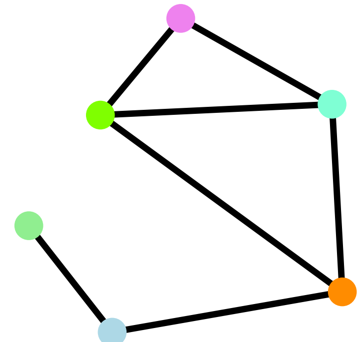

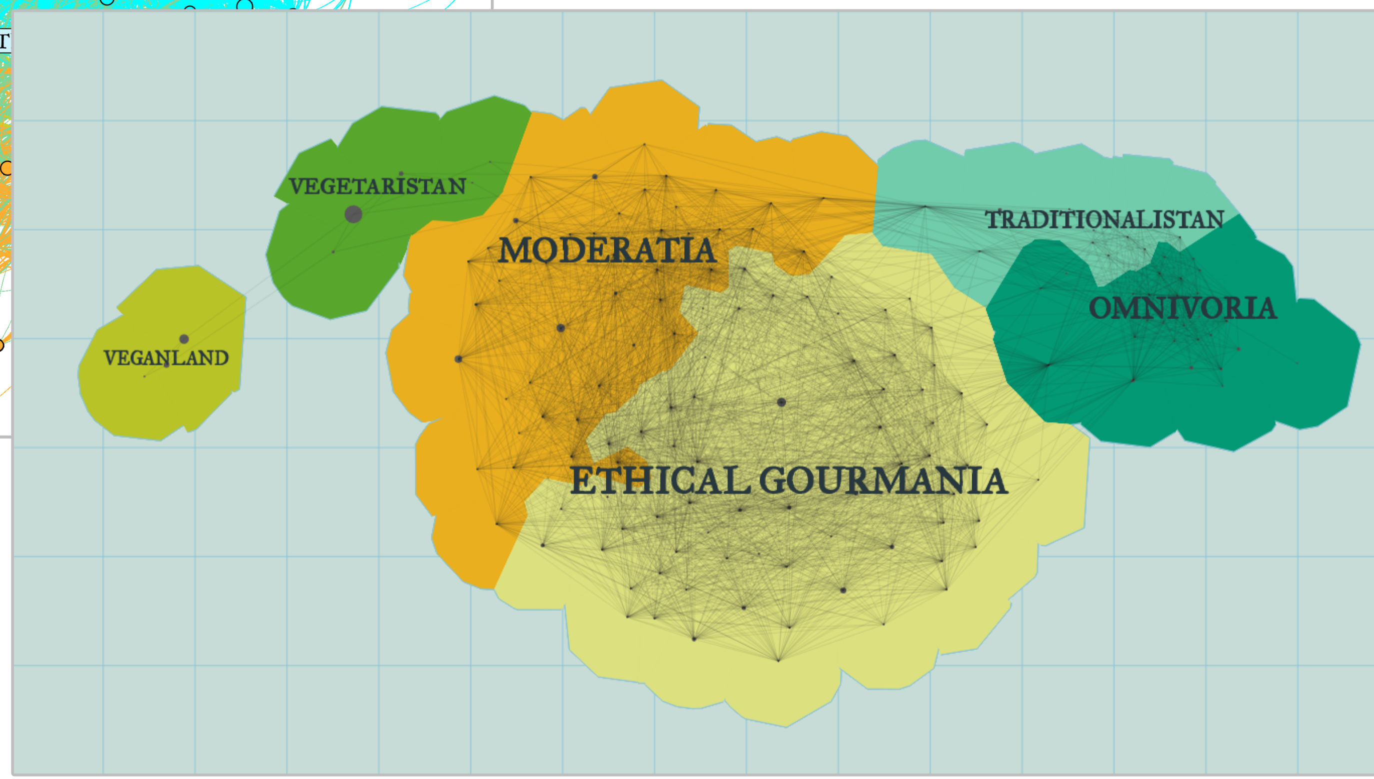

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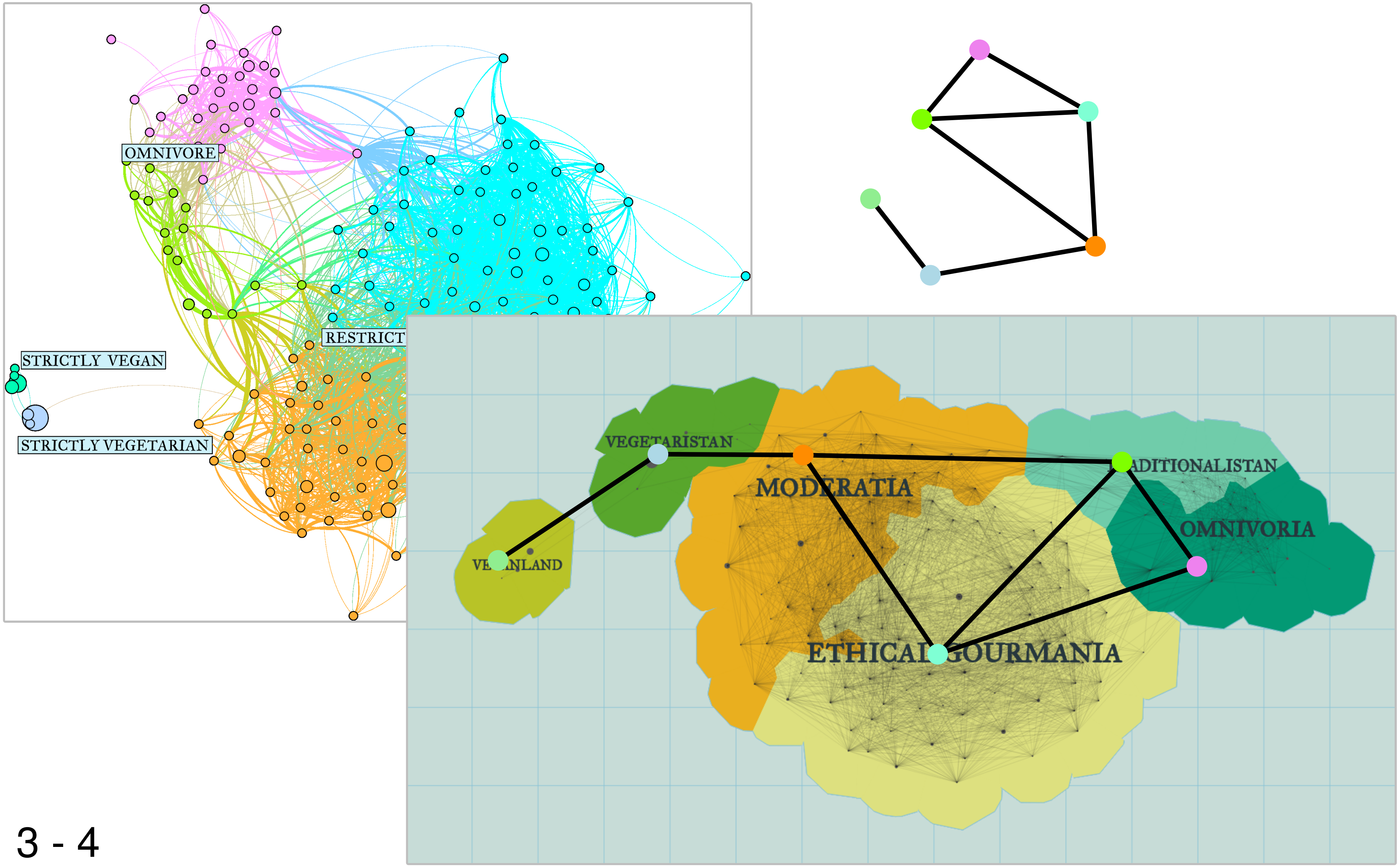

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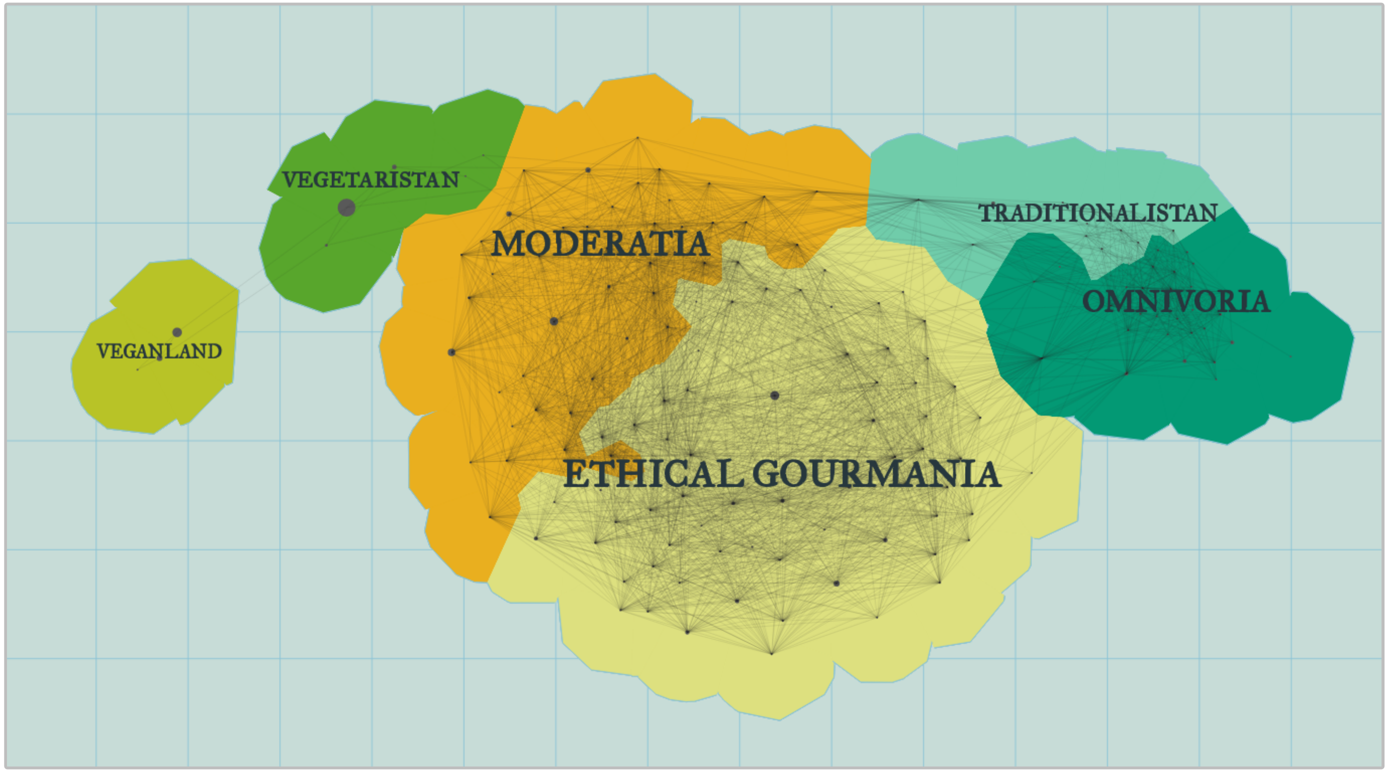
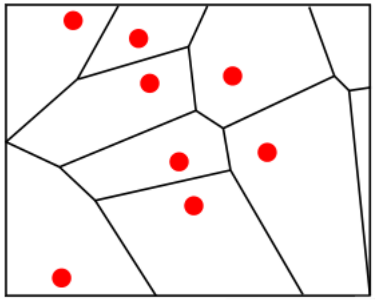
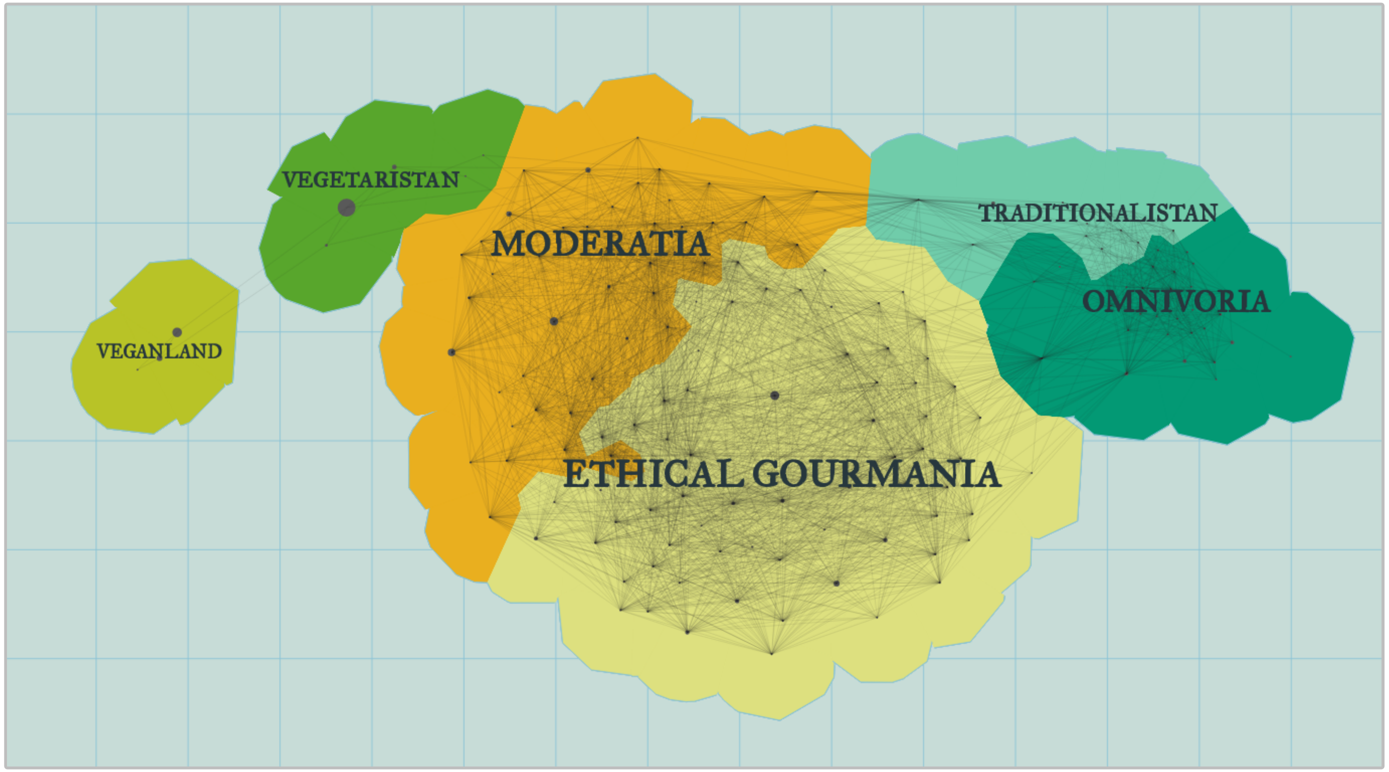


Application: visualization of a clustering

- Run force directed algorithm on the graph
- Compute voronoi diagram of the points representing the vertices





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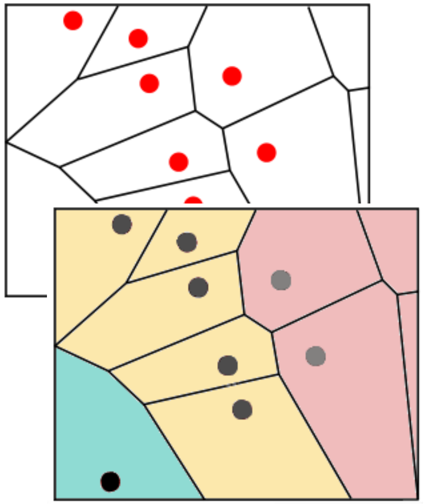
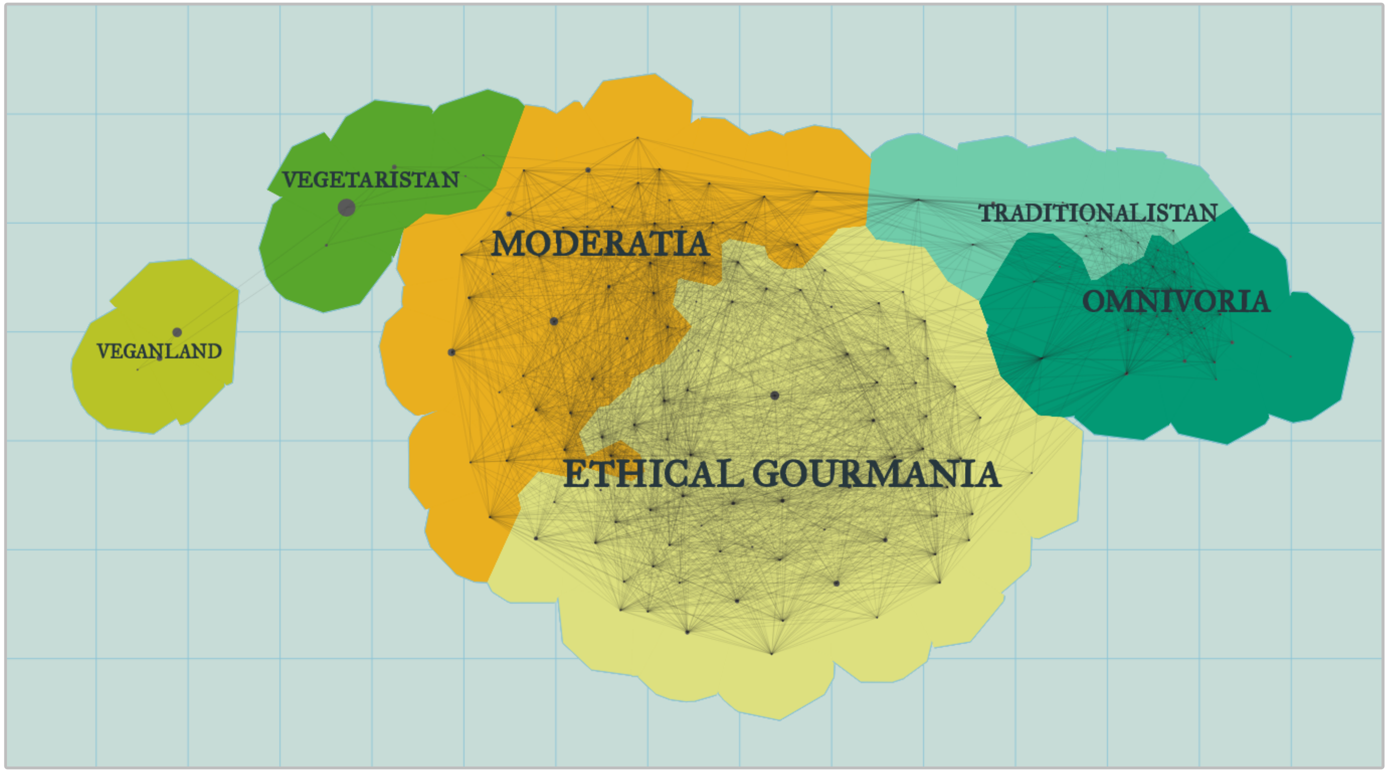
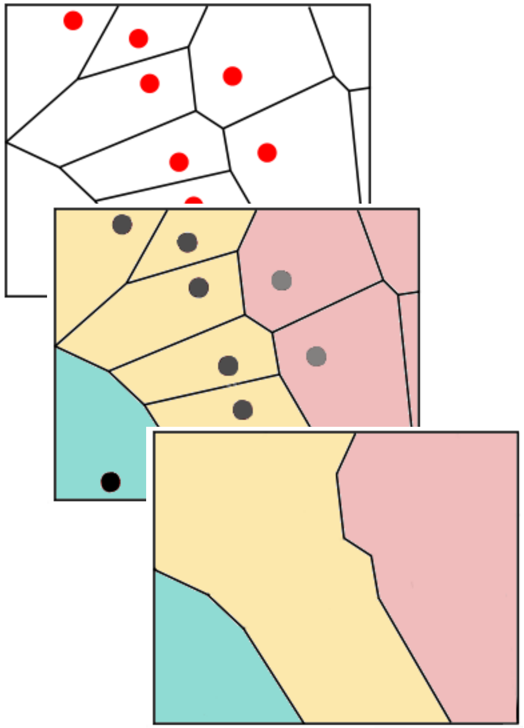
- Color voronoi cell according to the clustering



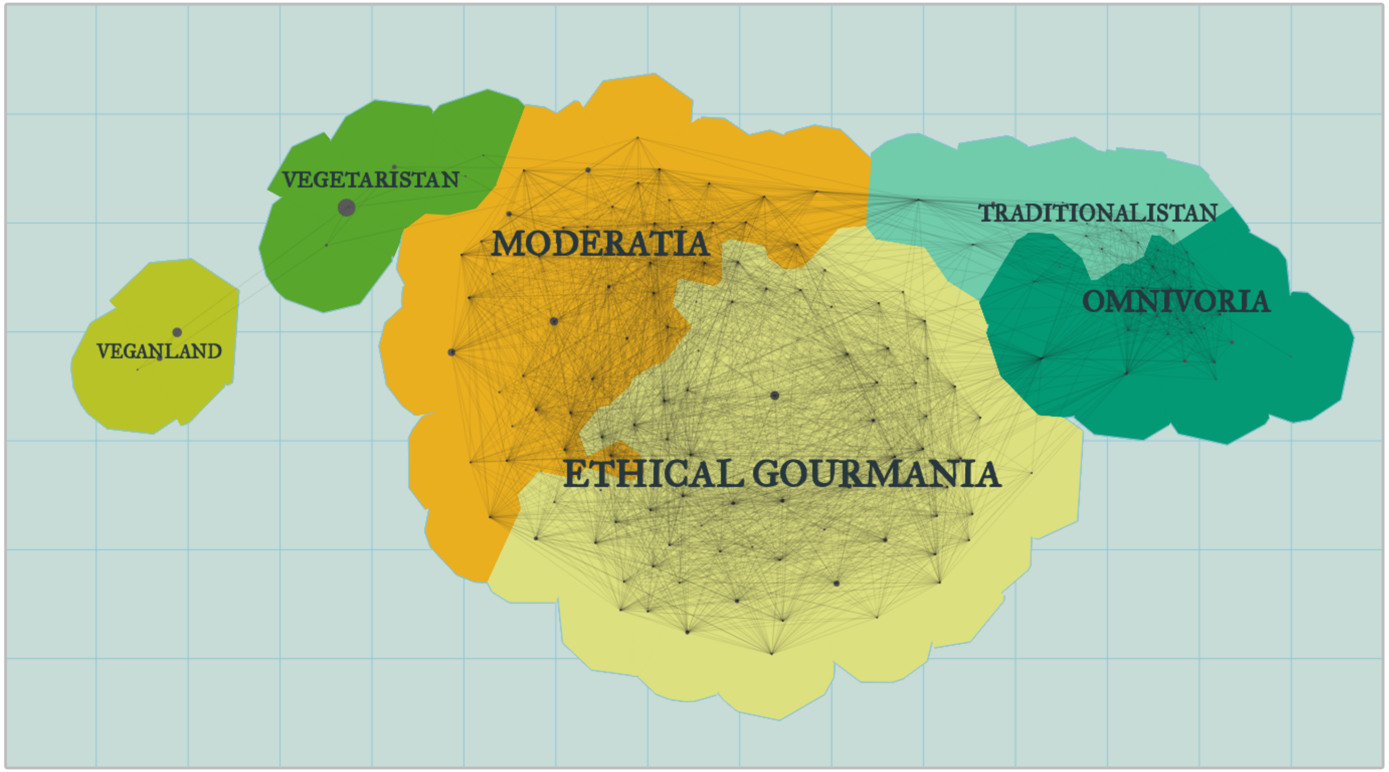


Application: visualization of a clustering

- Run force directed algorithm on the graph
- Compute voronoi diagram of the points representing the vertices
- Color voronoi cell according to the clustering
- Merge cells of the same cluster

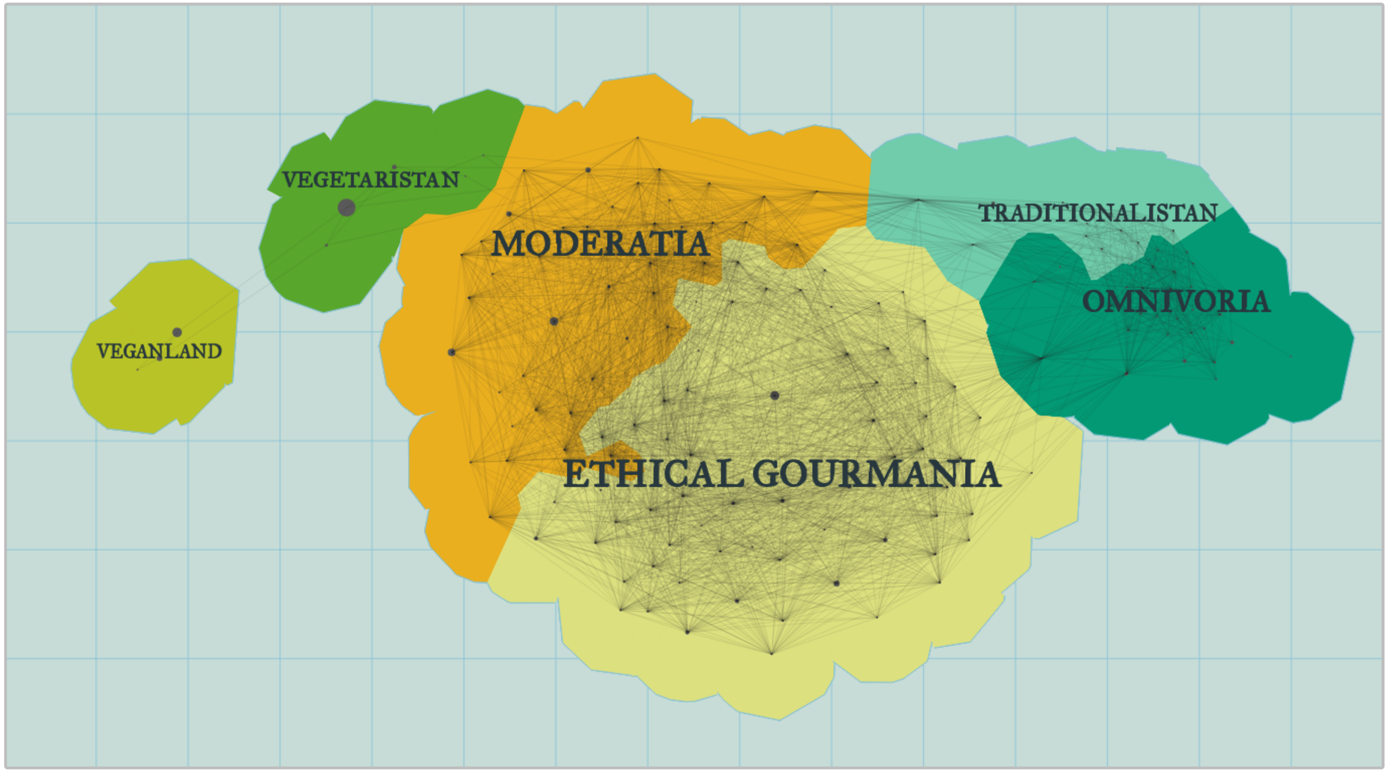


Inspired by GMap (Gansner et al.)



Application: visualization of a clustering

- Just a heuristic, there are no guarantees that:
  - The strong adjacencies between clusters are represented by contacts of the countries



Application: visualization of a clustering

- Just a heuristic, there are no guarantees that:
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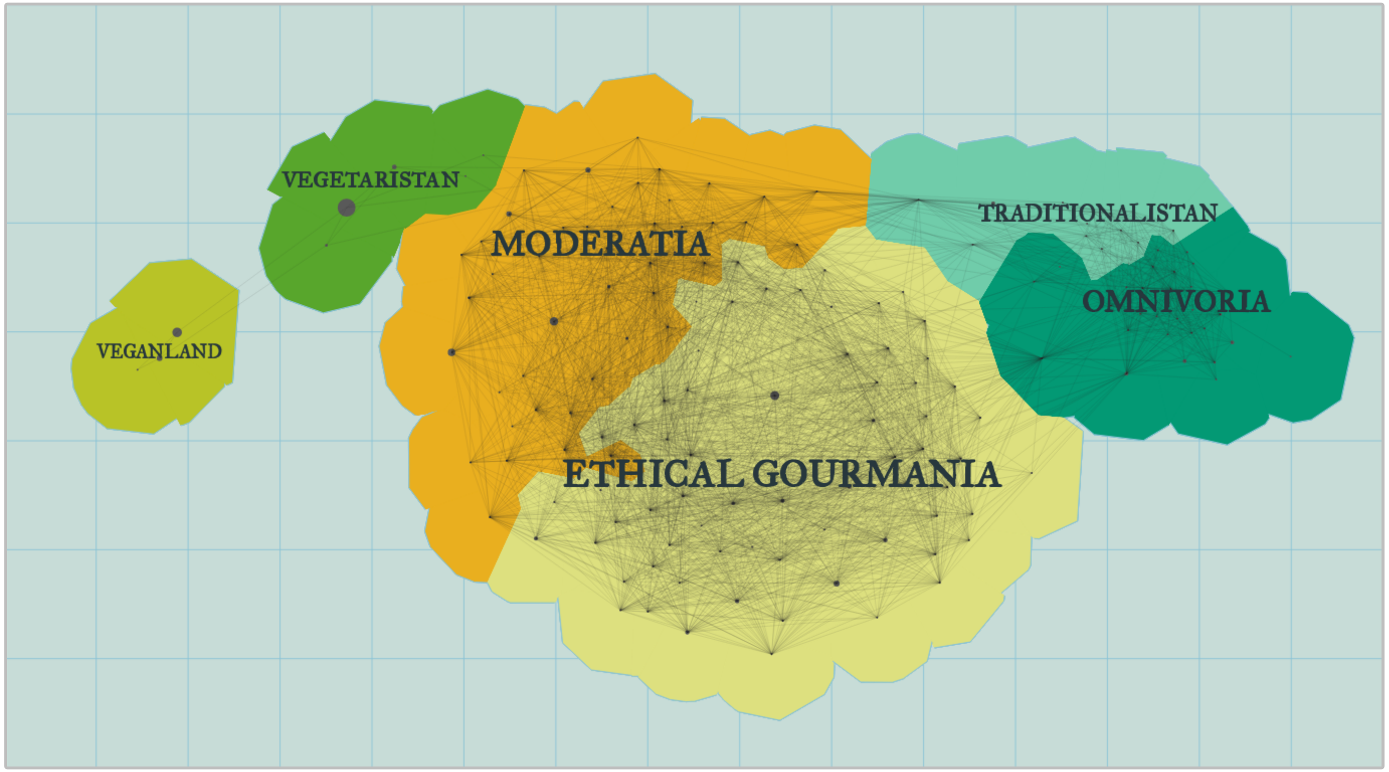
  - The length of the boundary is representing the strength of the adjacency



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- Just a heuristic, there are no guarantees that:
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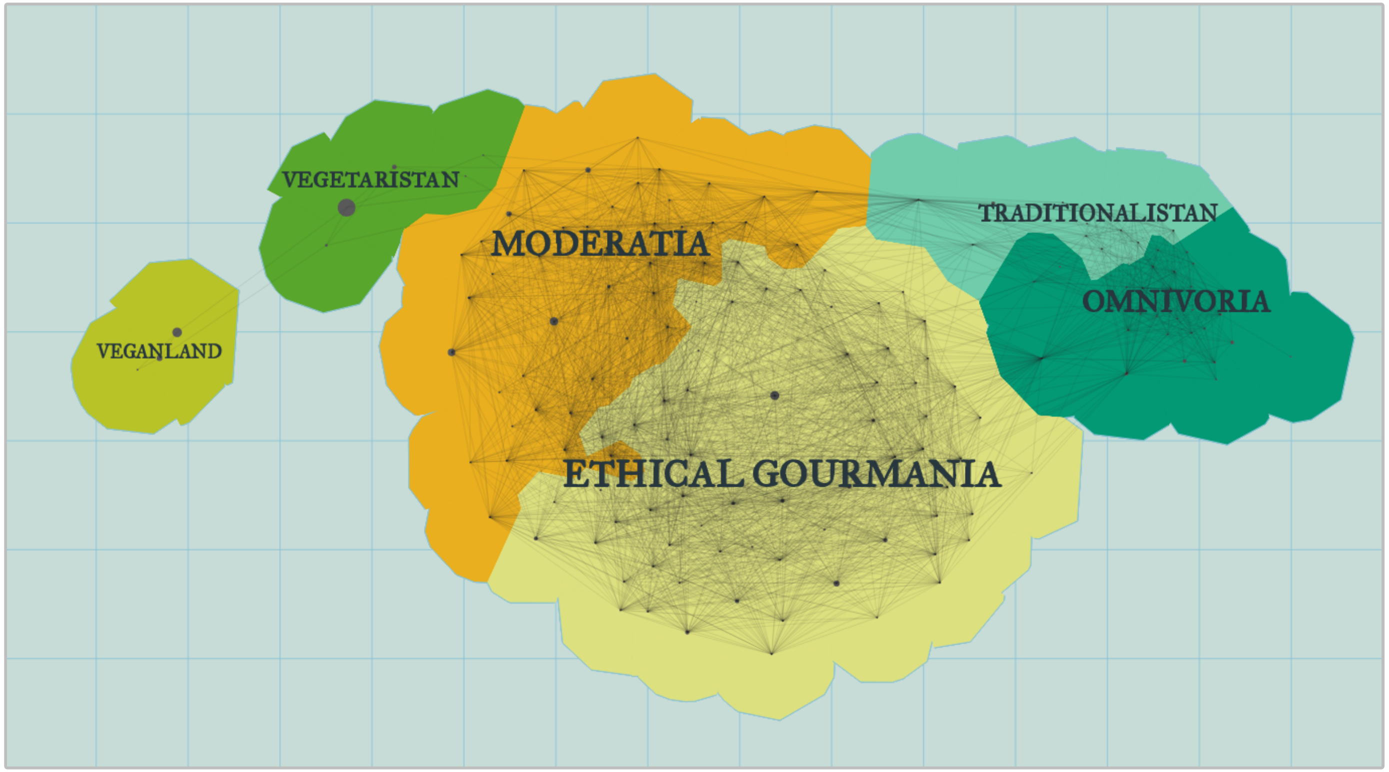

