Algorithms for Graph Visualization

Layered Layout – Part 2

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

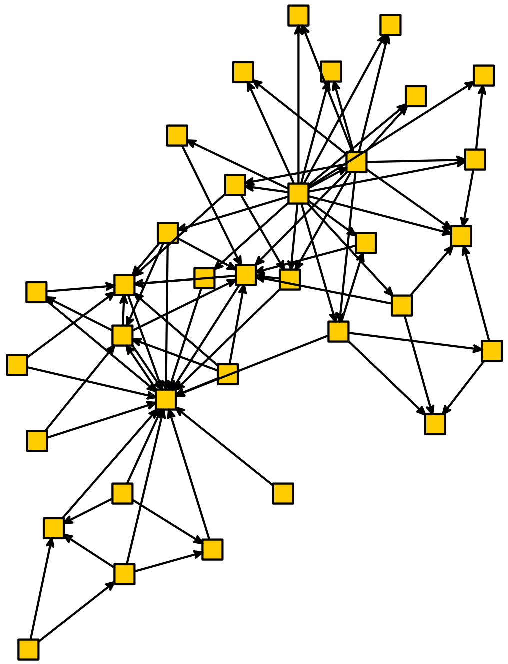
Tamara Mchedlidze

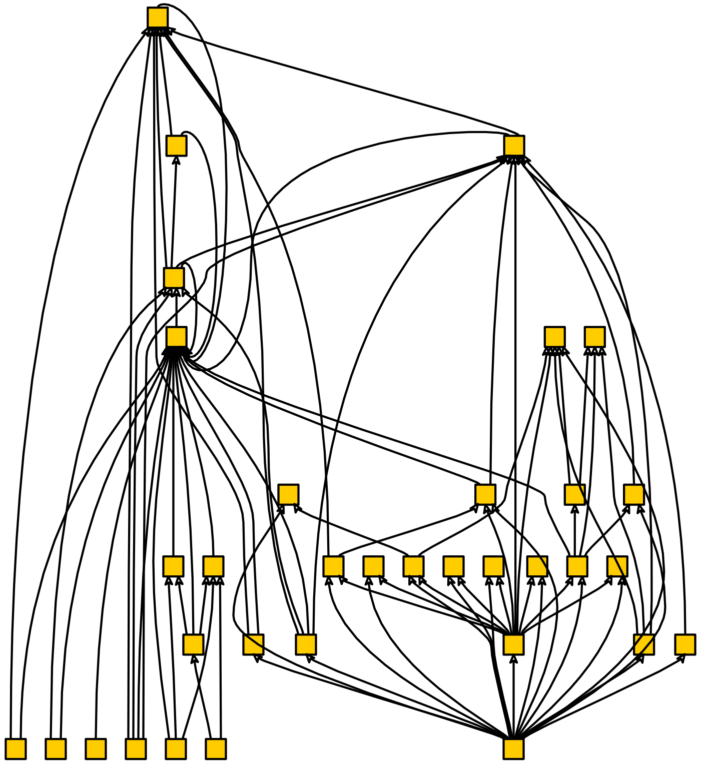
19.12.2016

Layered Layout

Given: directed graph $D = (V, A)$

Find: drawing of D that emphasized the hierarchy





Layered Layout

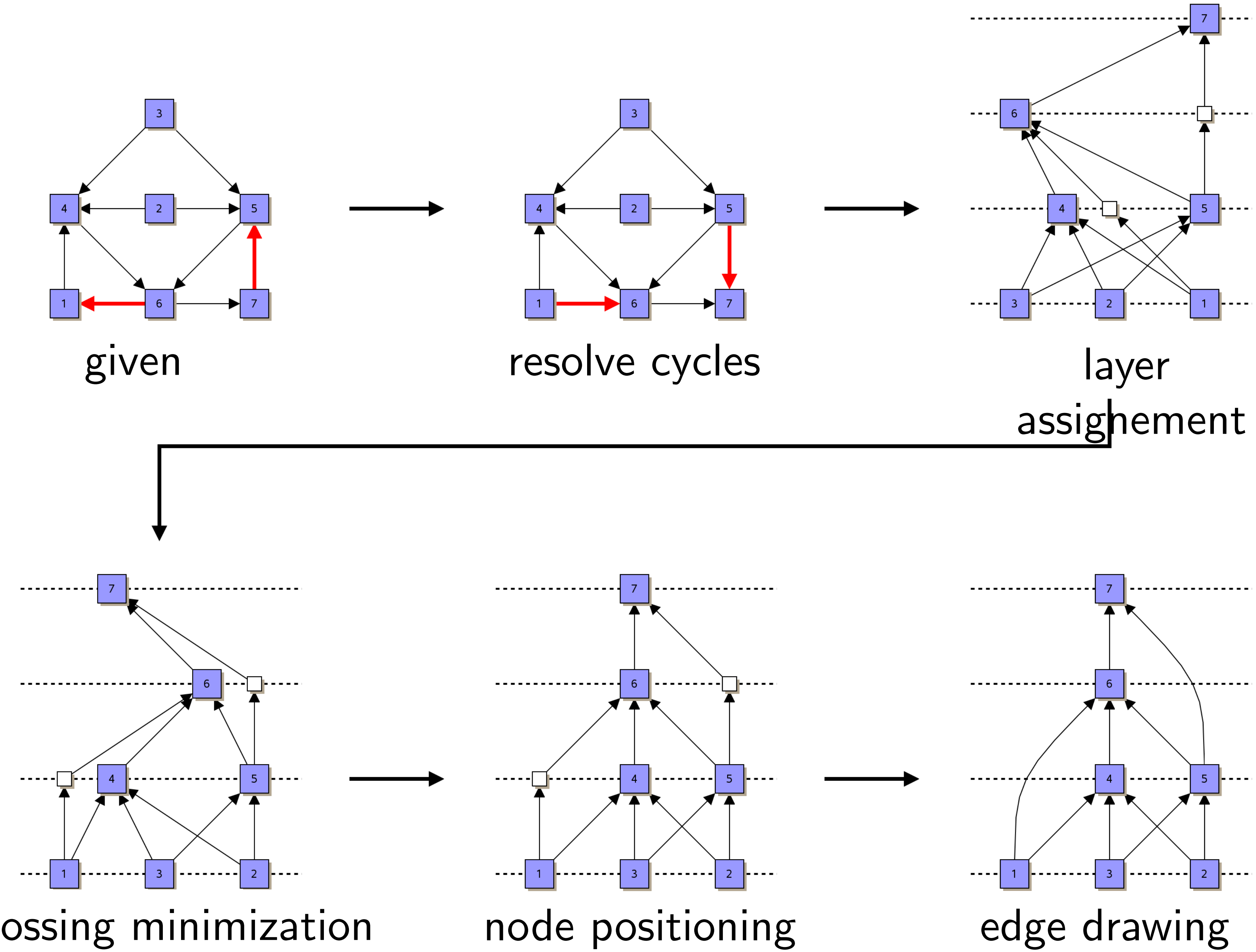
Given: directed graph $D = (V, A)$

Find: drawing of D that emphasized the hierarchy

Criteria:

- many edges pointing to the same direction
- edges preferably straght and short
- position nodes on (few) horizontal lines
- preferably few edge crossings
- nodes distributed evenly

Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



given

resolve cycles

layer assignement

crossing minimization

node positioning

edge drawing

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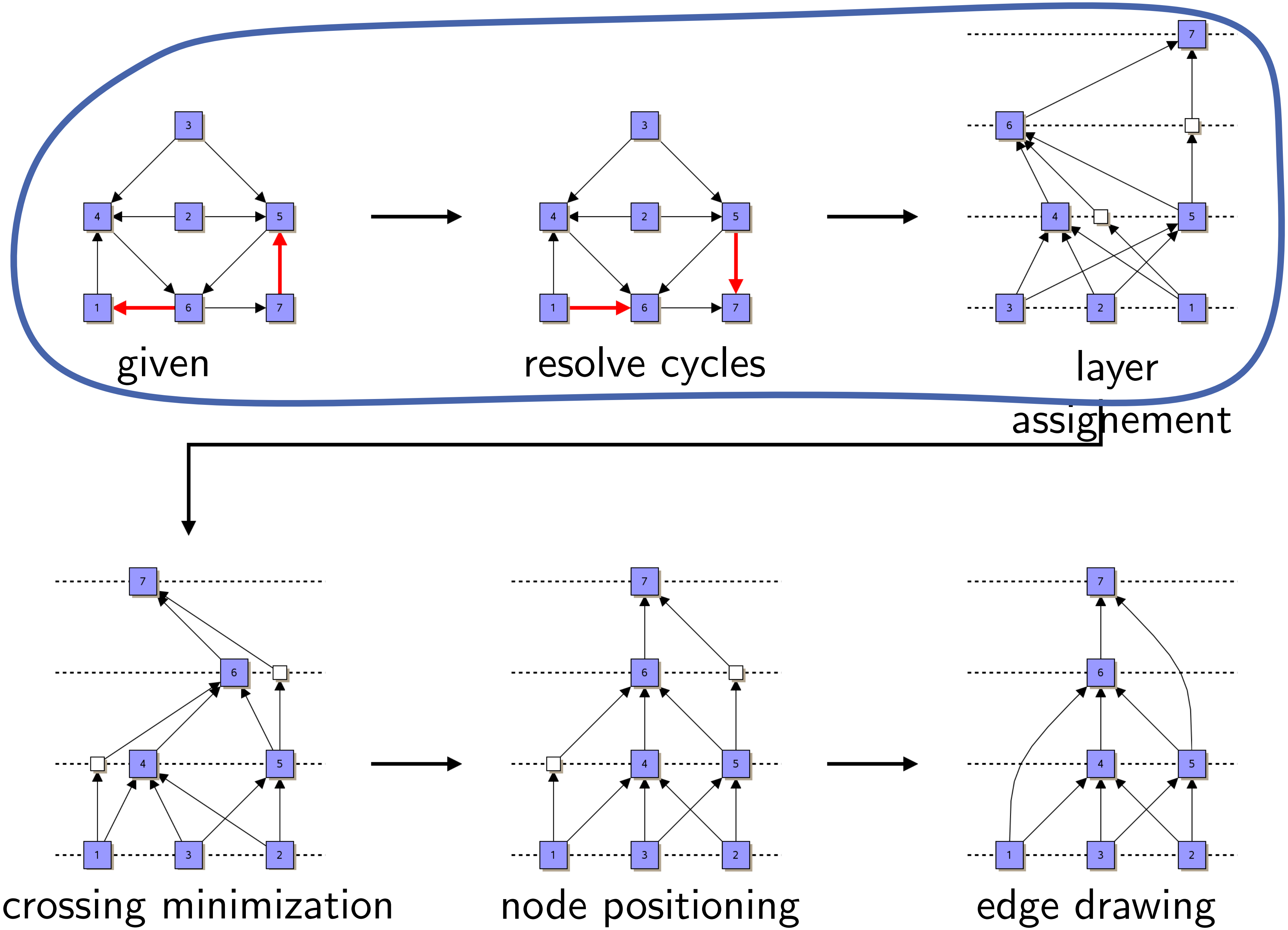

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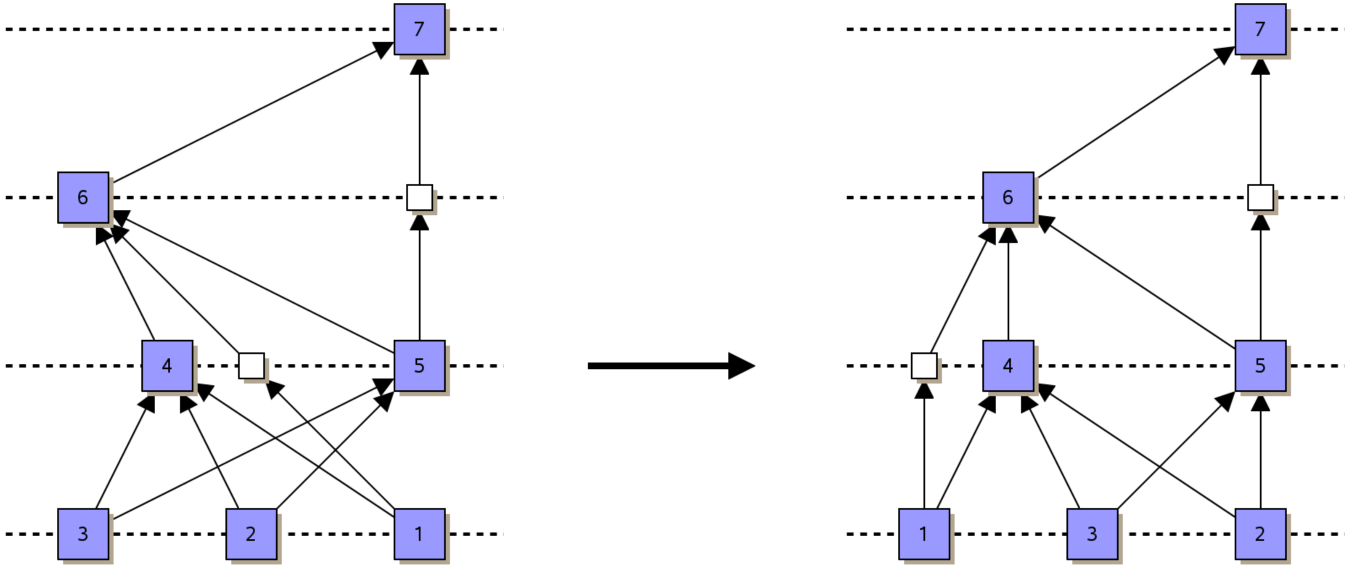
Step 3: Crossing Minimization



How would you proceed?

