

Algorithms for Graph Visualization

Flow Method: Upward Planarity

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Tamara Mchedlidze
25.01.2018



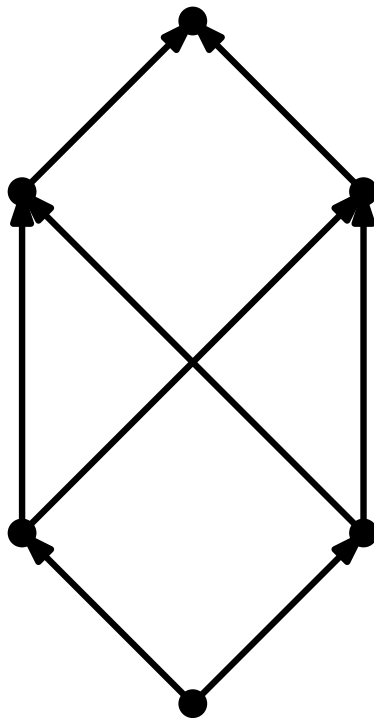
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Def: A directed acyclic graph $D = (V, A)$ is called **upward planar**, when D admits a drawing (vertices points, edges simple curves), which is planar and each edge is a monotone curve increasing in y -direction.

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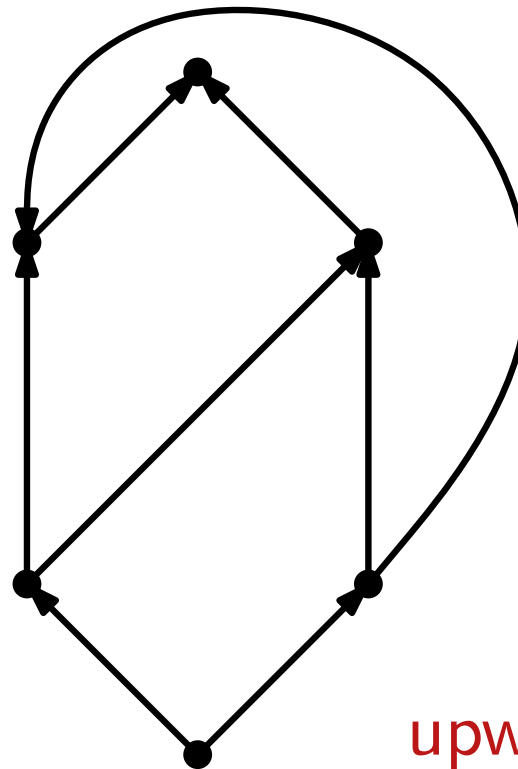
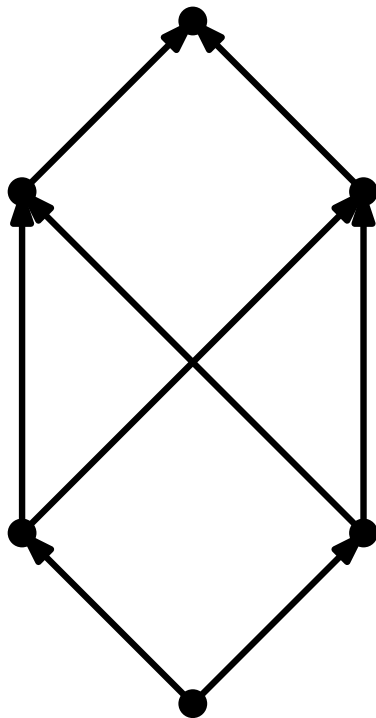
Example:



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Example:



planar!

upward planar? – NO!

Complexity

Thm 1: For a directed acyclic graph it is NP-hard to decide whether it is upward planar.

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Thm 4: For a directed graph $D = (V, A)$ the following statements are equivalent:

1. D is upward planar
2. D admits an upward planar straight-line drawing
3. D is the spanning subgraph of a planar st -digraph

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Pair, think and share:

How to do the augmentation in case of a disconnected graph?

3 min

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Pair, think and share:

Is the area produced by the algorithm described in the proof of (3) \Rightarrow (2) polynomial?

5 min

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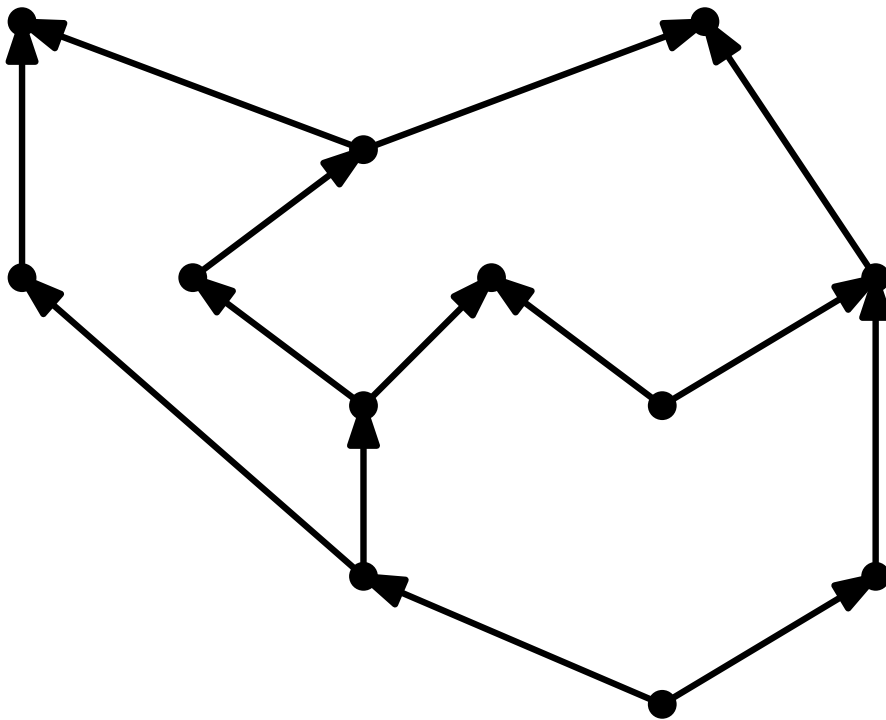
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- Step (3) \Rightarrow (2) implies a $O(n)$ algorithm to construct a planar straight-line drawing of an st -digraph.

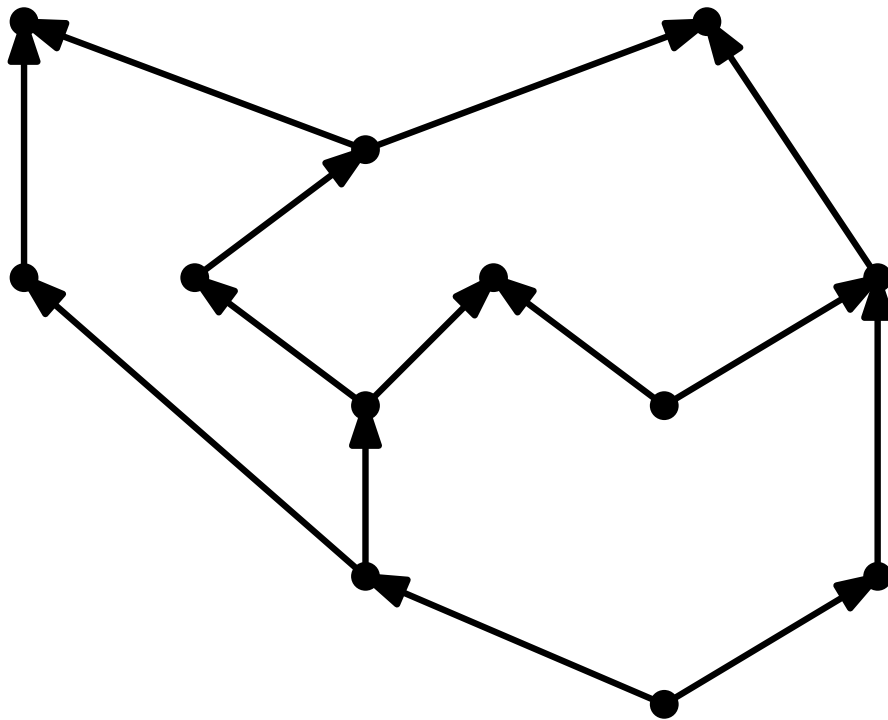
Fixed Outer Face: Angles

Problem: Consider a directed acyclic graph $D = (V, A)$ with embedding \mathcal{F}, f_0 . Test whether D, \mathcal{F}, f_0 is upward planar and construct corresponding drawing.