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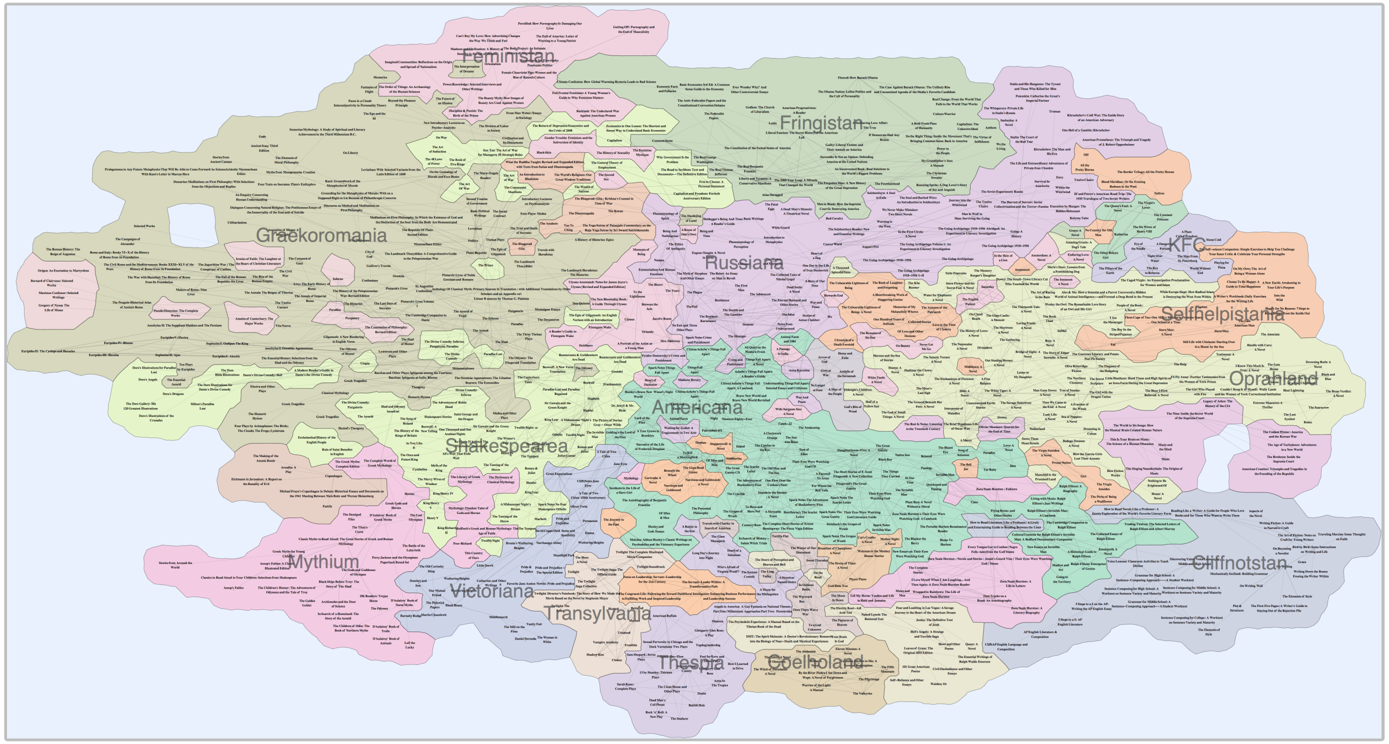
  - The size of the country is proportional to the population size
  - The countries may be fragmented



Gansner E., Hu Y., Kobourov S.: GMap: Drawing Graphs as Maps

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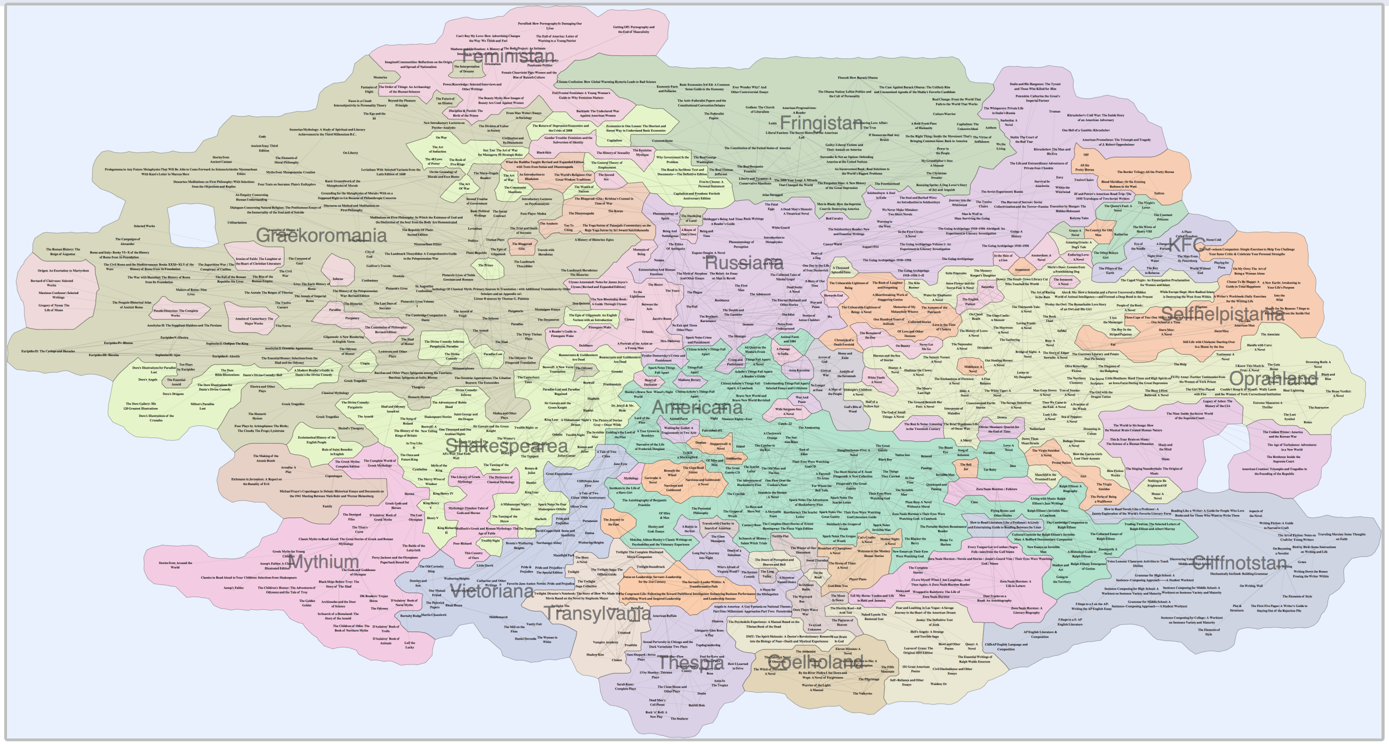

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Today

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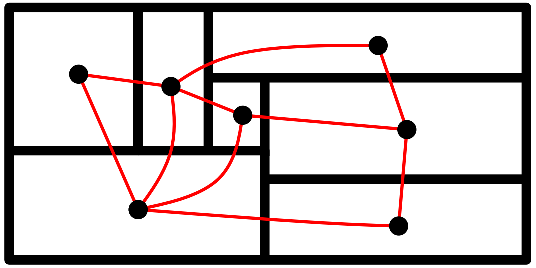
Contact representation with rectangles



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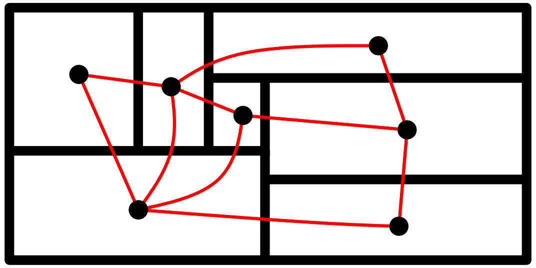

- Every 4-connected planar graph* admits a contact representation with rectangles (Xin He 1993*)
- A contact representation of G with rectangles, without holes and with rectangular outer boundary is called a **rectangular dual** of G

Rectangular Dual

- Which graphs have a rectangular dual?

Rectangular Dual

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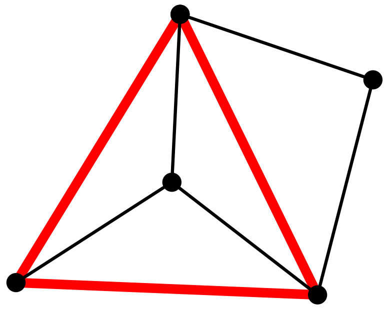
Separating triangle

Let G be a graph. A triangle C of G whose removal results in at least two disconnected components is called a separating triangle of G.



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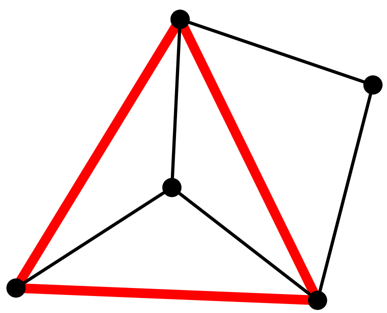

Does not have a rectangular dual!

(In order to enclose an area we need at least four boxes)

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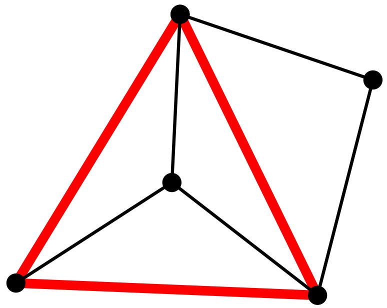

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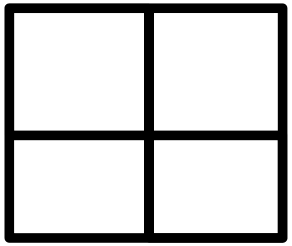
No four rectangles meet a a point!



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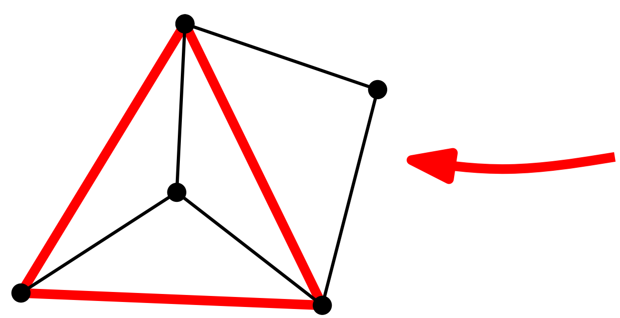

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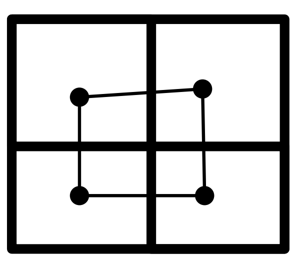
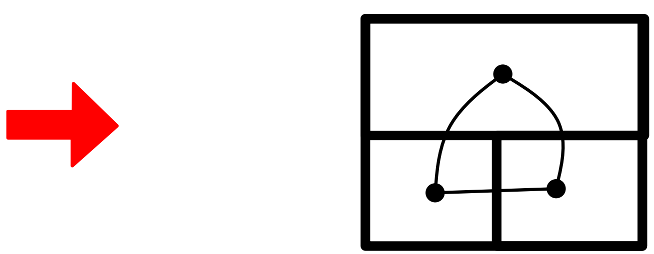
No four rectangles meet a a point!

Each face of G must be a triangle!





Rectangular Dual

Necessary conditions for a planar graph G to have a rectangular dual:

- G must have at least 4 vertices on the outer face
- G must have no separating triangle
- each internal face of G must be a triangle

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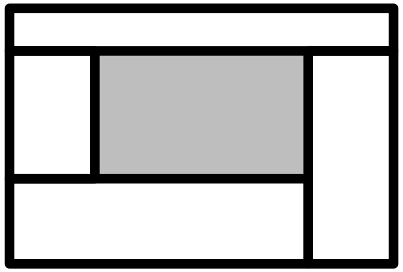
Theorem [1985,1986,1987,1988,1990,1993 by Xin He]

A planar graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following conditions hold:

- Every interior face of G is a triangle and the exterior face of G is a quadrangle;
- G has no separating triangles



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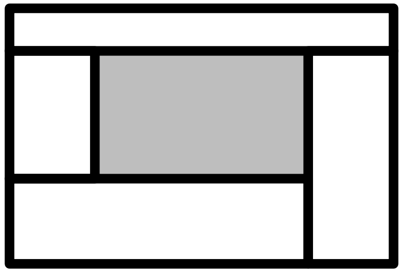
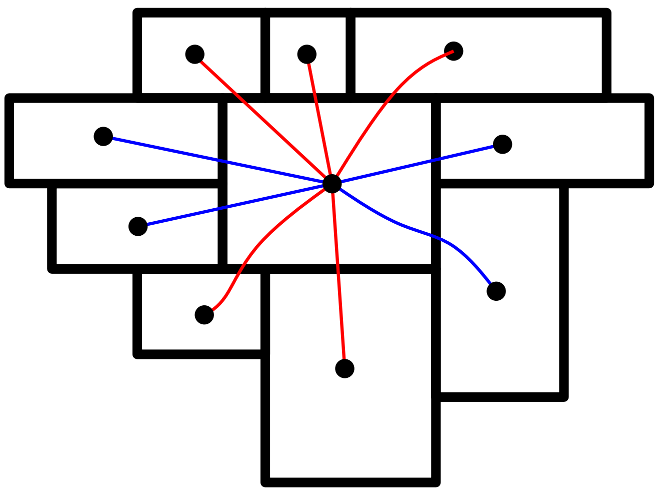
Proper Triangular Planar Graph (PTP)

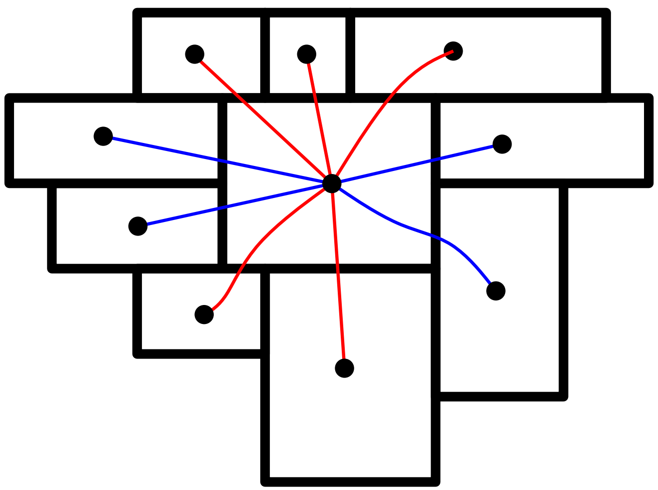


Rectangular Dual



In order to construct a rectangular dual we need to partition our edges on vertical and horizontal. Regular edge labeling (REL, for short) is a tool for that.

Rectangular Dual



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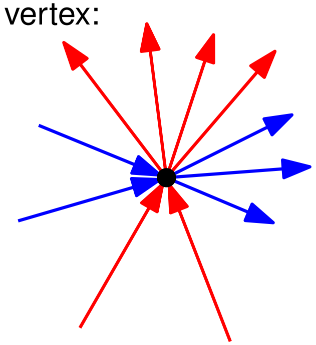
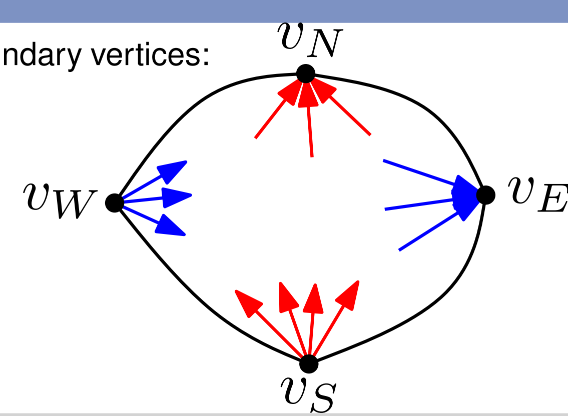
Regular edge labeling

For each internal vertex:



For the boundary vertices:

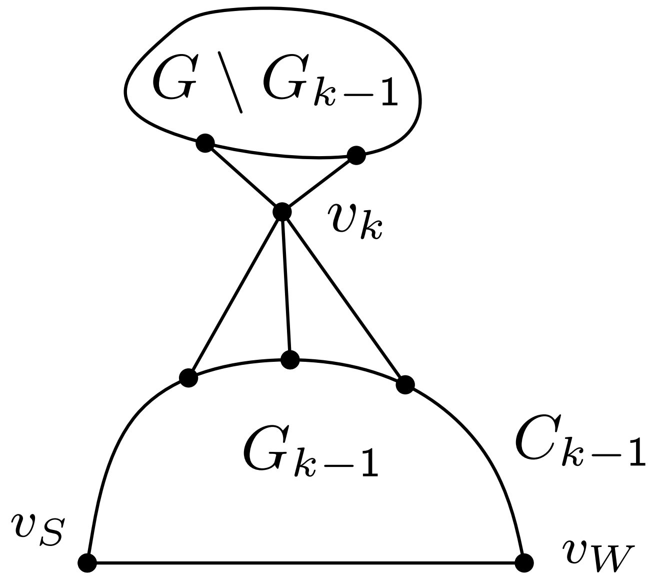


Rectangular Dual

Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k-2$, v_k has at least 2 neighbors in $G \setminus G_{k-1}$



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Canonical ordering with extra condition on v_k!

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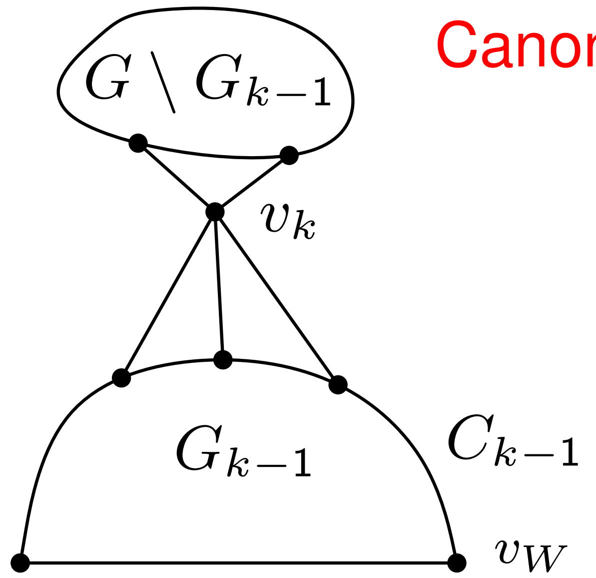

Rectangular Dual

Theorem (Refined canonical ordering)

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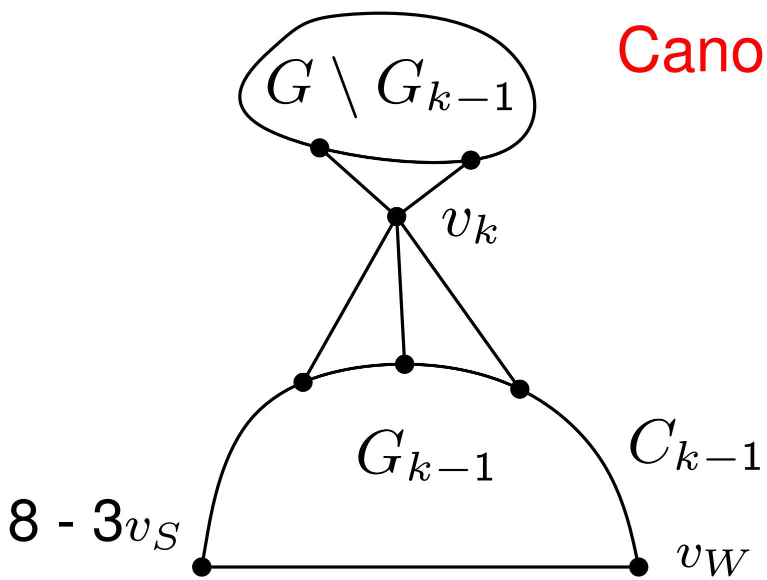
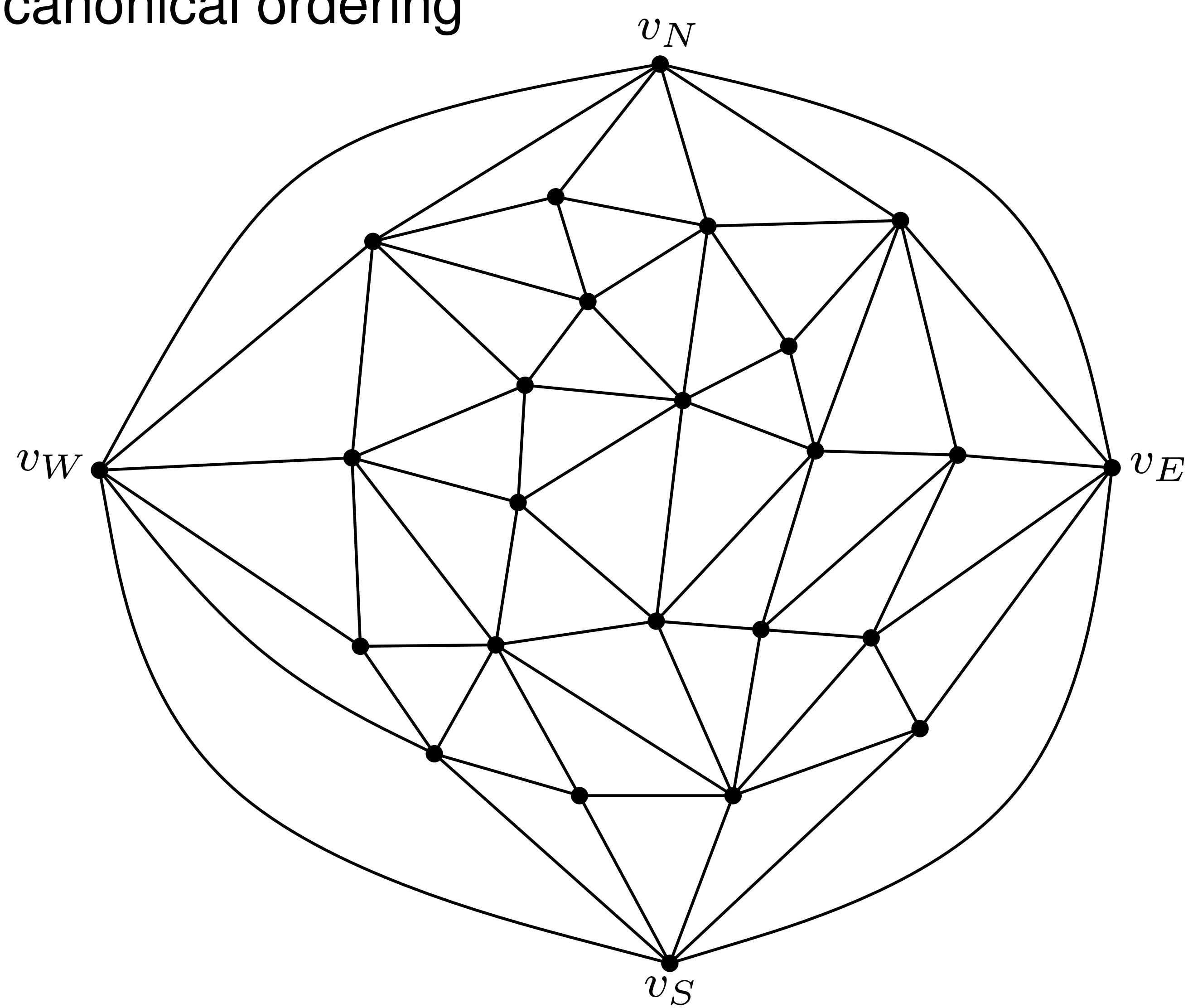


Canonical ordering with extra condition on v_k!

Rectangular Dual

Refined canonical ordering



Rectangular Dual

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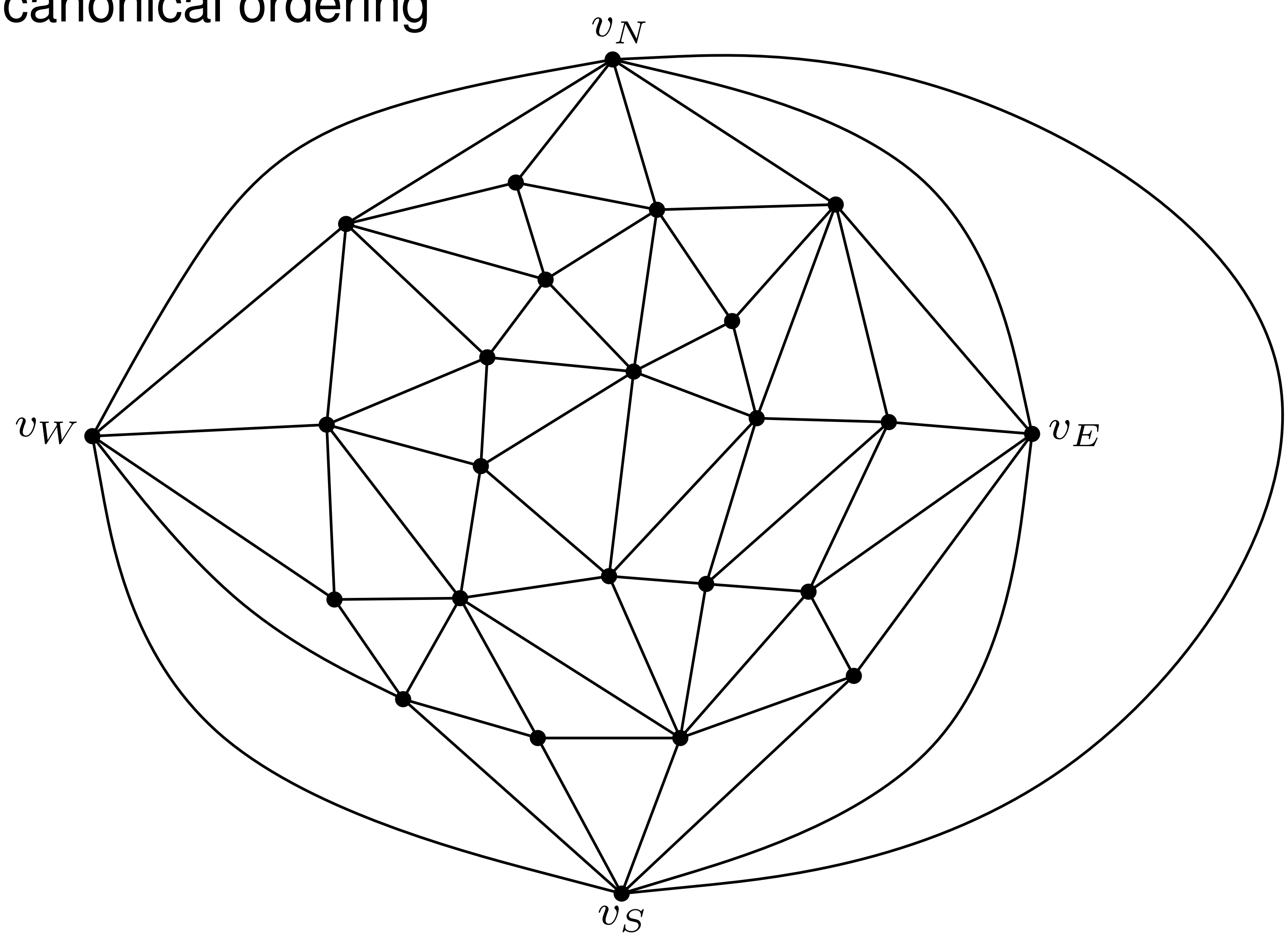

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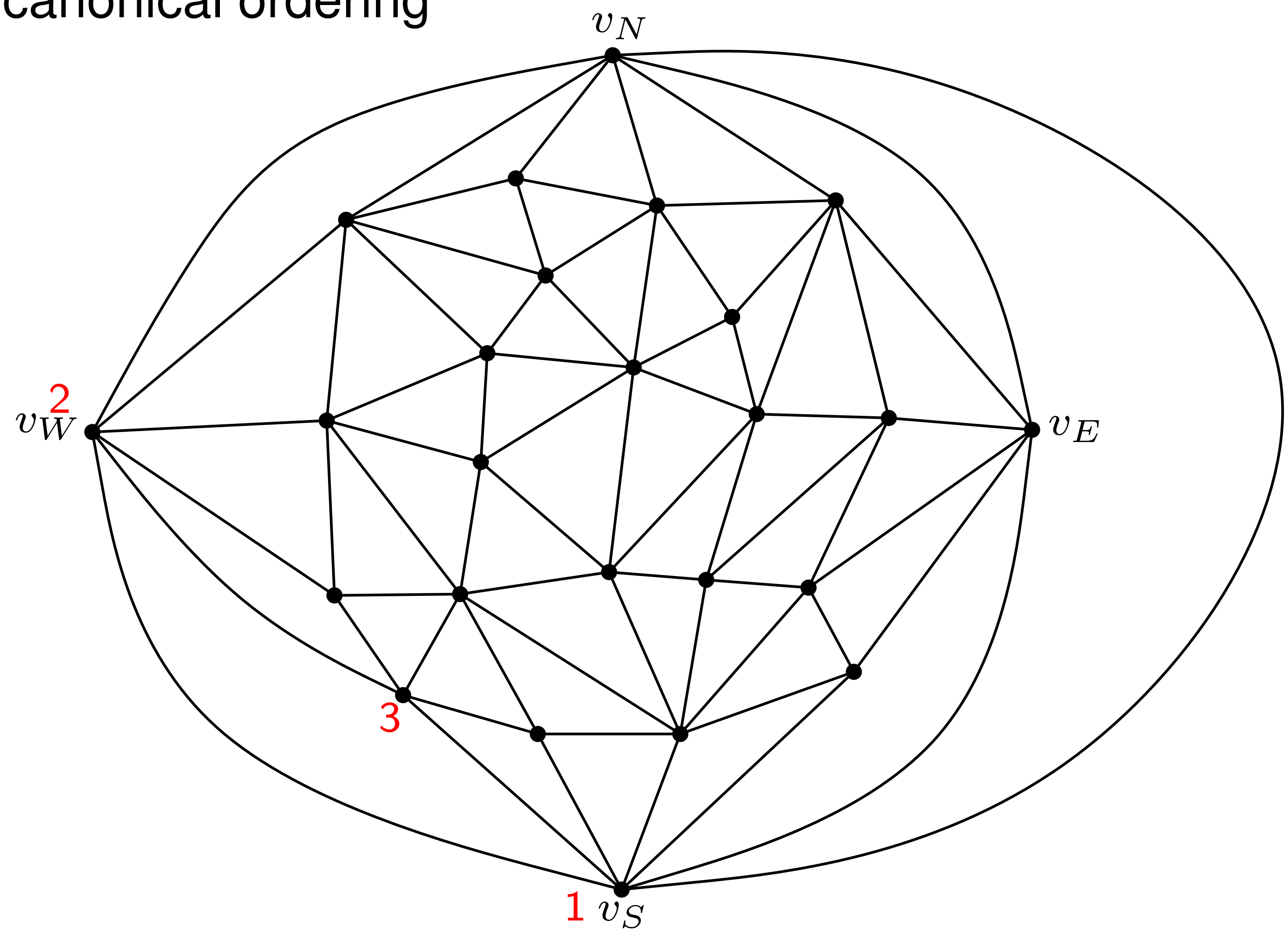

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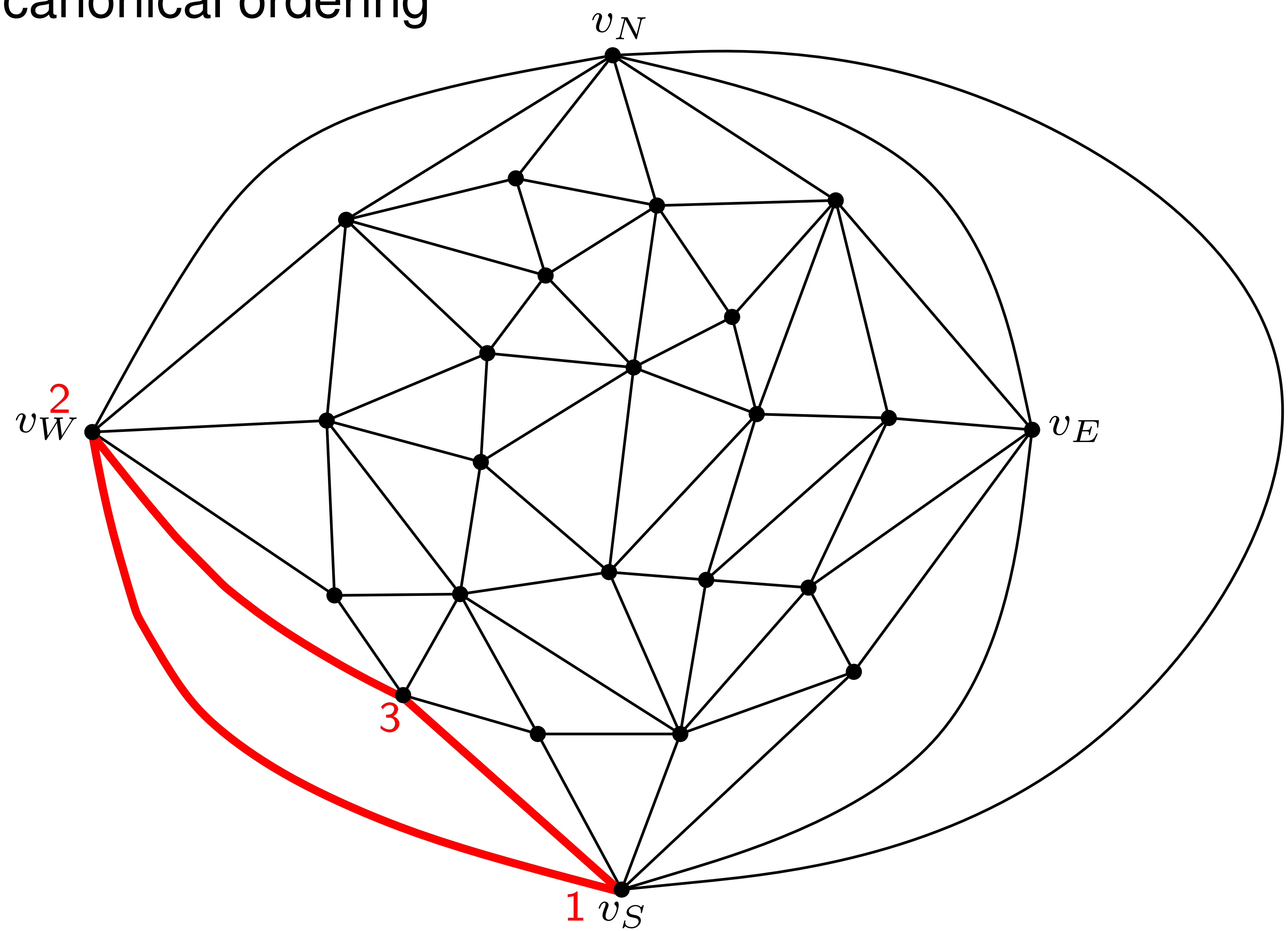