Problem Statement

Given: DAG $D = (V, A)$, nodes are partitioned in disjoint layers

Find: Order of the nodes on each layer, so that the number of crossing is minimized

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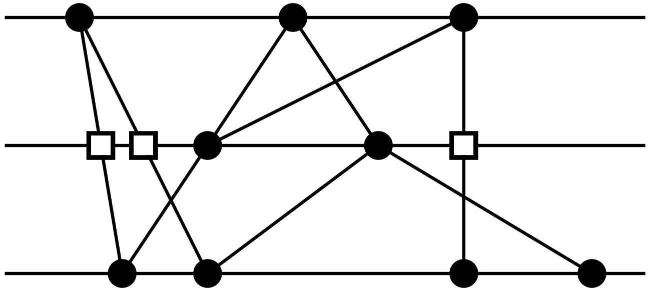
Properties

- Problem is NP-hard even for two layers (Bipartite Crossing Number [Garey, Johnson '83])
- Hardly any approach over several layers simultaneously
- Usually iterative optimization for two adjacent layers
- For that: insert dummy nodes at the intersection of edges with layers



One-sided Crossing Minimization (OSCM)

Given: 2-Layered-Graph $G = (L_1, L_2, E)$ and ordering of the nodes x_1 of L_1

Find: Node ordering x_2 of L_2, such that the number of crossings among E is minimum

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Observation:

- The number of crossings in 2-layered drawing of G depends only on x_1 and x_2, not from the exact positions
- for $u, v \in L_2$ the number of crossings among incident to them edges depends on whether $x_2(u) < x_2(v)$ or $x_2(v) < x_2(u)$ and not on the positions of other vertices

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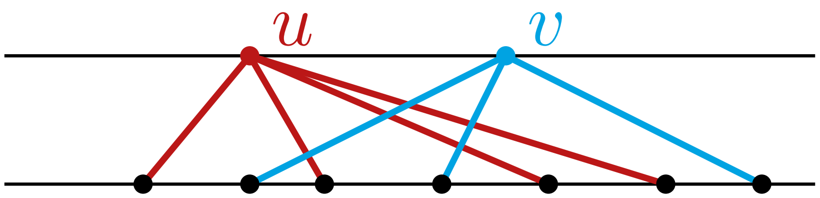
Def: $c_{uv} := |\{(uw, vz) \mid w \in N(u), z \in N(v), x_1(z) < x_1(w)\}|$ for $x_2(u) < x_2(v)$



$c_{uv} = 5$
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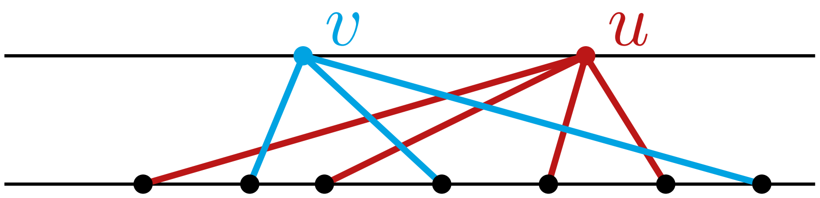

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Further Properties

Def: Crossing number of G with orders x_1 and x_2 for L_1 and L_2 is denoted by cr(G, x_1, x_2);
for fixed x_1 then $\mathrm{opt}(G, x_1) = \min_{x_2}$ cr(G, x_1, x_2)

Lemma 1: The following equalities hold:

- cr$(G, x_1, x_2) = \sum_{x_2(u) < x_2(v)} c_{uv}$
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Efficient computation of cr(G, x_1, x_2) see Exercise.

Iterative Crossing Minimization

Let $G = (V, E)$ be a DAG with layers $L_1, \dots, L_h$.

(1) compute a random ordering x_1 for layer L_1

(2) for $i = 1, \dots, h-1$ consider layers L_i and L_{i+1} and minimize cr(G, x_i, x_{i+1}) with fixed x_i ($\rightarrow$ **OSCM**)

(3) for $i = h-1, \dots, 1$ consider layers L_{i+1} and L_i and minimize cr(G, x_i, x_{i+1}) with fixed x_{i+1} ($\rightarrow$ **OSCM**)

(4) repeat (2) and (3) until no further improvement happens

(5) repeat steps (1)–(4) with another x_1

(6) return the best found solution

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Theorem 1: The One-Sided Crossing Minimization (OSCM) problem is NP-hard [Eades, Wormald 1994].

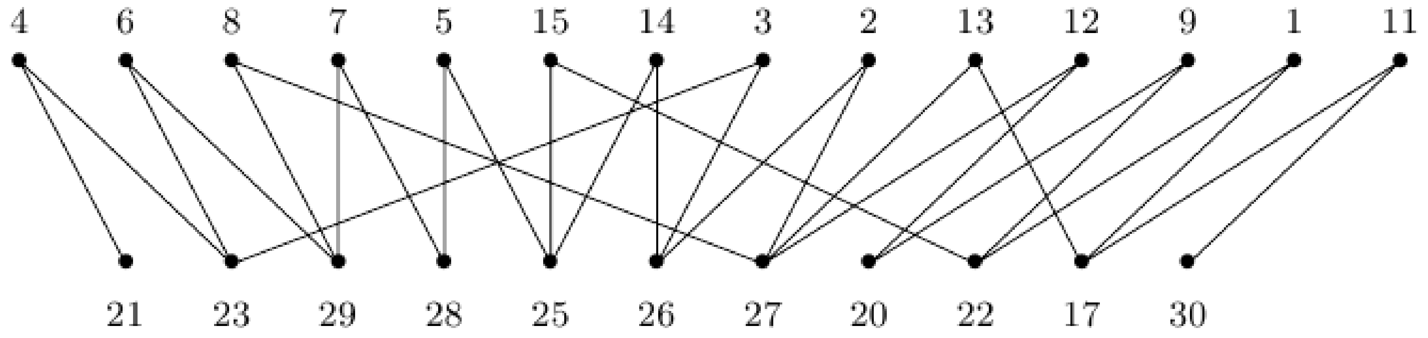
Algorithms for OSCM

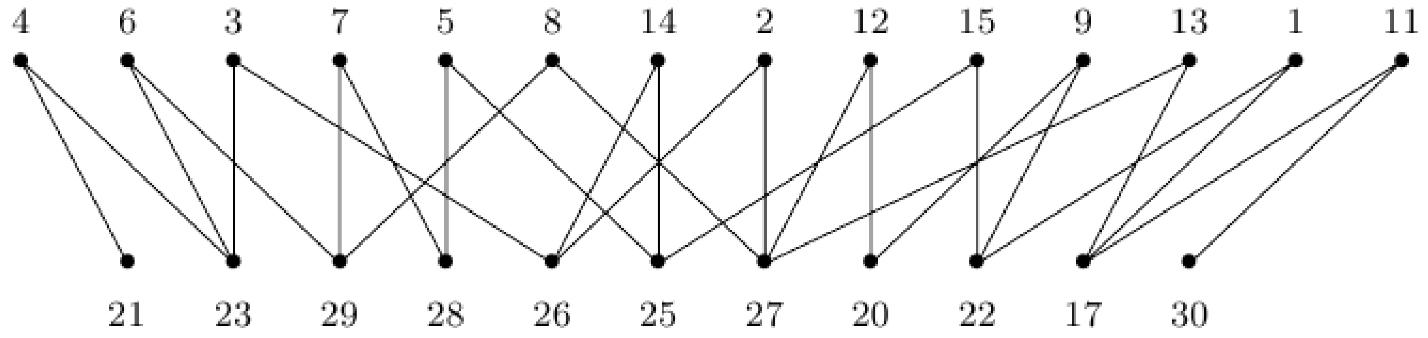
Heuristics:

- Barycenter
- Median
- . . .

Exact:

- ILP Model





Barycenter Heuristic (Sugiyama, Tagawa, Toda 1981)

Idea: few crossing when nodes are close to their neighbours

- set

$$x_2(u) = \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(u)$$

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Properties:

- trivial implementation
- fast
- usually very good results...
- finds optimum if $\operatorname{opt}(G, x_1) = 0$ (see Exercises)
- there are graphs on which it performs $\Omega(\sqrt{n})$ times worse than optimal

Median-Heuristic (Eades, Wormald 1994)

Idea: use the median of the coordinates of neighbours

- for a node $v \in L_2$ with neighbours $v_1, \ldots, v_k$ set $x_2(v) = \mathrm{med}(v) = x_1(v_{\lfloor k/2 \rfloor})$ and $x_2(v) = 0$ if $N(v) = \emptyset$
- if $x_2(u) = x_2(v)$ and u, v have different parity, place the node with odd degree to the left
- if $x_2(u) = x_2(v)$ and u, v have the same parity, place an arbitrary of them to the left
- Runs in time $O(|E|)$

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Properties:

- trivial implementation
- fast
- mostly good performance
- finds optimum when $\mathrm{opt}(G, x_1) = 0$
- **Factor-3 Approximation**

Approximation Factor

Theorem 2: Let $G = (L_1, L_2, E)$ be a 2-layered graph and x_1 an arbitrary ordering of L_1. Then it holds that

$$\mathrm{med}(G, x_1) \leq 3\,\mathrm{opt}(G, x_1).$$

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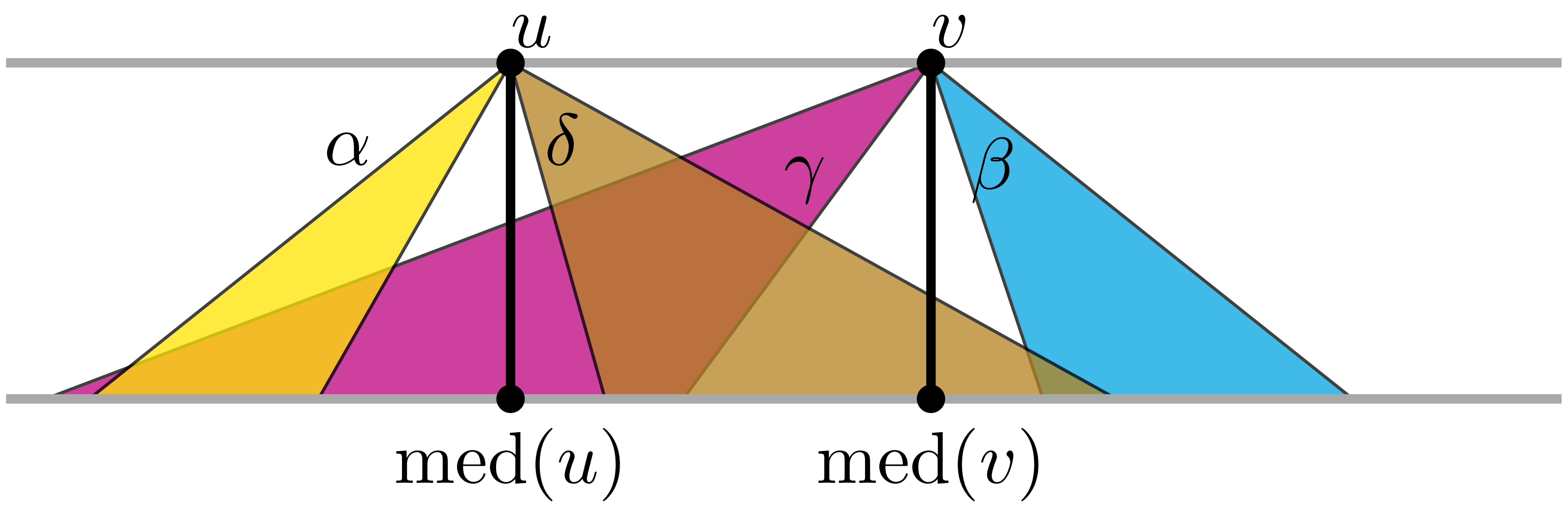

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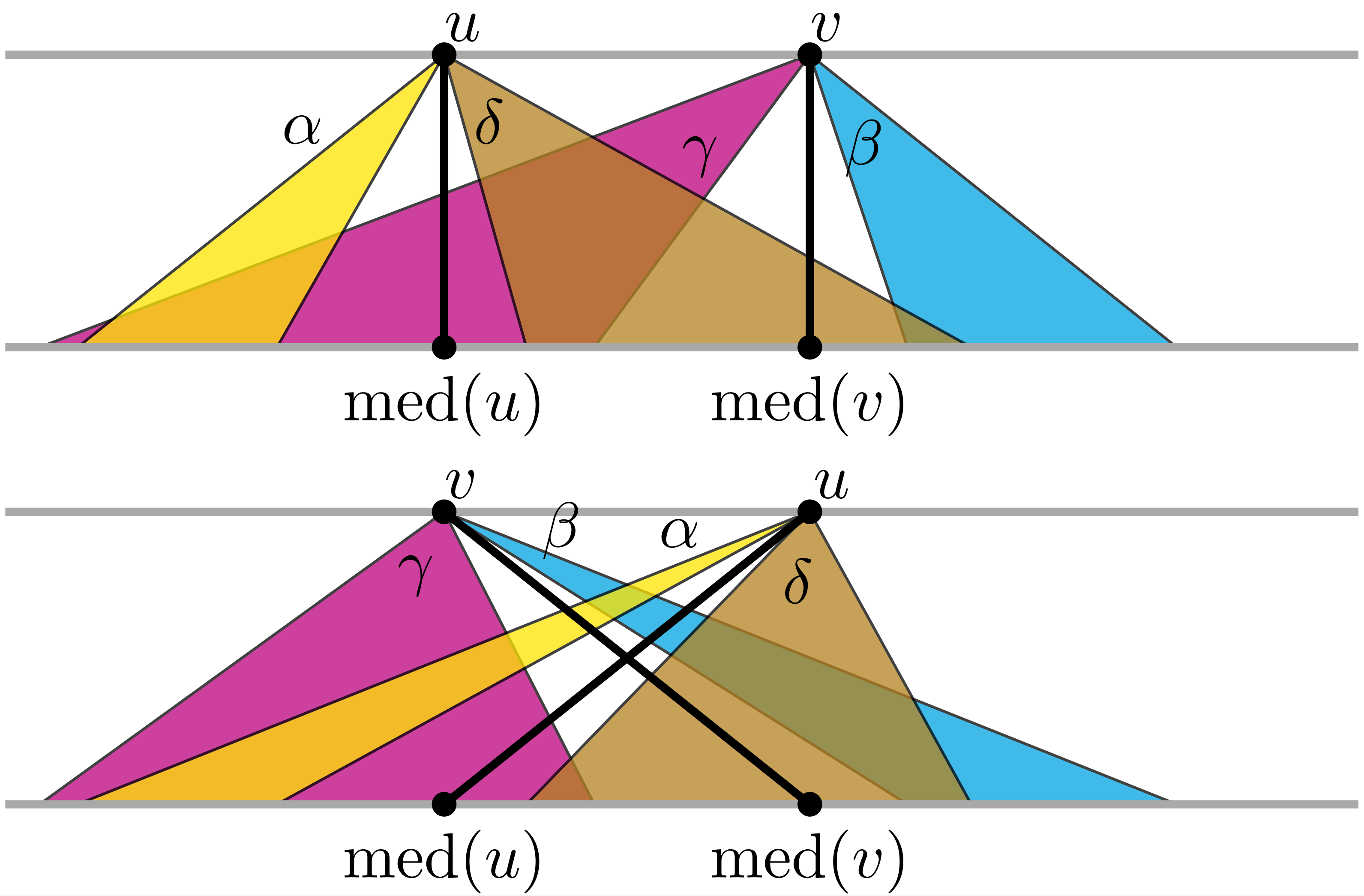

Integer Linear Programming

Properties:

- branch-and-cut technique for DAGS of limited size
- useful for graphs of small to medium size
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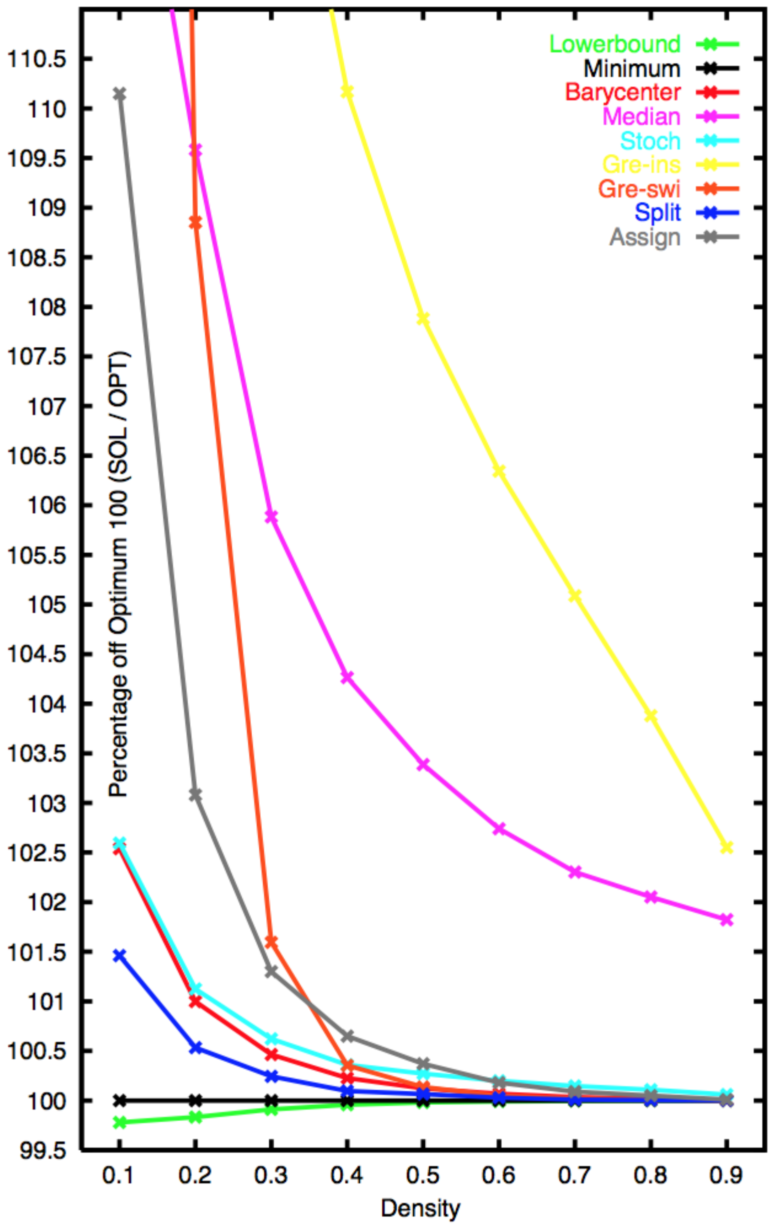
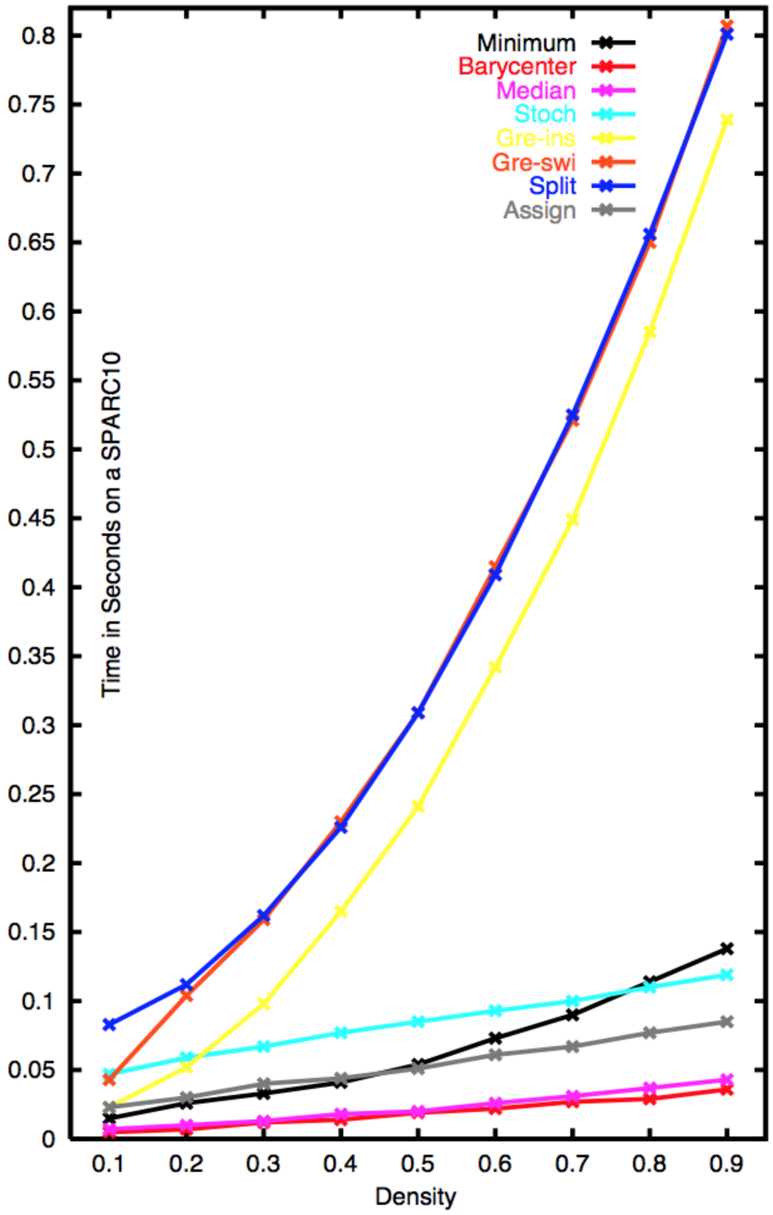
Modell: see Blackboard

Experimental Evaluation (Jünger, Mutzel 1997)



Results for 100 instances on 20 + 20 nodes with increasing density



Time for 100 instances on 20 + 20 nodes with increasing density

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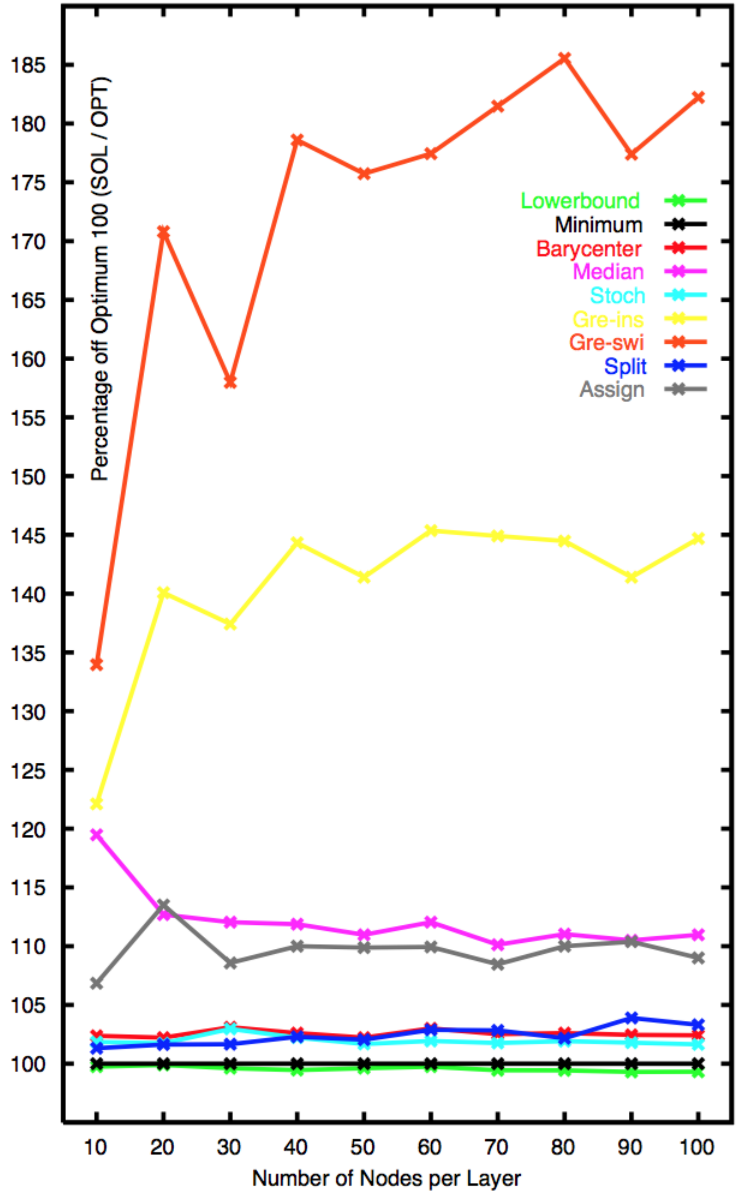
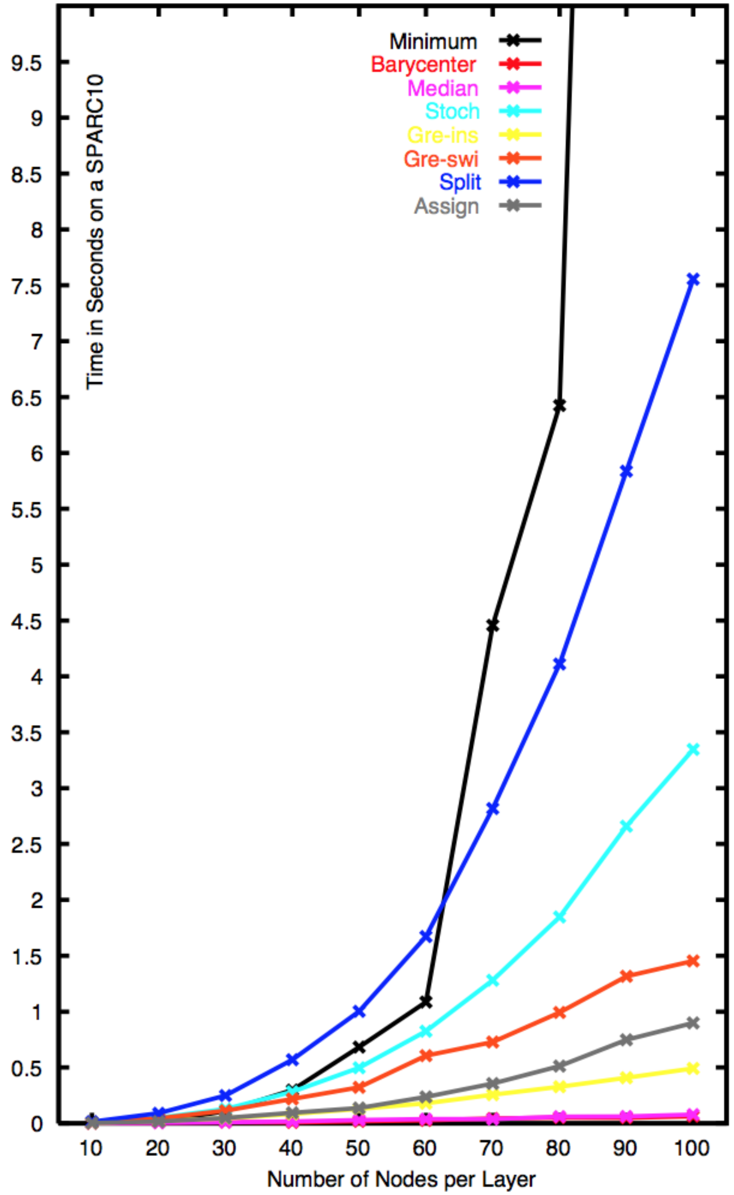
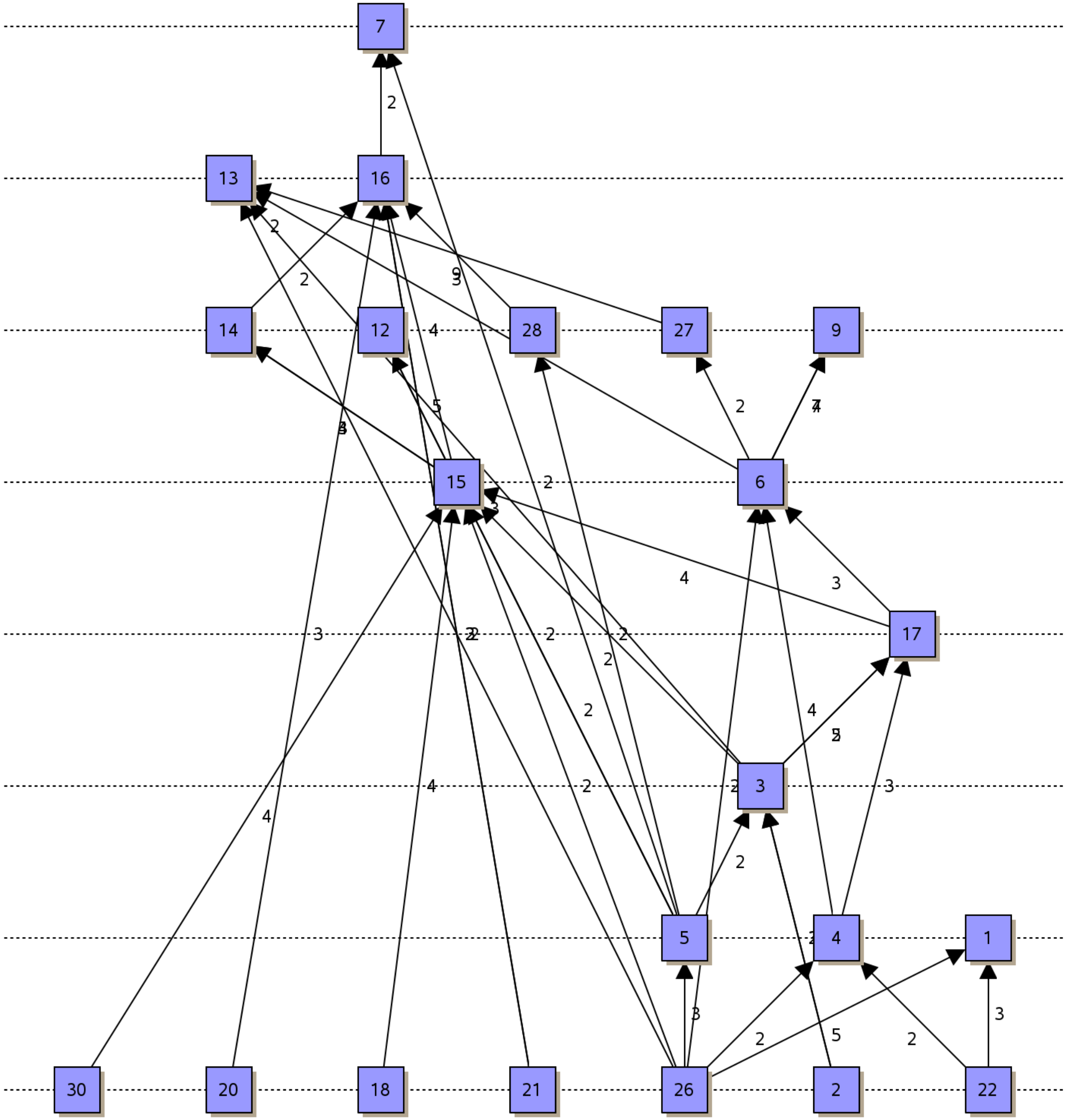