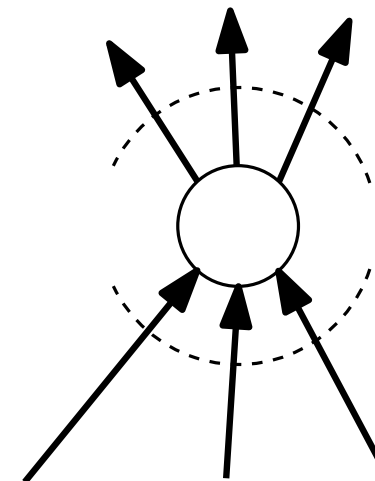
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Embedding is **bimodal**

if for each node:

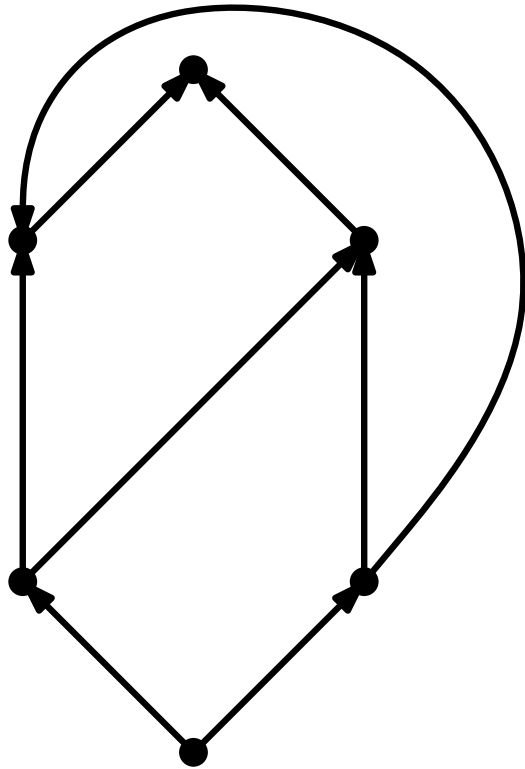


outgoing

incomming

Fixed Outer Face: Observations

- Bimodality is necessary but not sufficient



Fixed Outer Face: Observations

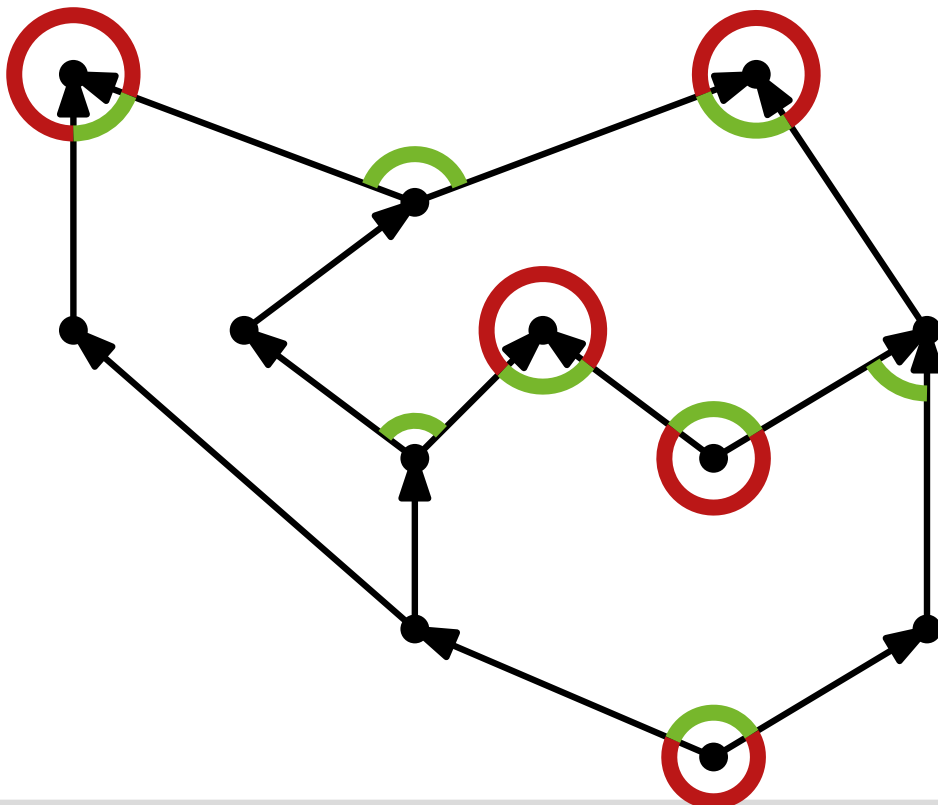
- Bimodality is necessary but not sufficient
- measure angles between two **incoming** or two **outgoing** edges

Angle α is **large** when $\alpha > \pi$, **small** otherwise

$L(v) := \#$ large angles at node v

$L(f) := \#$ large angles in face f

$S(v)$ resp. $S(f)$: $\#$ **small** angles



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Lemma 1: In any upward layout of D holds:

$$(1) \forall v \in V : L(v) = \begin{cases} 0 & v \text{ non source/sink} \\ 1 & v \text{ source/sink} \end{cases}$$

$$(2) \forall f \in \mathcal{F} : L(f) - S(f) = \begin{cases} -2 & f \neq f_0 \\ 2 & f = f_0 \end{cases}$$

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Pair, think and share:

Think about property (2). Why it holds?

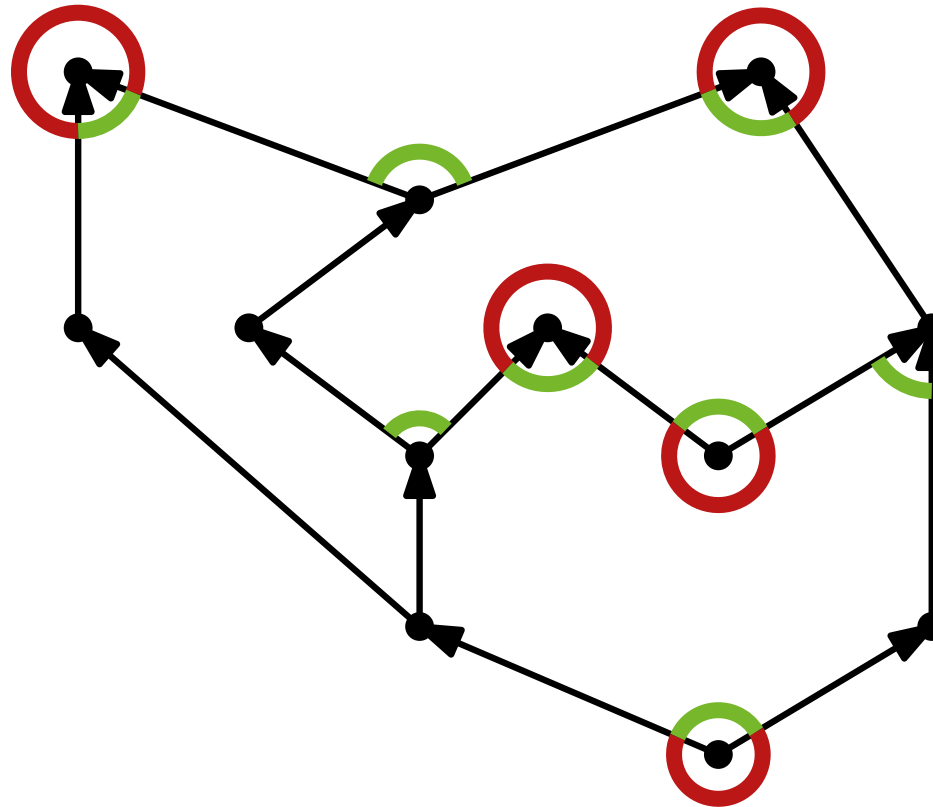
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- $A(f) := \#$ sources in face f (equal to the number of sinks)