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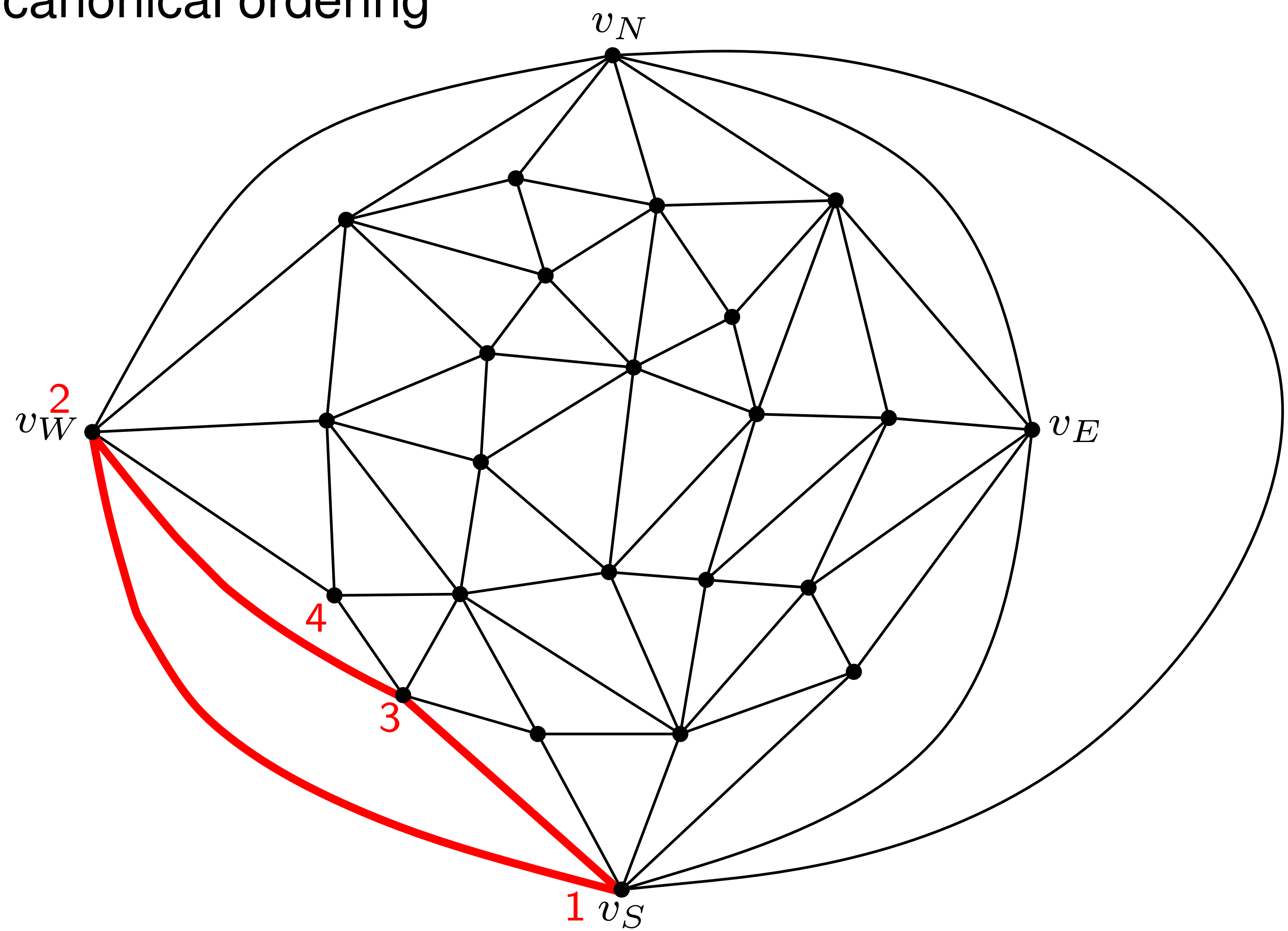
Rectangular Dual

Refined canonical ordering



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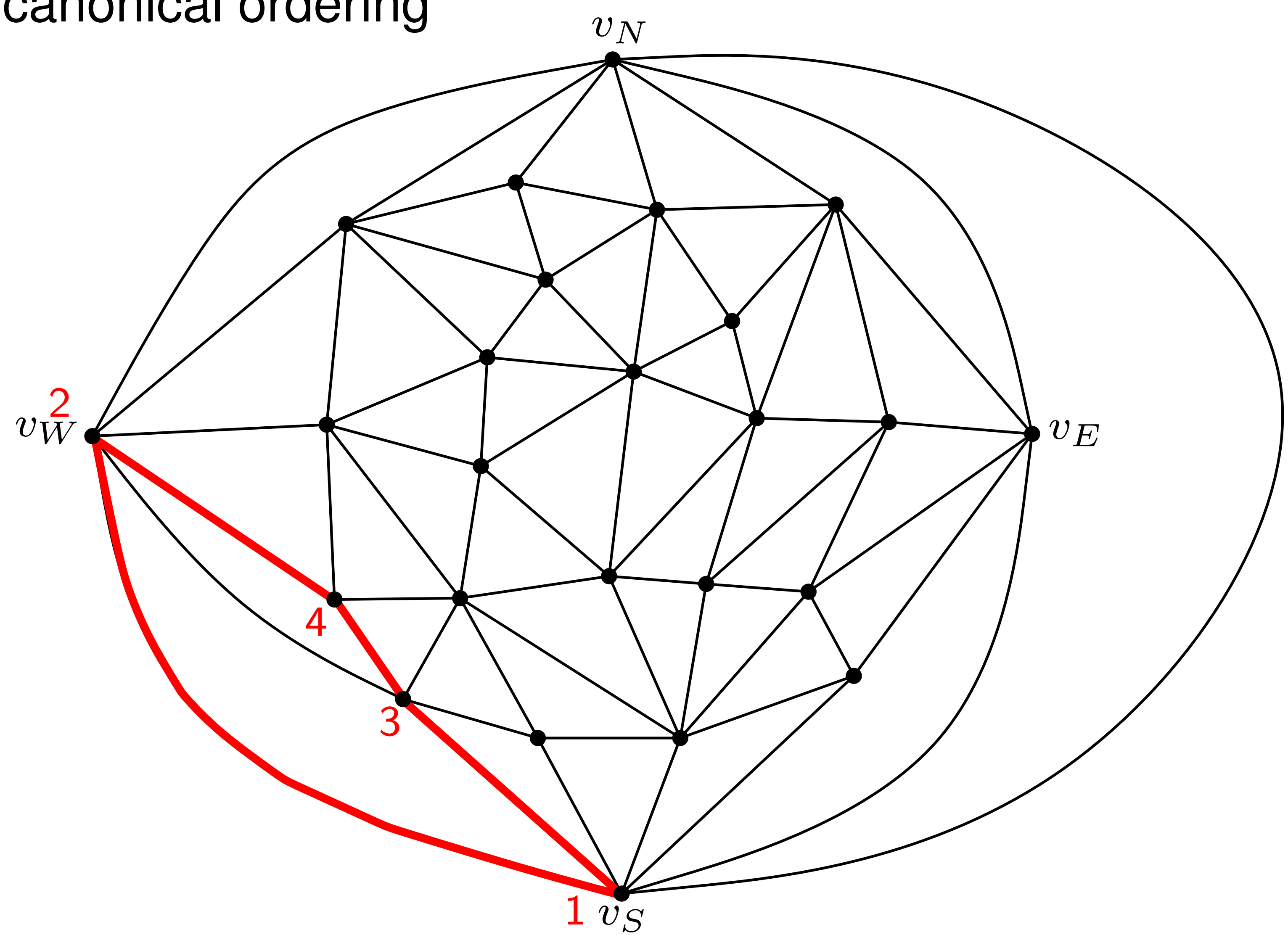

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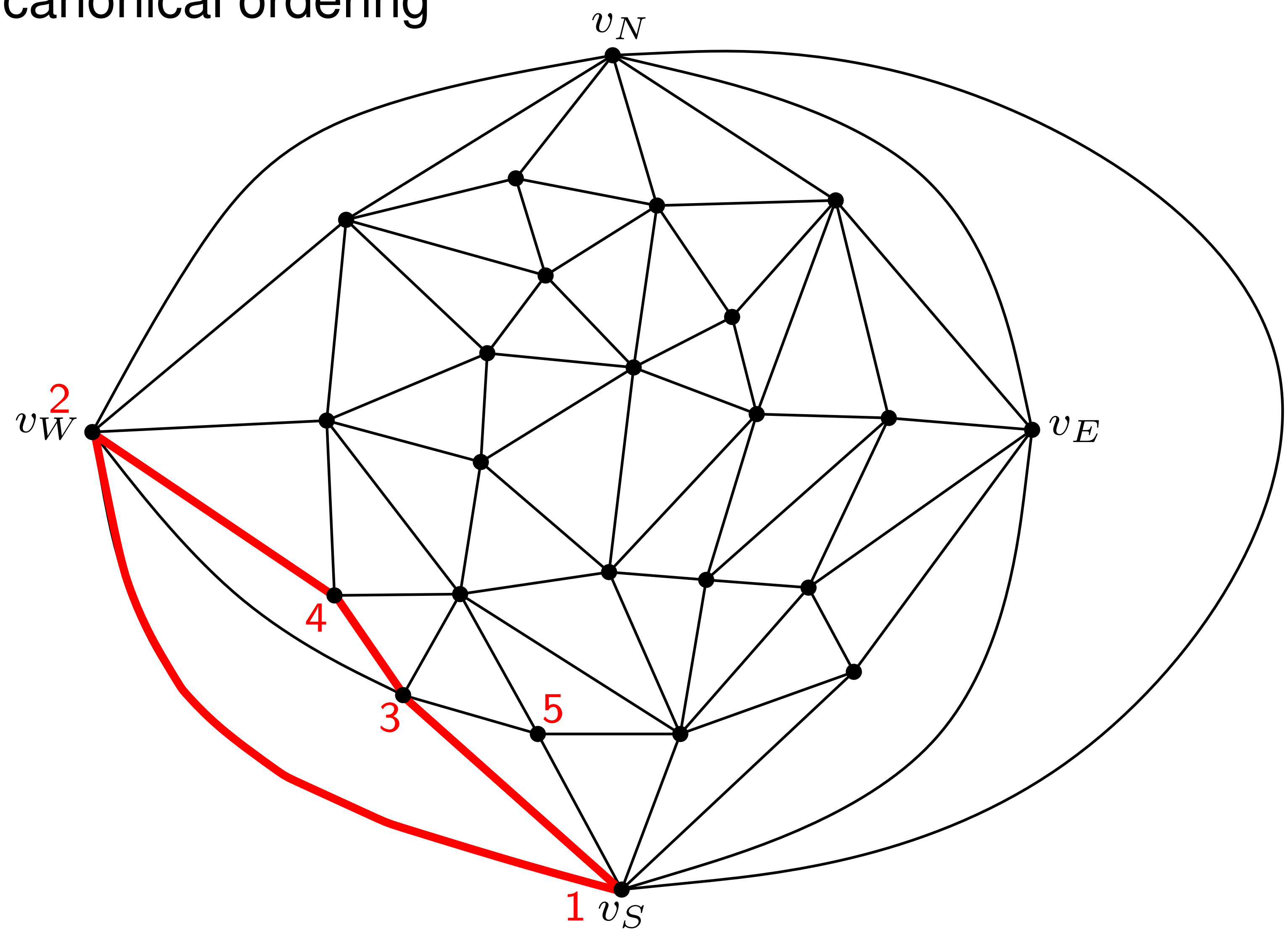

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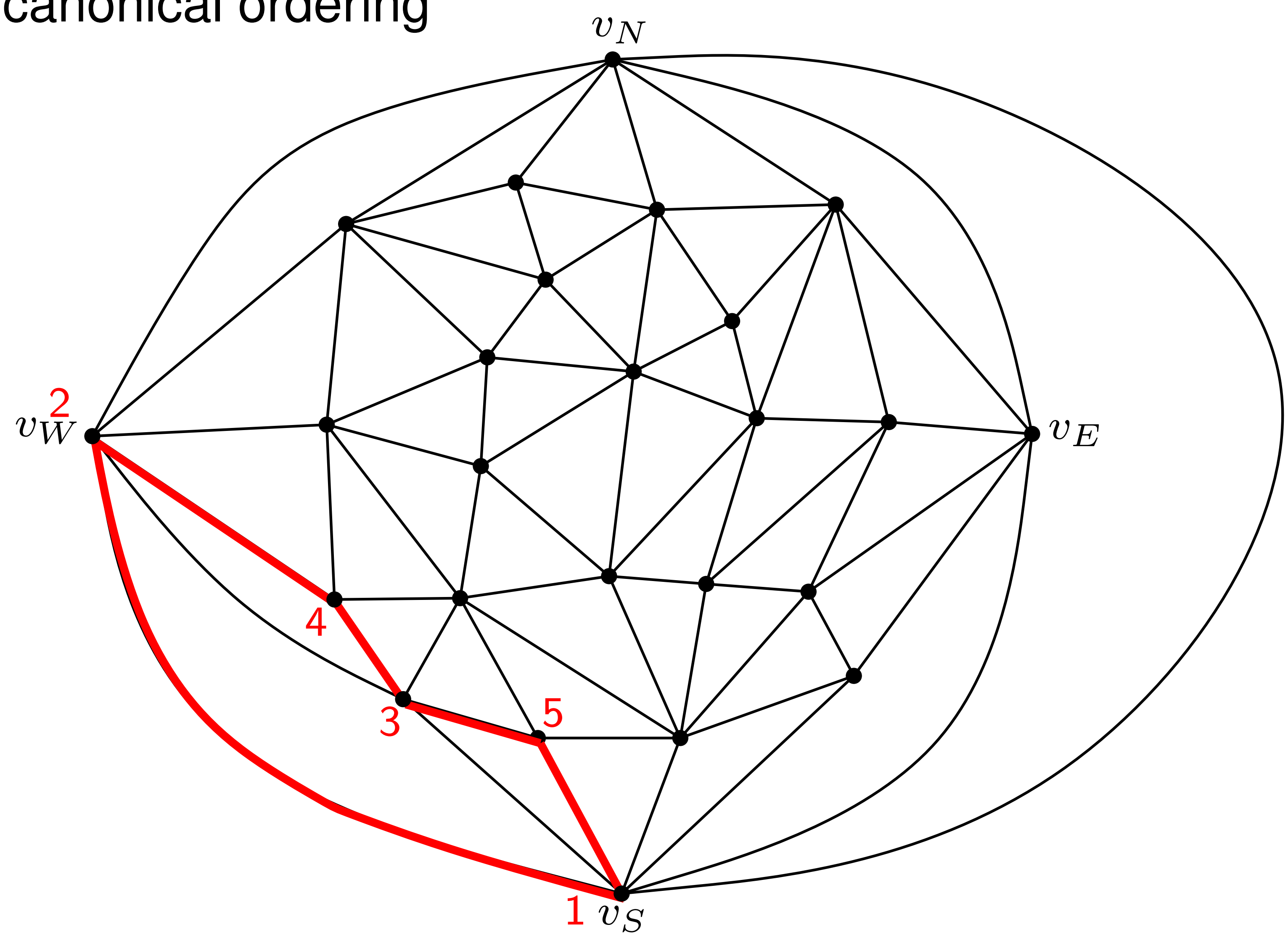

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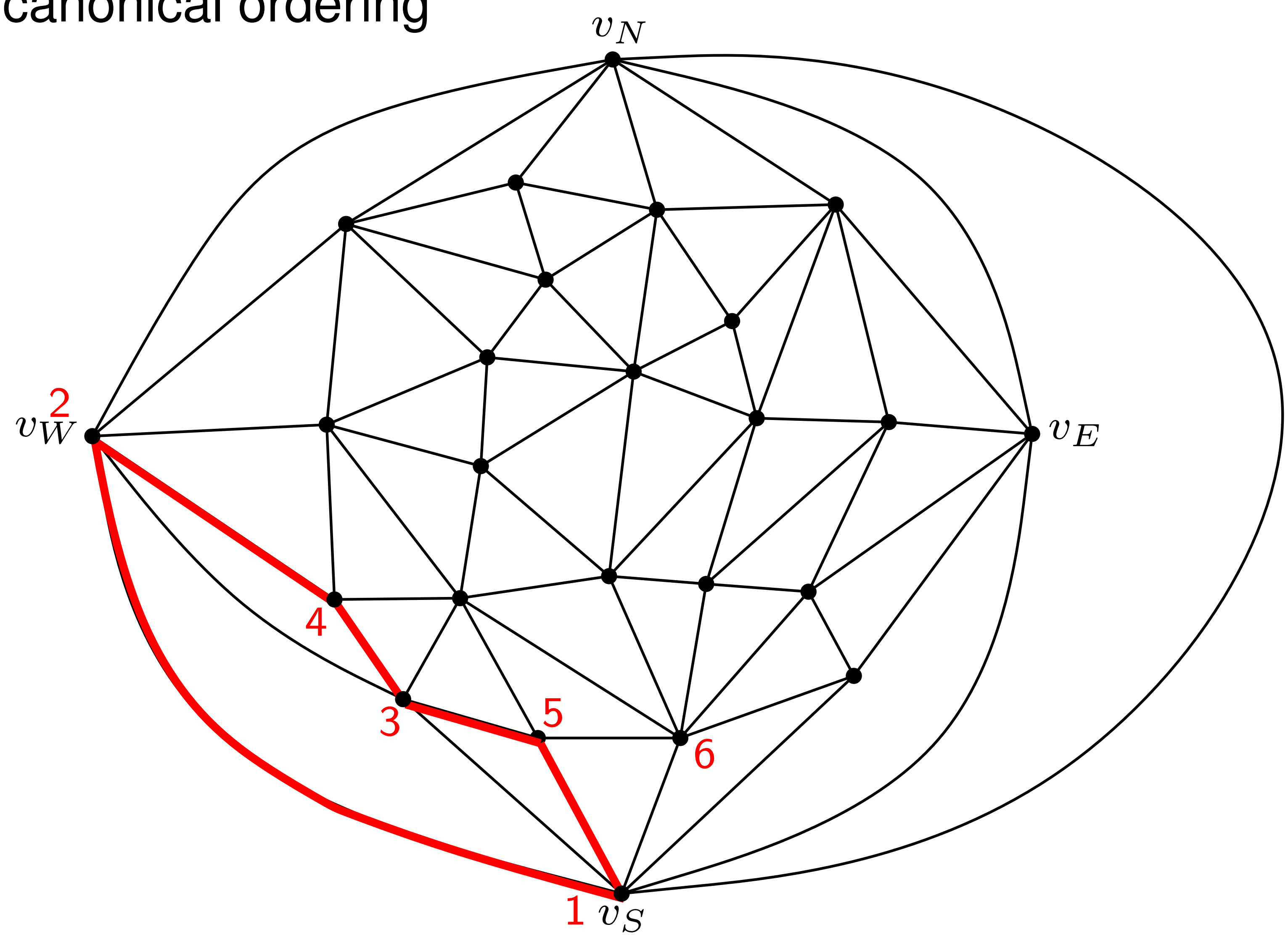

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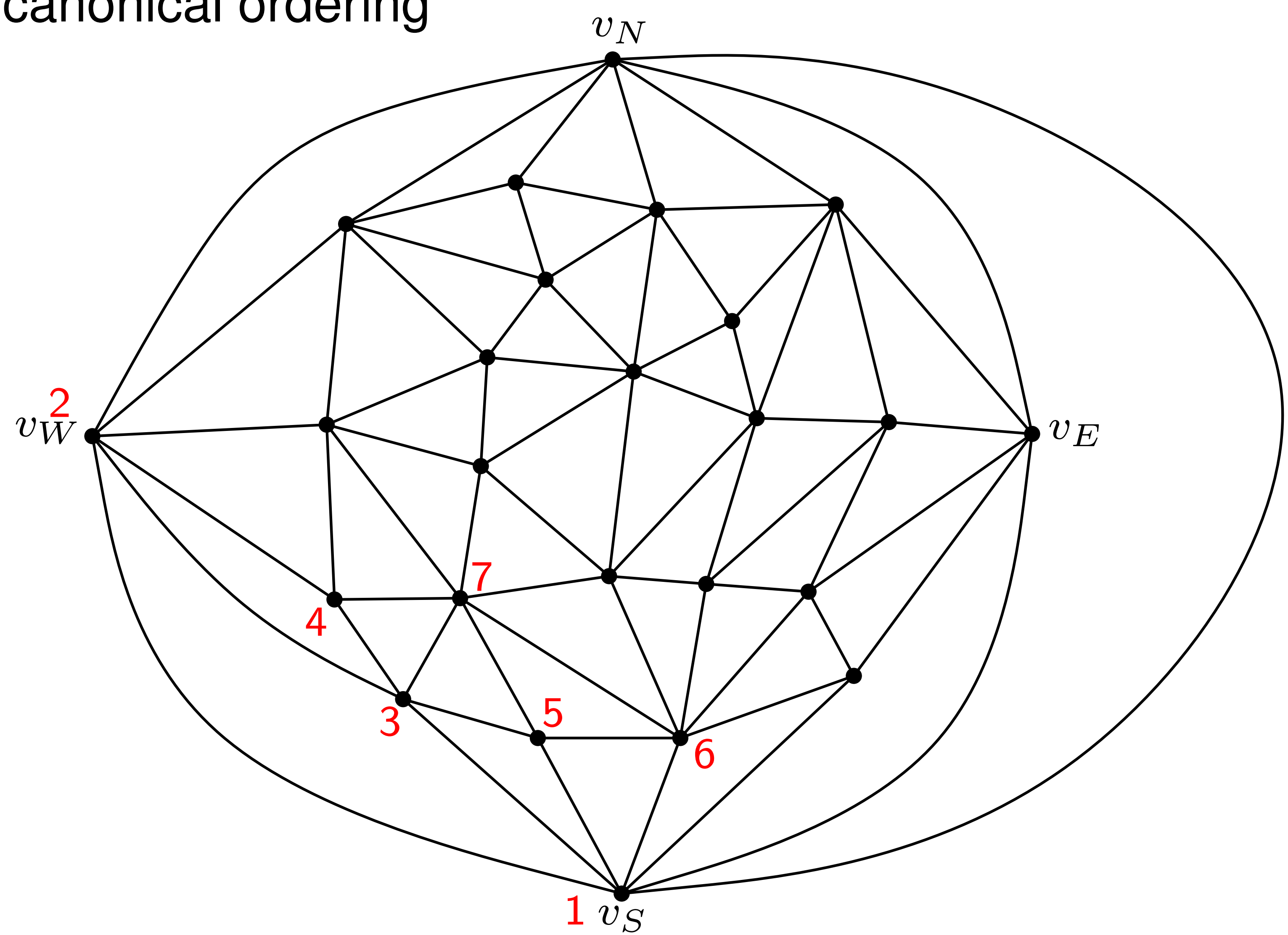

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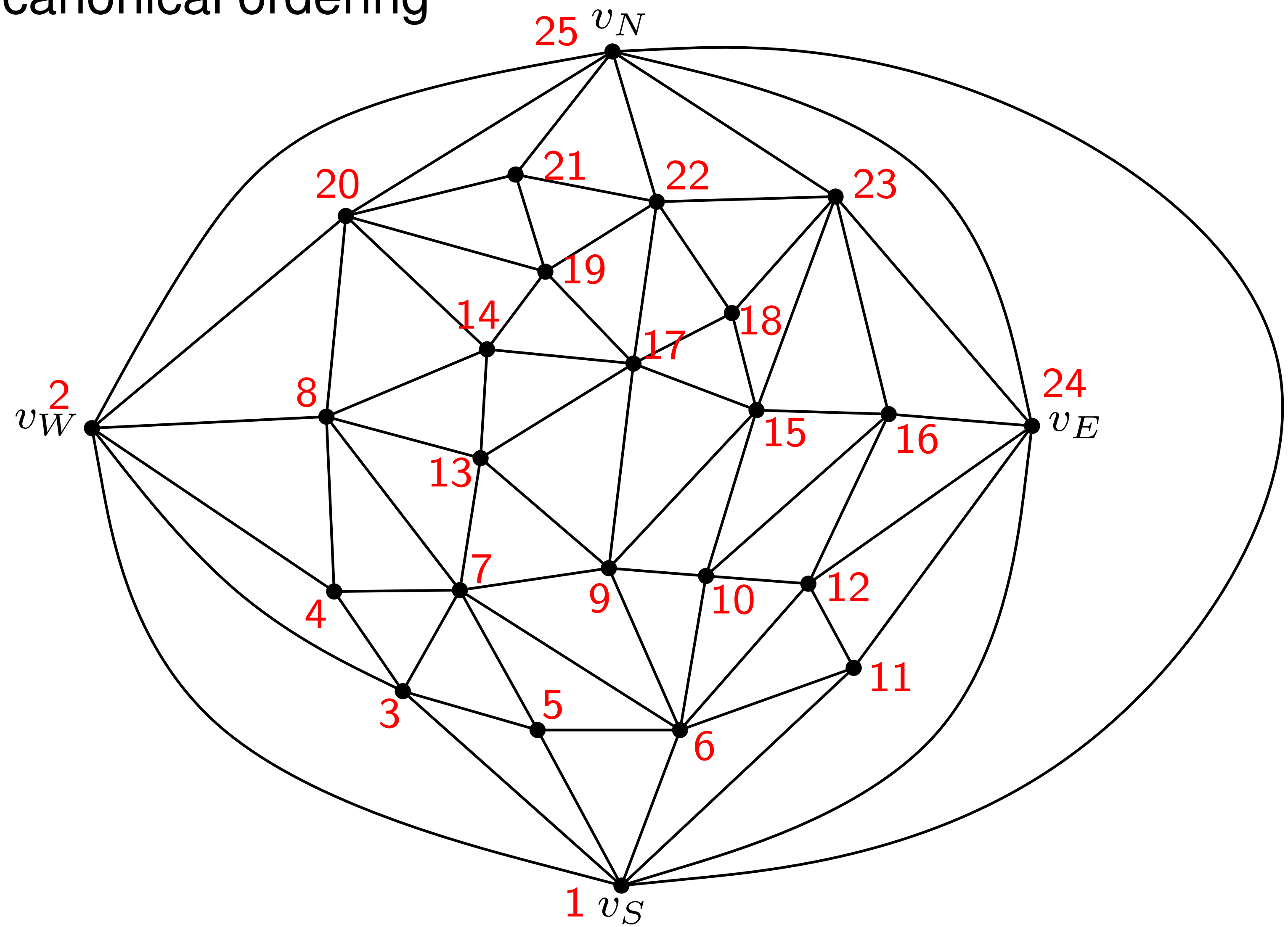

9 - 11

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Given a refined canonical ordering of G we construct a REL as follows:

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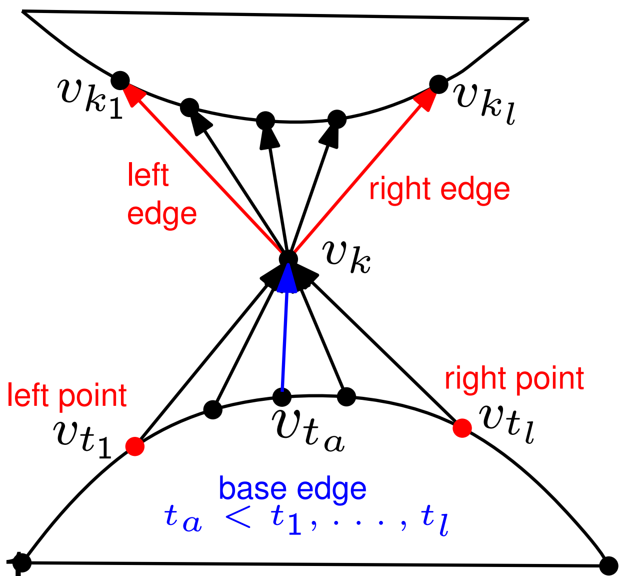

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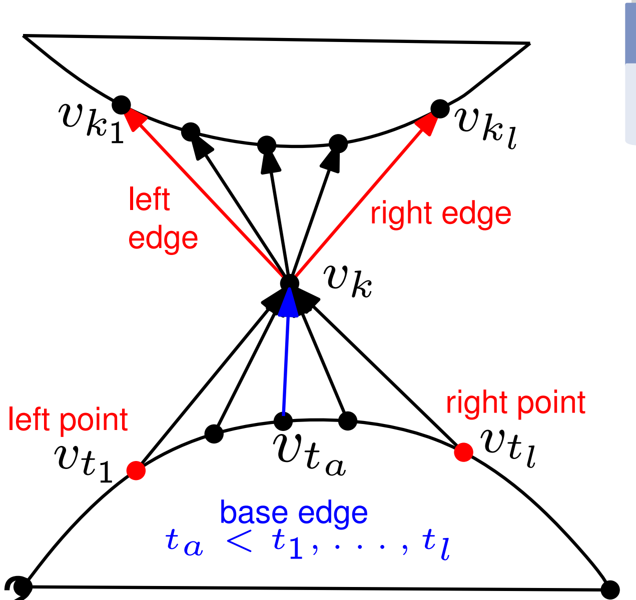

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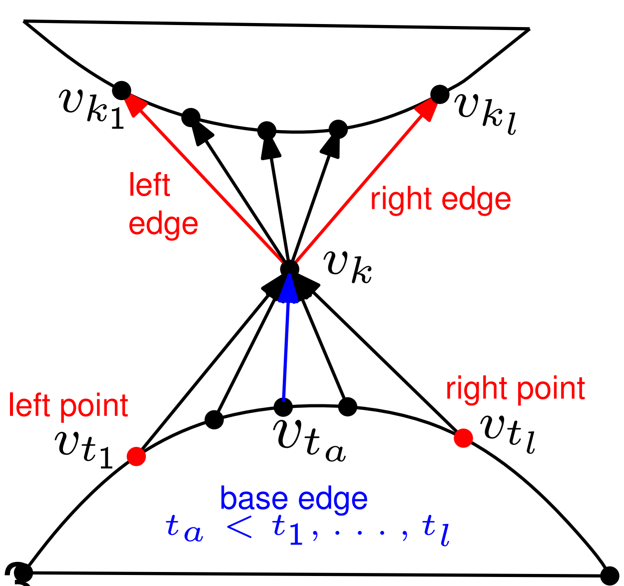

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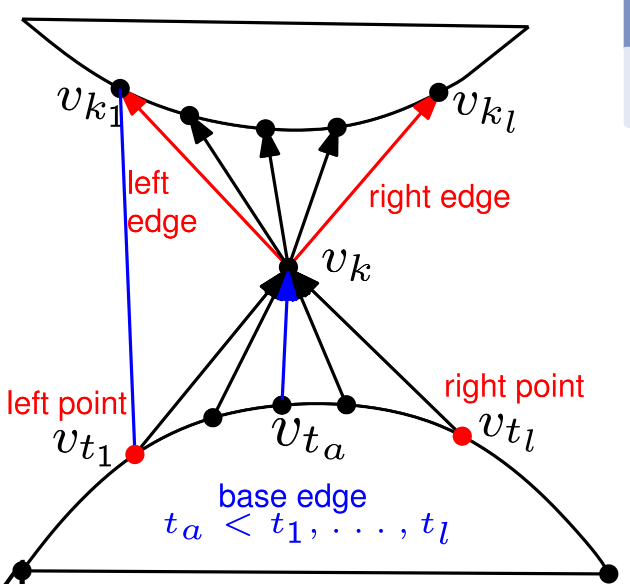

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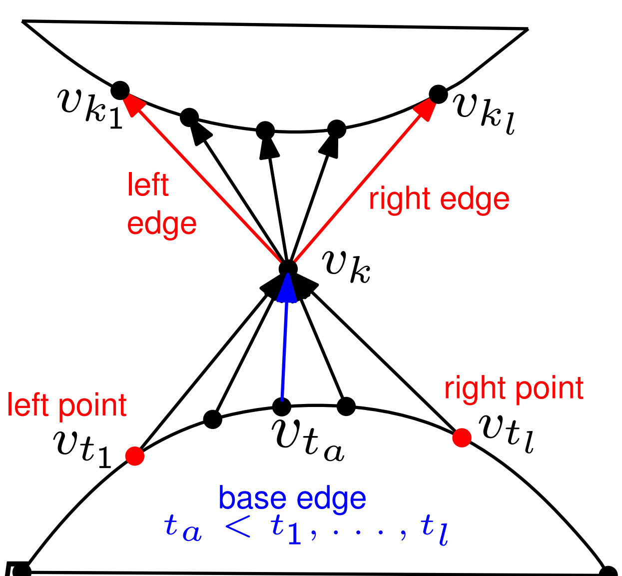

Lemma 2

An edge is either a left edge, a right edge or a base edge.