Results for 10 instances of sparse graphs with increasing size



Time for 10 instances of sparse graphs with increasing size

Example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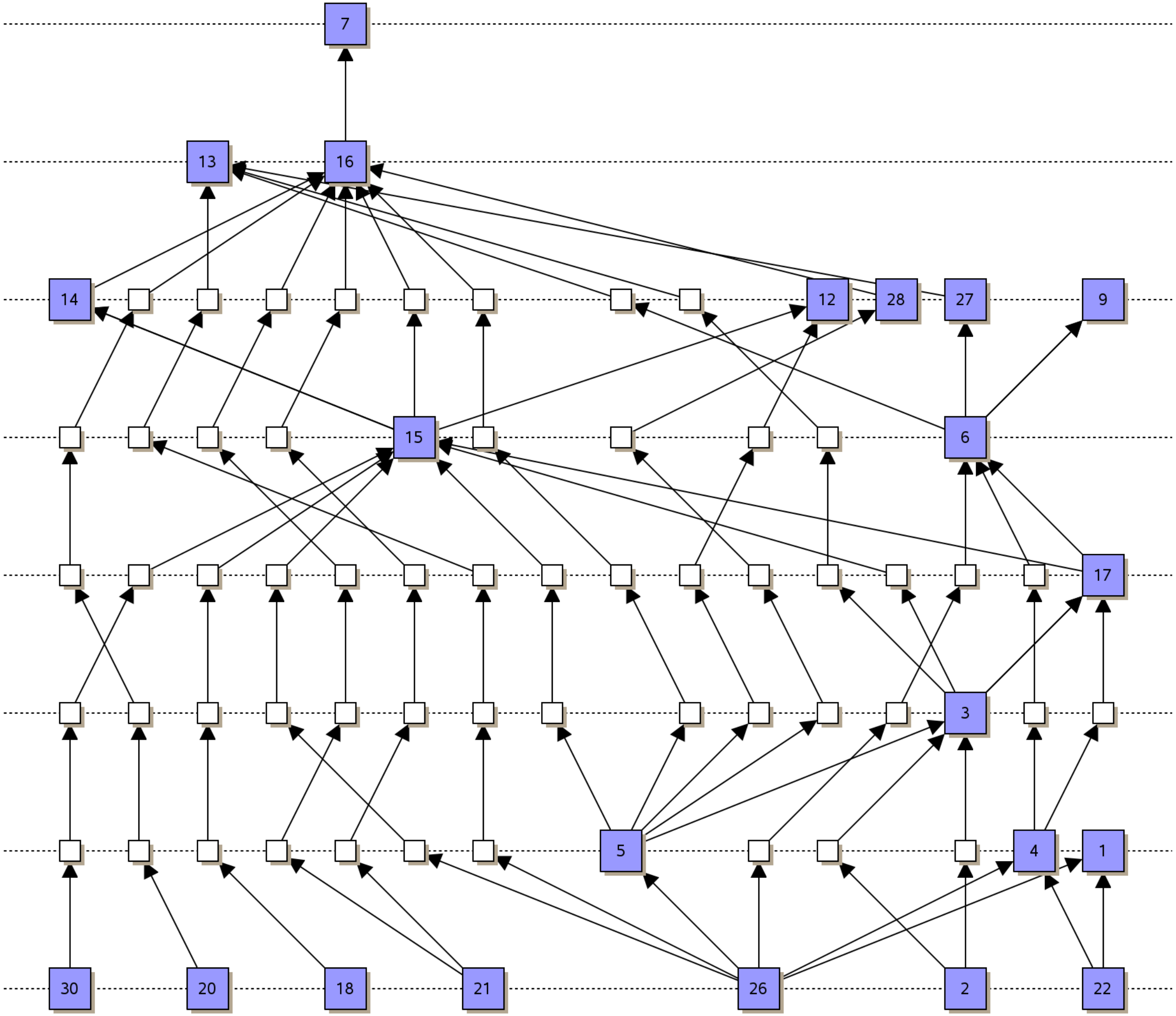
Example



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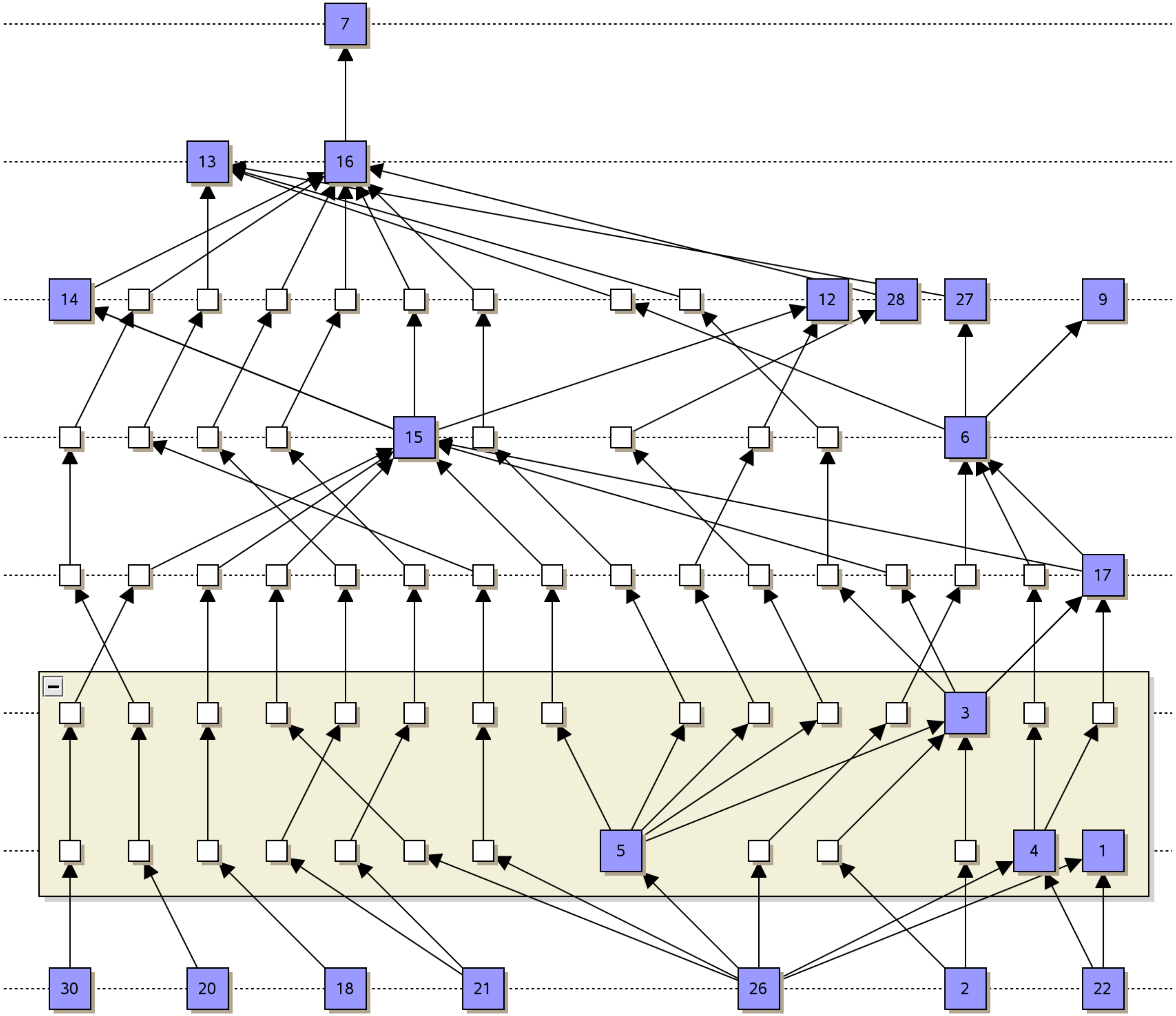

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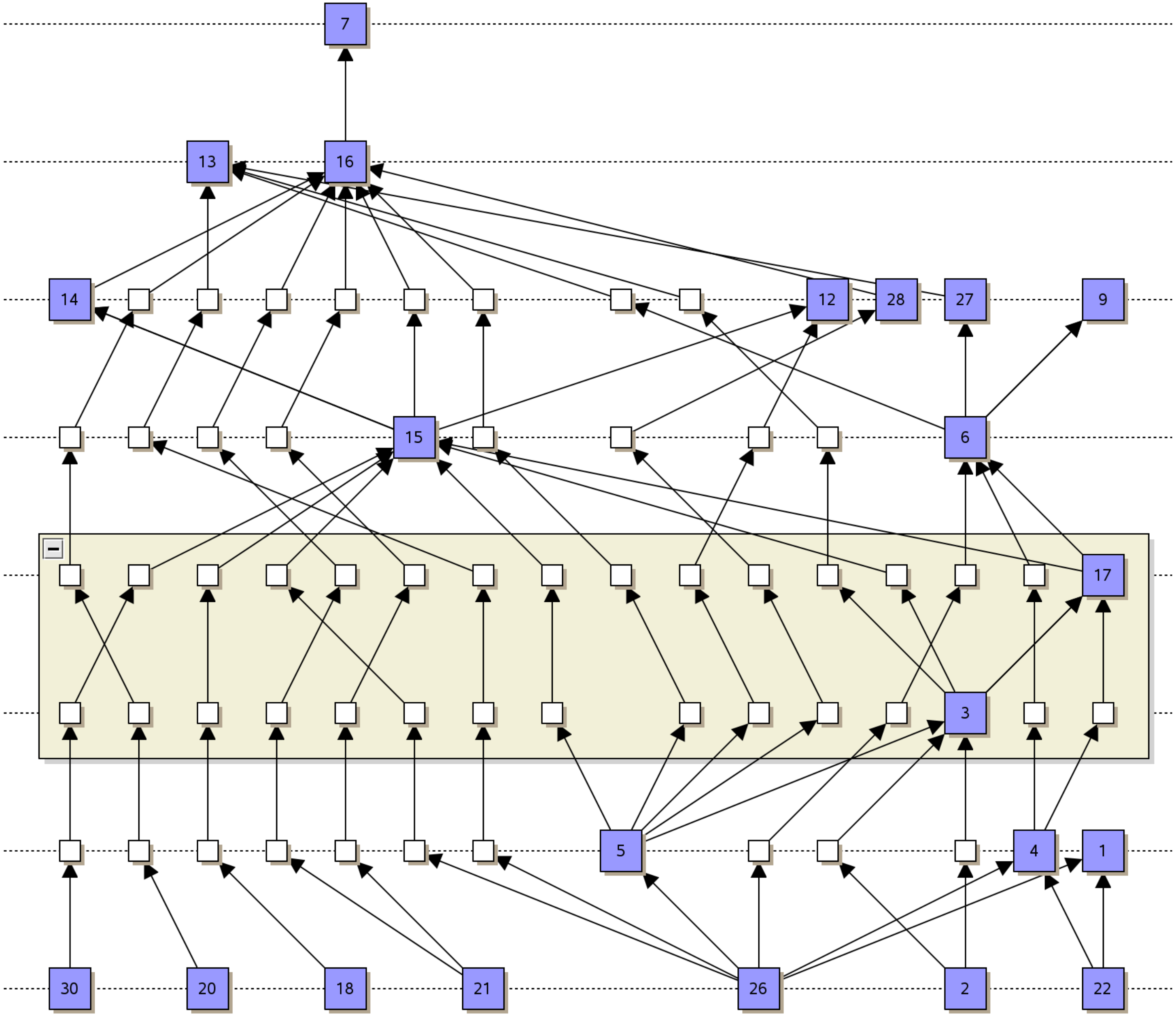

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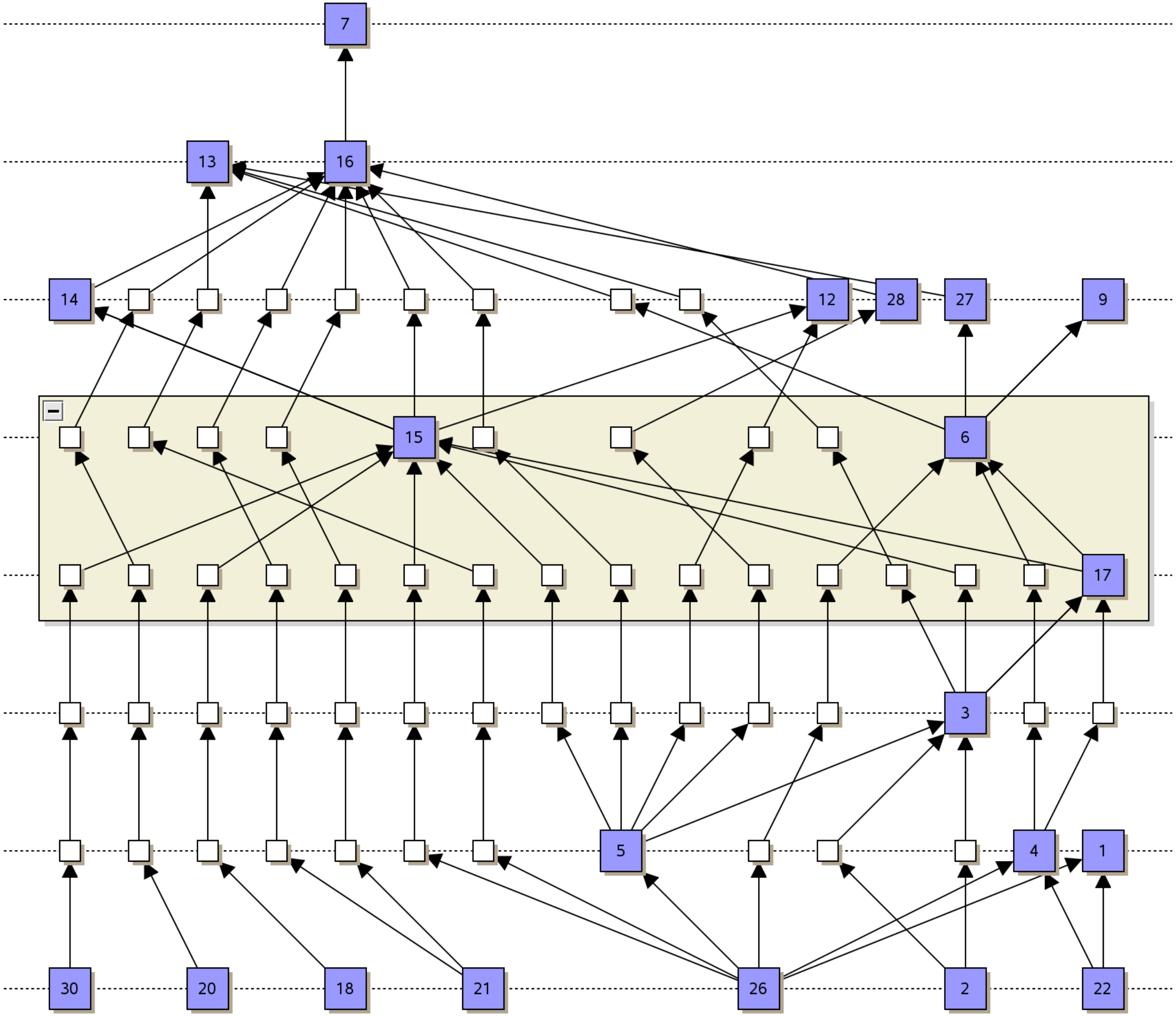