It holds that: $L(f) + S(f) = 2A(f)$ for all faces.

- in any upward planar layout of D holds:

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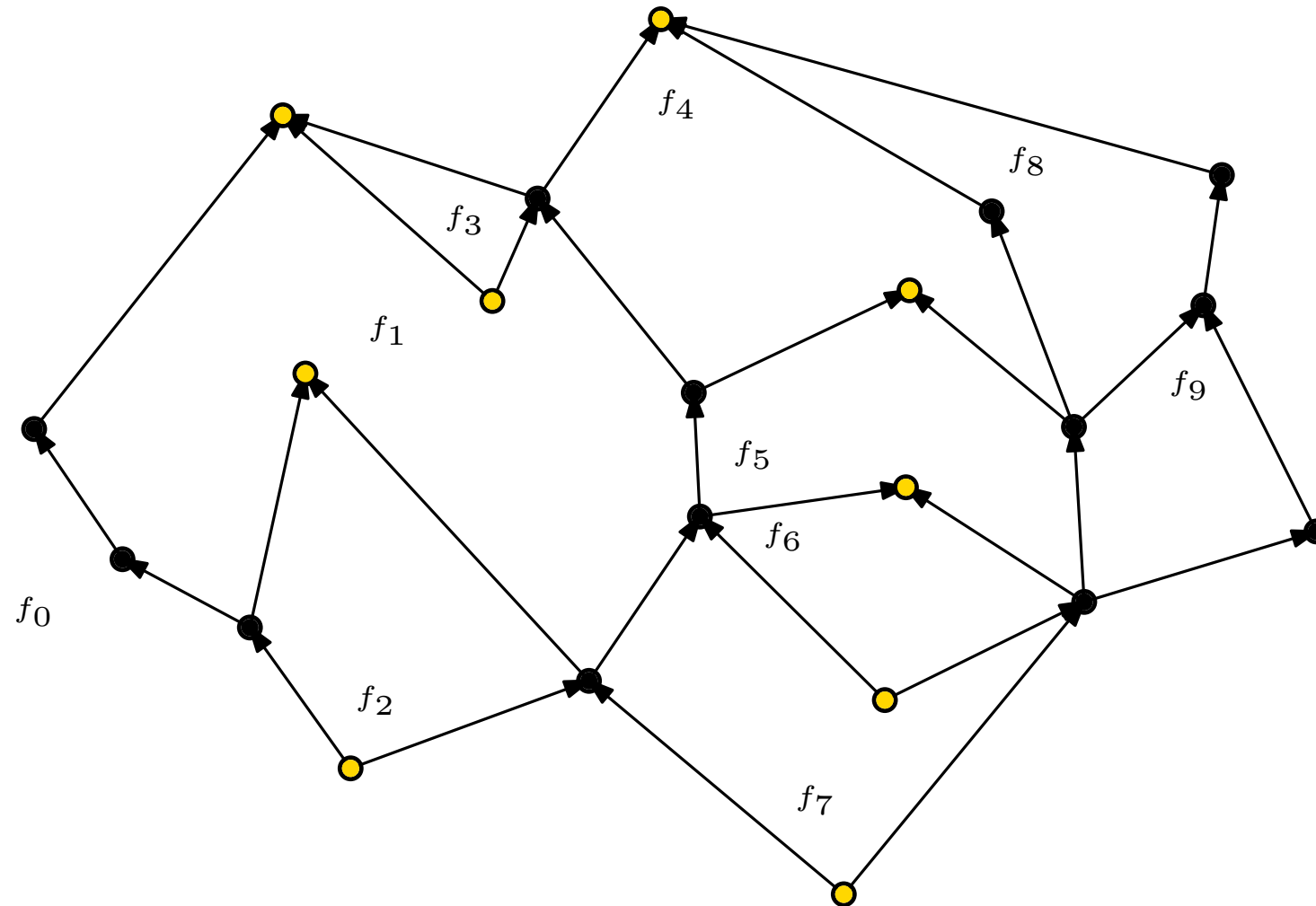
- Define assignement $\Phi : S \cup T \rightarrow \mathcal{F}$

(S set of sources, T sinks), where

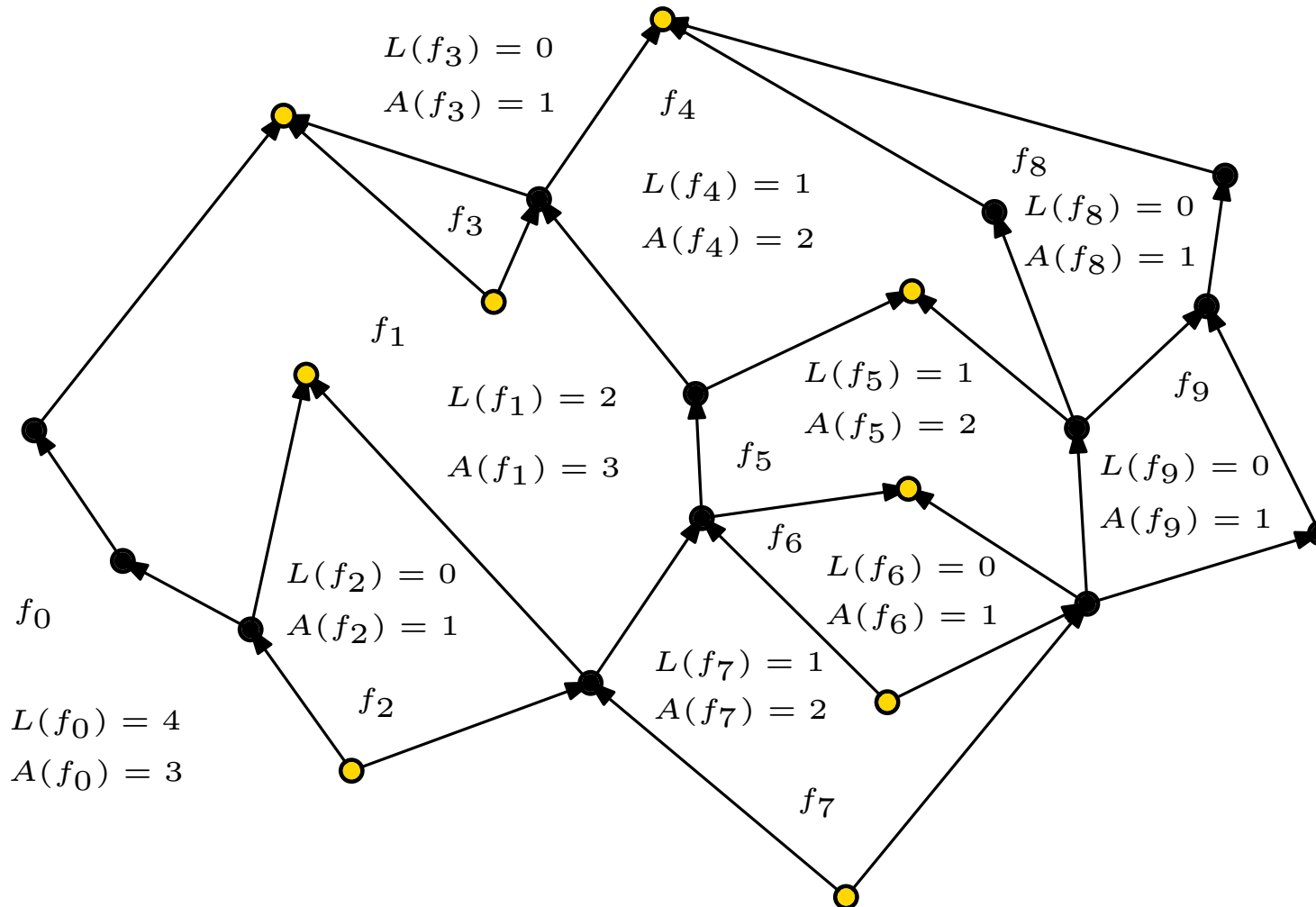
$\Phi : v \mapsto$ incident face, where v is forms large angle