Proof:

- The exclusive "or" follows from Lemma 1.
- Let (v_{t_a}, v_k) be base edge of v_k.
- v_{t_a} is right point of $v_{t_{a-1}}$, $v_{t_{a-1}}$ is right point of $v_{t_{a-2}}$, generally $v_{t_{i+1}}$ is right point of v_{t_i}, $1 \leq i < a-1$
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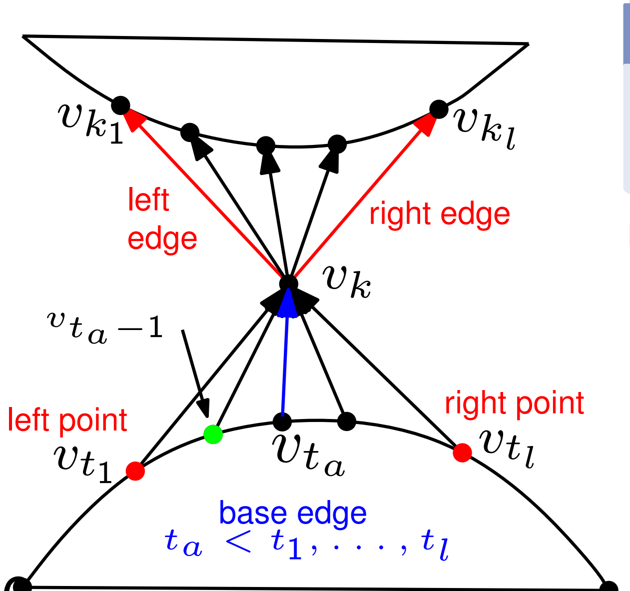

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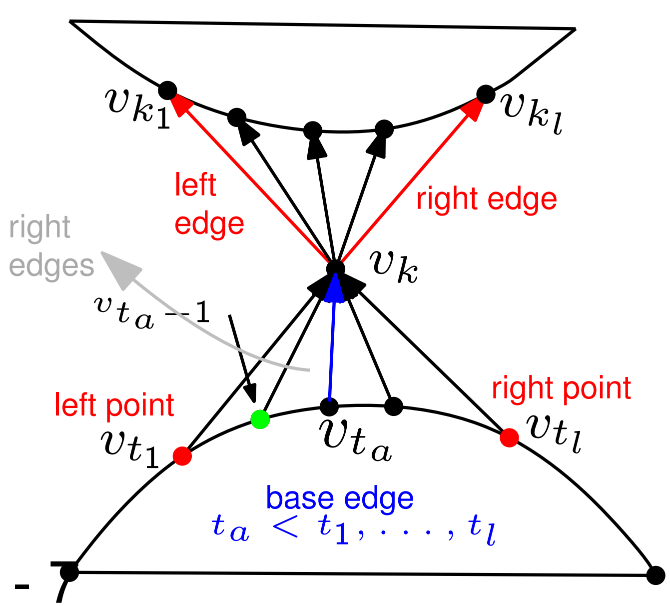

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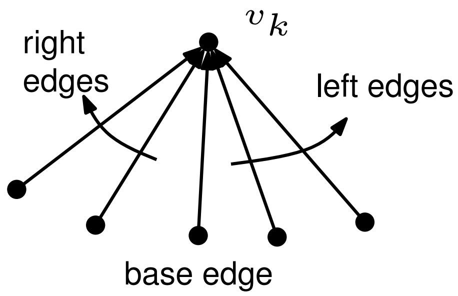

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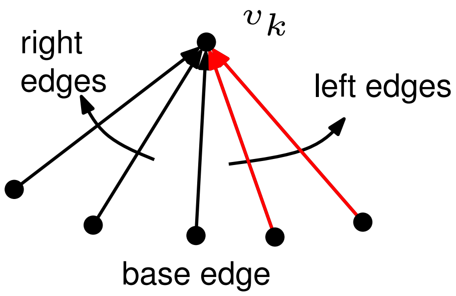

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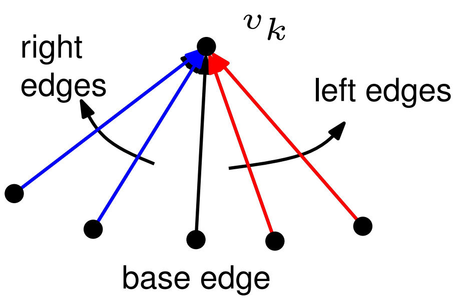

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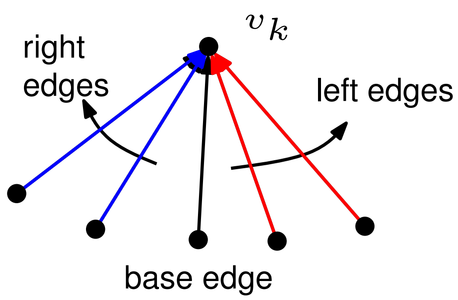

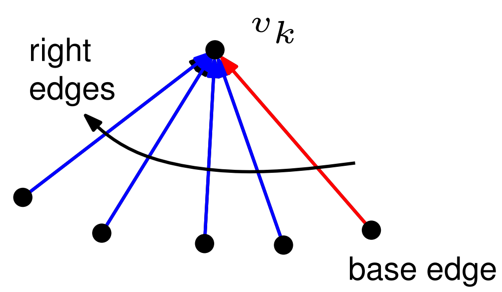

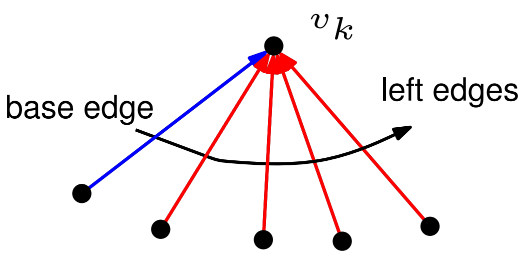

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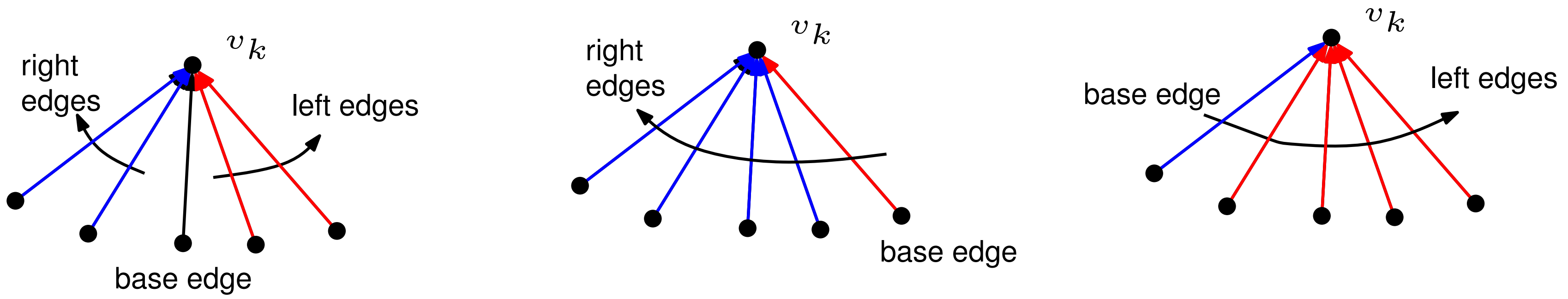
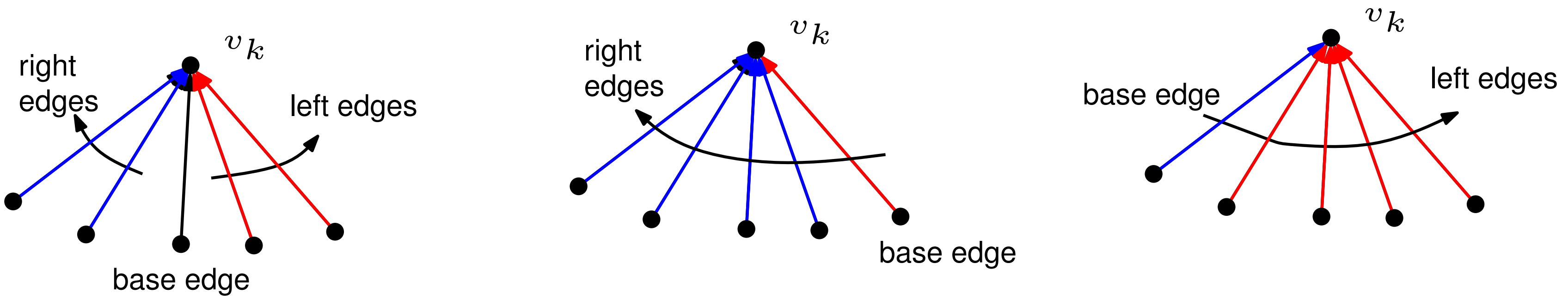


We call T_b blue edges and T_r red edges.

Rectangular Dual



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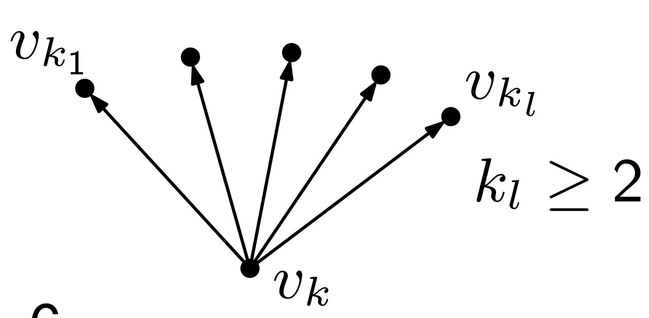

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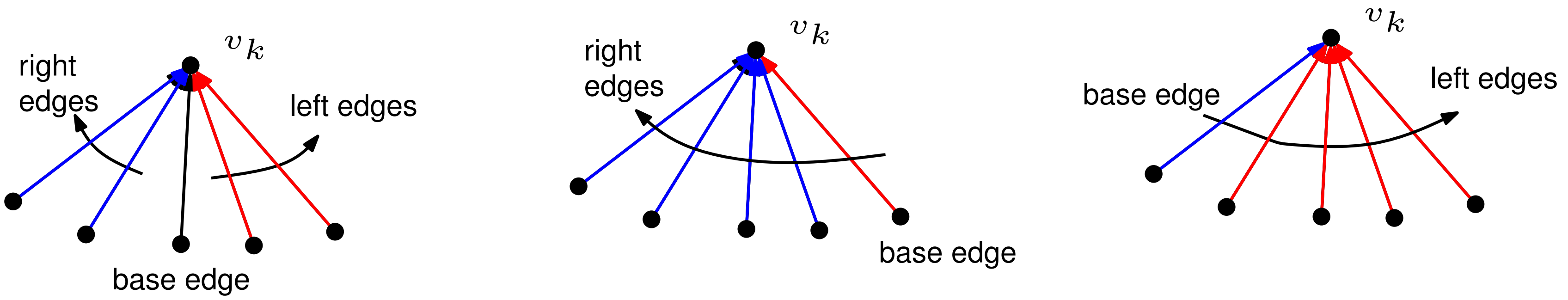

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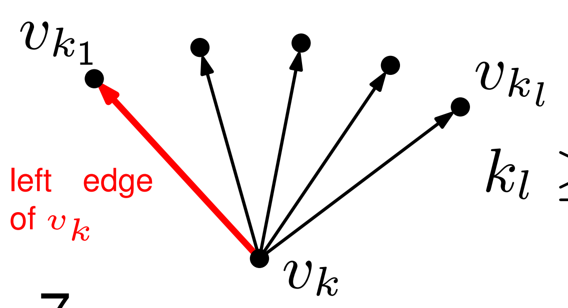
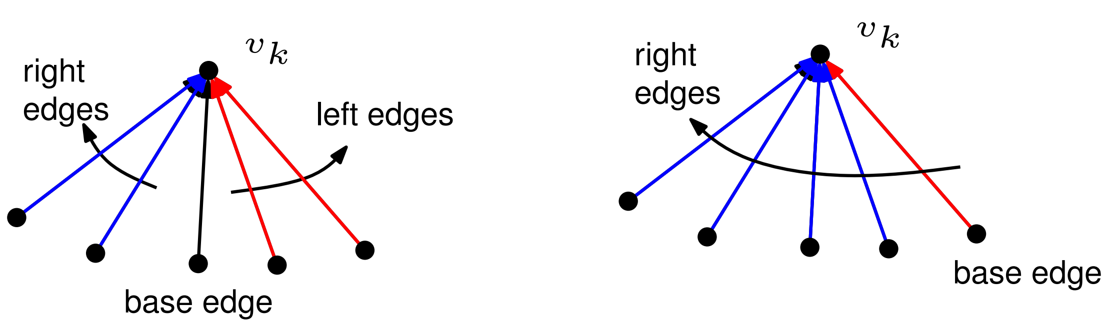


$k_l \geq 2$

Rectangular Dual



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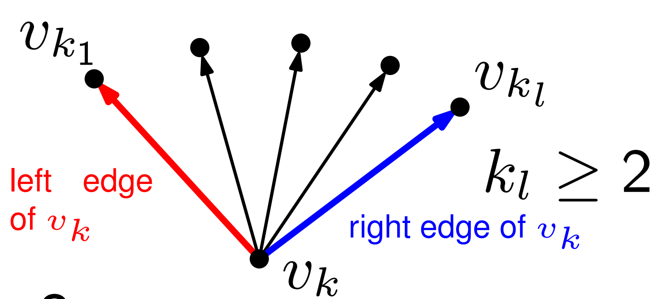

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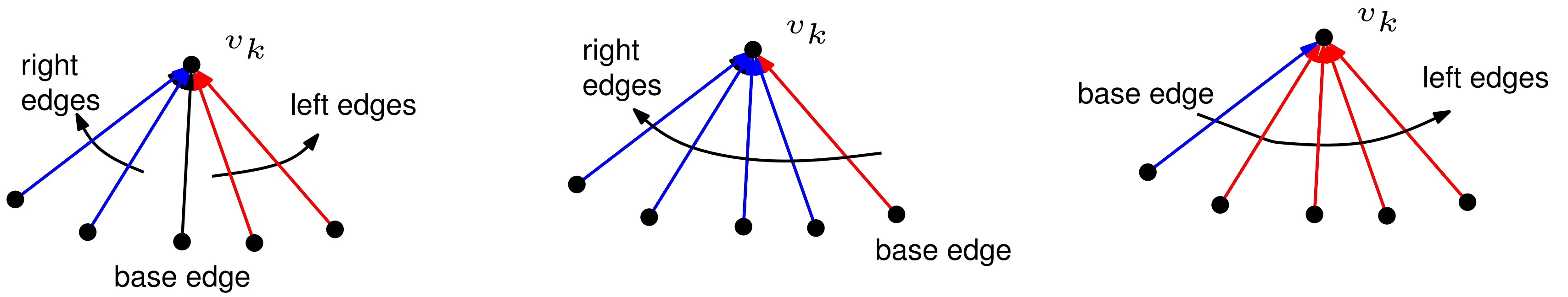

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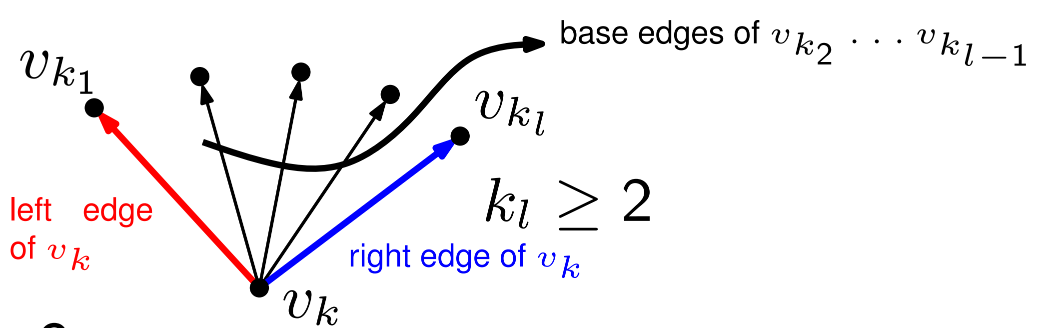

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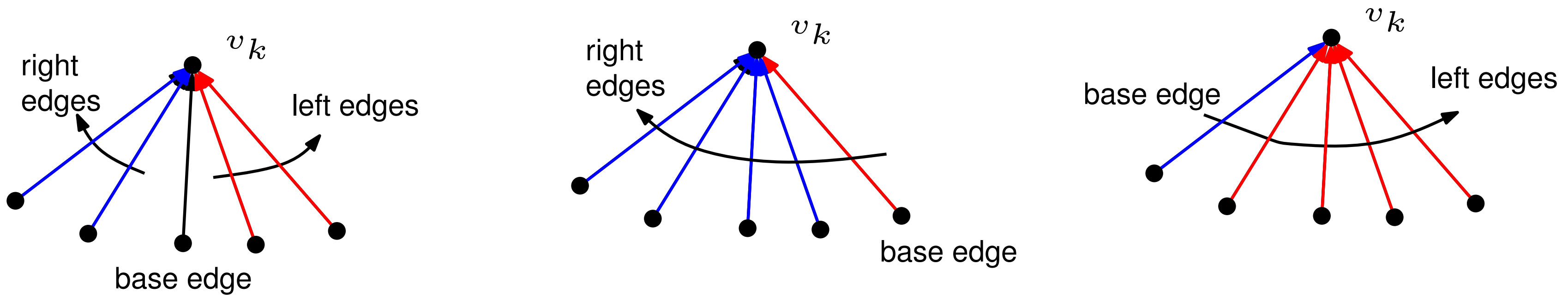

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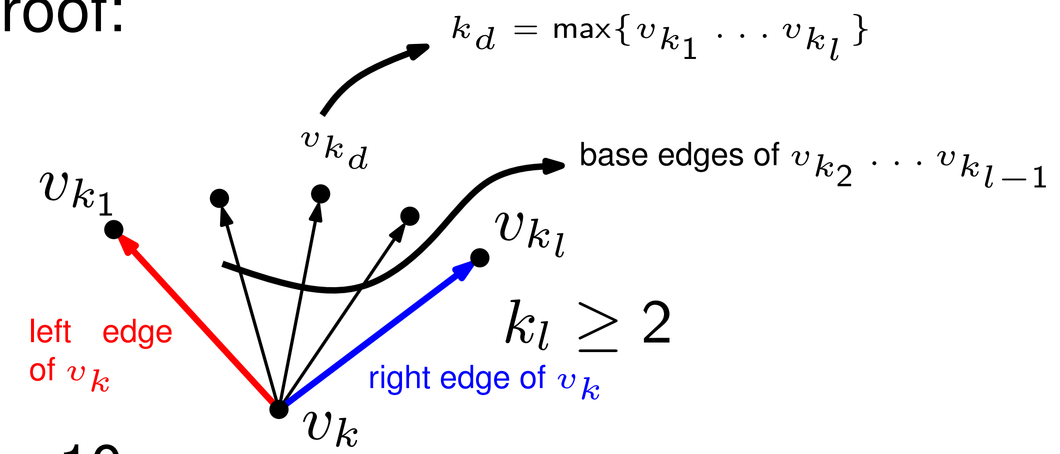

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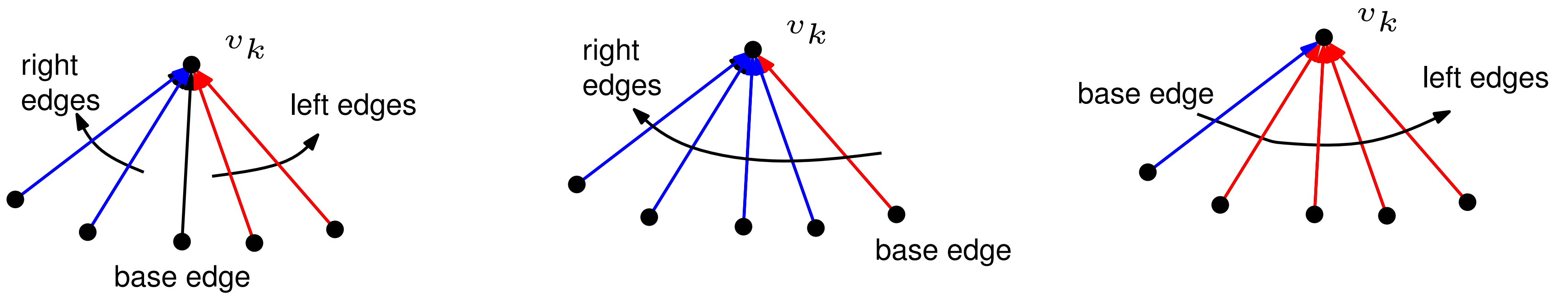

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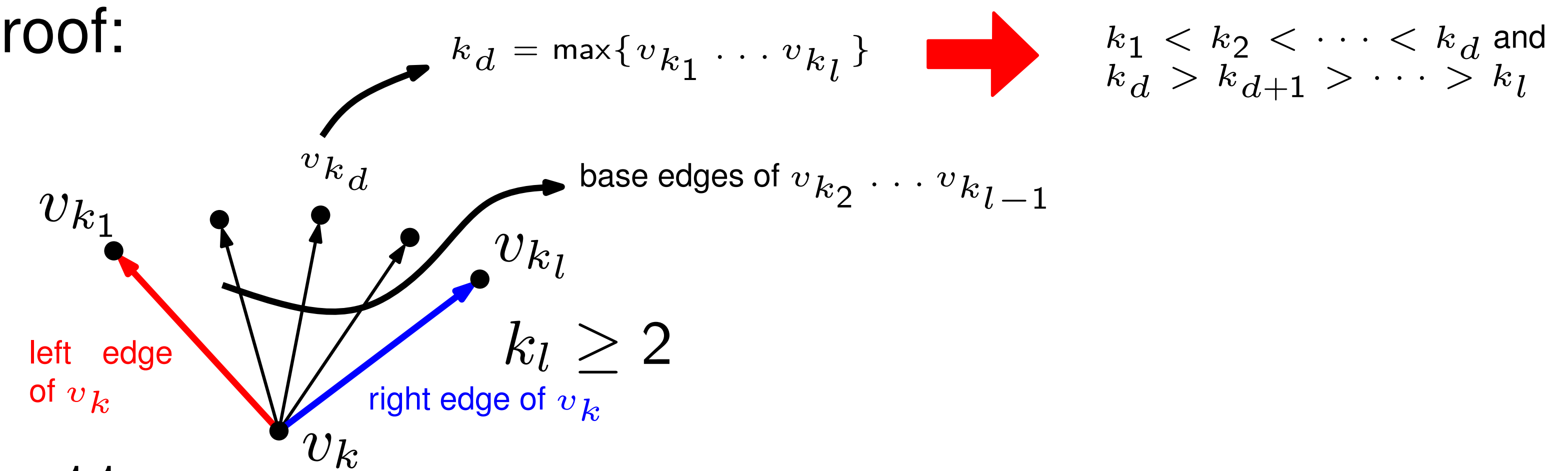

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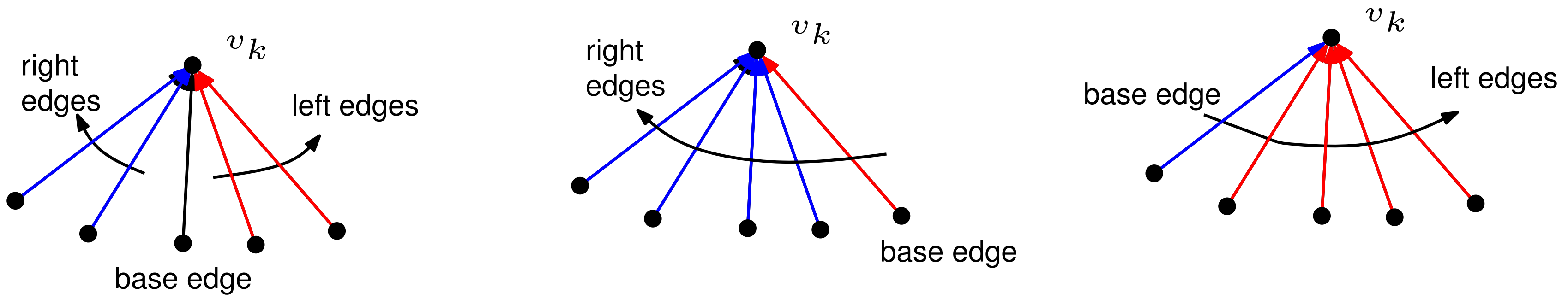

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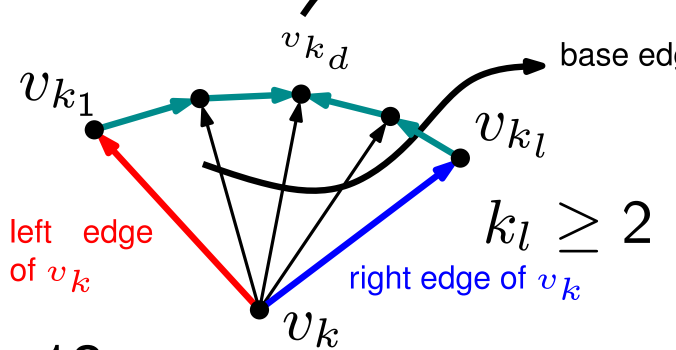
$k_d = \max\{v_{k_1} \dots v_{k_l}\}$ ⇒ $k_1 < k_2 < \cdots < k_d$ and $k_d > k_{d+1} > \cdots > k_l$



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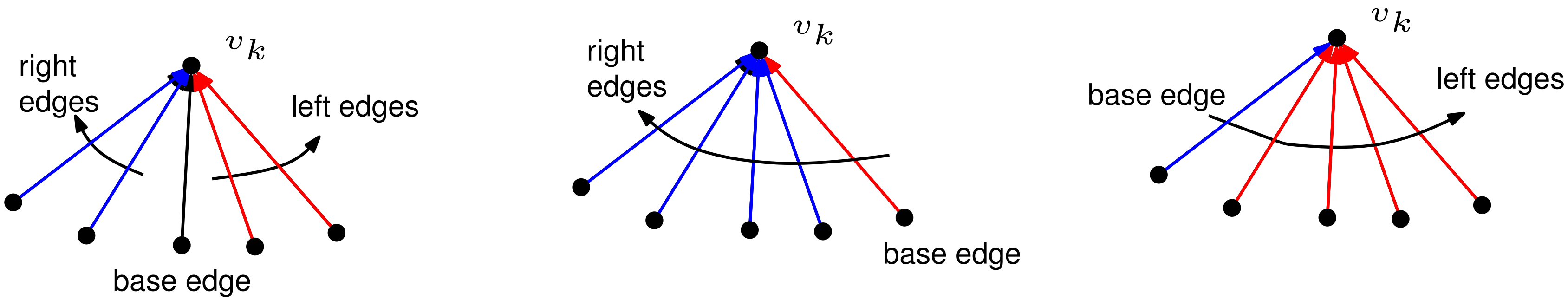

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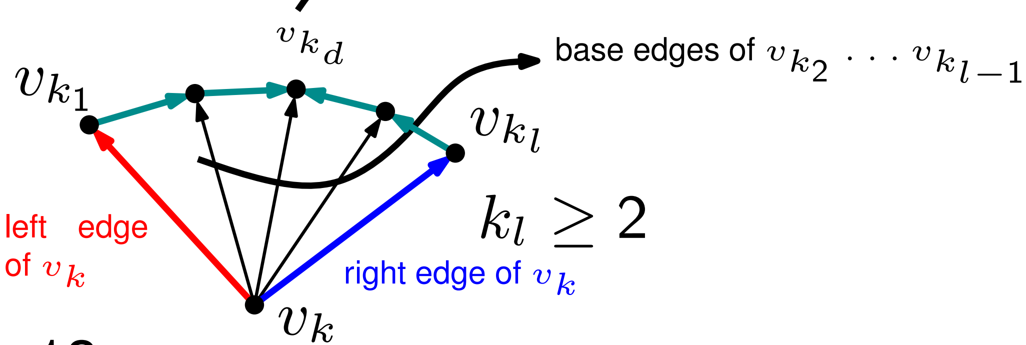

(v_k, v_{k_i}), $2 \leq i \leq d-1$ are red
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edge (v_k, v_{k_d}) is either red or blue

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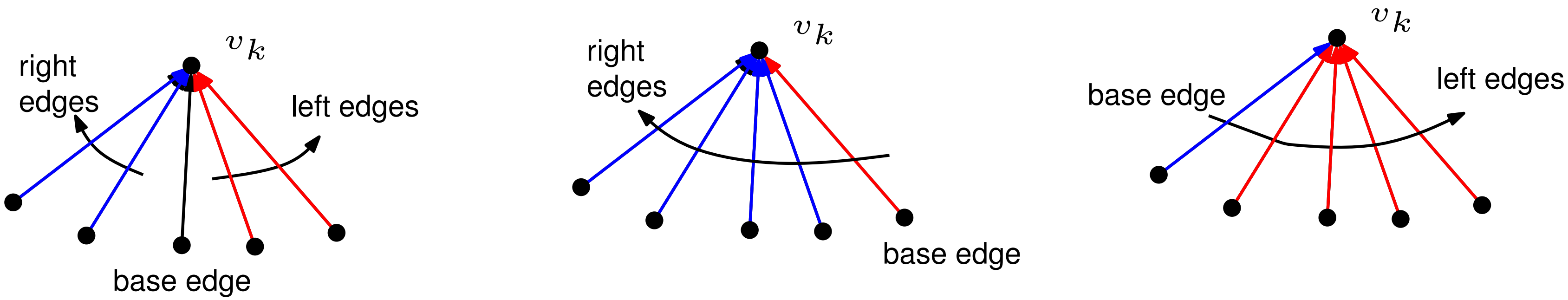

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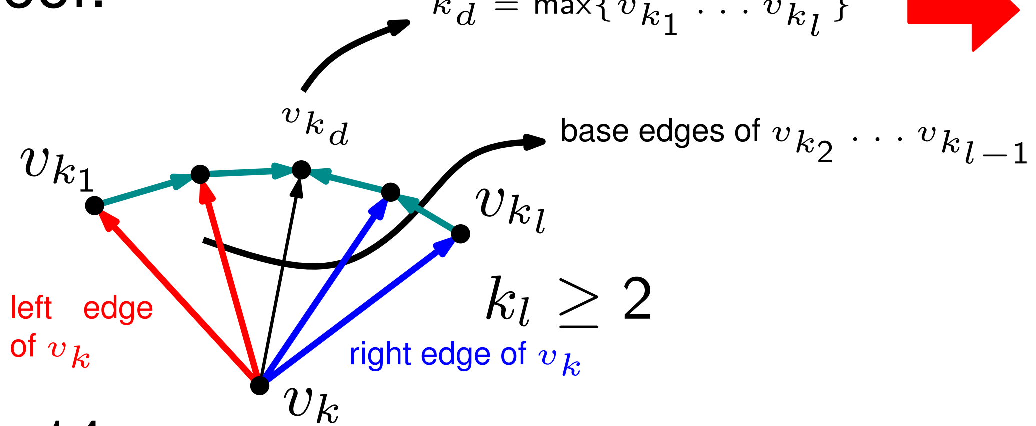

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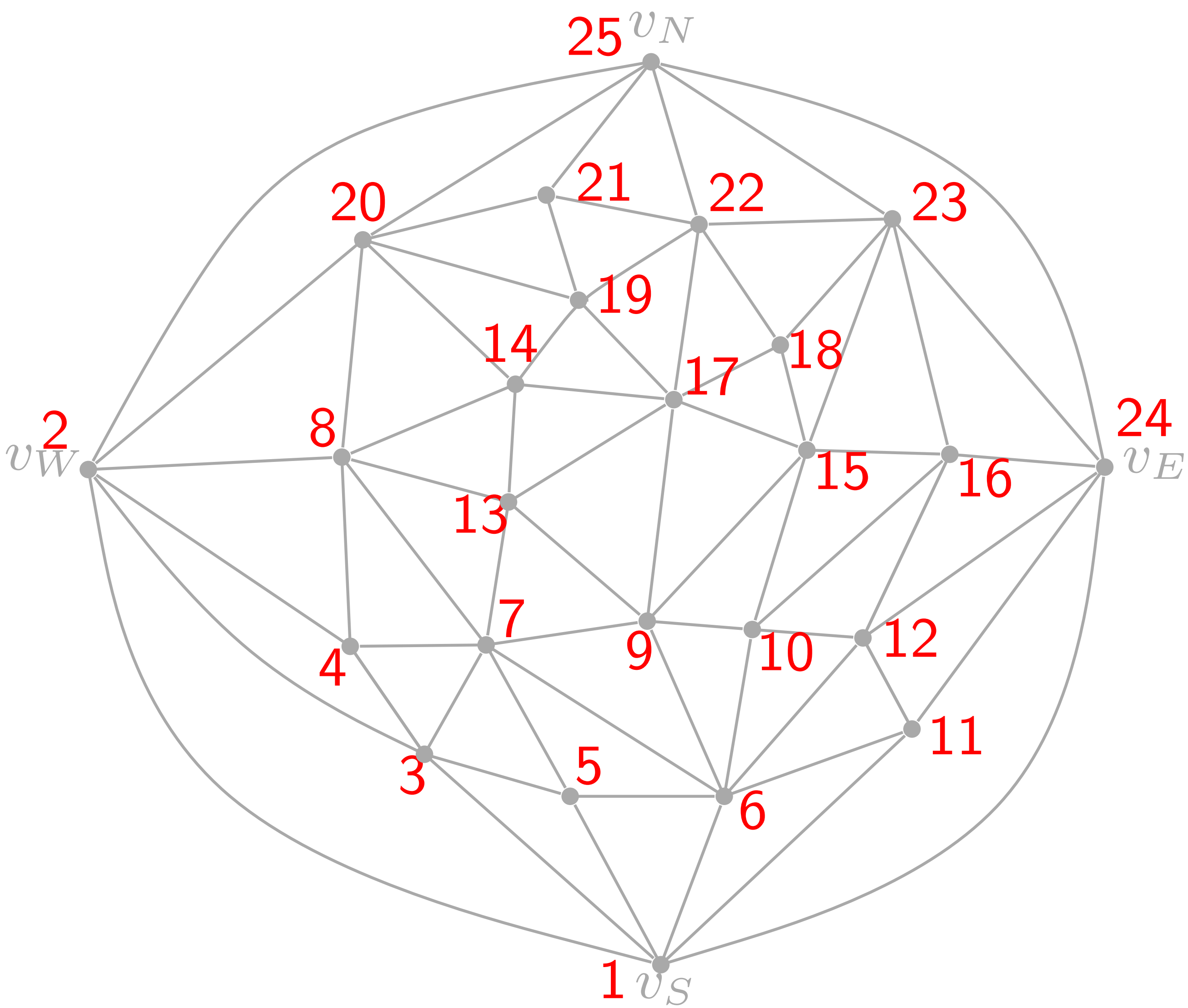

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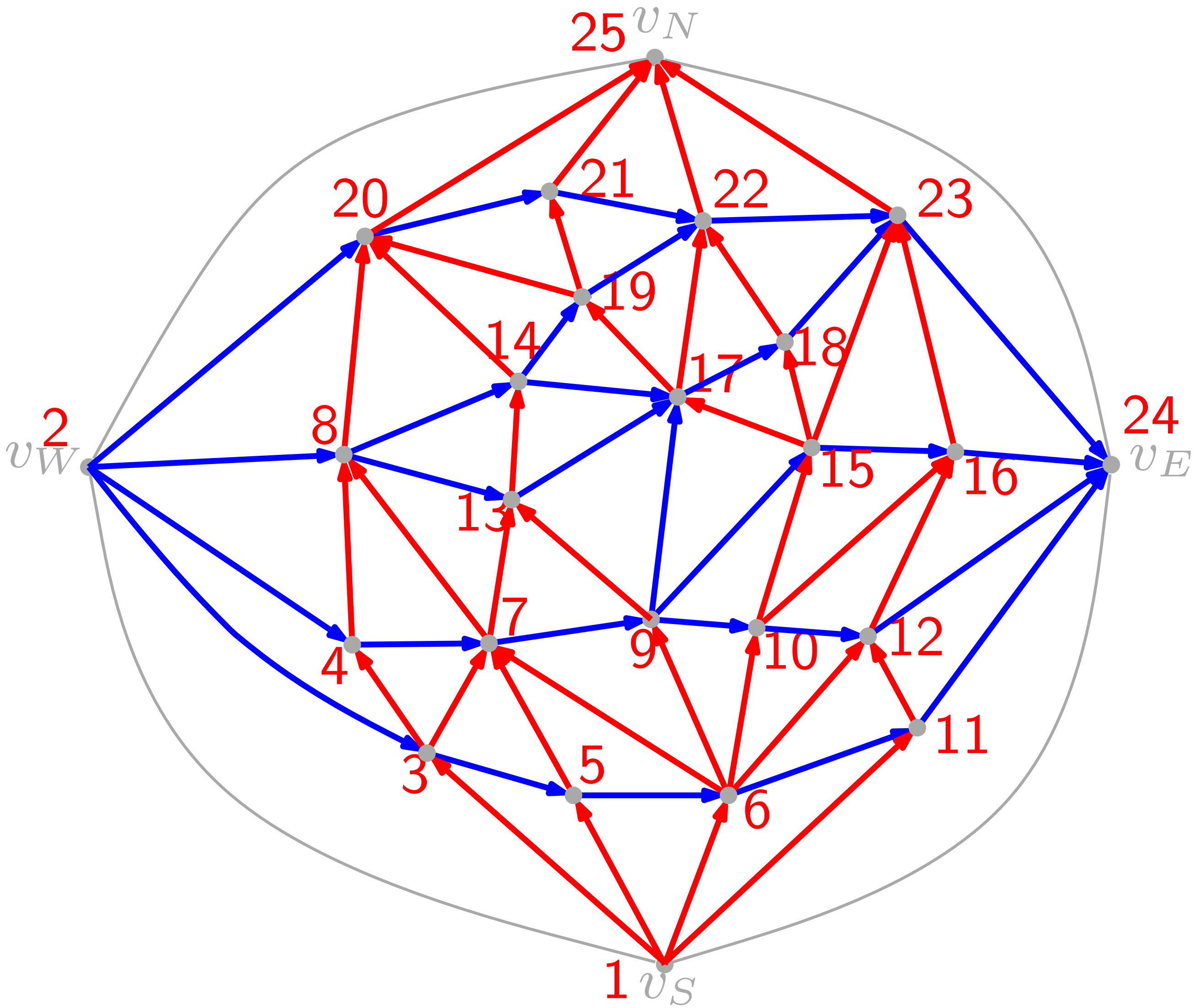