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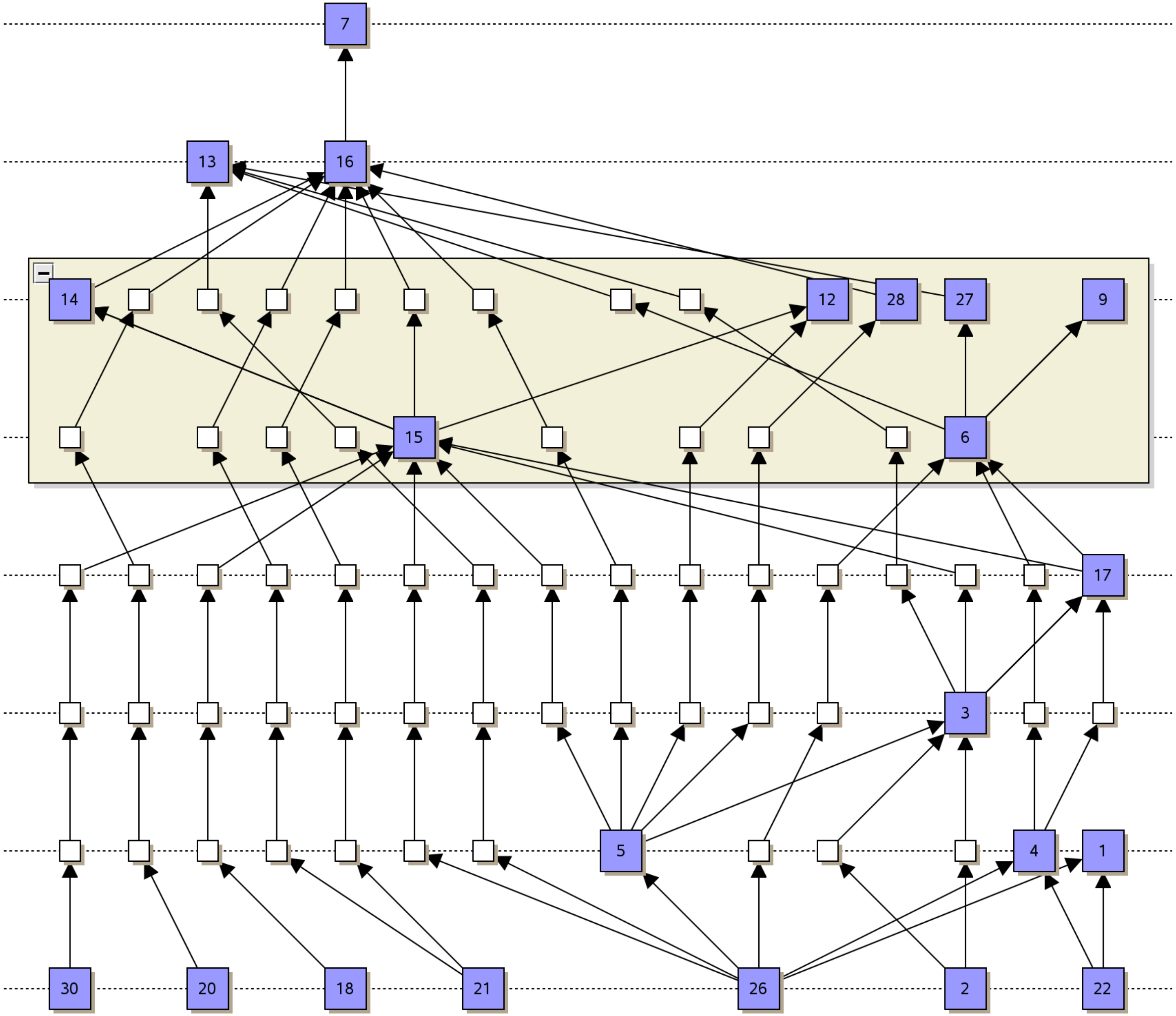

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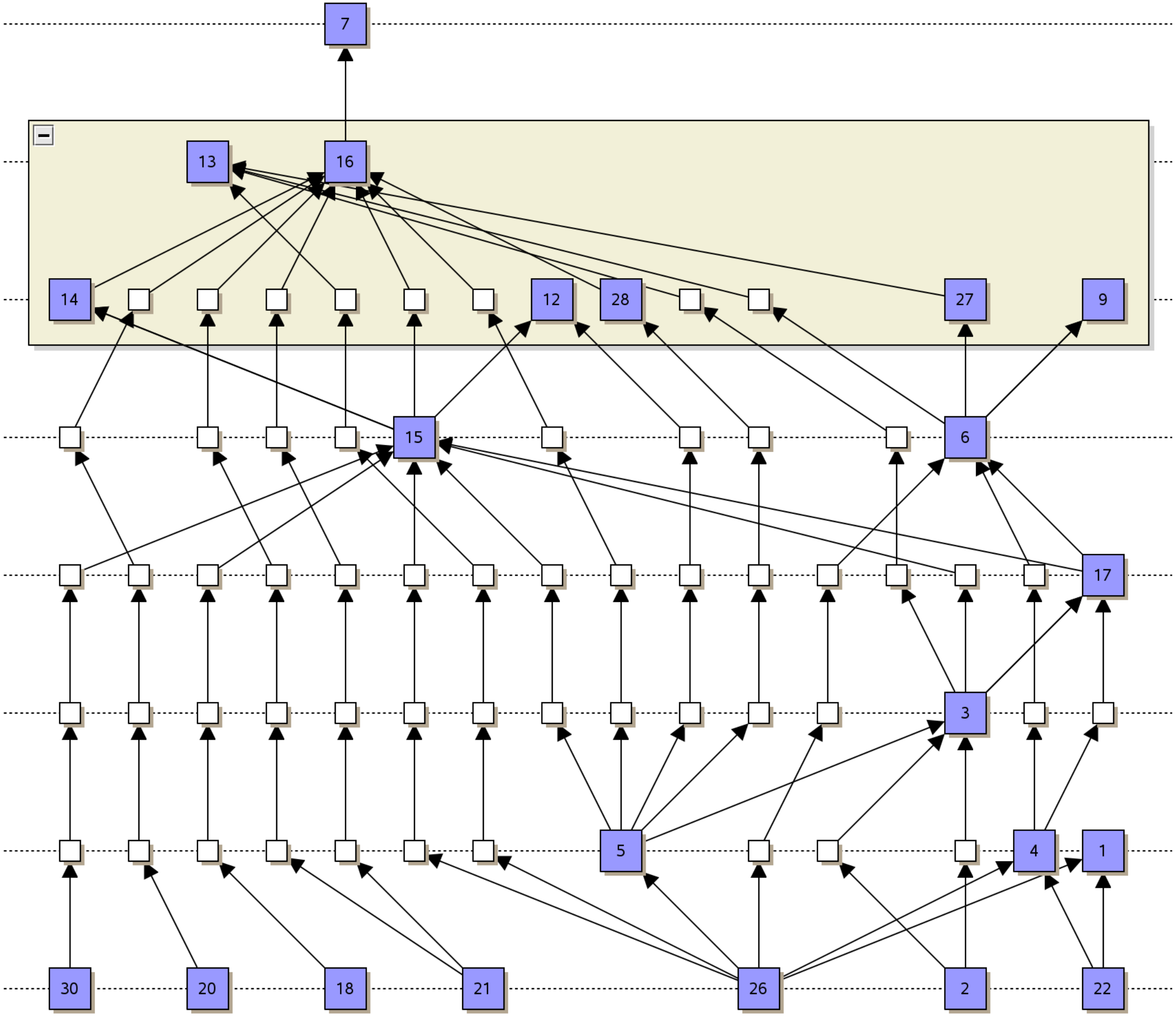

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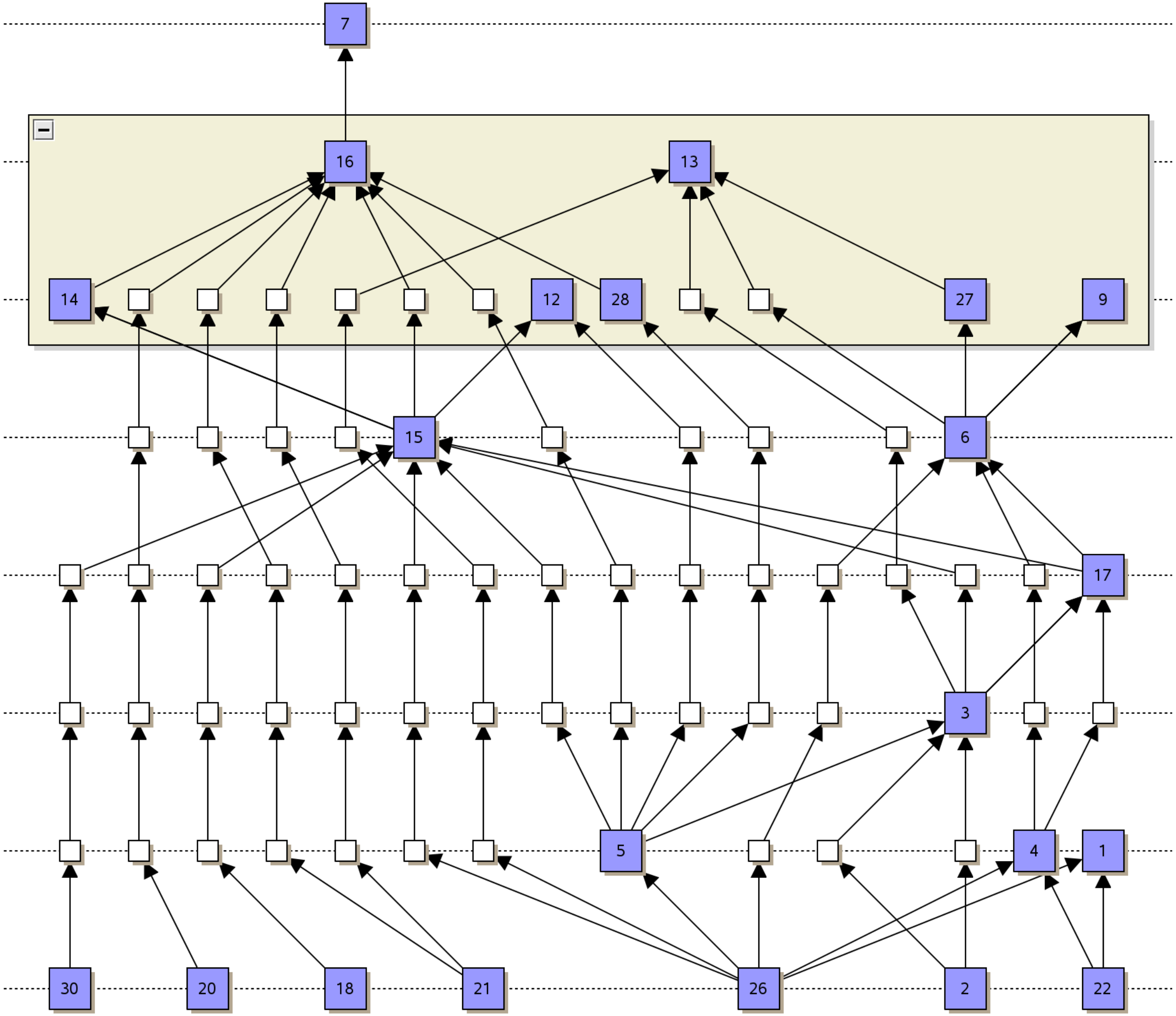

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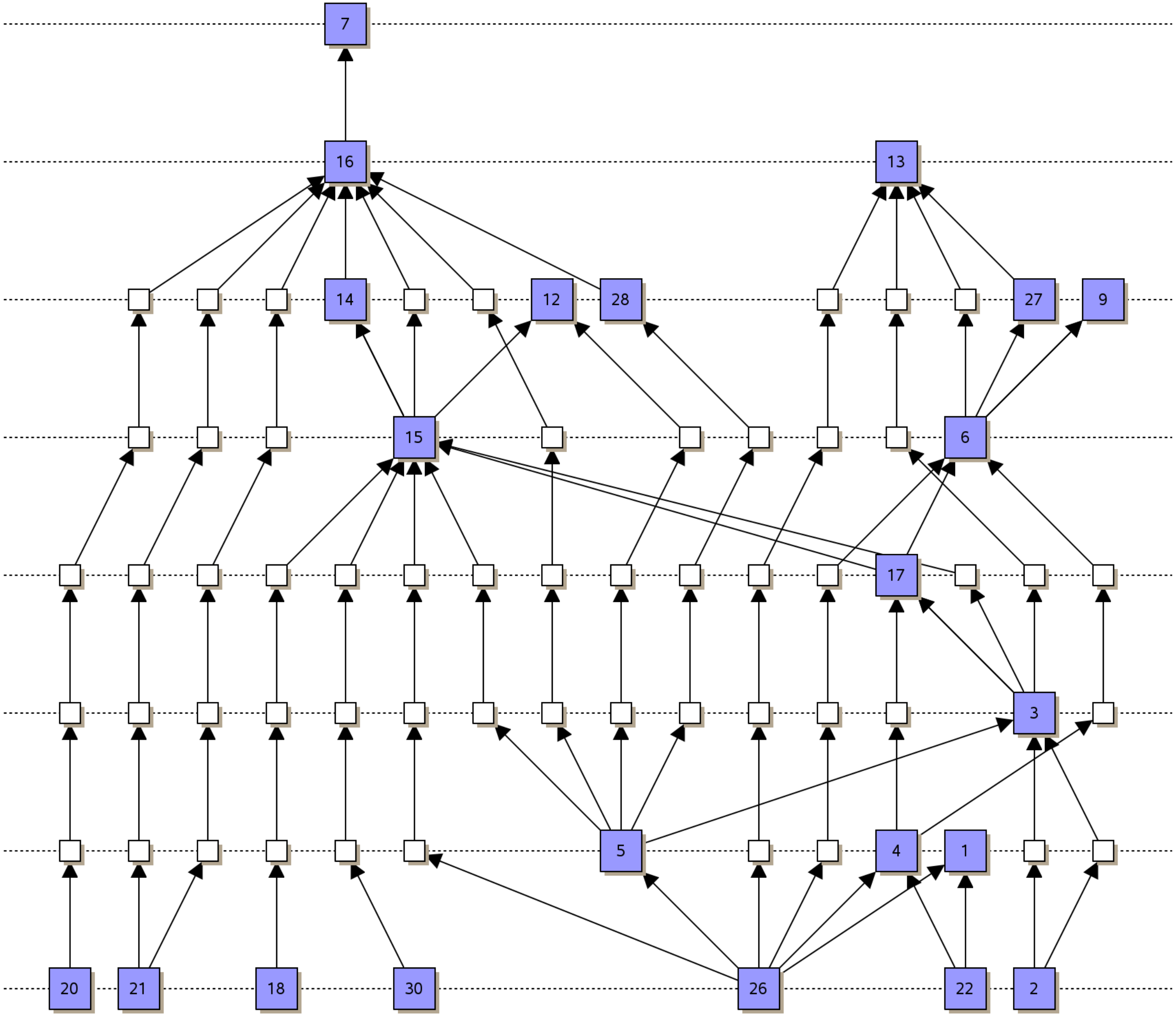
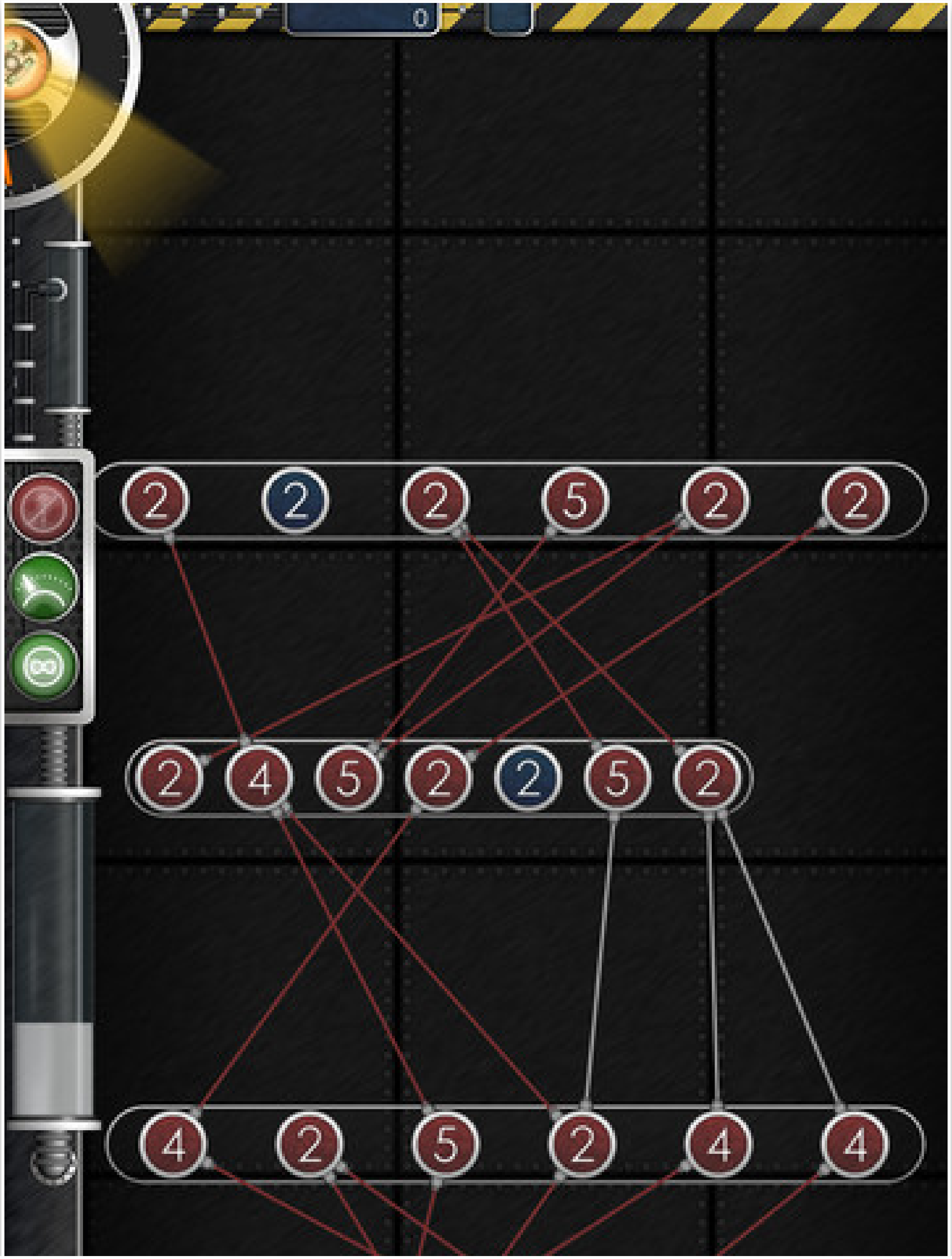