- Φ is called **consistent**, if: $|\Phi^{-1}(f)| = \begin{cases} A(f) - 1 & f \neq f_0 \\ A(f) + 1 & f = f_0 \end{cases}$

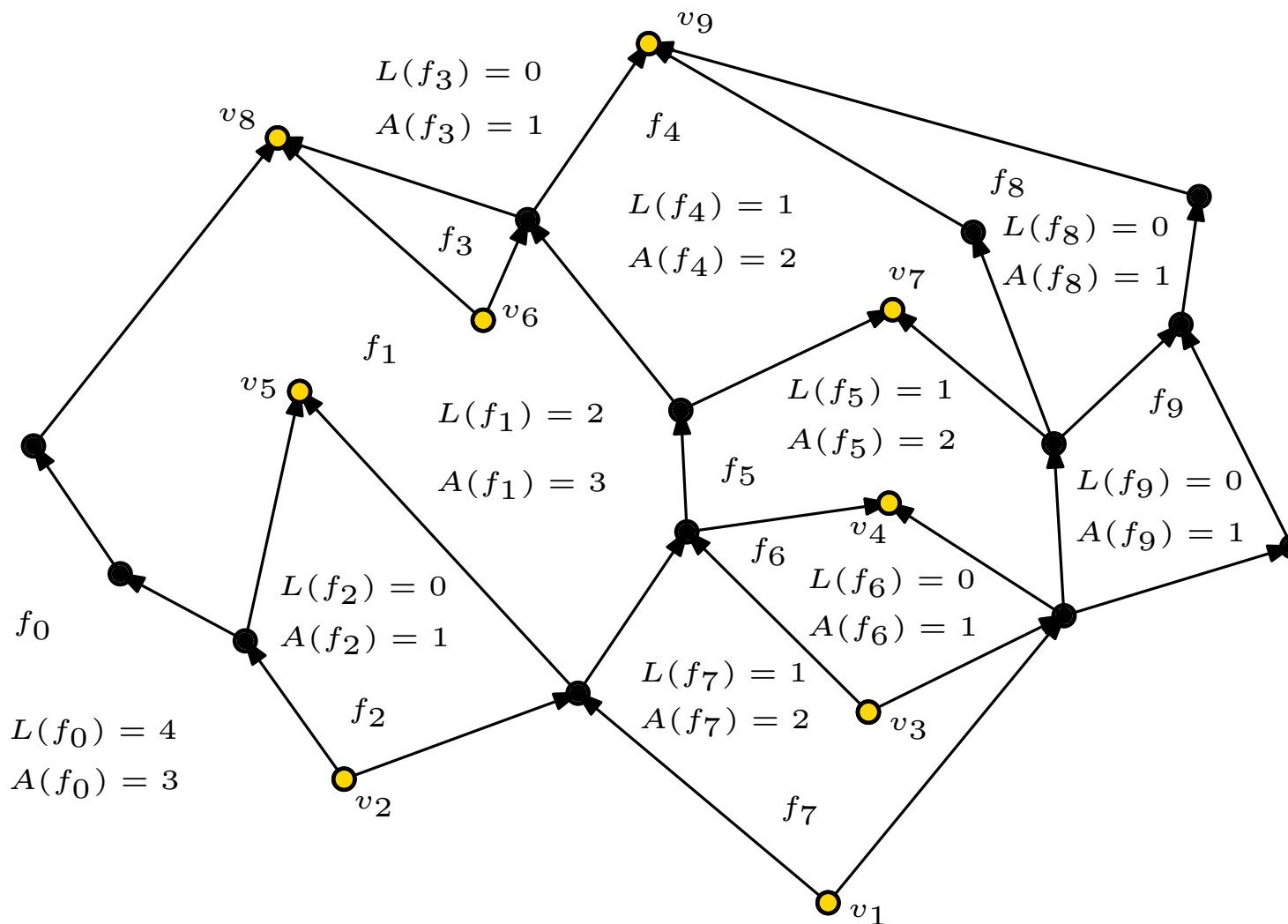
Example: Vertex-Face Assignment



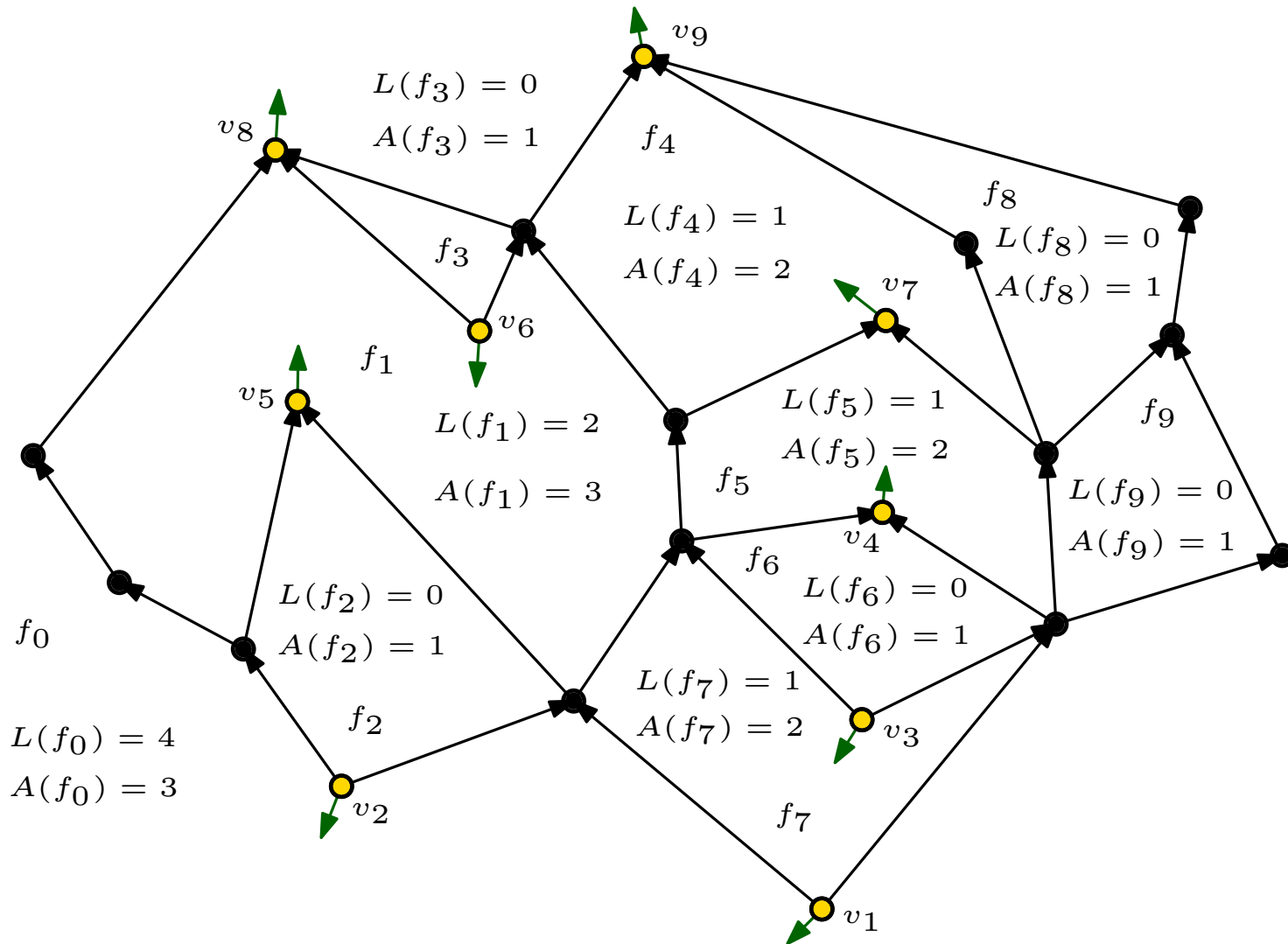
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- $\Phi(v_1) = f_0$
- $\Phi(v_2) = f_0$
- $\Phi(v_3) = f_7$
- $\Phi(v_4) = f_5$
- $\Phi(v_5) = f_1$
- $\Phi(v_6) = f_1$
- $\Phi(v_7) = f_4$
- $\Phi(v_8) = f_0$
- $\Phi(v_9) = f_0$

Thm 5: For a directed acyclic graph $D = (V, A)$ with combinatorial embedding \mathcal{F}, f_0 it holds:

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Characterization

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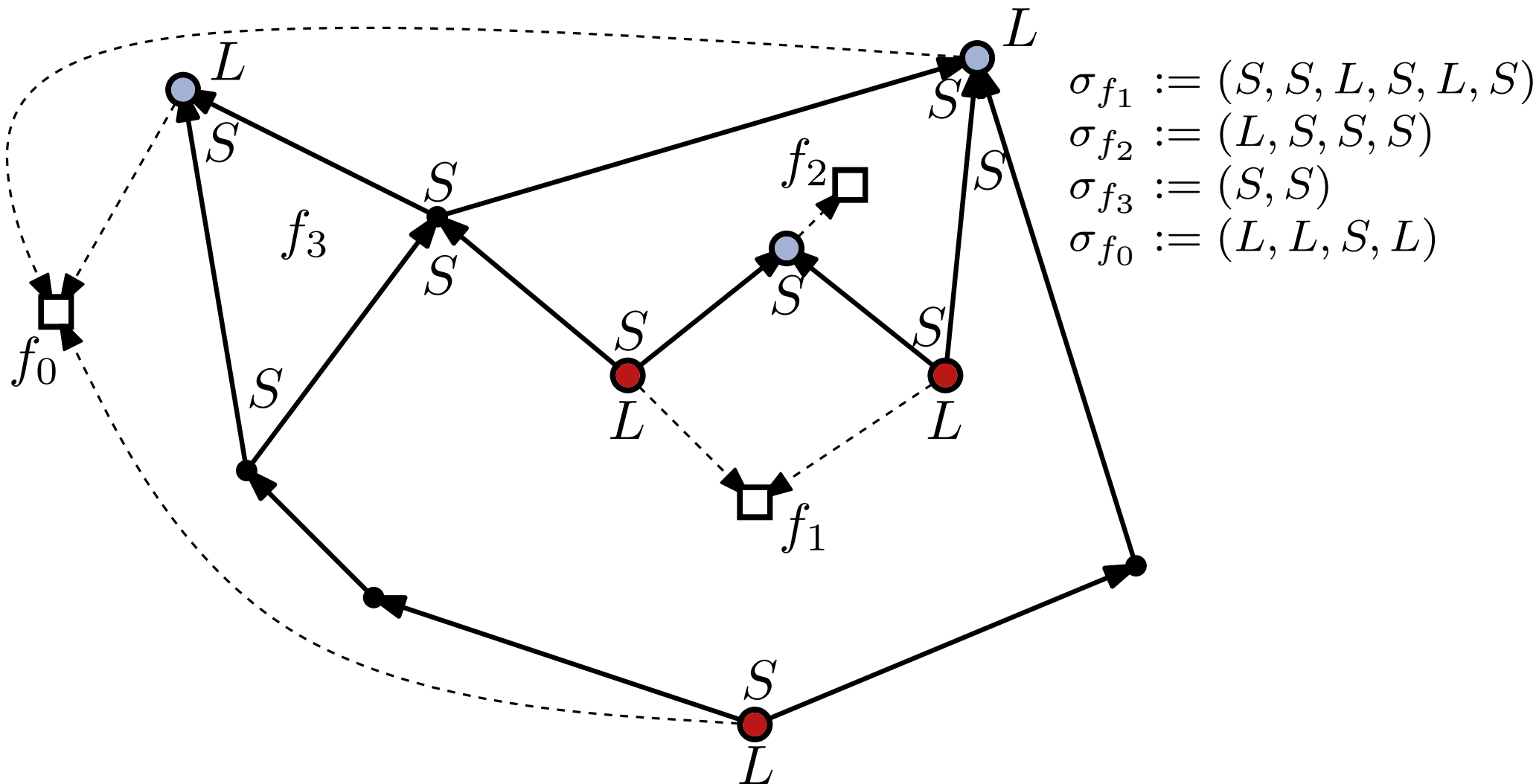
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\Leftarrow construct an st -digraph that contains D as spanning subgraph:

- insert edges in faces until they have single source and sink
- proof acyclicity, planarity and bimodality

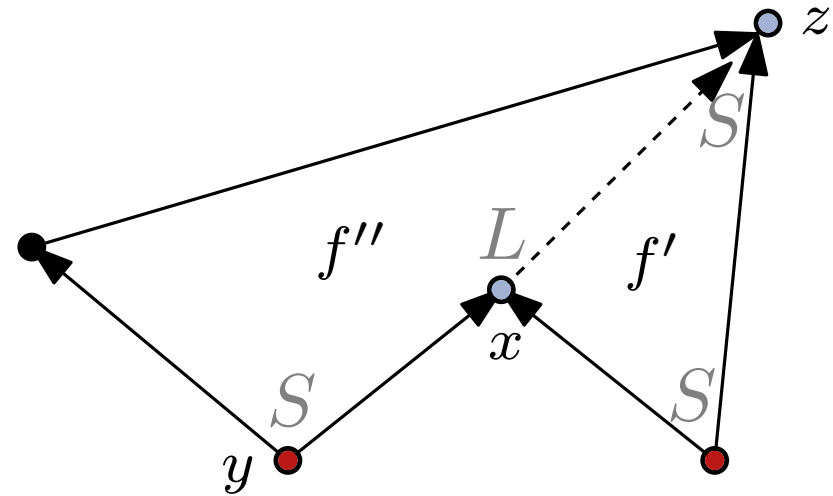
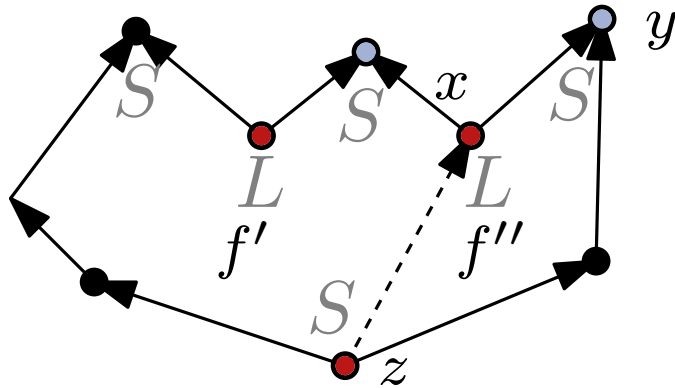
Proof of Theorem 5

Assign labels s_L, t_L, s_S, t_S to each source/sink of each face f .
Sequence σ_f .