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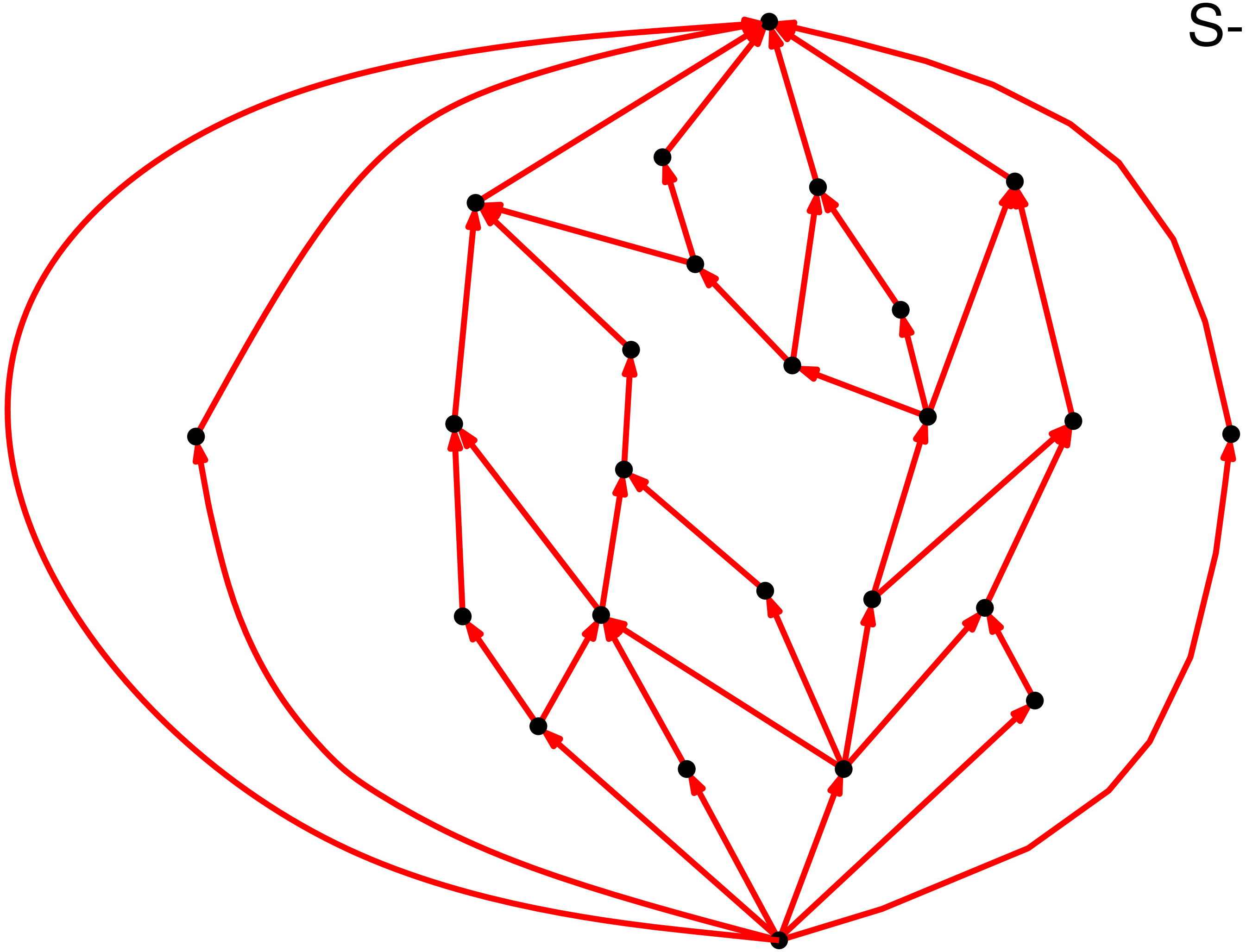
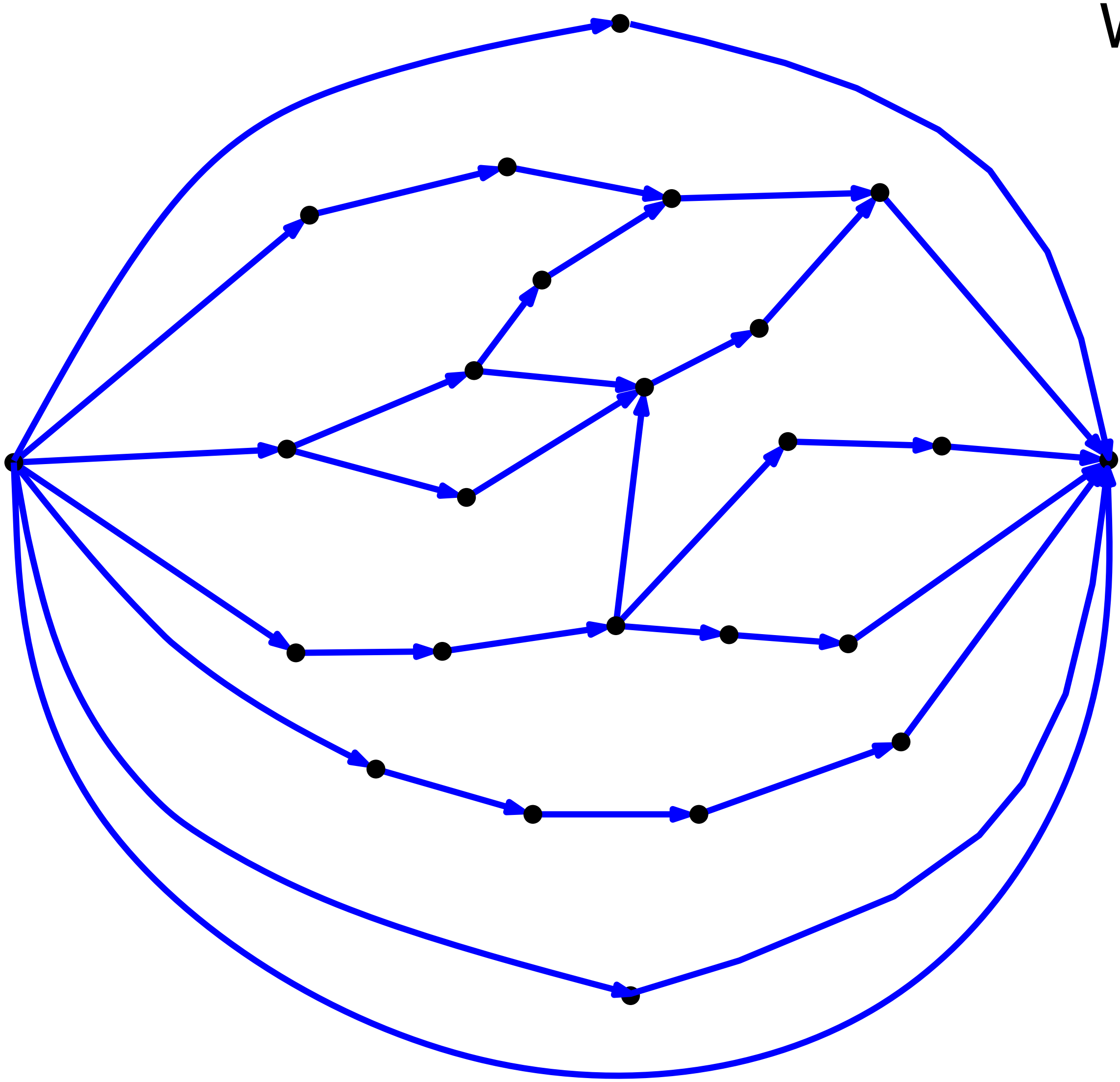
Rectangular Dual

S-N net G_{S-N}



Rectangular Dual

W-E net G_{W-E}



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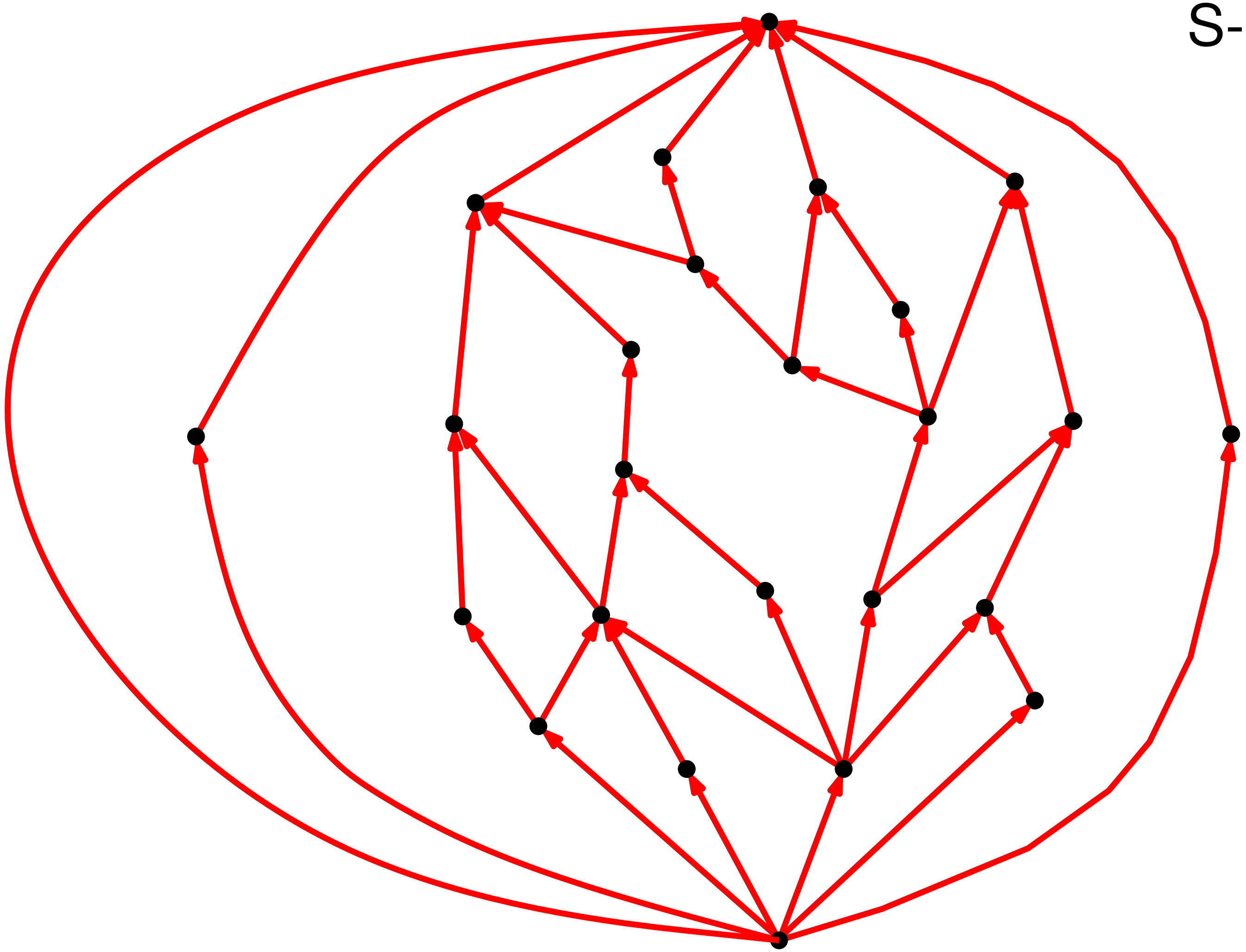

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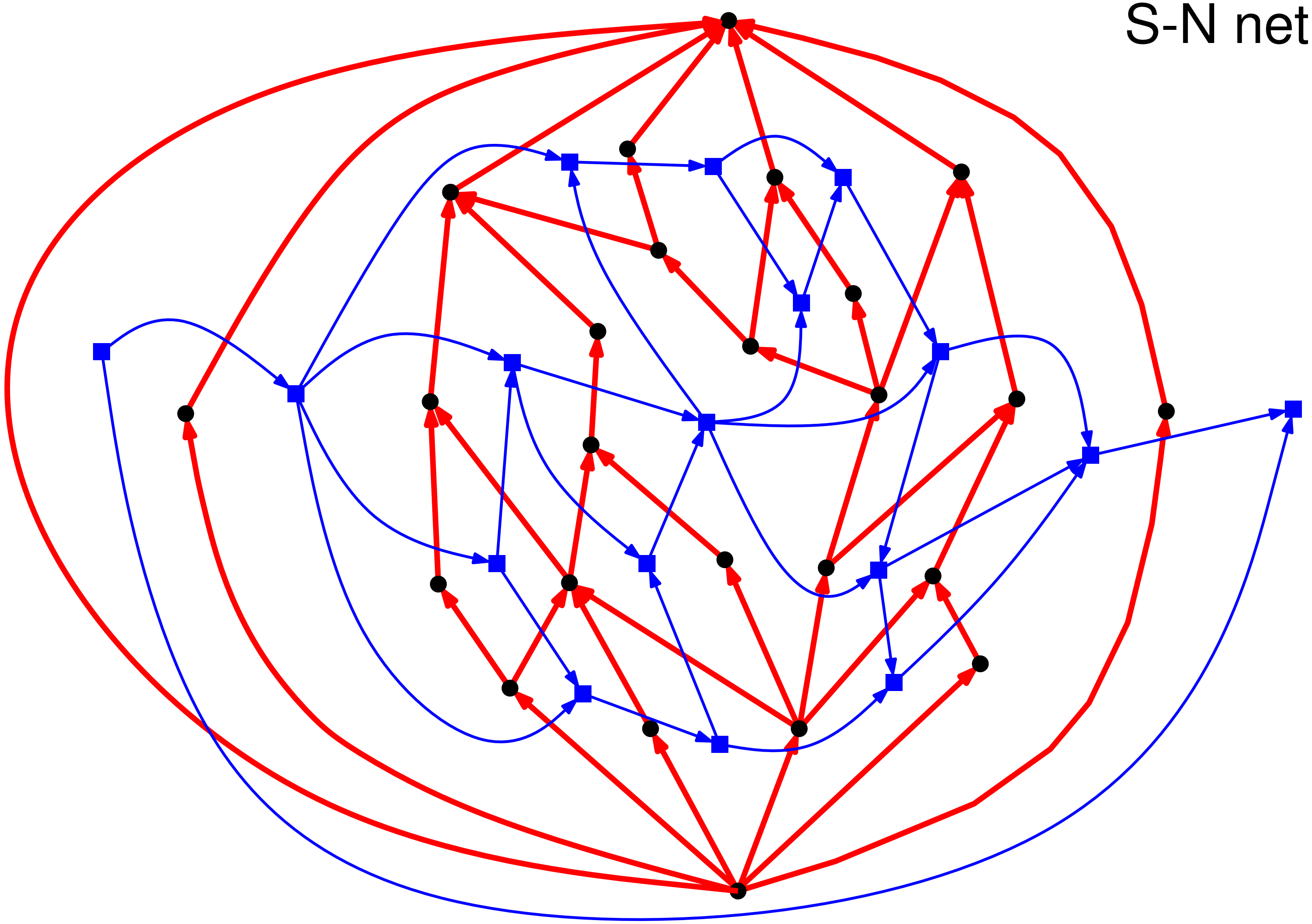

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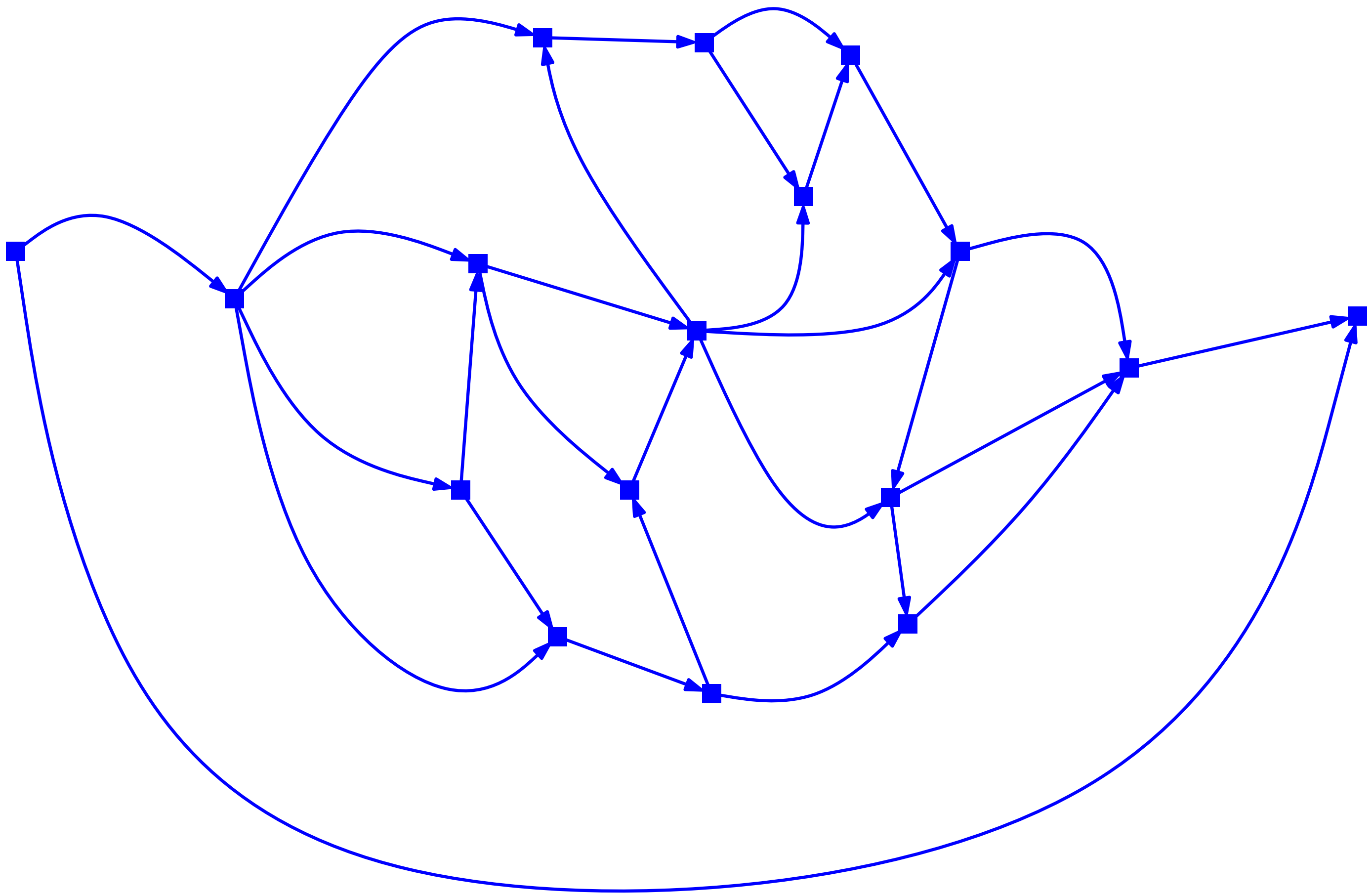

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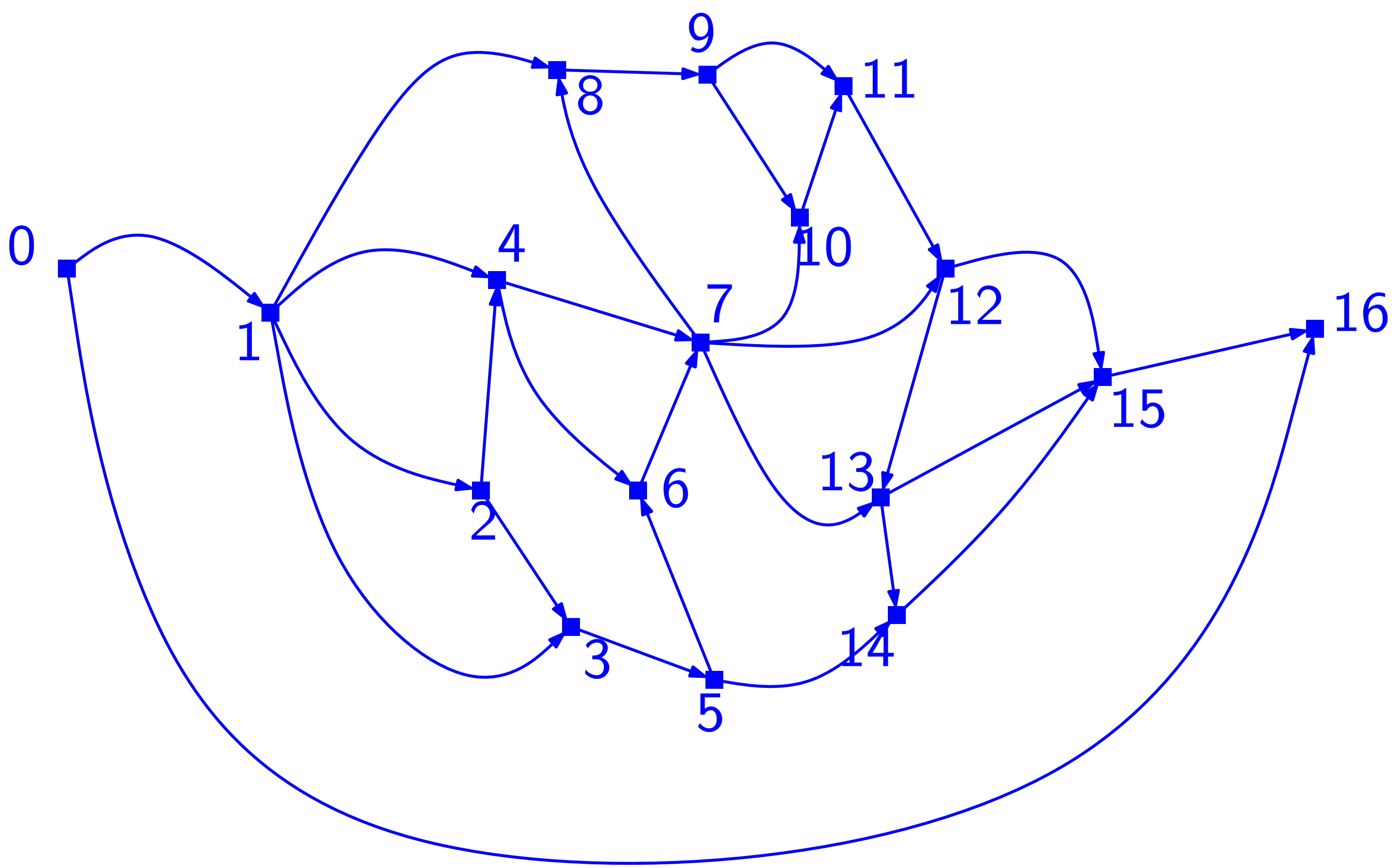

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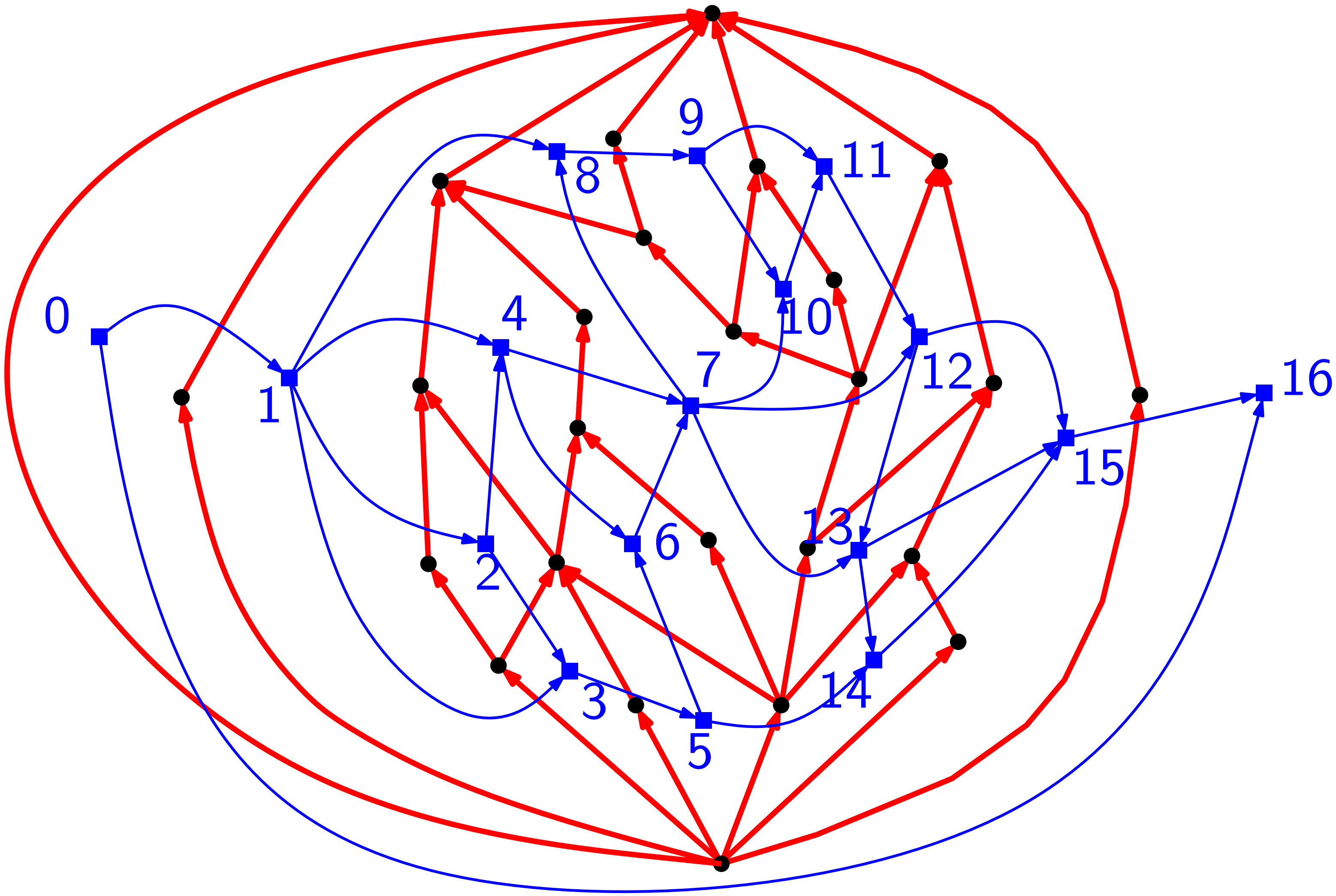

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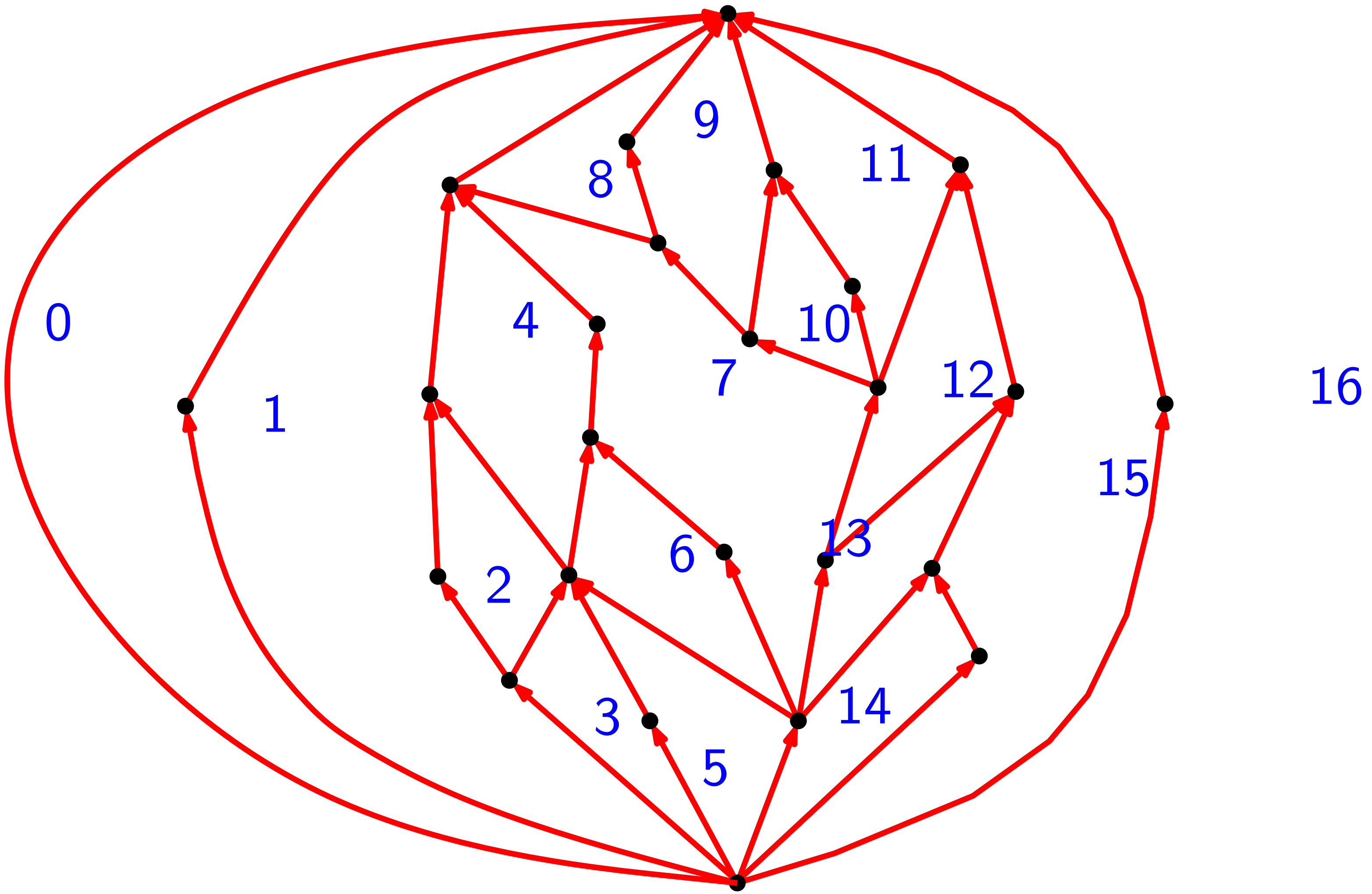
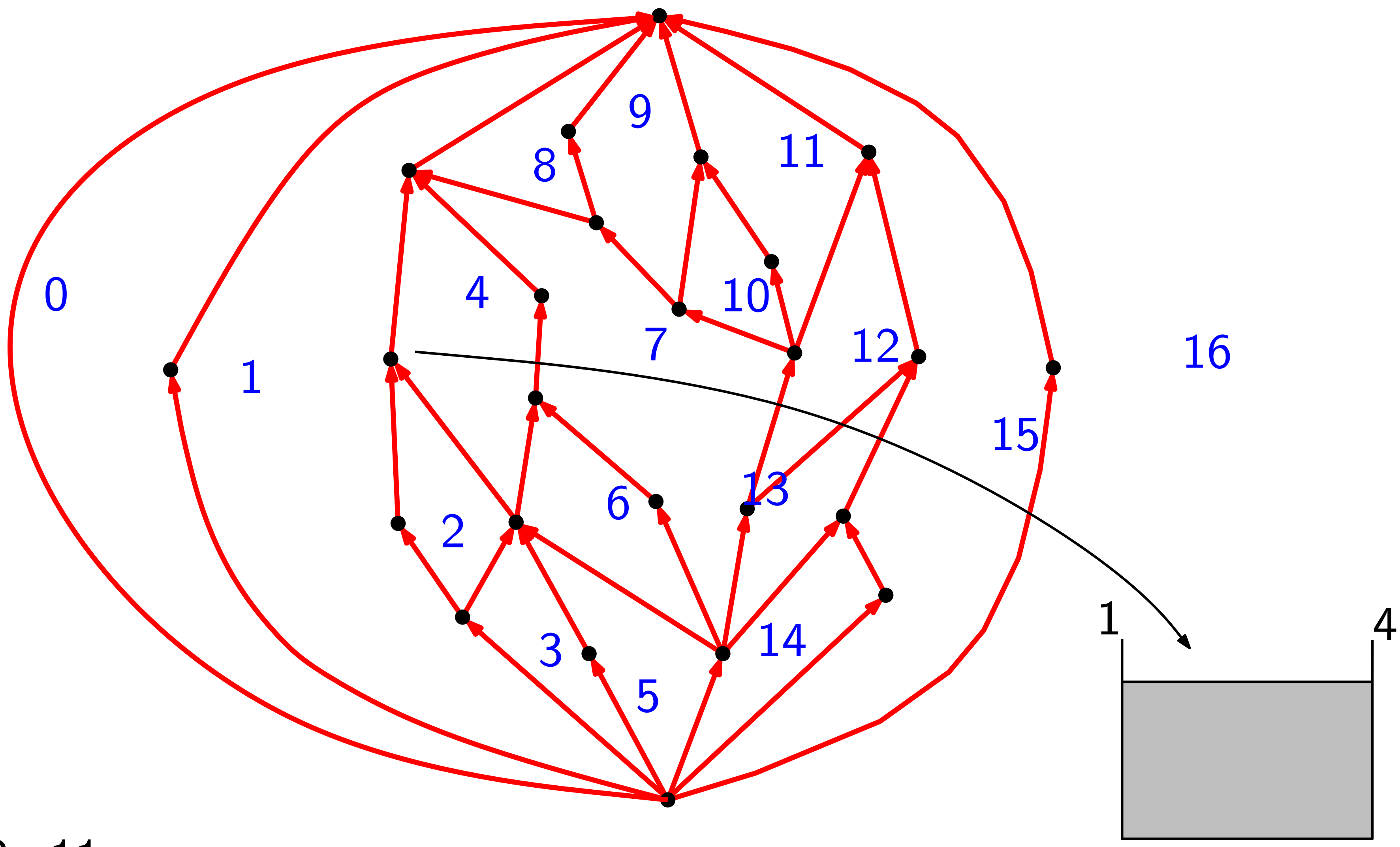
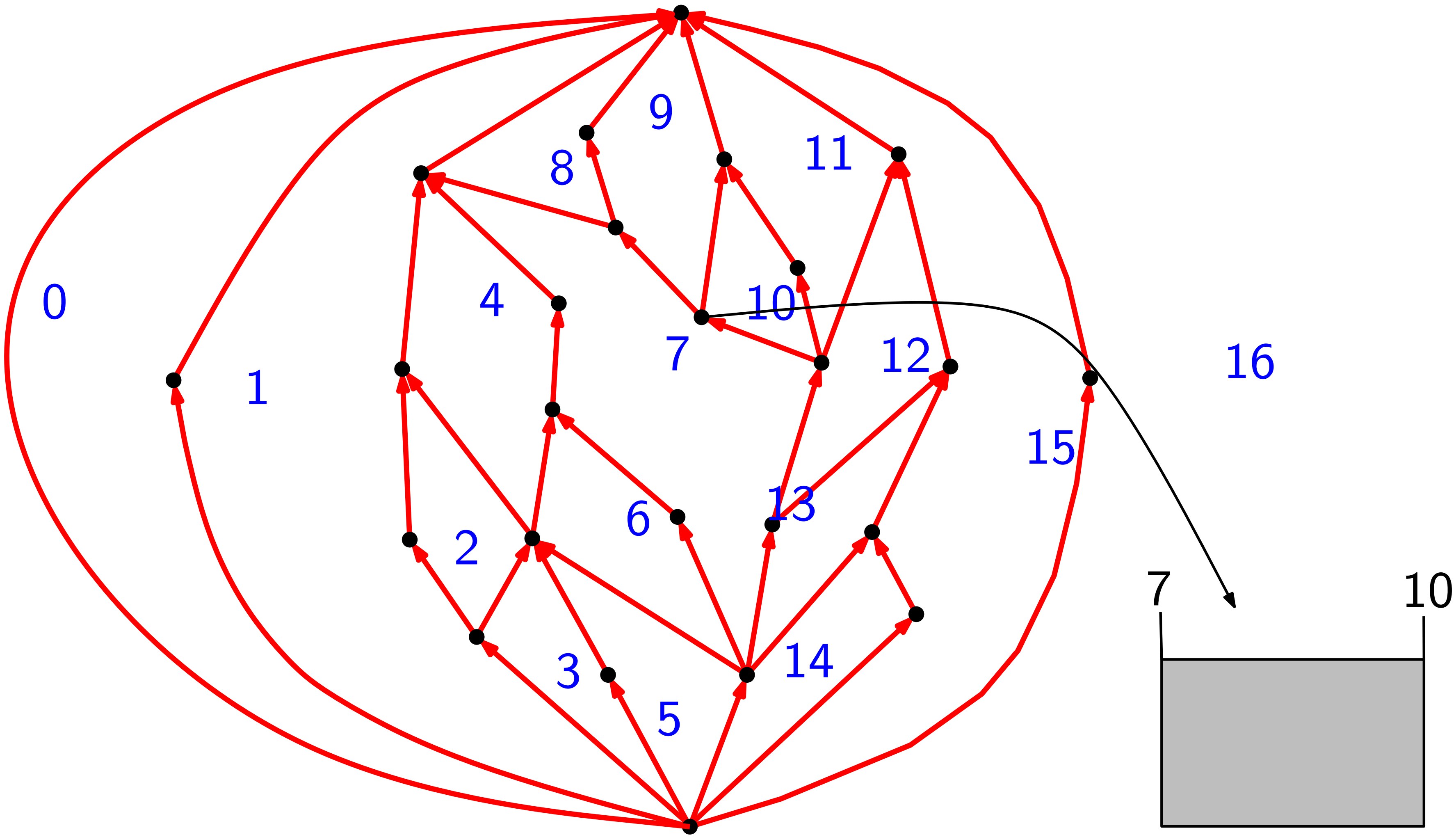