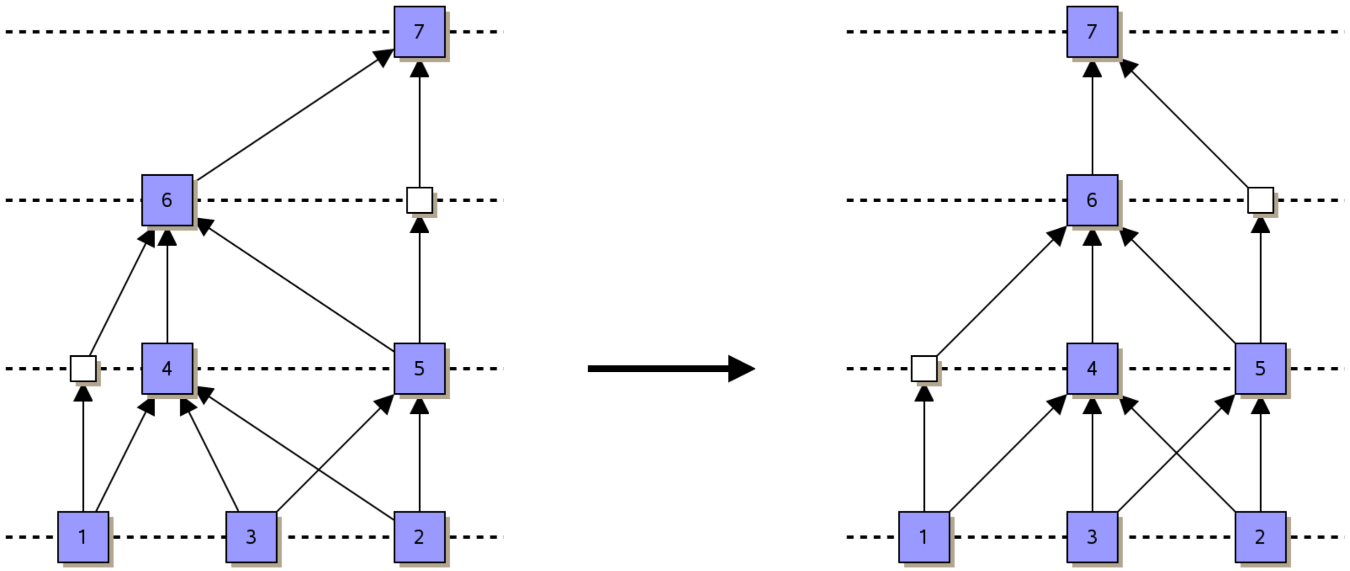
CrossingX



There was even an iPad game **CrossingX** for the OSCM Problem!

Winner of Graph Drawing Game Contest 2012

Step 4: Coordinate Computation



Which could be the goals?

Steightening Edges

Goal: minimize deviation from a straight-line for the edges with dummy-nodes

Idea: use quadratic Program

- let $p_{uv} = (u, d_1, \ldots, d_k, v)$ path with k dummy nodes betwen u and v
- let $a_i = x(u) + \frac{i}{k+1}(x(v) - x(u))$ the x-coordinate of d_i when (u, v) is straight
- minimize $\sum_{i=1}^{k}(x(d_i) - a_i)^2$ for all paths
- constraints: $x(w) - x(z) \geq \delta$ for consecutive nodes on the same layer, w right from z (δ distance parameter)

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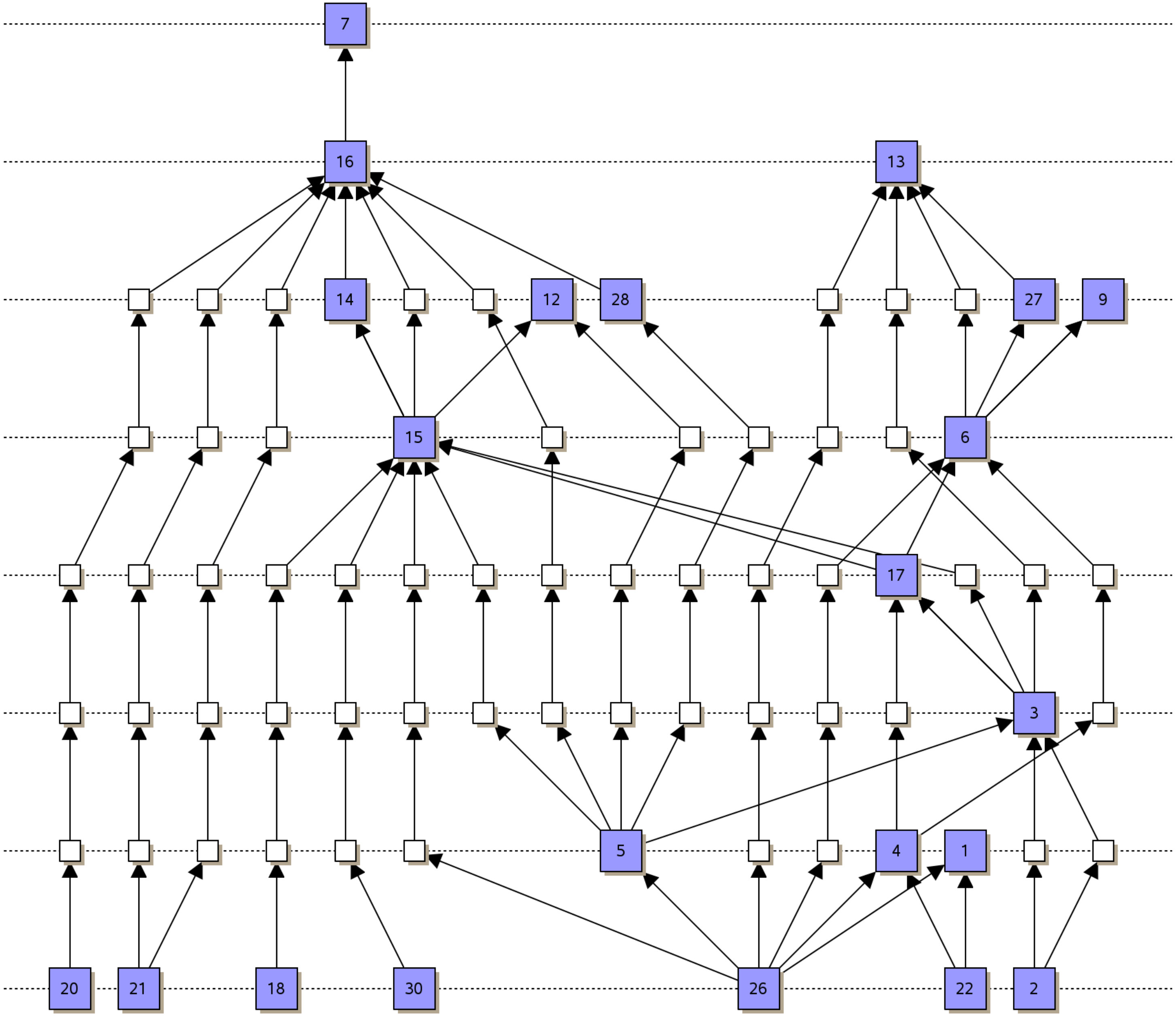
Properties:

- quadratic program is time-expensive
- width can be exponential
- optimization function can be adapted to optimize "verticality"