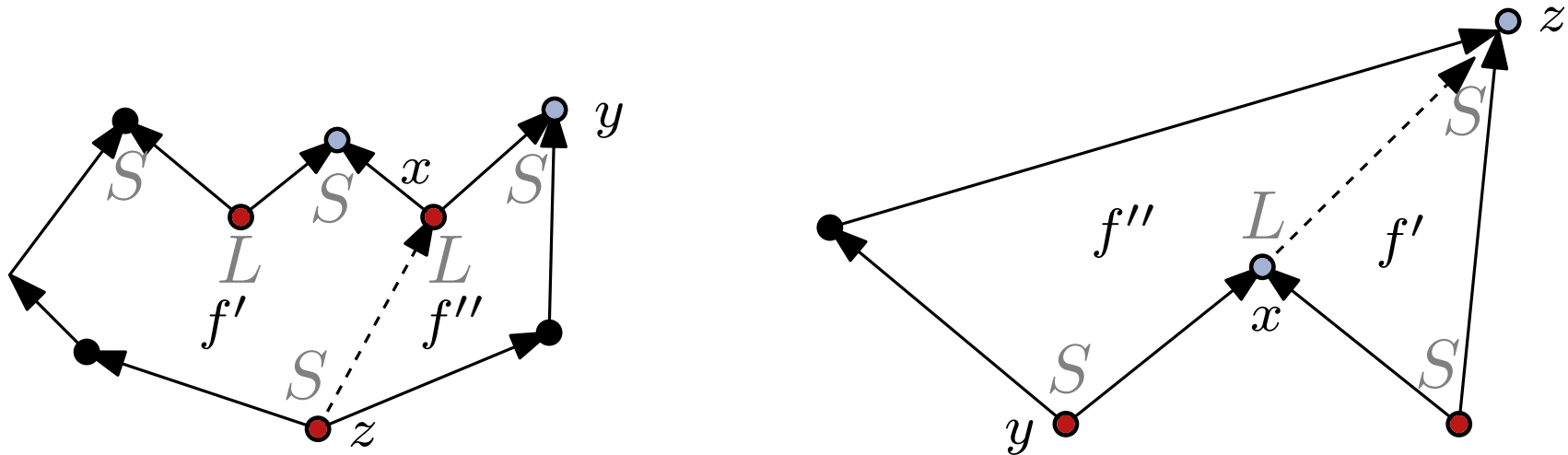
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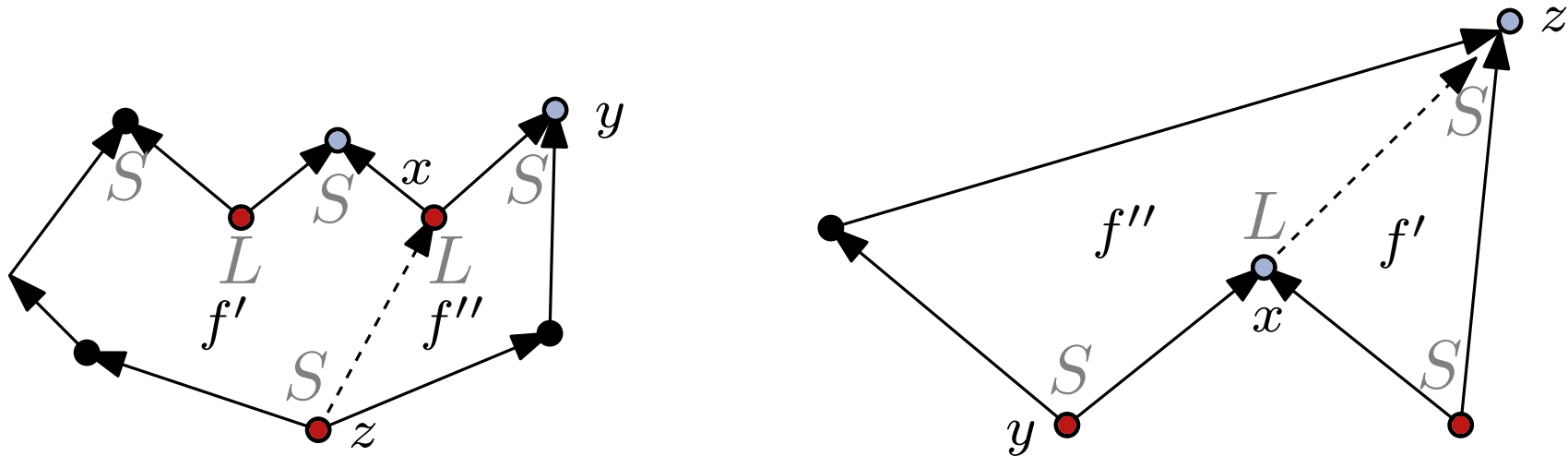
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- In the outface: select super source (resp. super sink) and add edges to (from) other sources (resp. sinks)

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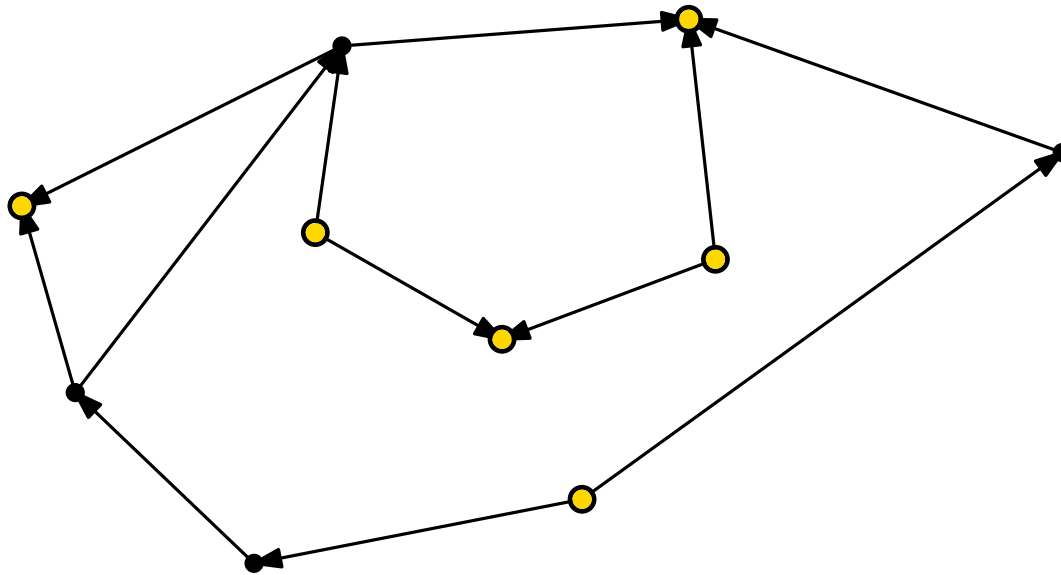
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How to check whether a consistent assignment exists?