Rectangular Dual



12 - 11

Rectangular Dual



Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a S-N net G_{S-N} of G (consists of T_r plus outer edges)
- Construct the dual $G^{\star}_{S-N}$ of G_{S-N} and compute a topological ordering f_{sn} of $G^{\star}_{S-N}$
- For each vertex $v \in V$, let f and g be the face on the left and face on the right of v. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a net of G (consists of plus outer edges)
- Construct the dual o and compute a topological ordering of
- For each vertex $v \in V$, let f and g be the face and face v. Set $= f_{sn}(f)$ and $= f_{sn}(g)$.
- Define and

13 - 2

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a $\boxed{\text{W-E}}$ net $\boxed{G_{W-}}$ of G (consists of $\boxed{T_b}$ plus outer edges)
- Construct the dual o and compute a topological ordering of
- For each vertex $v \in V$, let f and g be the face and face v. Set $= f_{sn}(f)$ and $= f_{sn}(g)$.
- Define and

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a W-E net G_{W-} of G (consists of T_b plus outer edges)
- Construct the dual $G^{\star}_{W-E}$ of G_{W-E} and compute a topological ordering f_{we} of $G^{\star}_{W-E}$
- For each vertex $v \in V$, let f and g be the face and face v. Set $= f_{sn}(f)$ and $= f_{sn}(g)$.
- Define and

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a W-E net G_{W-} of G (consists of T_b plus outer edges)
- Construct the dual $G^\star_{W-E}$ o G_{W-} and compute a topological ordering f_{we} of $G^\star_{W-E}$
- For each vertex $v \in V$, let f and g be the face below and face above v. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$.
- Define and

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a W-E net G_{W-E} of G (consists of T_b plus outer edges)
- Construct the dual $G^\star_{W-E}$ of G_{W-E} and compute a topological ordering f_{we} of $G^\star_{W-E}$
- For each vertex $v \in V$, let f and g be the face below and face above v. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$.
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$

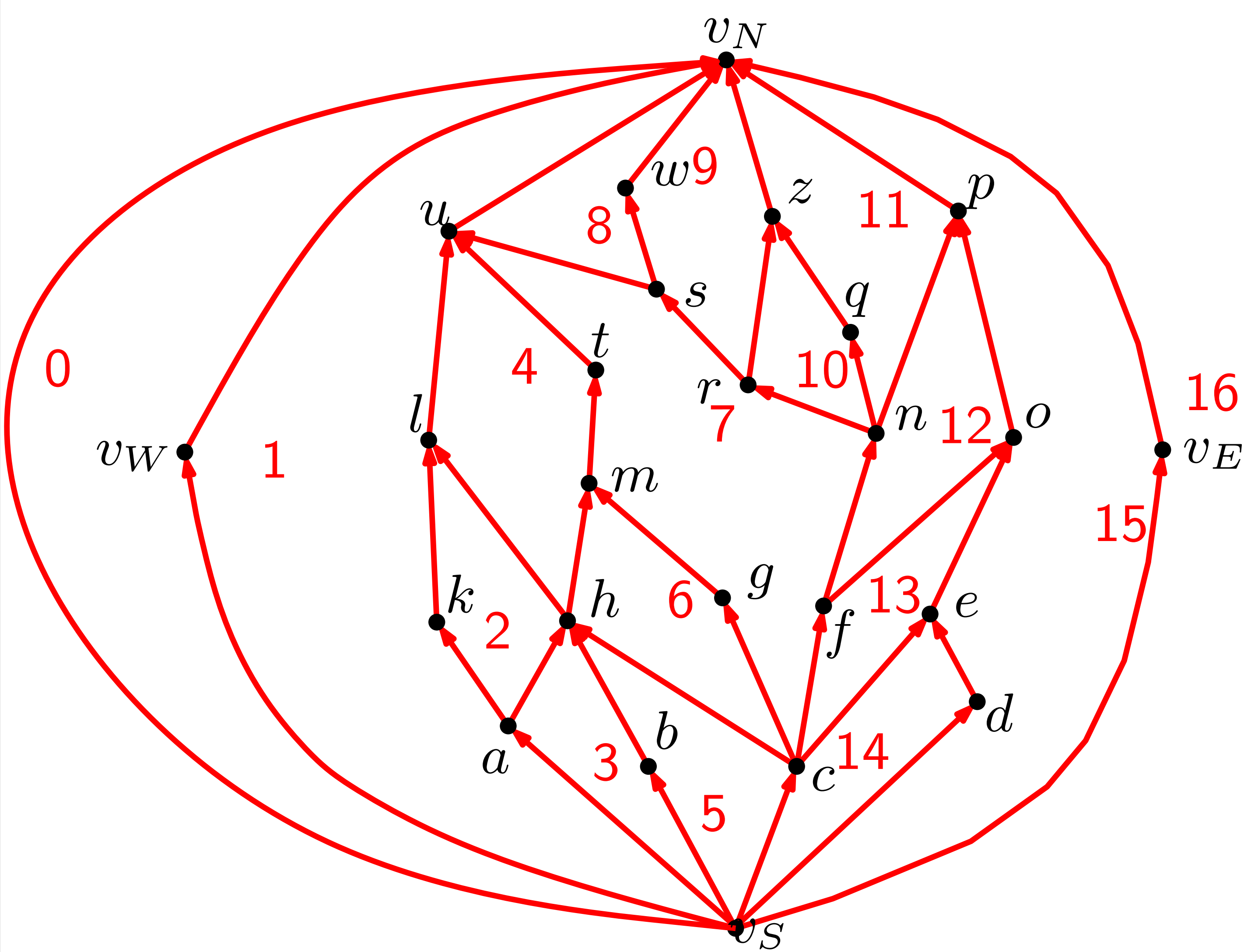
Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

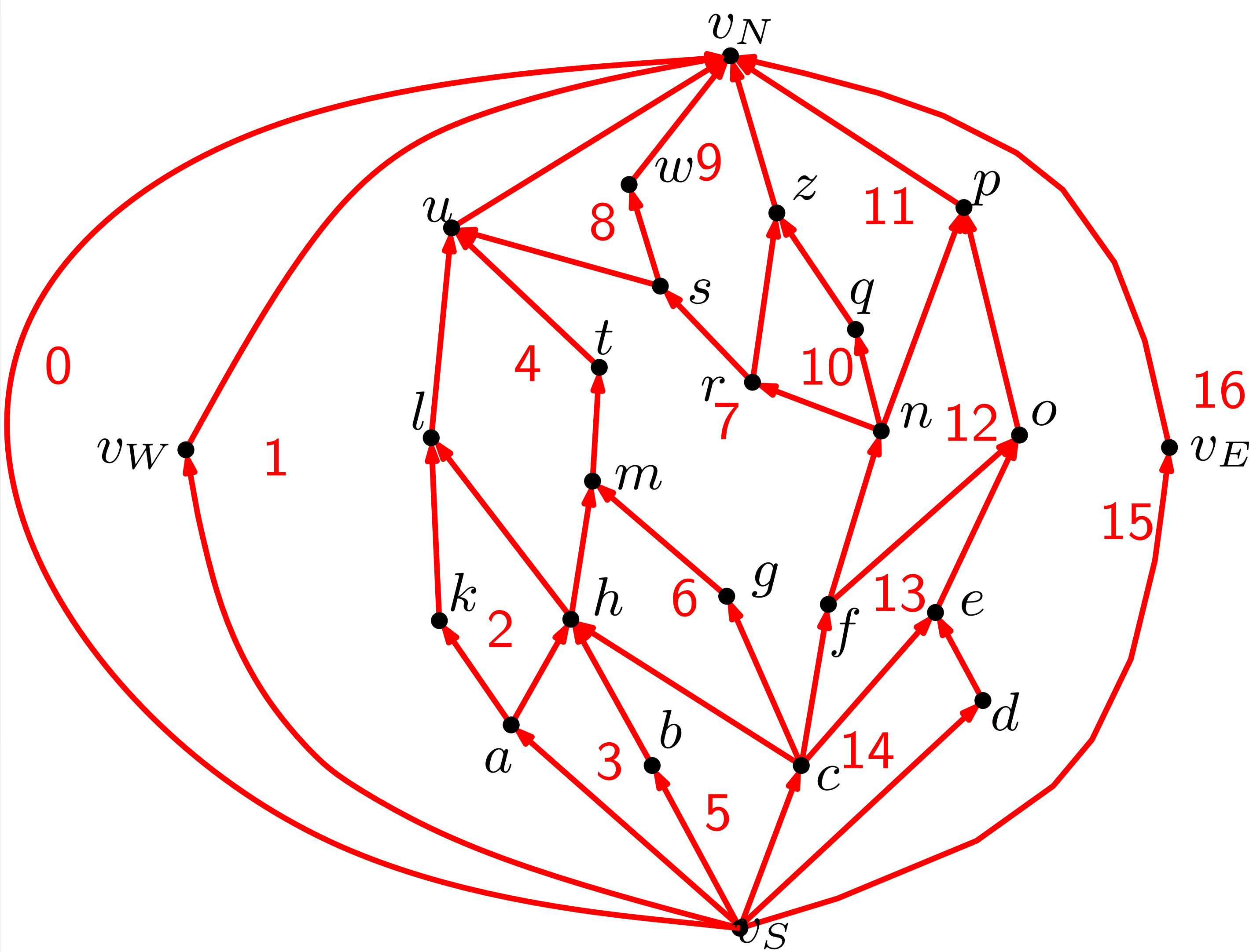
- Find a REL T_r, T_b of G;
- Construct a W-E net G_{W-E} of G (consists of T_b plus outer edges)
- Construct the dual $G^\star_{W-E}$ of G_{W-E} and compute a topological ordering f_{we} of $G^\star_{W-E}$
- For each vertex $v \in V$, let f and g be the face below and face above v. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$.
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$
- For each $v \in V$, assign a rectangle $R(v)$ bounded by x-coordinates $x_1(v),\ x_2(v)$ and y-coordinates $y_1(v),\ y_2(v)$.

13 - 7

Rectangular Dual

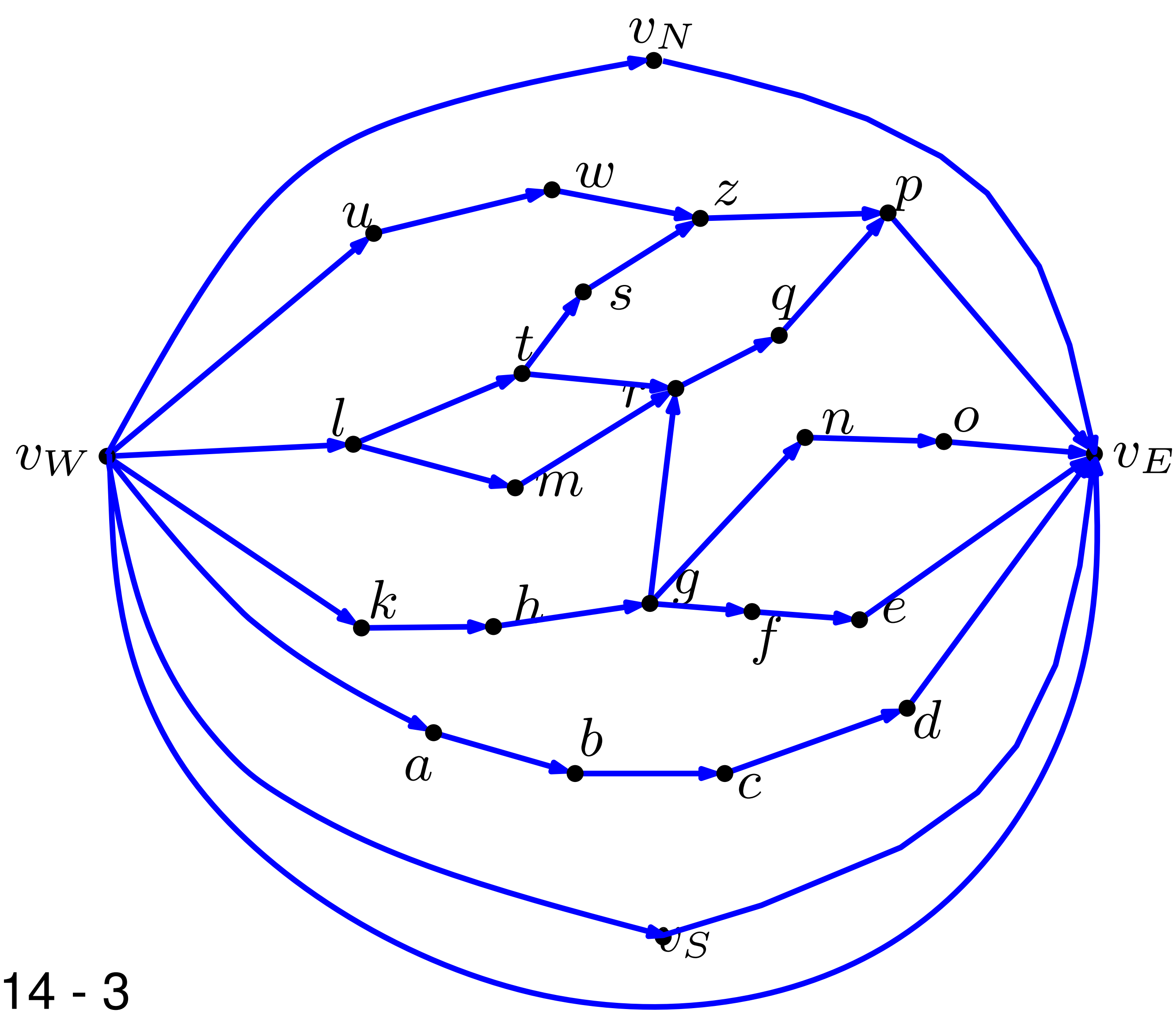


Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

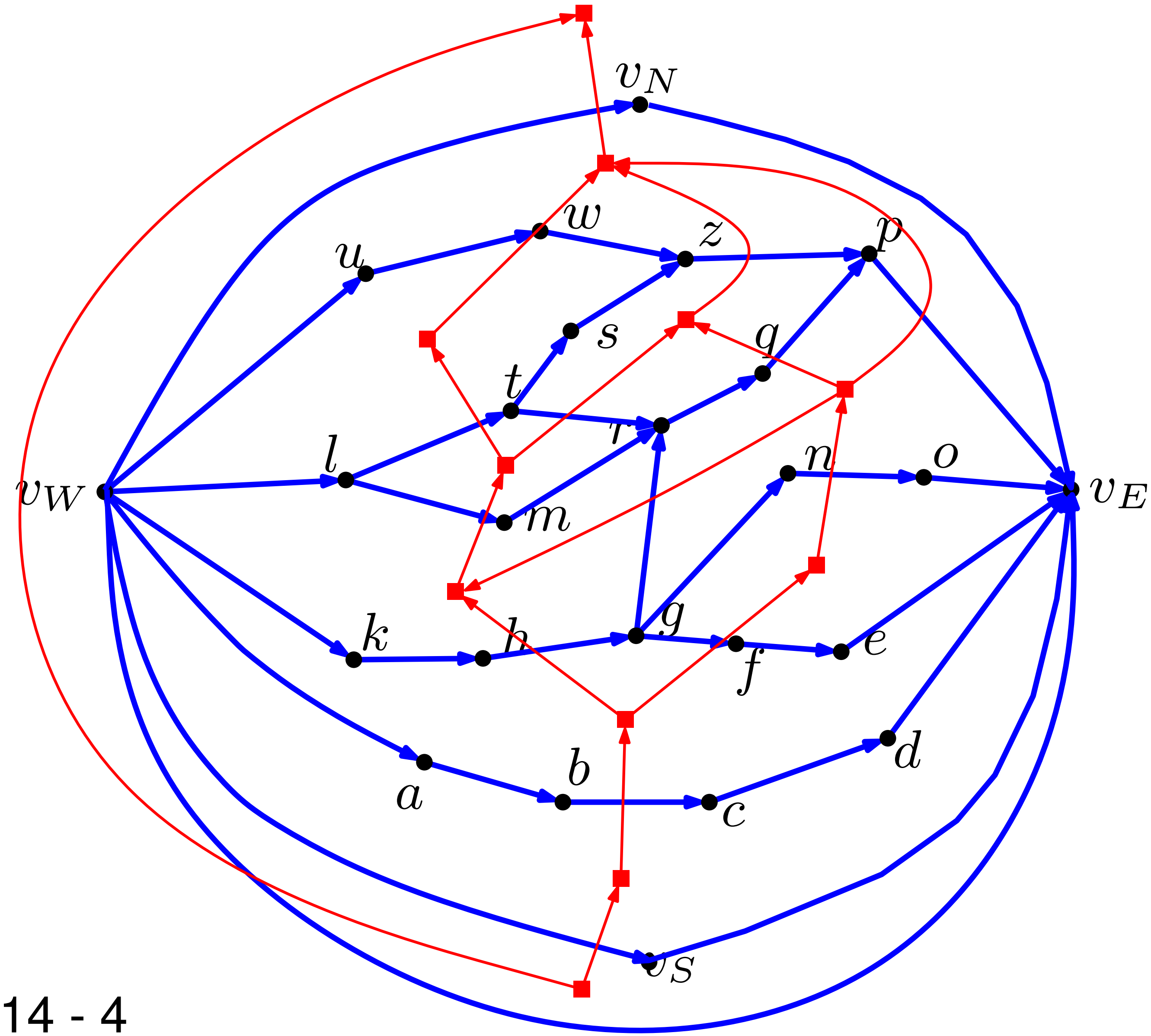
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

14 - 3

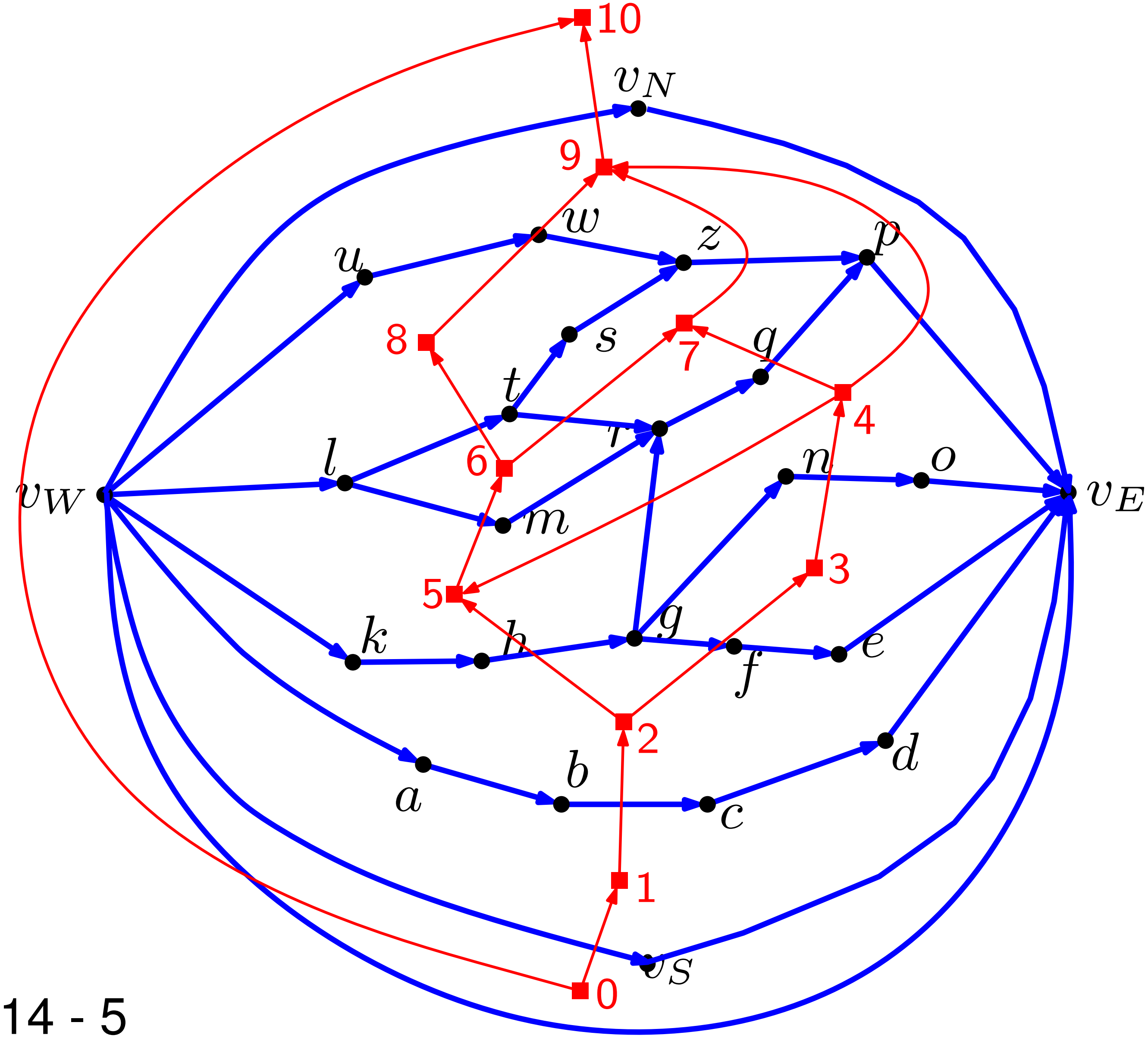
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

14 - 4

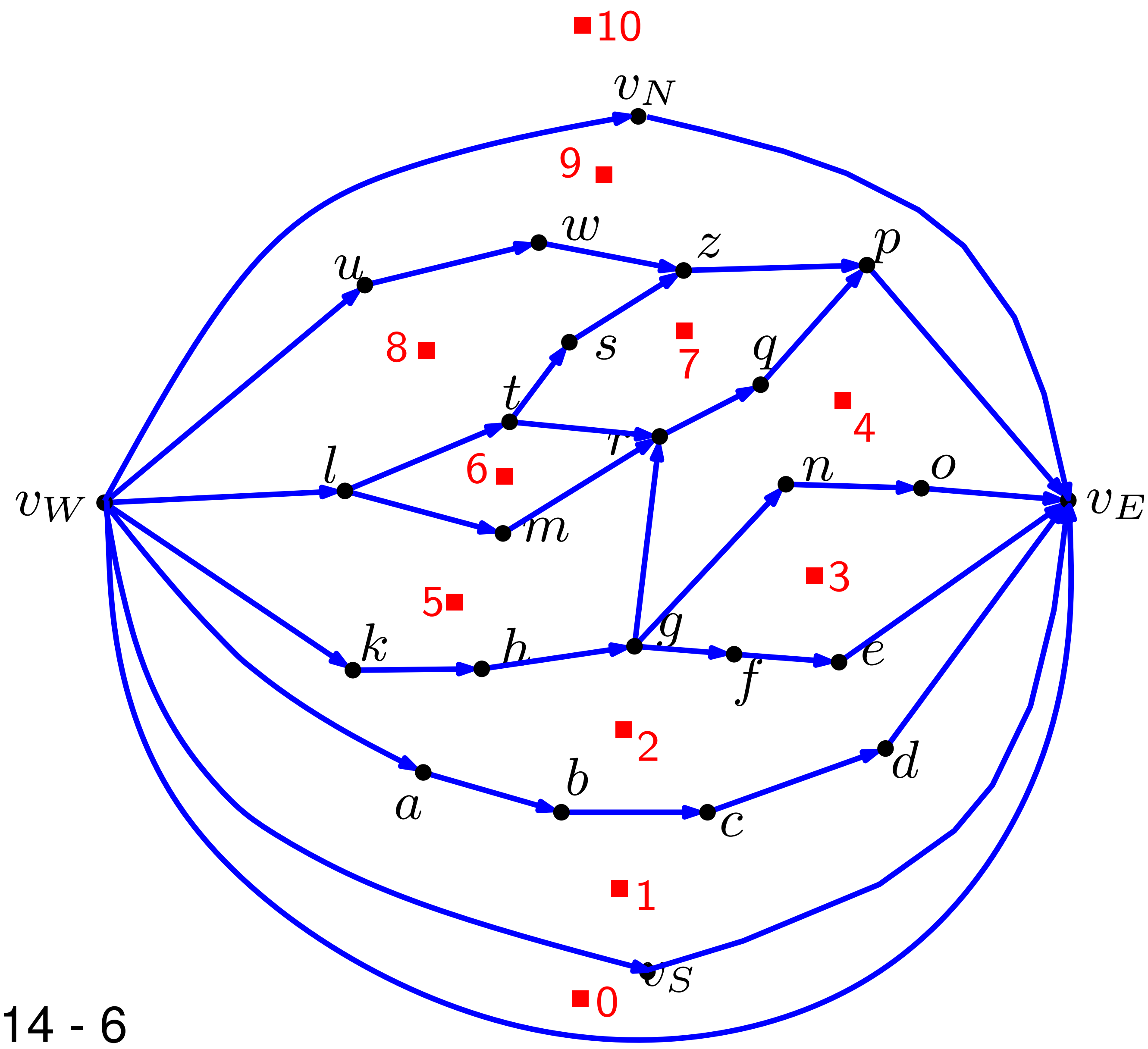
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

14 - 5

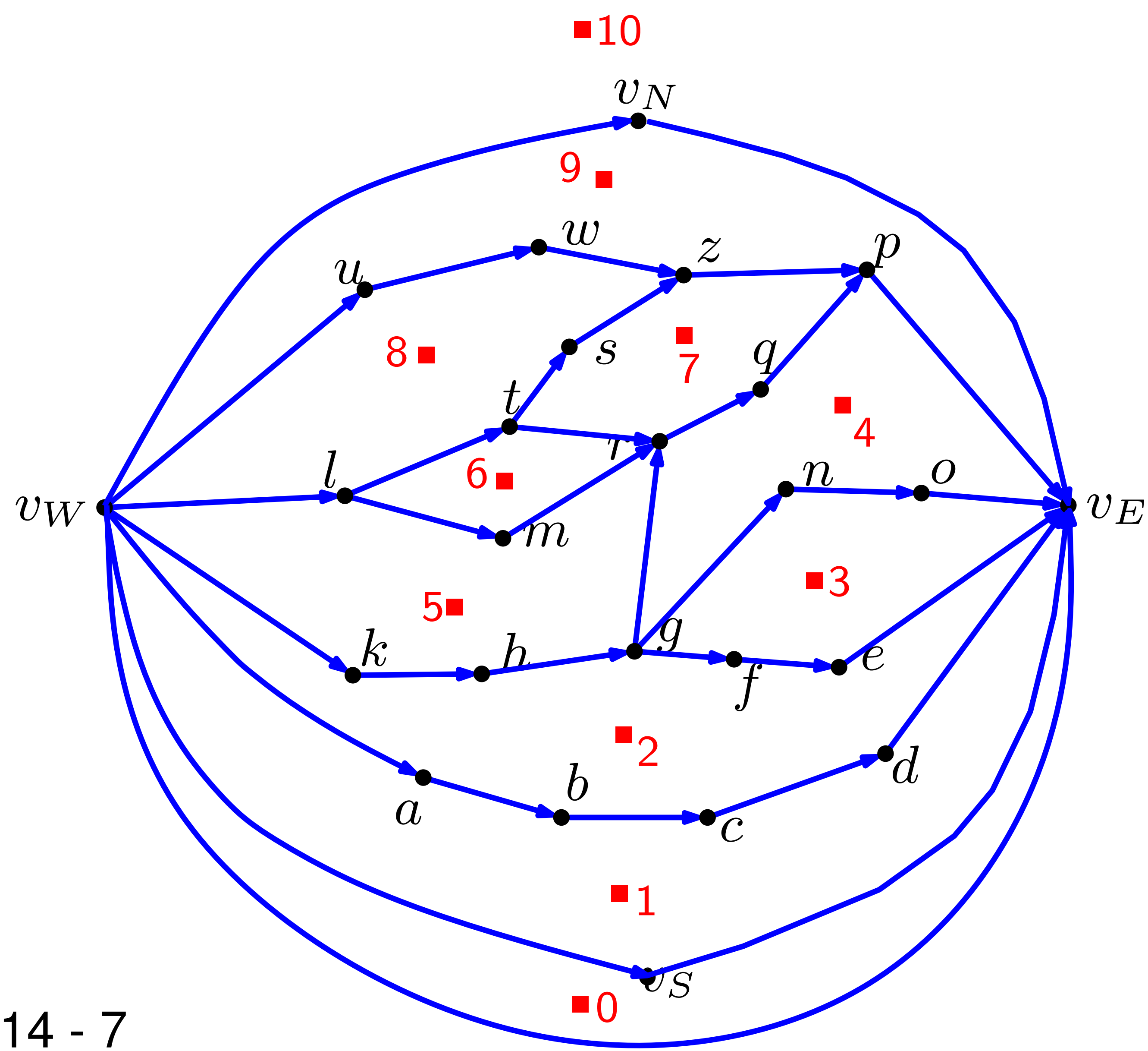
Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

14 - 6

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

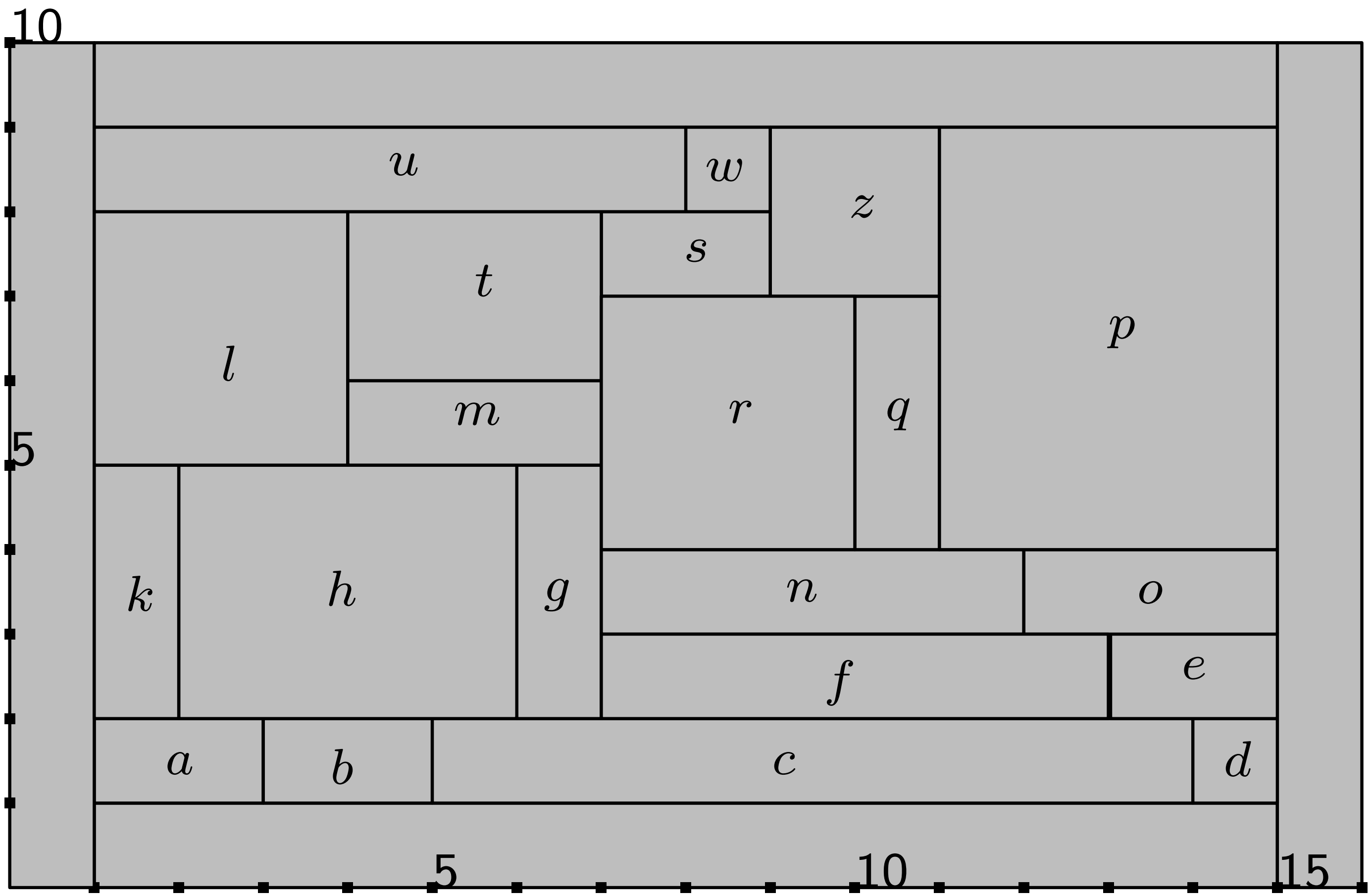
$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual

$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual



$$x_1(v_N) = 1,\ x_2(v_N) = 15$$
$$x_1(v_S) = 1,\ x_2(v_S) = 15$$
$$x_1(v_W) = 0, x_2(v_W) = 1$$
$$x_1(v_E) = 15,\ x_2(v_E) = 16$$
$$x_1(a) = 1,\ x_2(a) = 3$$
$$x_1(b) = 3,\ x_2(b) = 5$$
$$x_1(c) = 5,\ x_2(c) = 14$$
$$x_1(d) = 14,\ x_2(d) = 15$$
$$x_1(e) = 13,\ x_2(e) = 15$$