Example



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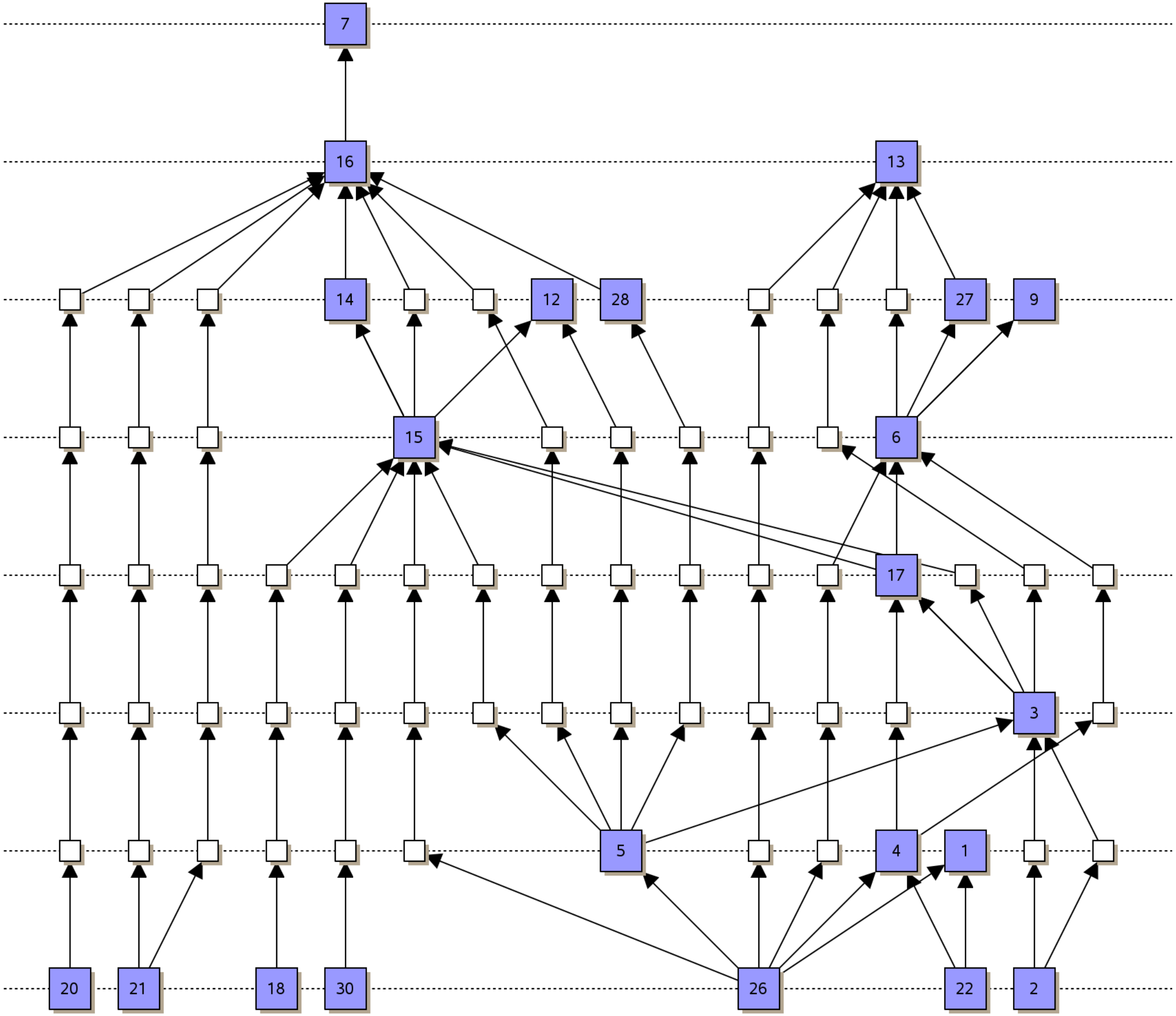
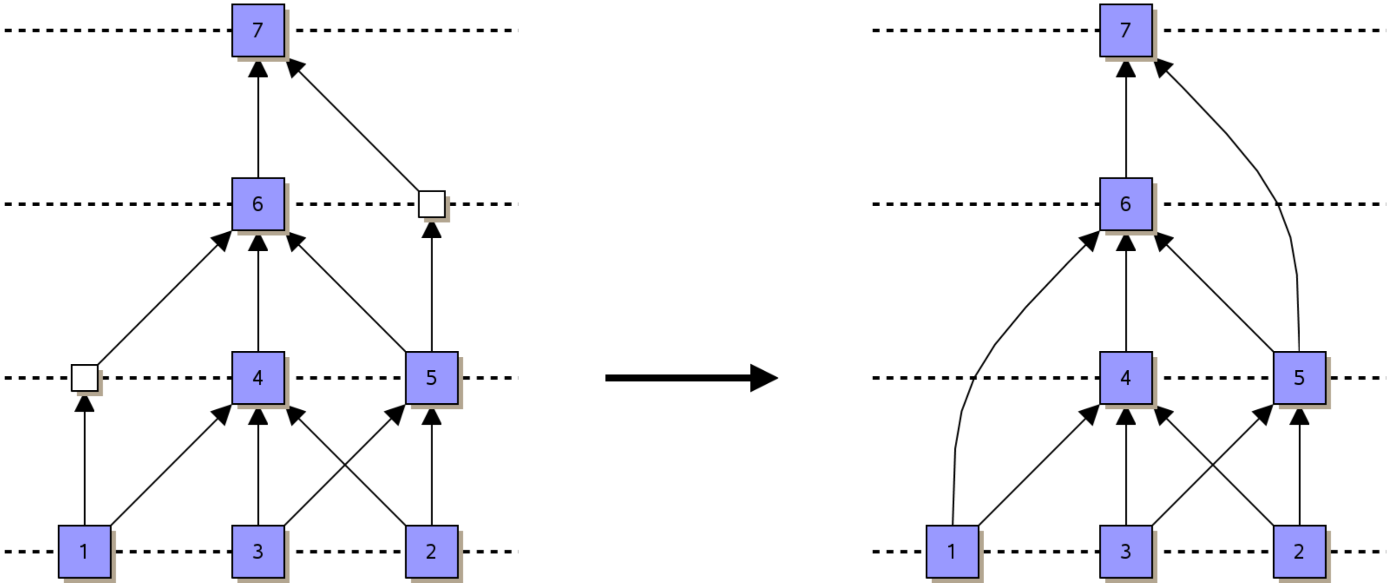
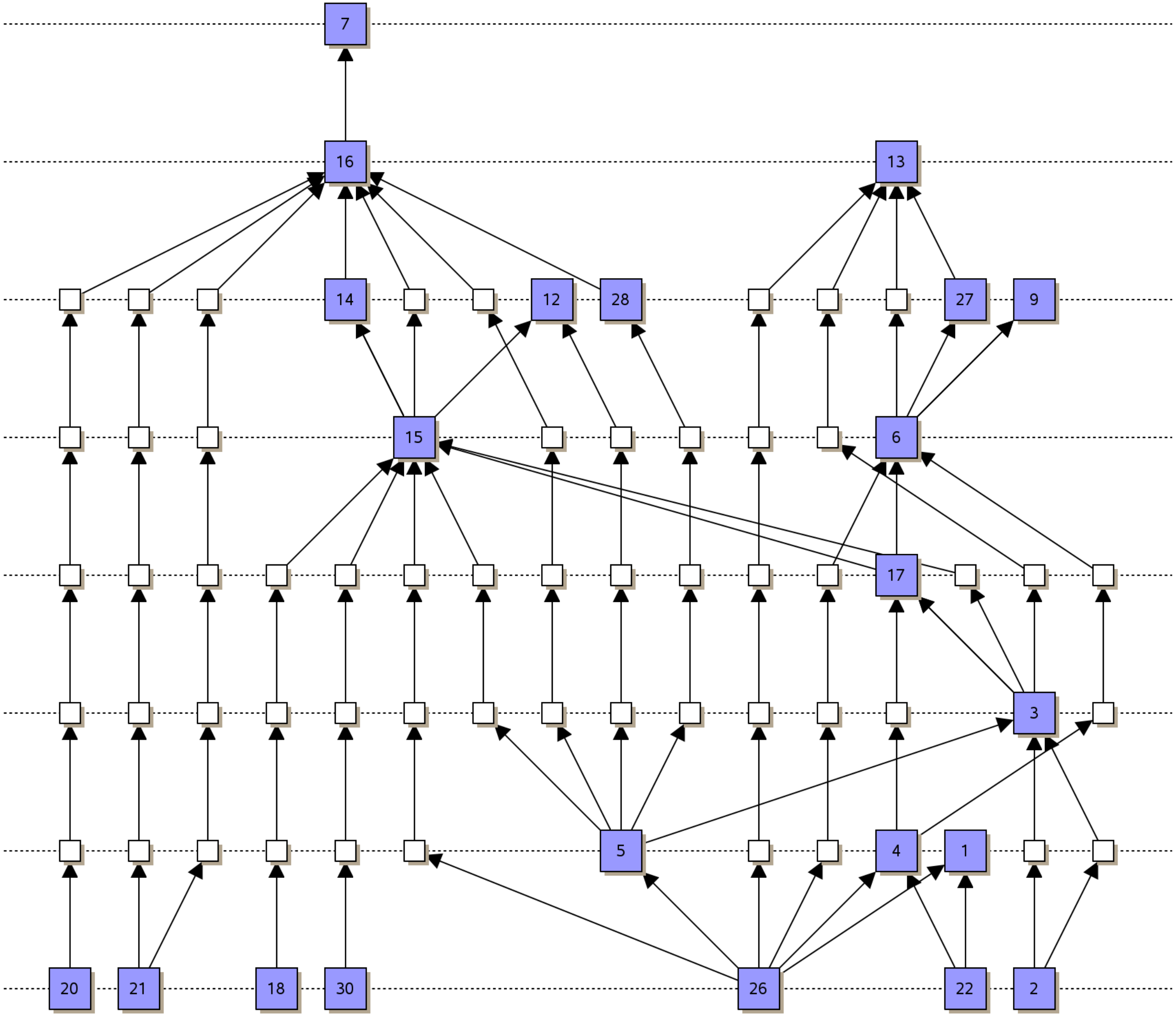


Step 5: Drawing edges



Possibility: Substitute polylines by Bézier curves

Example



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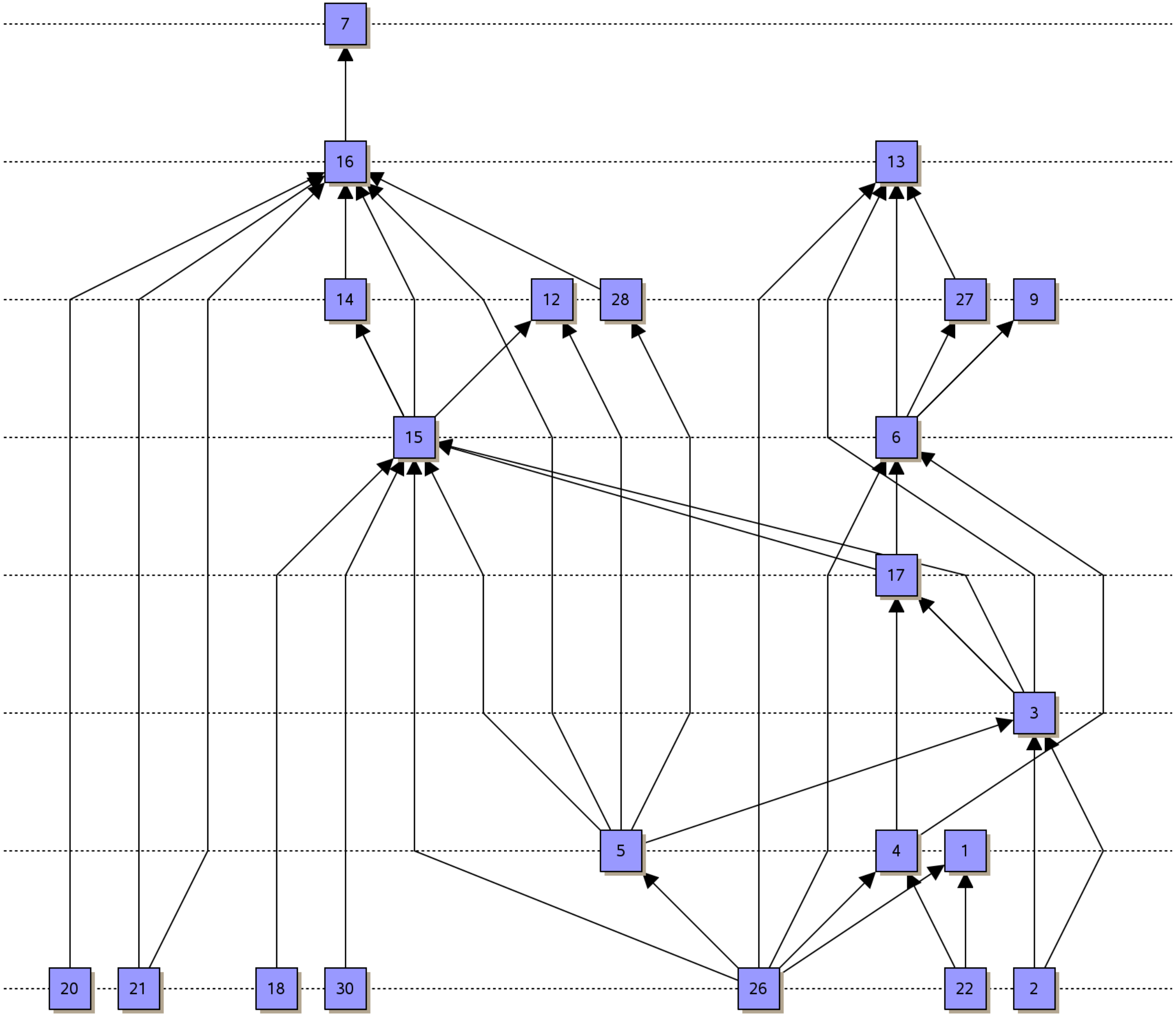

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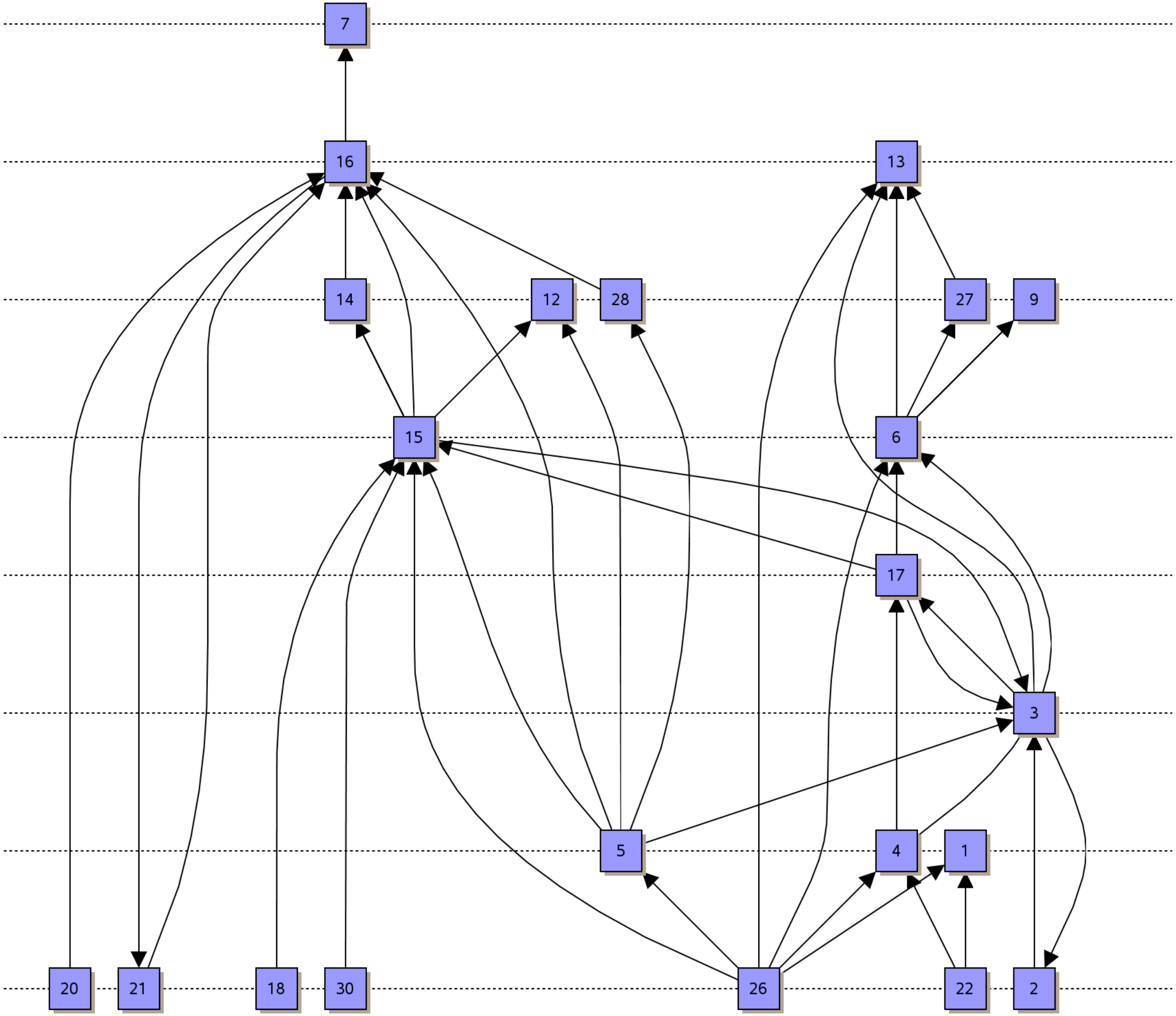
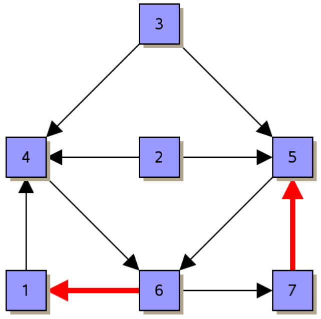
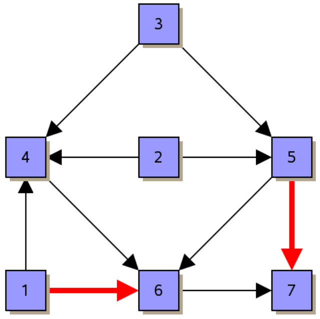