Def: Flow network $N(D, \mathcal{F}, f_0) = ((W, A_N); \ell; u; b)$

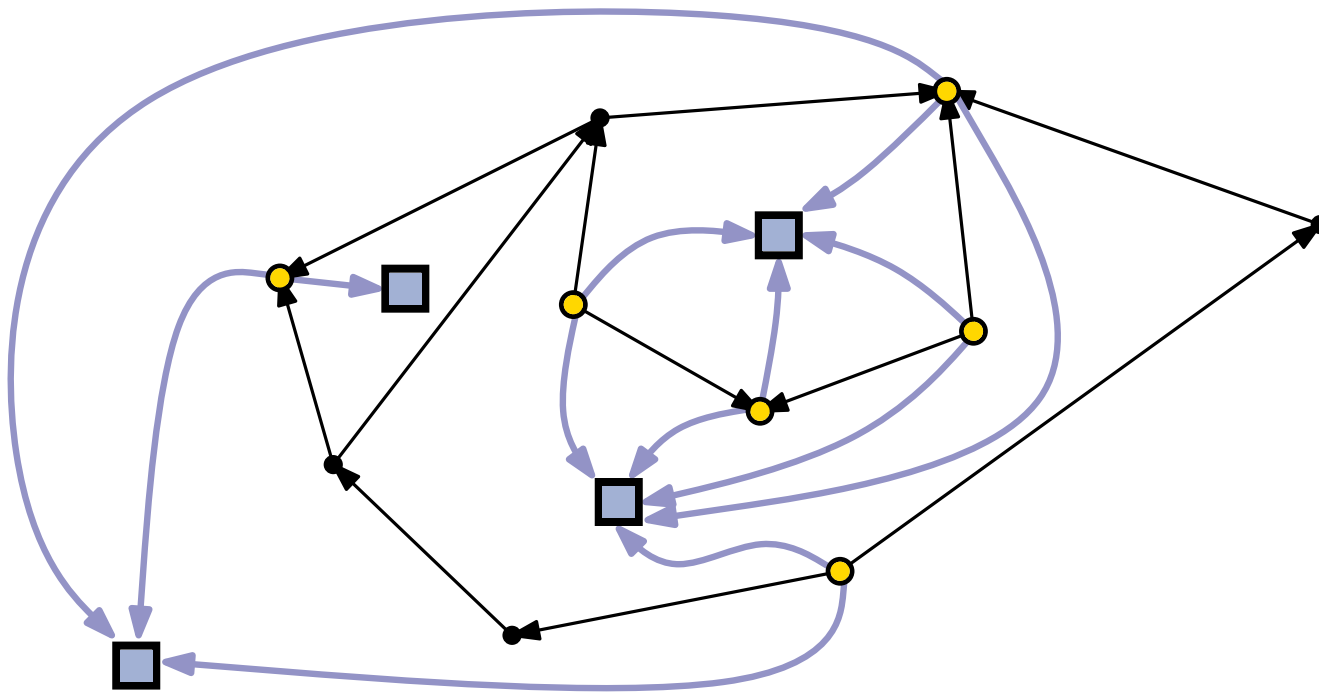
- $W = \{v \in V \mid v \text{ is source or sink}\} \cup \mathcal{F}$
- $A_N = \{(v, f) \mid v \text{ incident to } f\}$
- $l(a) = 0 \quad \forall a \in A_N$
- $u(a) = 1 \quad \forall a \in A_N$
- $b(q) = \begin{cases} 1 & \forall q \in W \cap V \\ -(A(q) - 1) & \forall q \in \mathcal{F} \setminus \{f_0\} \\ -(A(q) + 1) & q = f_0 \end{cases}$