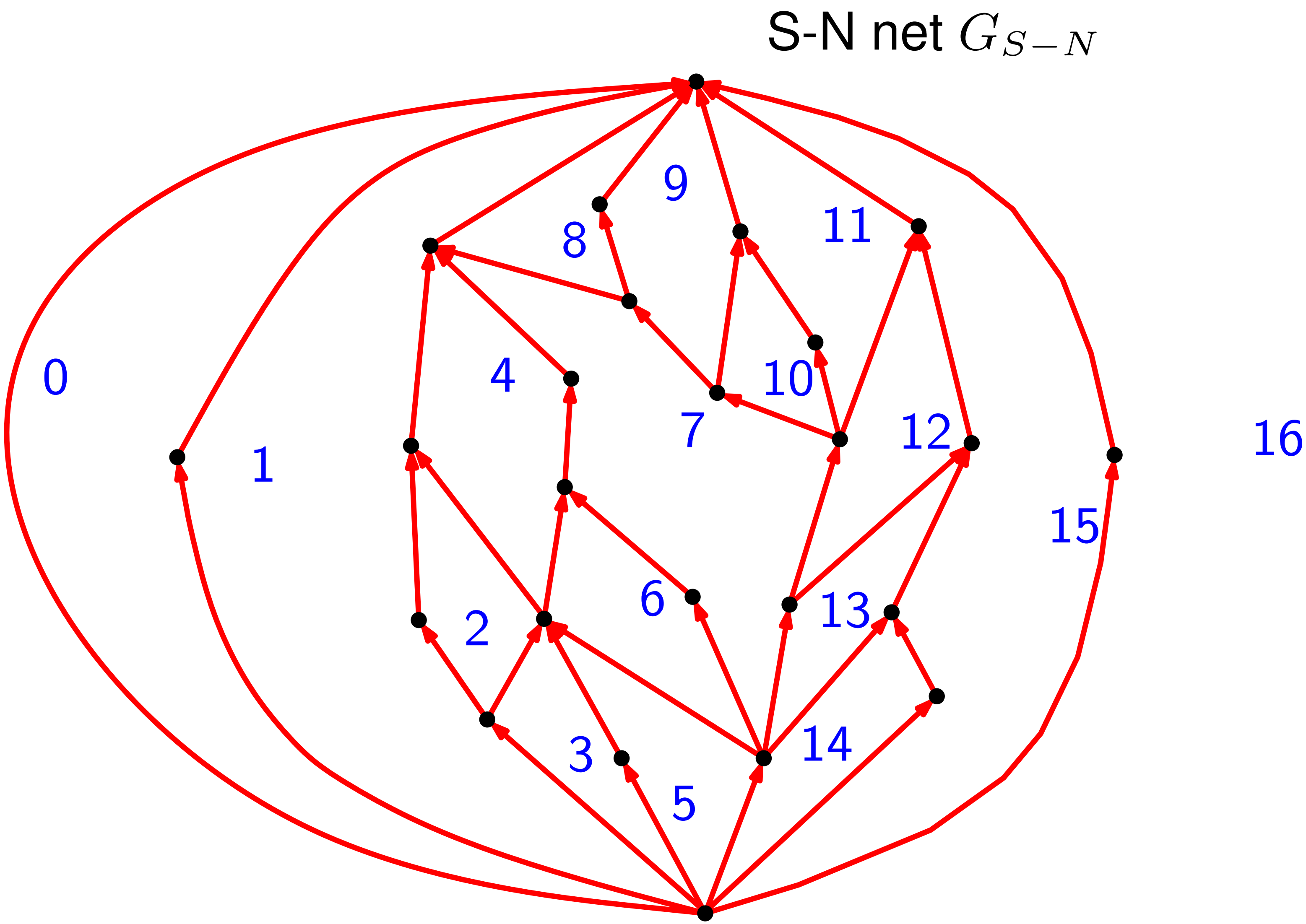
$$y_1(v_W) = 0, y_2(v_W) = 10$$
$$y_1(v_E) = 0,\ y_2(v_E) = 10$$
$$y_1(v_N) = 9,\ y_2(v_N) = 10$$
$$y_1(v_S) = 0,\ y_2(v_S) = 1$$
$$y_1(a) = 1,\ y_2(a) = 2$$
$$y_1(b) = 1,\ y_2(b) = 2$$
$$y_1(c) = 1,\ y_2(c) = 2$$
$$y_1(d) = 1,\ y_2(d) = 2$$
$$y_1(e) = 2,\ y_2(e) = 3$$

Rectangular Dual

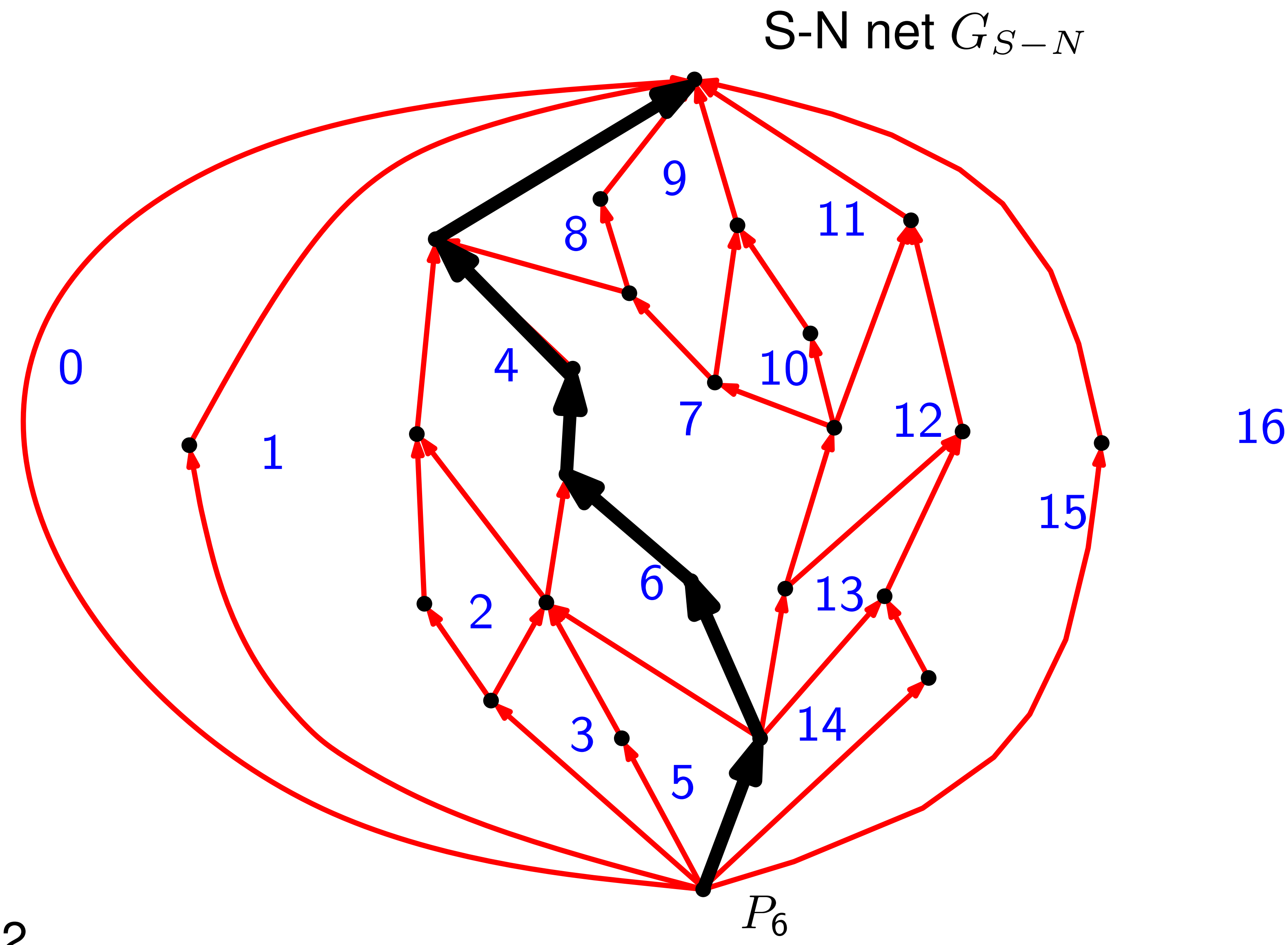
In the following we prove that presented algorithm constructs a rectangular dual of G.

- Let G be a PTP graph and let G_{S-N} (resp. G_{W-E}) be its S-N net, let $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) be the dual of G_{S-N} (resp. G_{W-E})
- Let $f_1, \ldots, f_k$ be the faces of $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) , enumerated according to st-numbering f_{sn} (resp. f_{we})
- Let G^i_{S-N} (resp. G^i_{W-E}) denote the subgraph of G that is induced by vertices and edges of $f_1, \ldots, f_i$
- We denote P_i (reps. Q_i) the right (resp. top) boundary of G^i_{S-N} (resp. G^i_{W-E}) .

Rectangular Dual



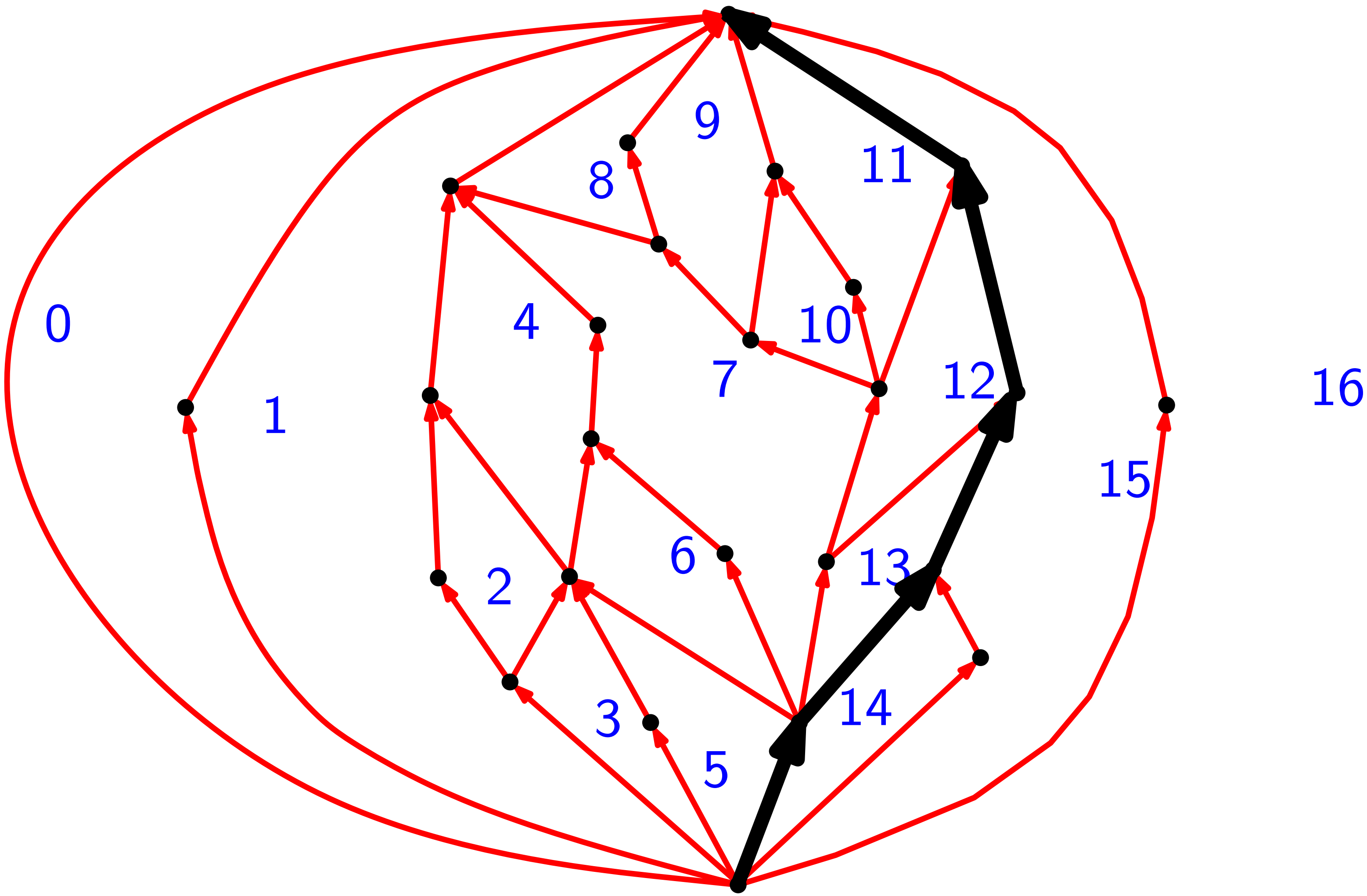
Rectangular Dual



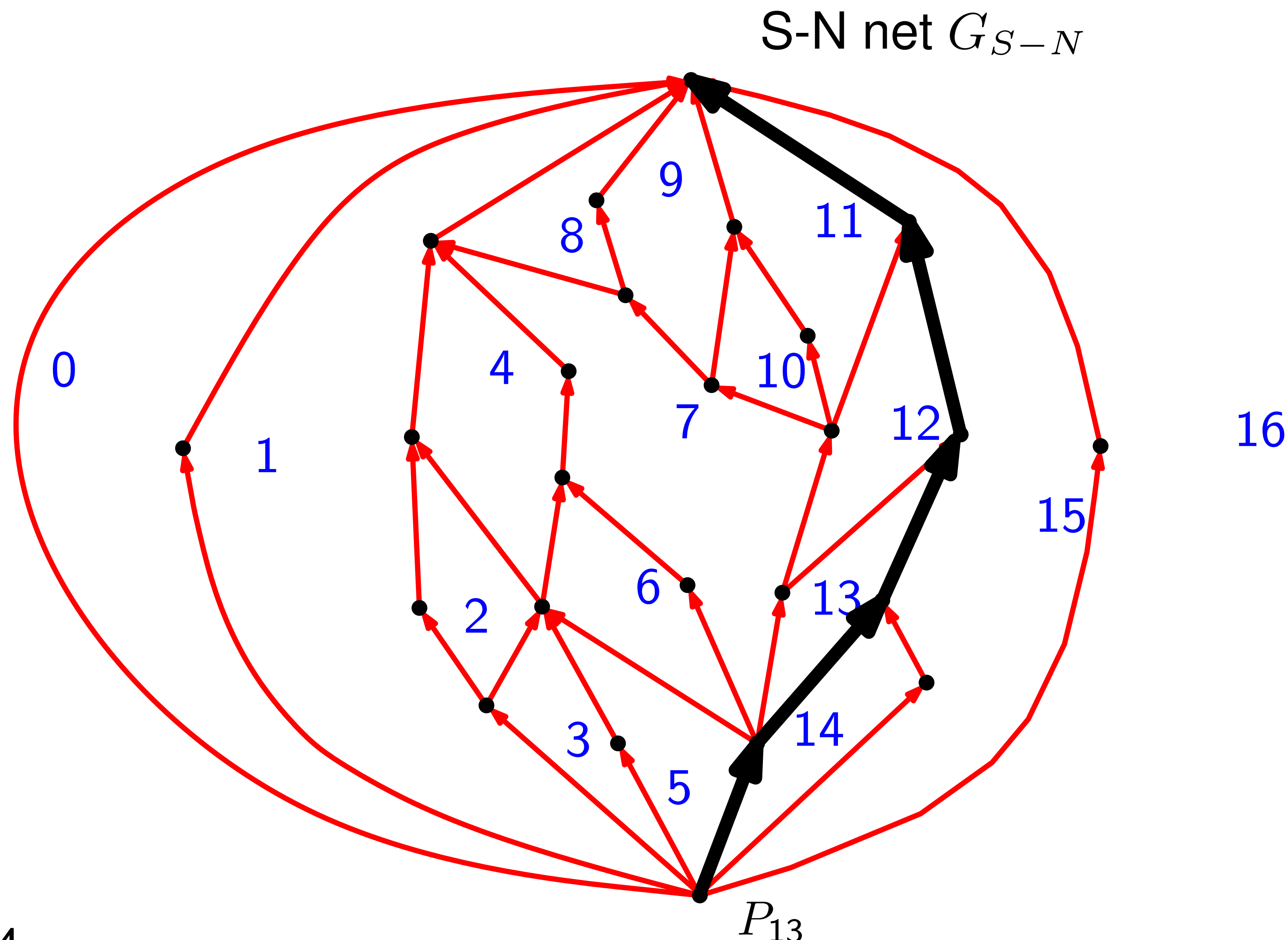
16 - 2

Rectangular Dual

S-N net G_{S-N}



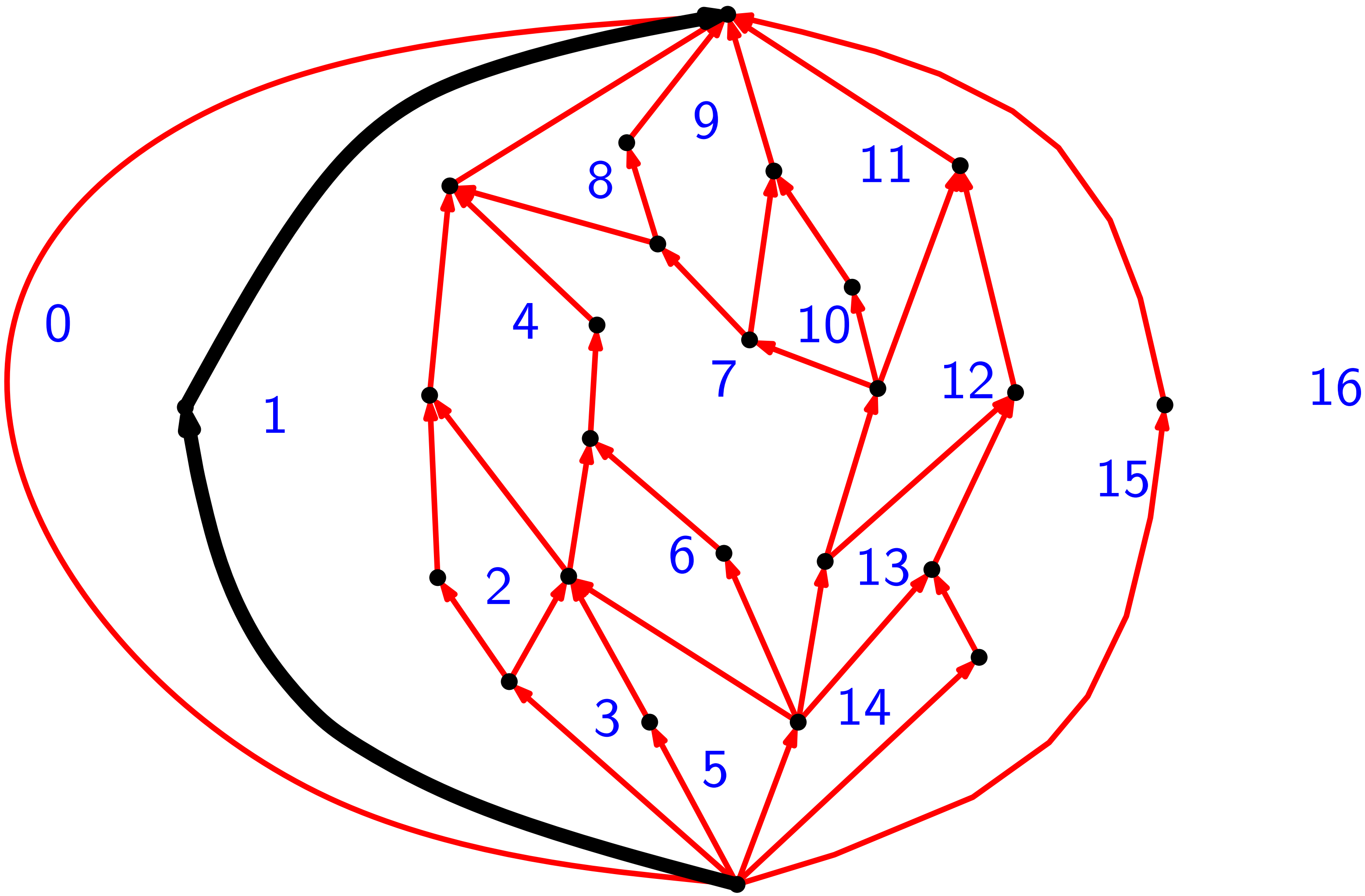
Rectangular Dual



16 - 4

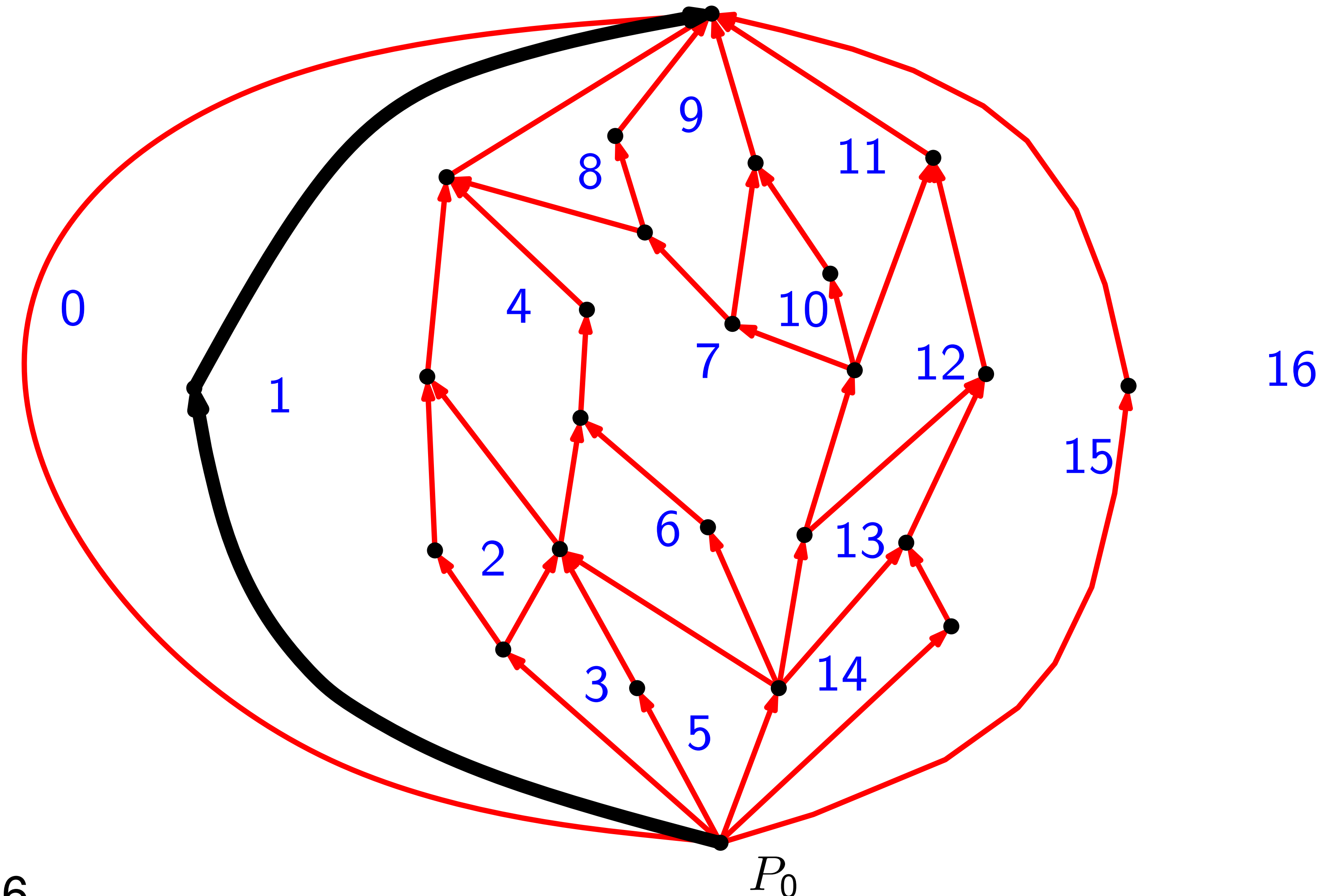
Rectangular Dual

S-N net G_{S-N}



Rectangular Dual

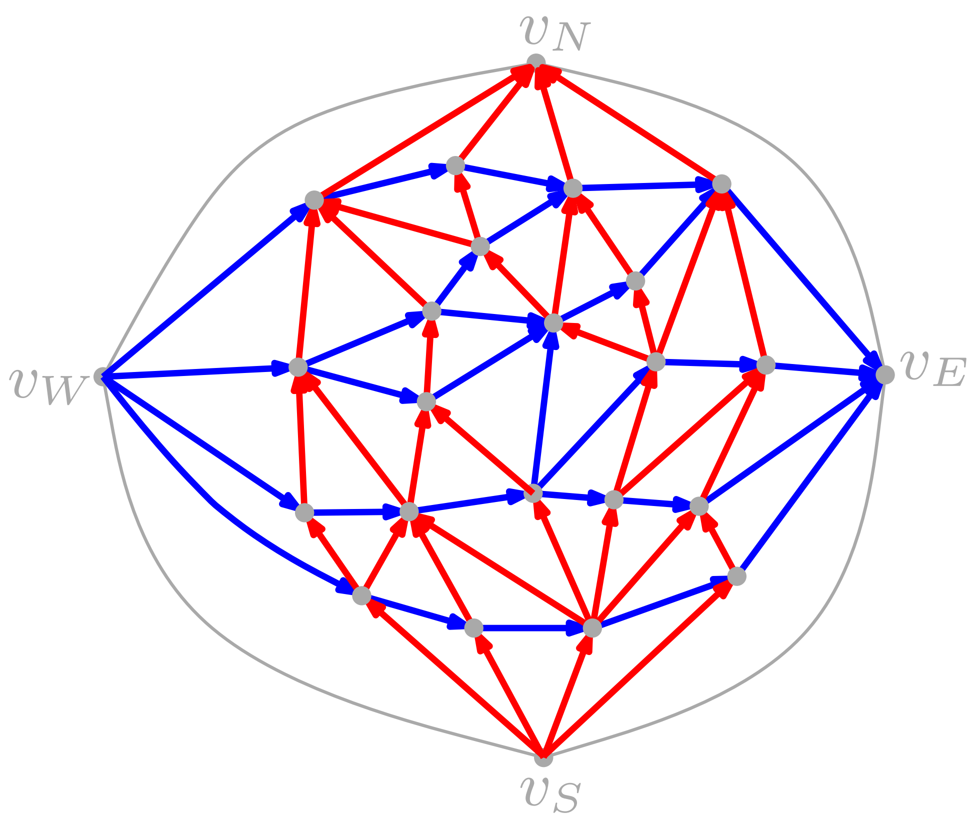
S-N net G_{S-N}



16 - 6

Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.



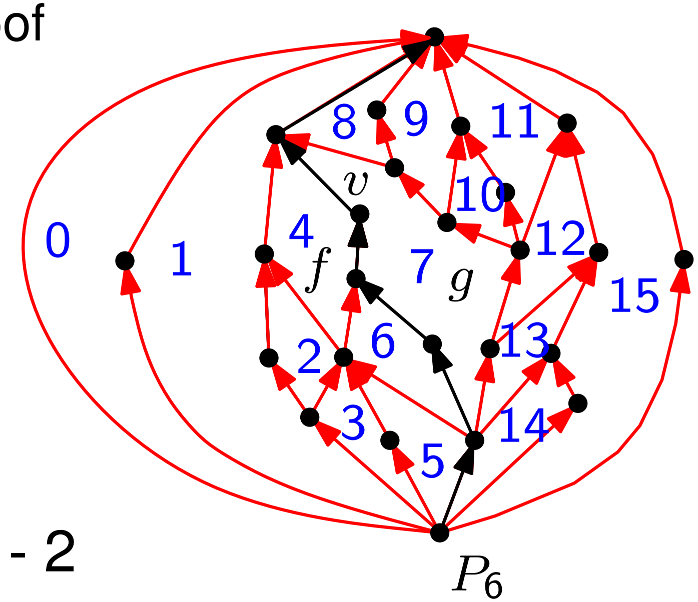
Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

Lemma 4

Let $v \in V$, f and g are the left and the right face of v. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Proof



- $f_{sn}(f) \leq i$ and $f_{sn}(g) \geq i + 1$

Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

Lemma 4

Let $v \in V$, f and g are the left and the right face of v. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Lemma 5

Let $v \in V$, f and g are the fases below and above v in G_{W-E}. Let $y_1(v) = f_{we}(f)$ and $y_2(v) = f_{we}(g)$. Vertex v belongs to path Q_j if and only if $y_1(v) \leq j \leq y_2(v) - 1$.

Proof (identical)

Rectangular Dual

Lemma 6

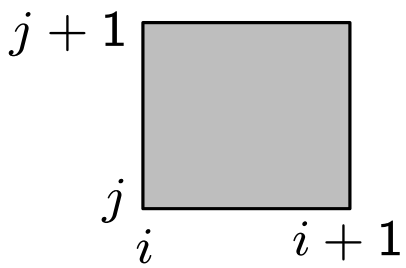
The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:

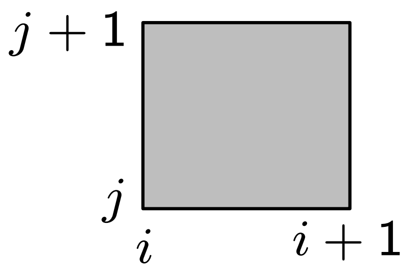


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there **exists a vertex** over this box: $u \in P_i \cap Q_j$

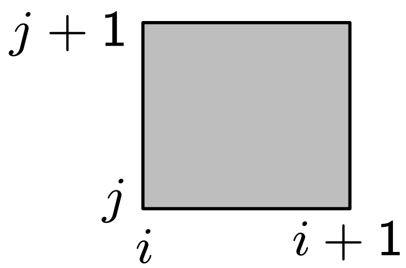


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box

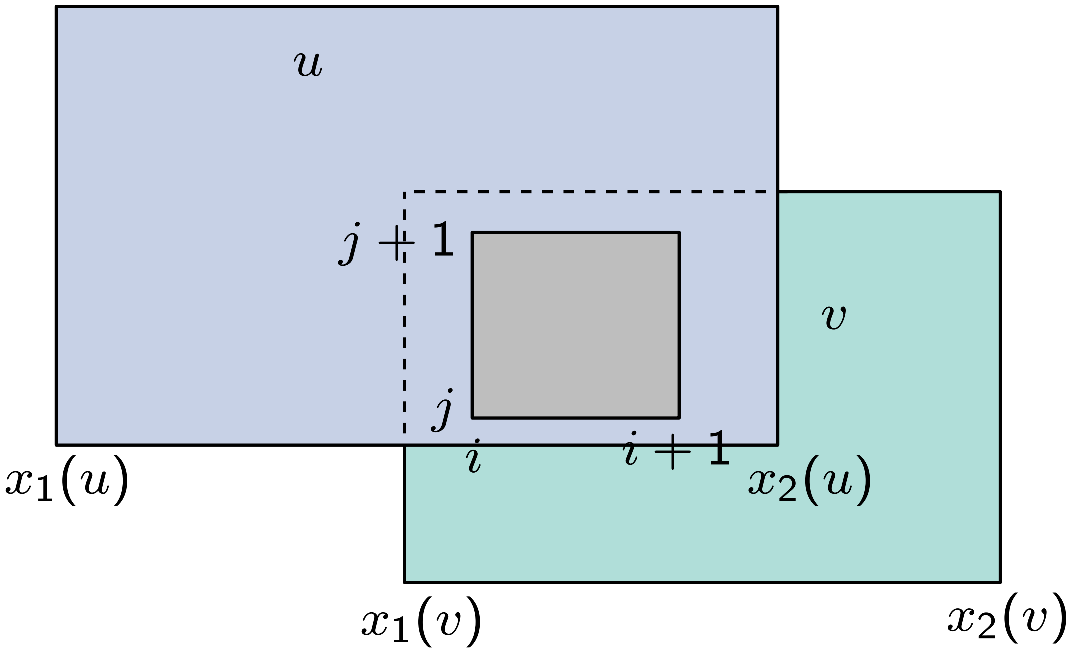


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

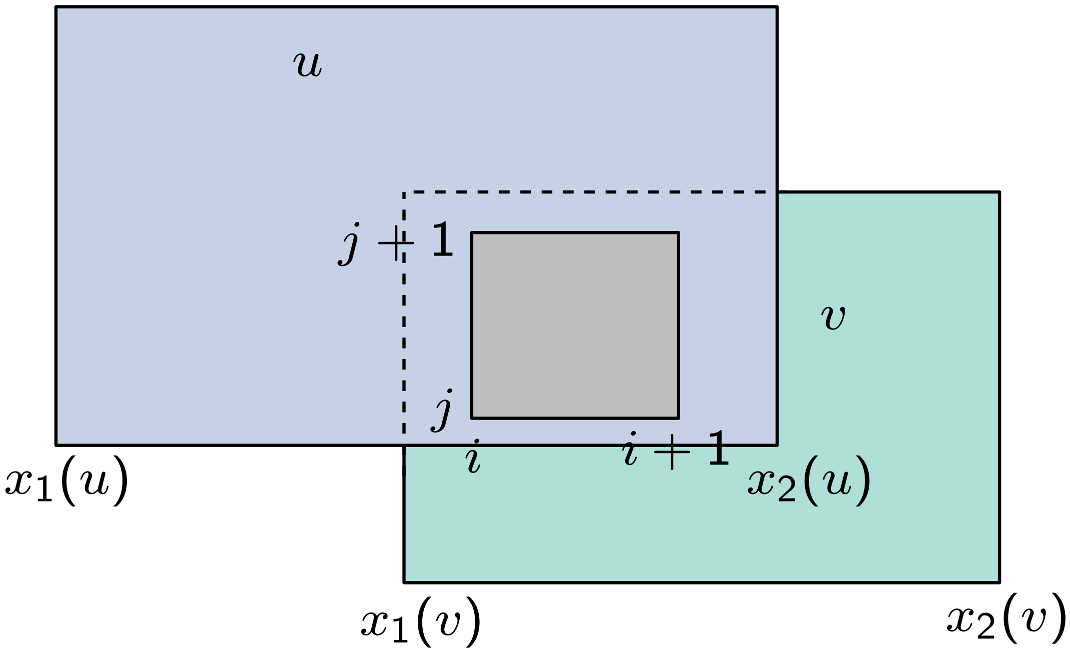
Proof: Show that there is **at most one vertex** over this box



Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box

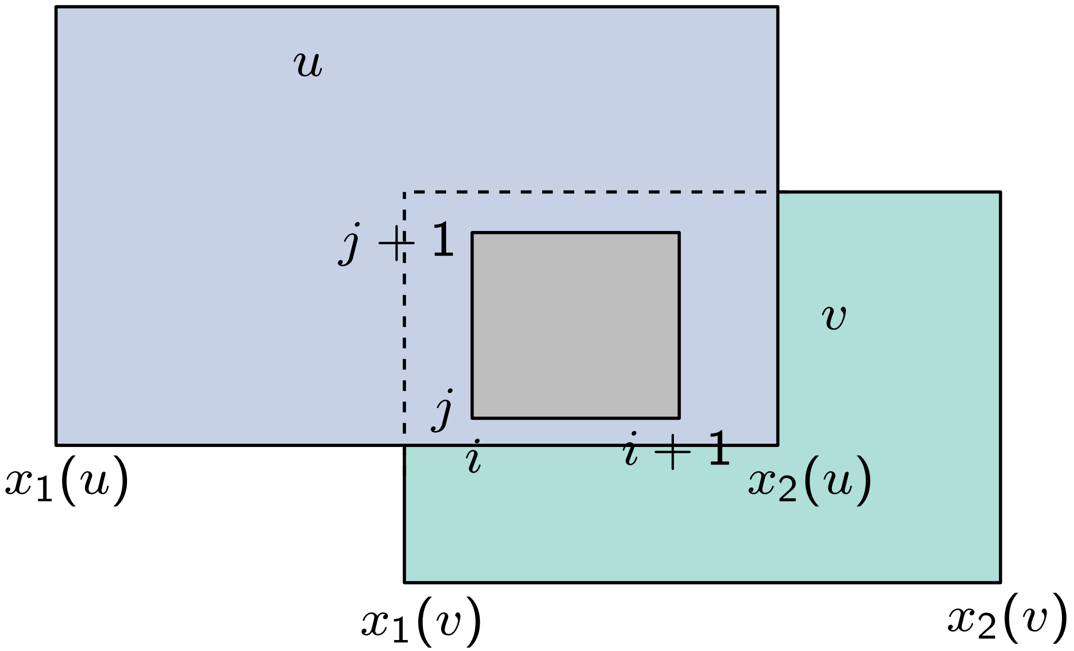


$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

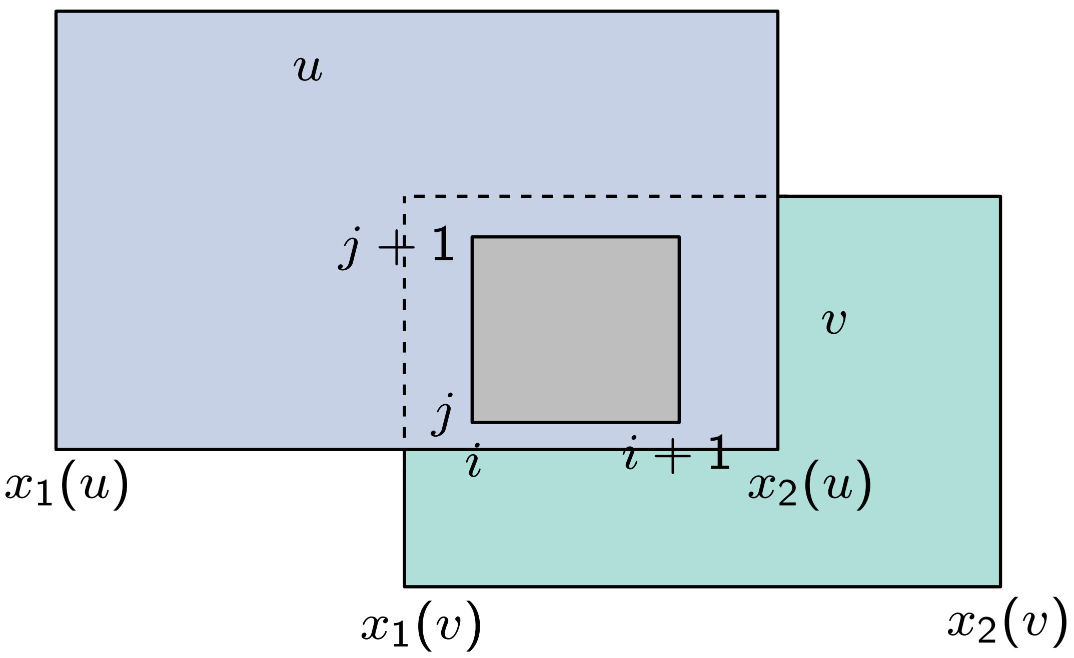
u belongs to P_i

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

$\Downarrow$ (Lemma 4)

u belongs to P_i

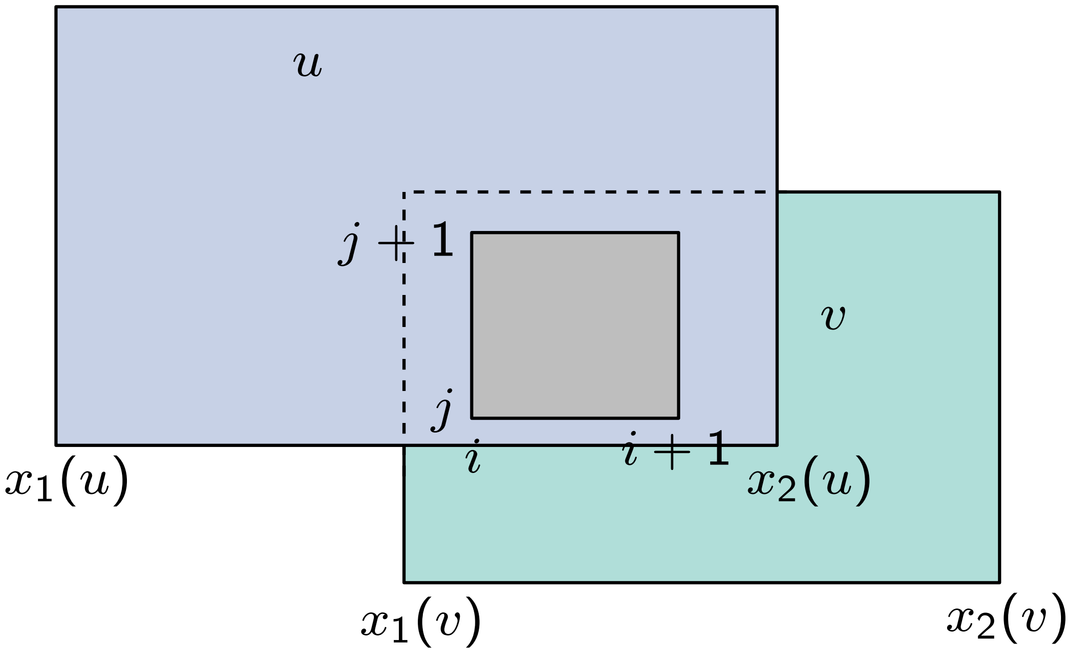
Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i{+}1 \leq x_2(u)$

(Lemma 4)

u belongs to P_i

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

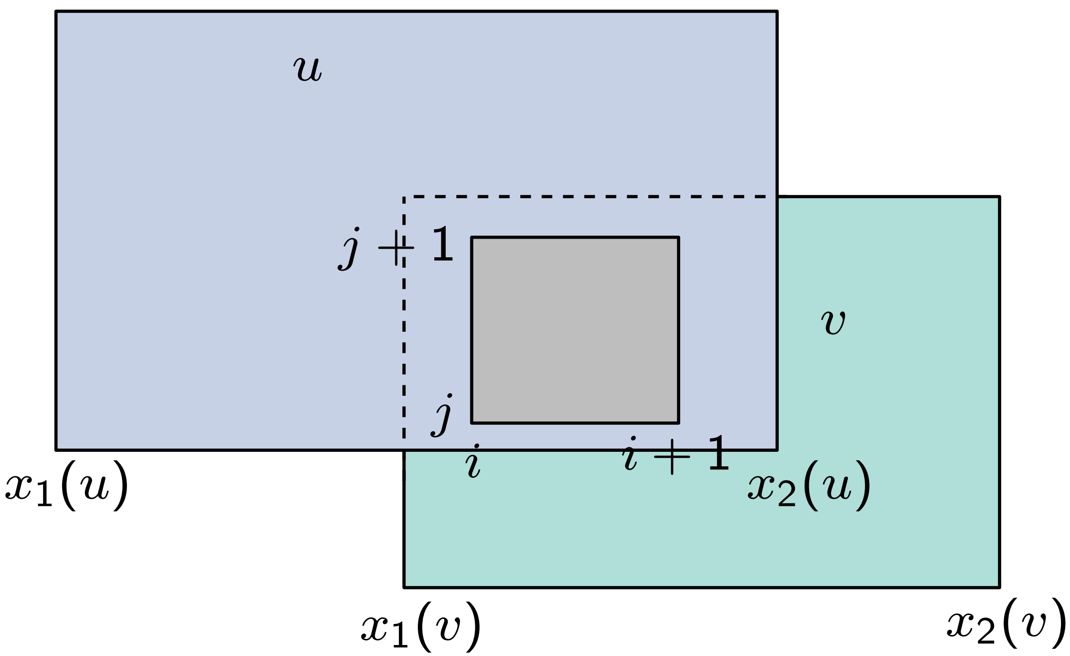
Paths P_i and Q_j intersect at two vertices u and v.

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

u belongs to P_i

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Paths P_i and Q_j intersect at two vertices u and v.

Which is a contradiction to the property of paths P_i, Q_j except for the cases when: (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$ (corner boxes).

Rectangular Dual

Lemma 7

Let G_{S-N} and G_{W-E}. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
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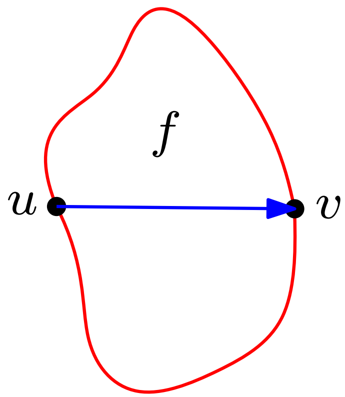

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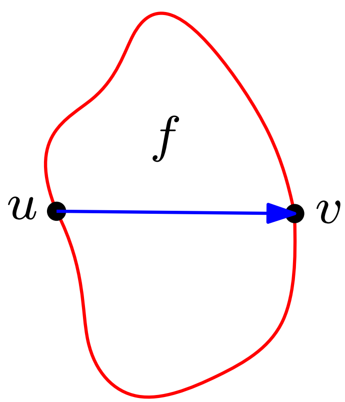

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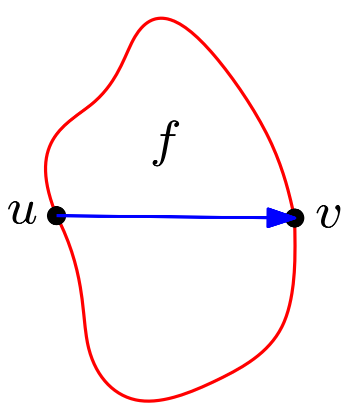

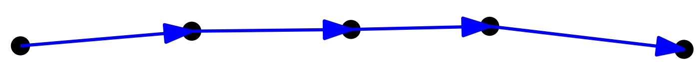

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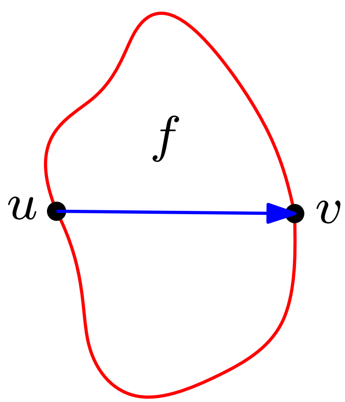

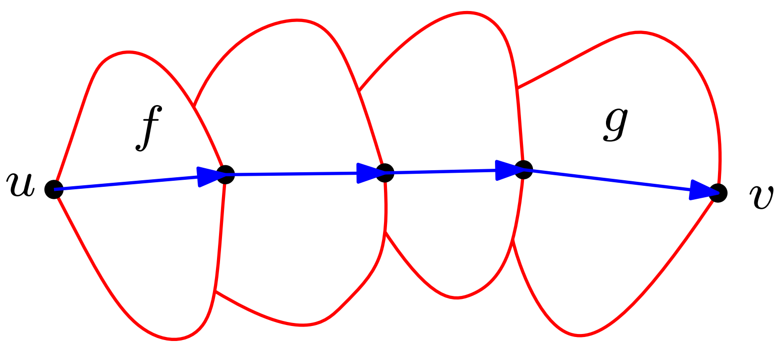

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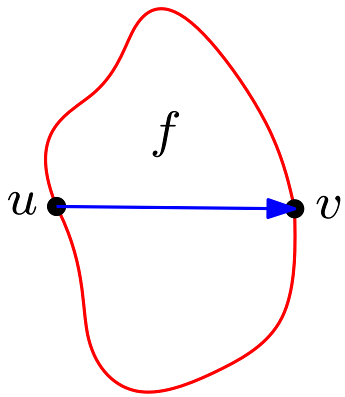
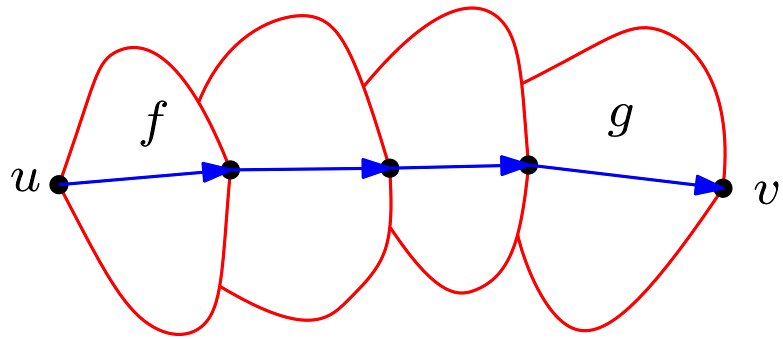


- $x_2(u) = f_{sn}(f)$
- $x_1(v) = f_{sn}(g)$



Rectangular Dual

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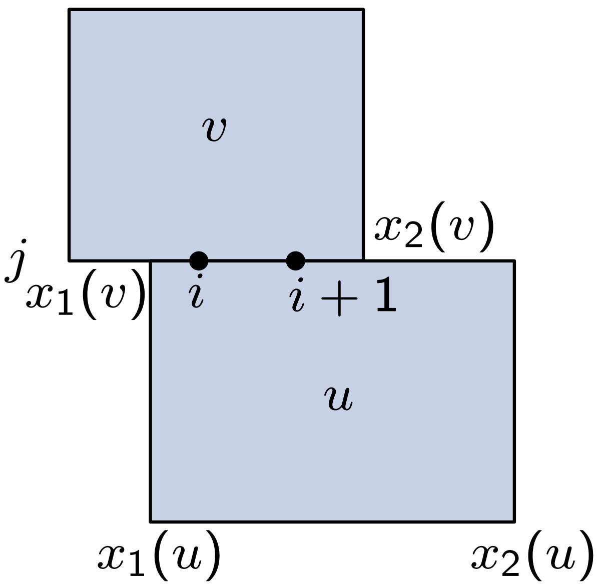
Lemma 8

The assignment provided by the algorithm has the following property: rectangles assigned to vertices u and v have a common segment if and only if there exists edge (u, v) in the graph.

Proof:

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



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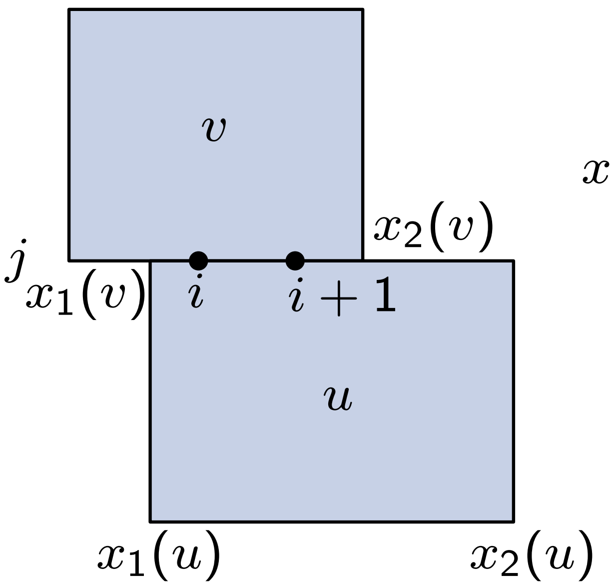

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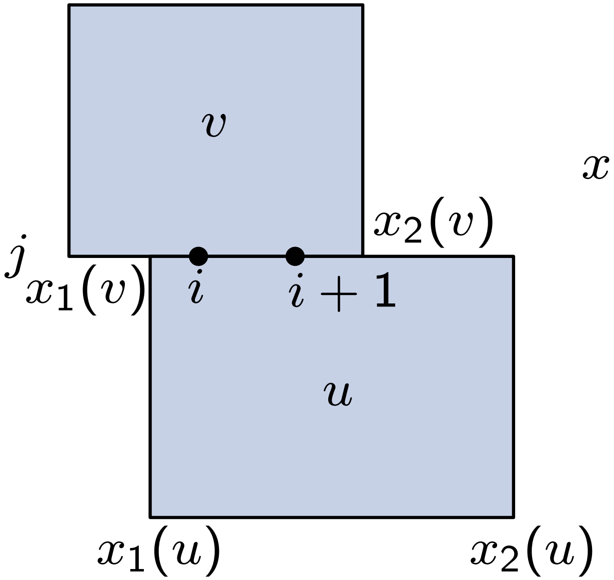

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u, v belong to P_i

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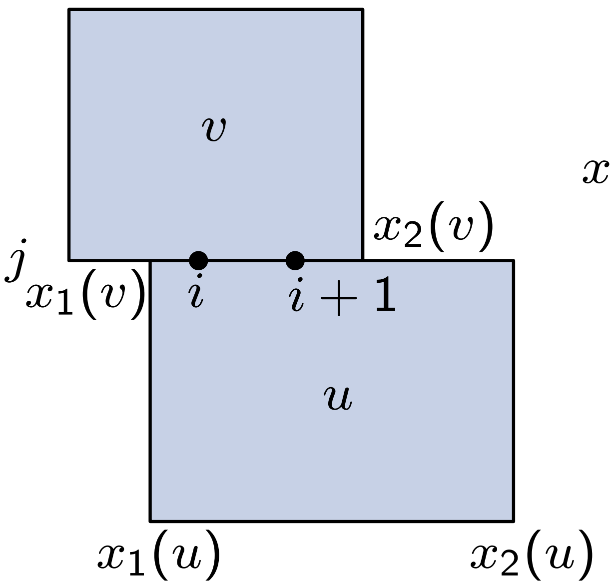

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If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

20 - 4

Rectangular Dual

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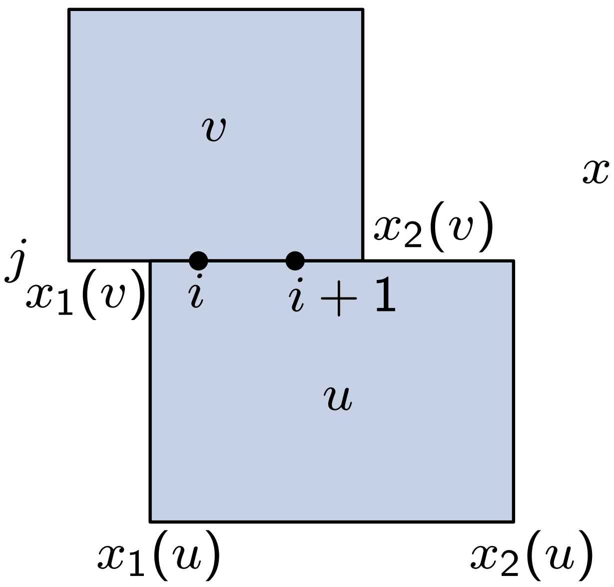

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If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

A contradiction to the hypothesis!

20 - 5

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.

$$\Downarrow$$

- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$

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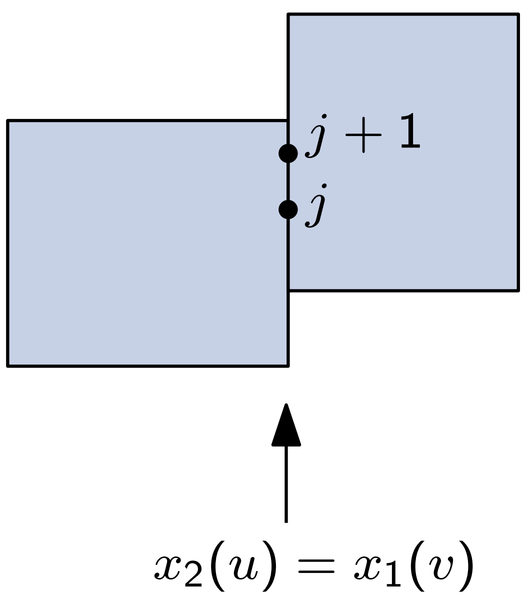

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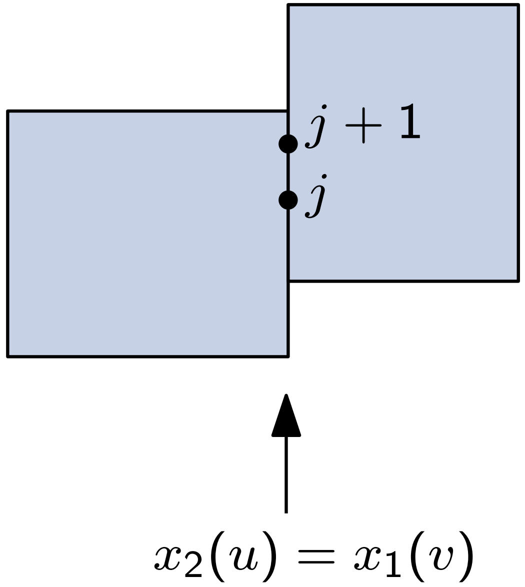

Lemma 8 is proved!

Rectangular Dual

Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

Rectangular Dual

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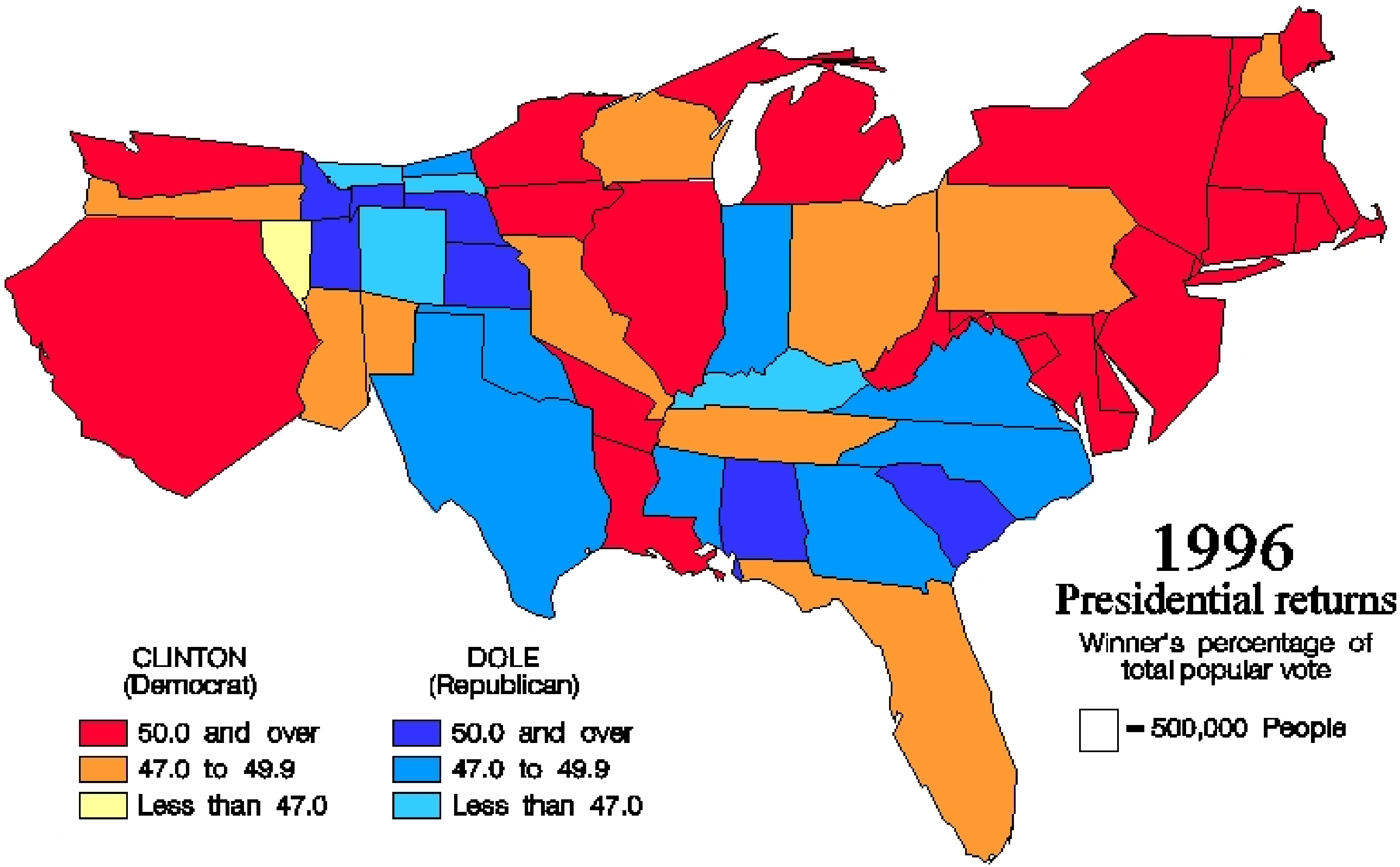
Every PTP graph G has a rectangular dual which can be computed in linear time.

- Compute a planar embedding of G
- Compute a revised canonical ordering of G
- Traverse the graph and color the edges, construct G_{S-N} and G_{E-W}
- Construct the duals $G^{\star}_{S-N}$ and $G^{\star}_{E-W}$ of G_{S-N} and G_{E-W}, respectively
- Compute a topological ordering of $G^{\star}_{S-N}$ and $G^{\star}_{E-W}$
- Assing coordinates to the rectangles representing vertices.

Discussion



23 Proportional Cartogram. Source: `http://www.ncgia.ucsb.edu`

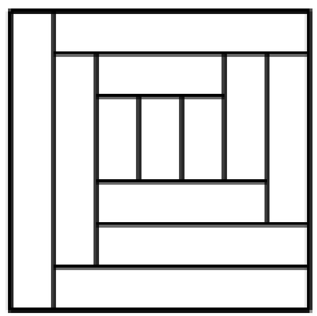
Discussion

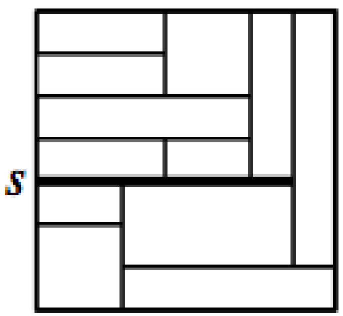
- A layout is area-universal if any assignment of areas to rectangles can be realized by a combinatorially equivalent rectangular layout.

- A rectangular layout is **area-universal** if and only if it is **one-sided**.

[Eppstein et al. SIAM J. Comp. 2012]

one-sided





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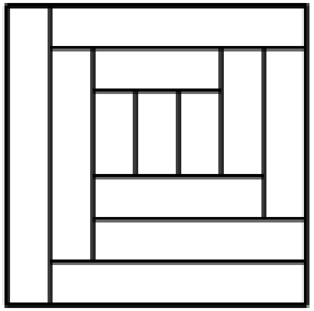
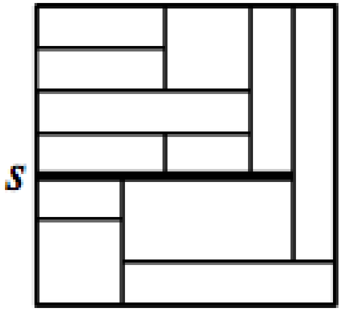
one-sided  

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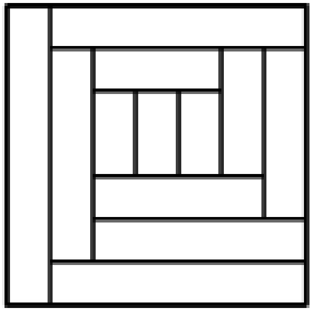
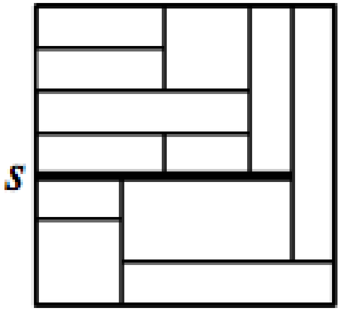
one-sided   not one-sided

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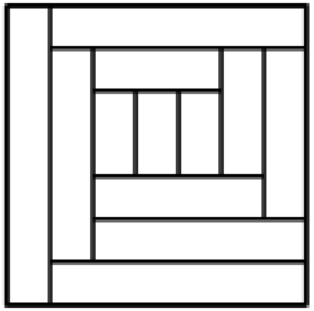
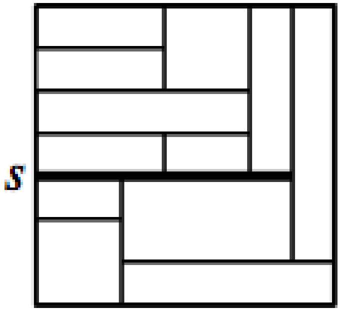
one-sided  

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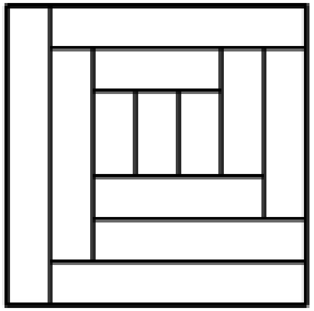

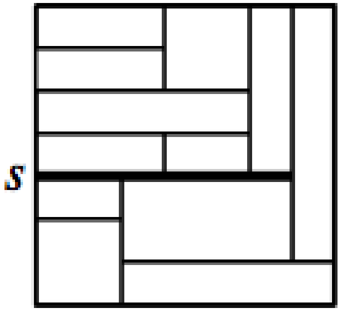

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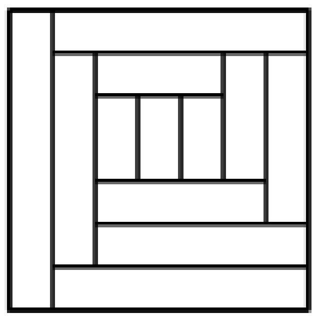
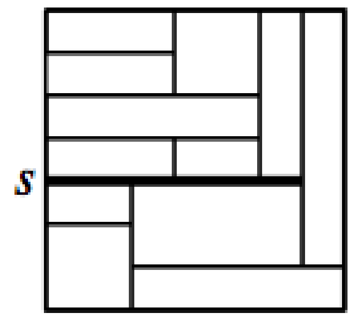
one-sided  

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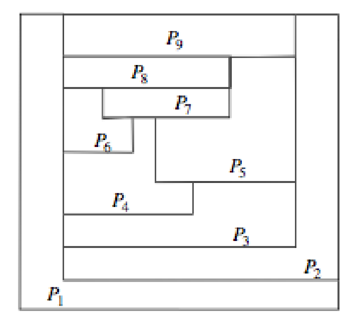
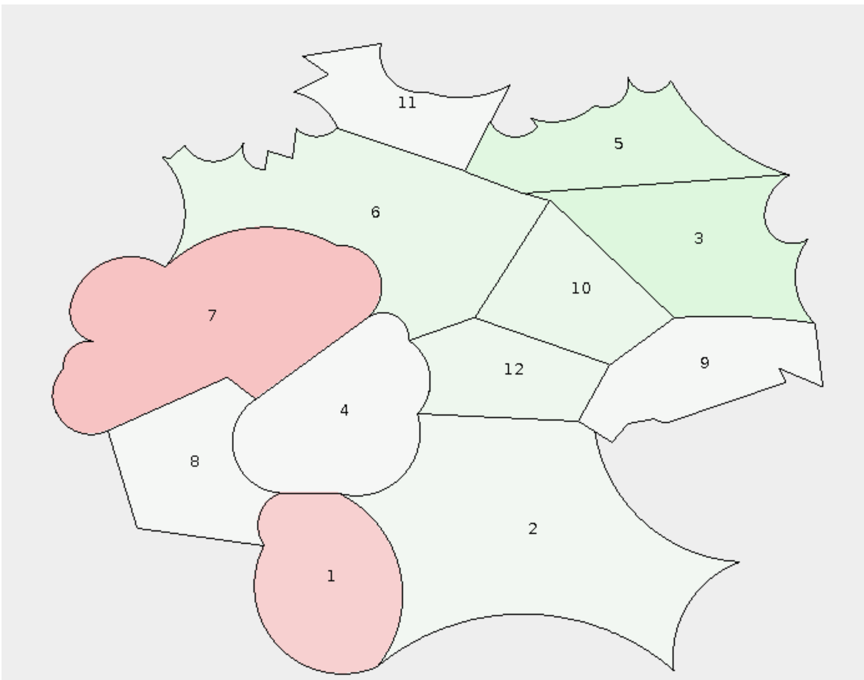
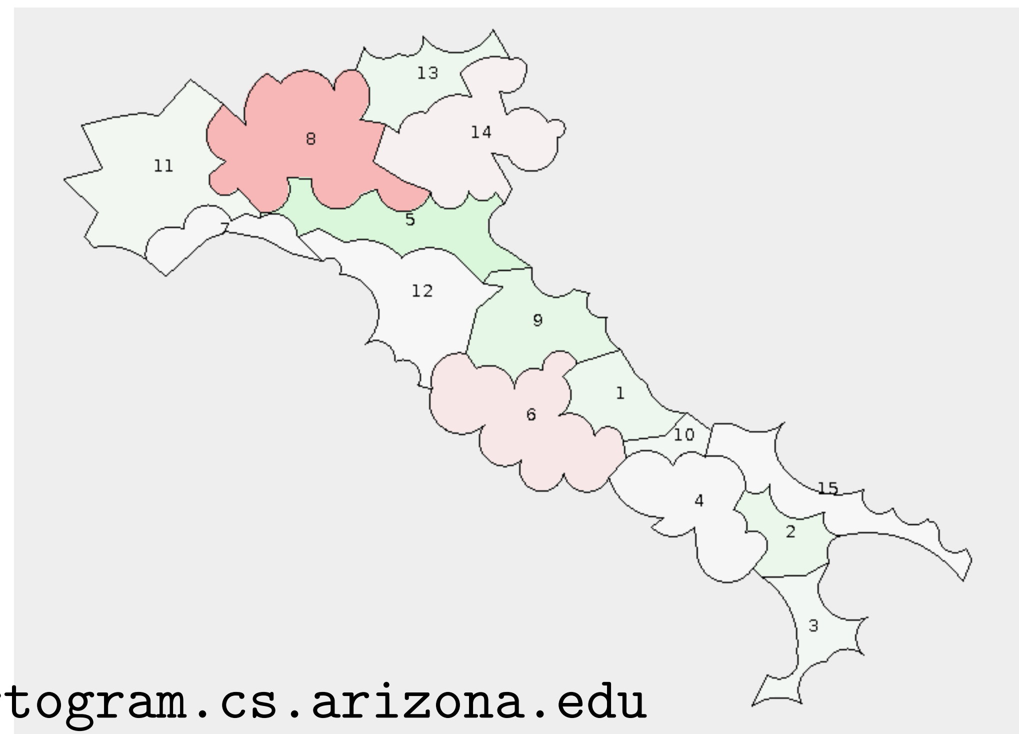


Discussion

- Circular Arc Cartograms [Kämper, Kobourov, Nöllenburg. IEEE PasViz 2013]





Source: `http://cartogram.cs.arizona.edu`