Example



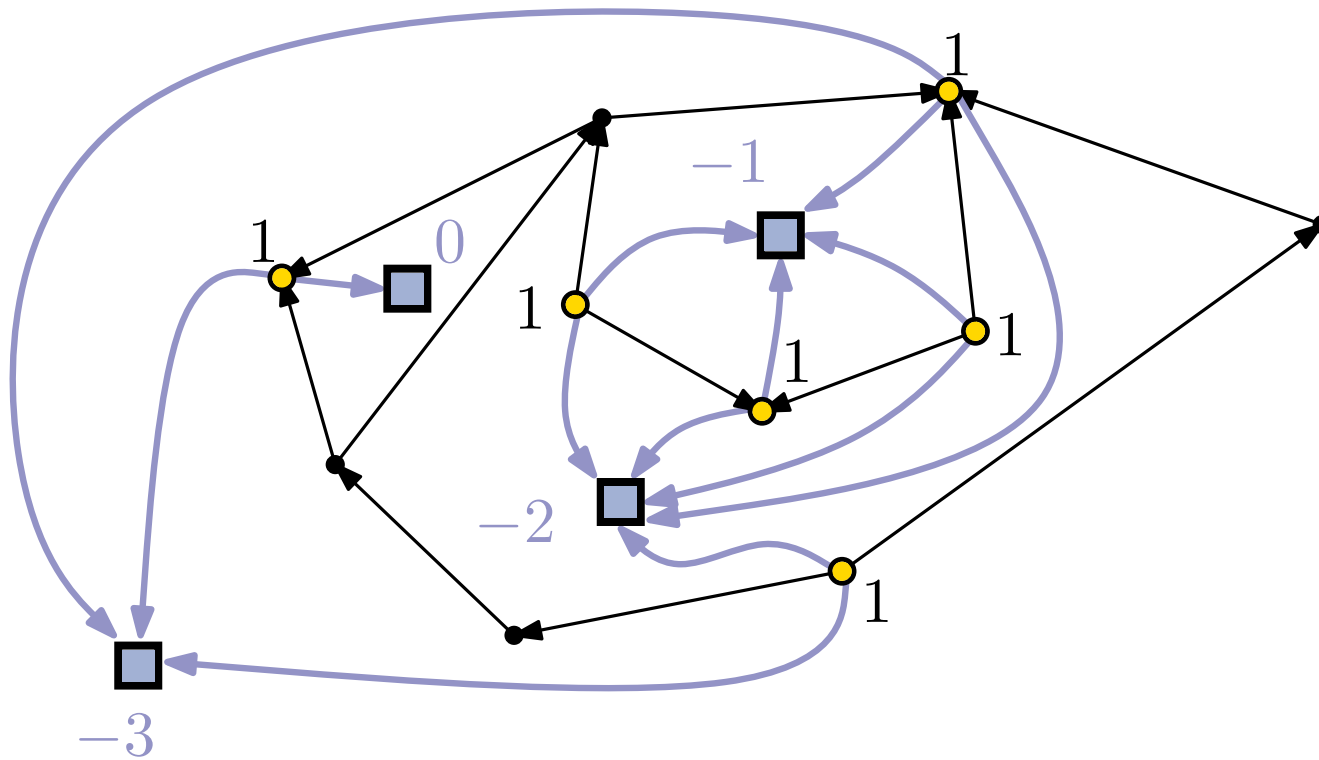
- normal nodes
- sources/sinks

Example



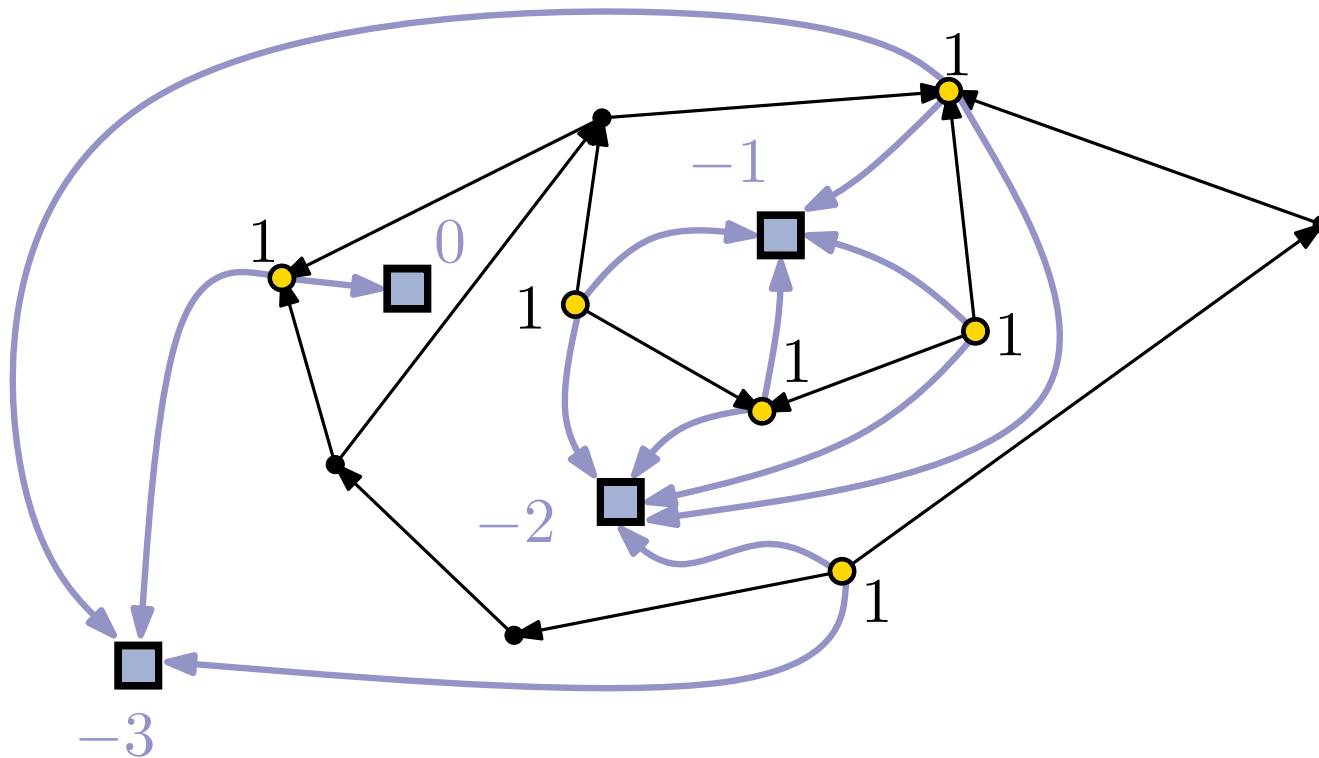
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Example



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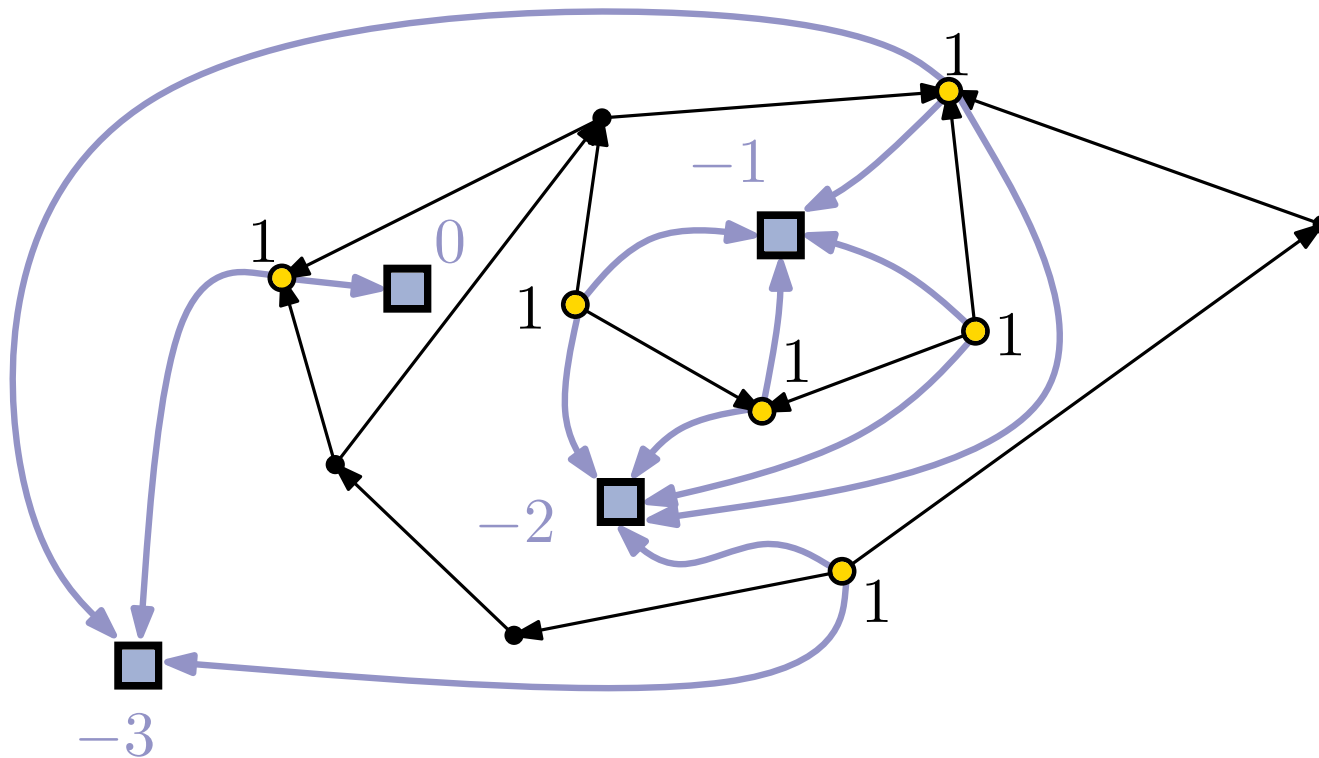
Example



- normal nodes
- sources/sinks
- face nodes

Thm 6: Let G be a directed acyclic digraph with embedding F and outer face f_0 . Bipartite flow network $N(D, \mathcal{F}, f_0)$ admits a valid flow of value r ($\#$ of sources/sinks) iff G has a consistent assignment of sources and sinks to faces.

Example



- normal nodes
- sources/sinks
- face nodes

- start with zero flow
- search for augmenting path (r times for total of r sources and sinks)

Final Remarks

- $O(rn)$ to decide whether consistent assignment exists
- works also without fixed outer face f_0 : first compute all faces as internal and then add two units of demand to a face vertex and test whether the total flow can be augmented by two units. Do it for every face.

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The layout can be constructed in the same time: $O(n)$ to augment to st -digraphs and $O(n)$ to draw the st -digraph

Discussion

- There exists a fixed parameter tractable algorithm to test upward planarity, with parameter number of triconnected components

[Healy, Lynch SOFSEM 2005]

- The decision of Theorem 2 can be done in $O(n + r^{1.5})$ time where $r = \#$ sources/sinks

[Abbasi, Healy, Rextin IPL 2010]

- many related concepts have been studied recently: quasi-planarity, upward drawings of mixed graphs