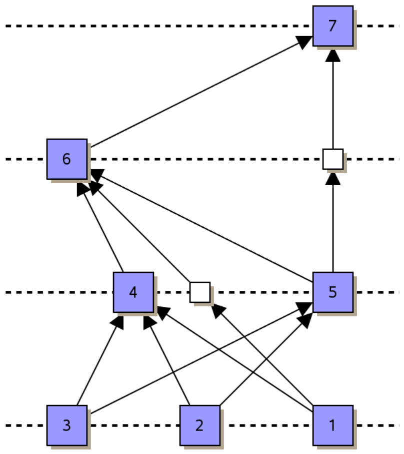
Summary



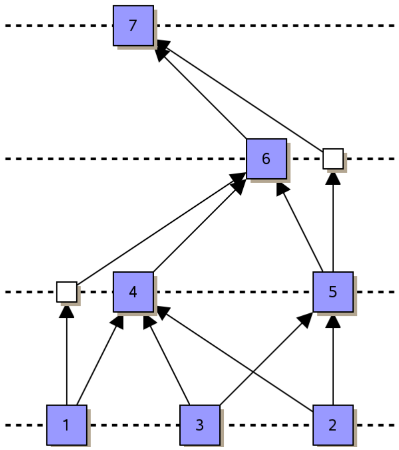
given



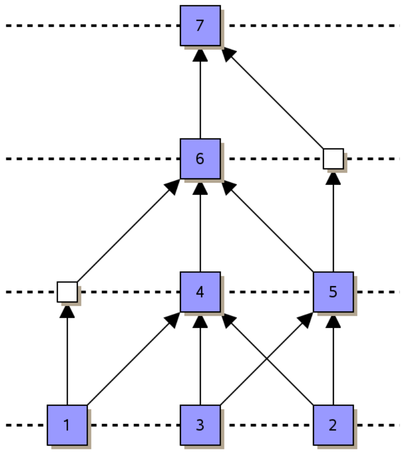
resolve cycles



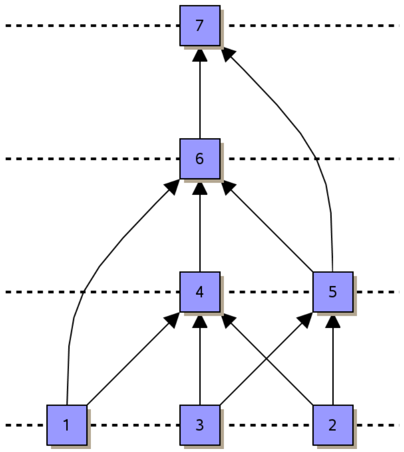
layer assignement



crossing minimization



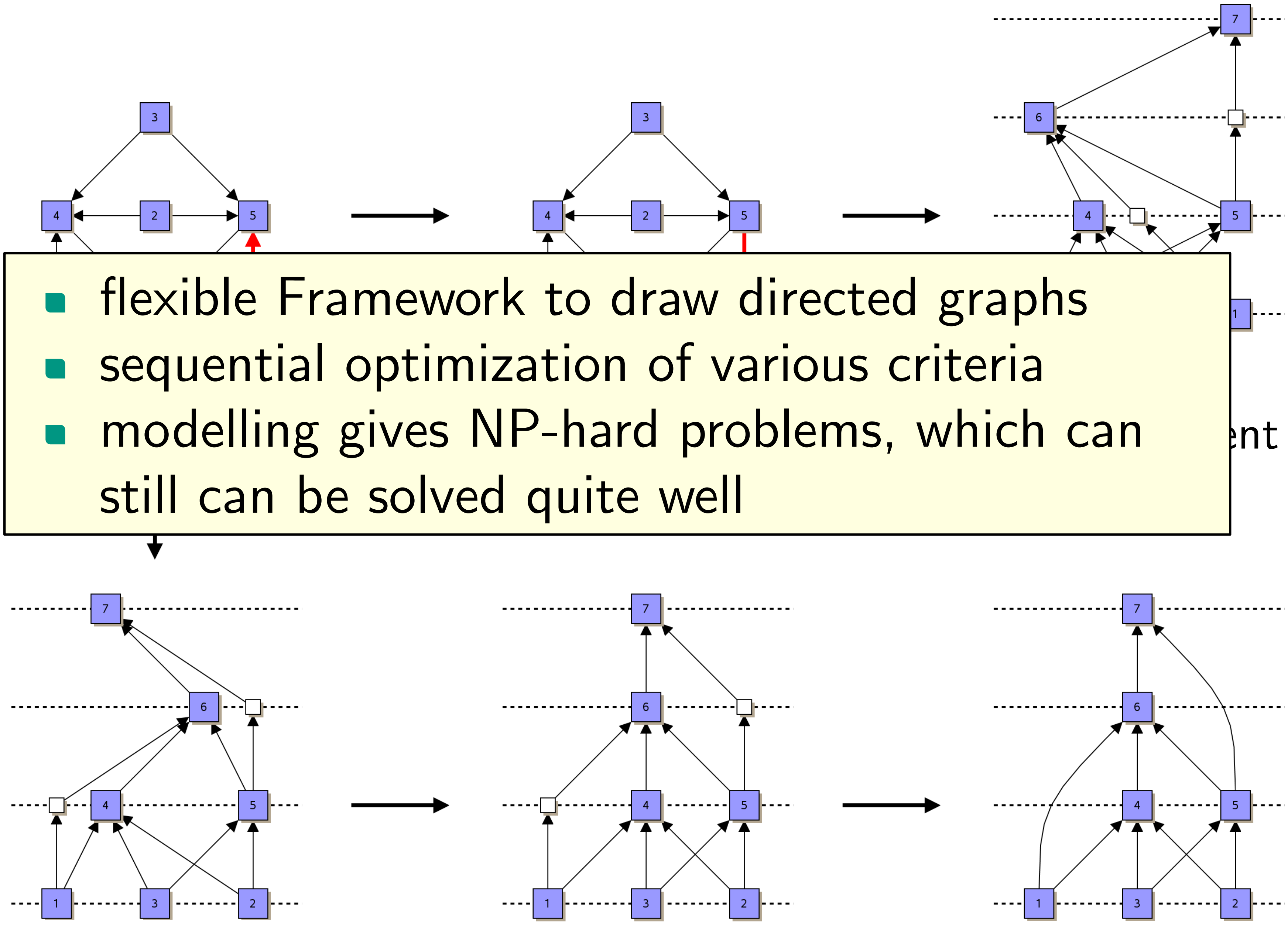
node positioning



edge drawing

Summary

- flexible Framework to draw directed graphs
- sequential optimization of various criteria
- modelling gives NP-hard problems, which can still can be solved quite well



crossing minimization

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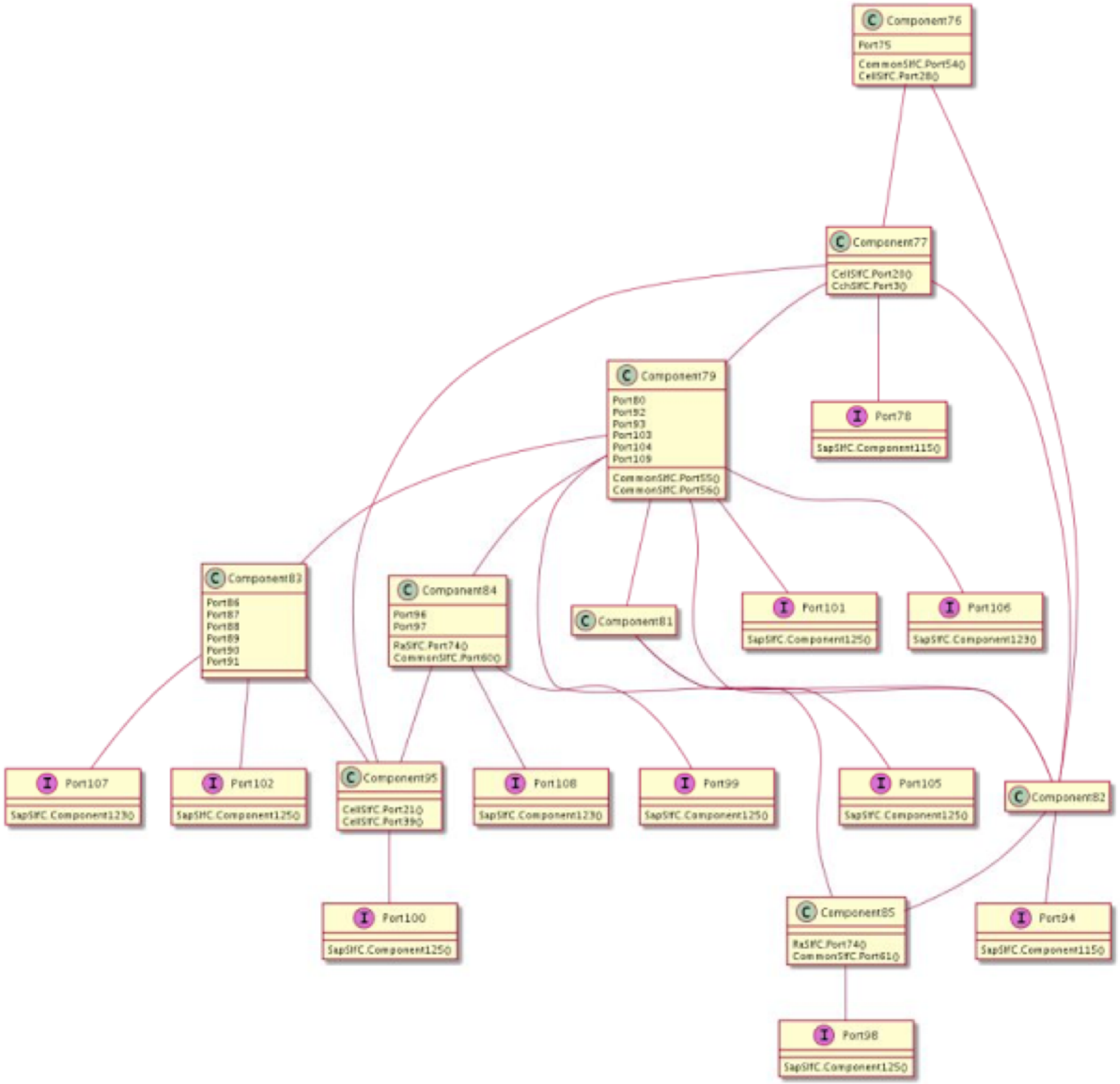
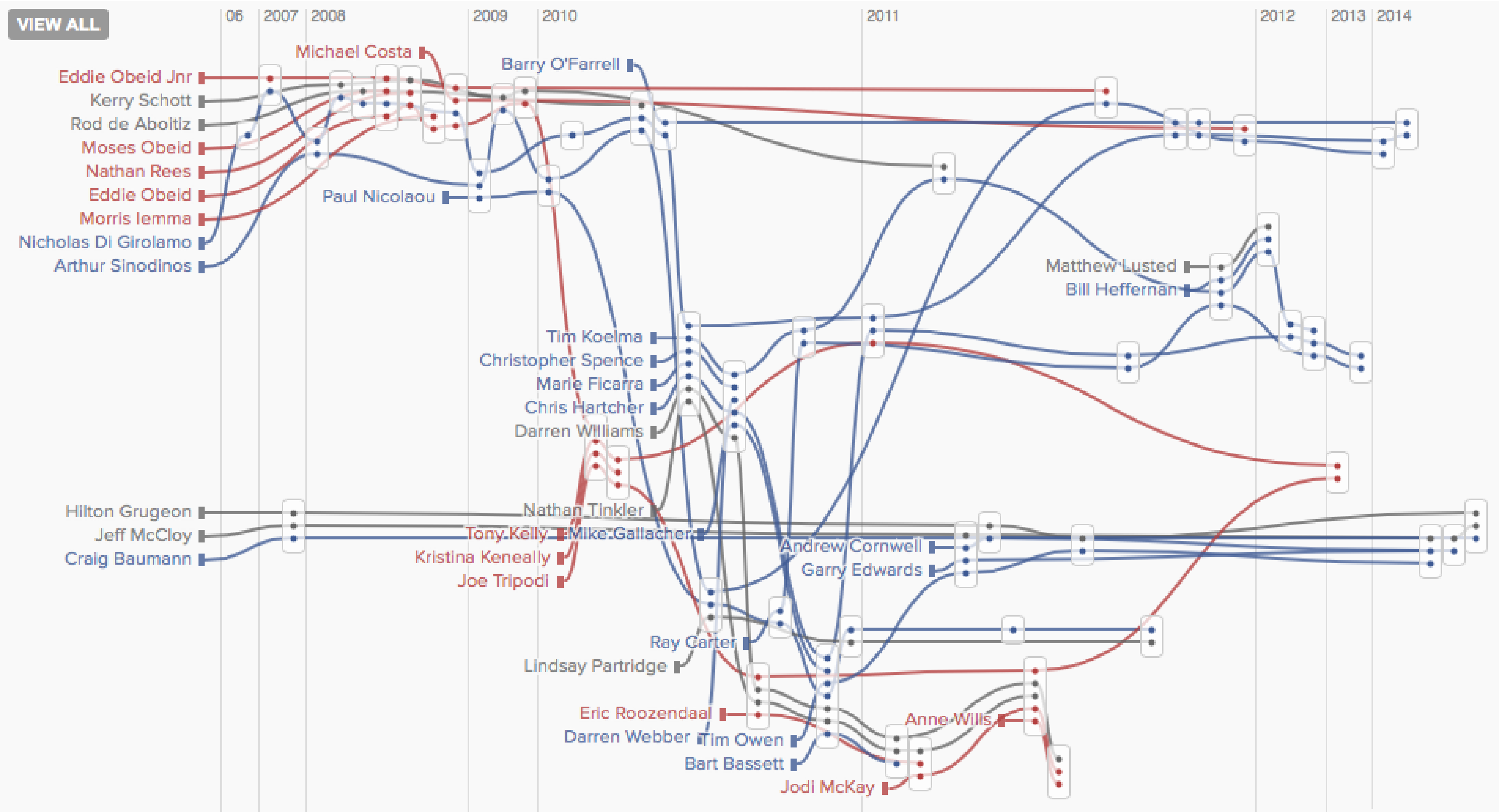
Applications: UML diagrams



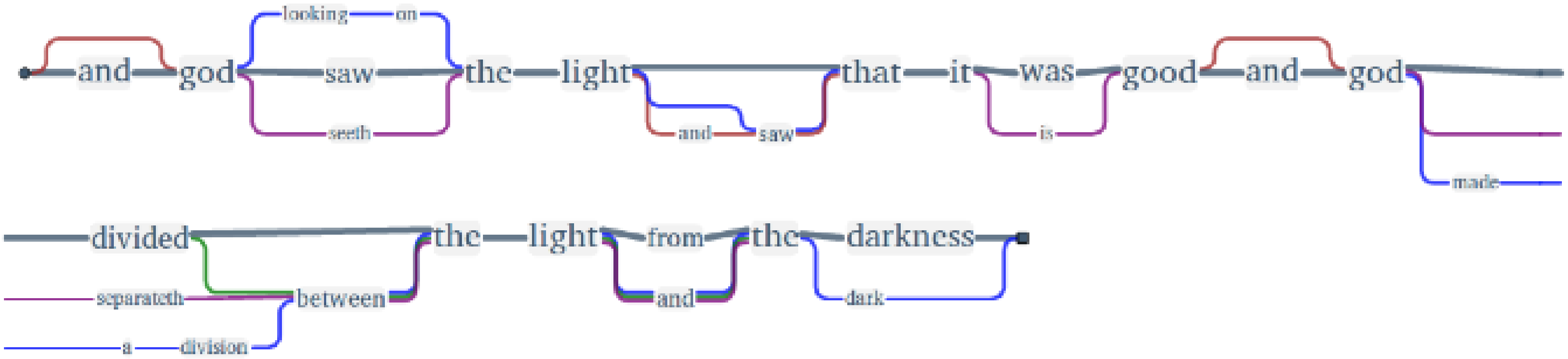
Source: http://betterumldiagrams.blogspot.de

Applications: Storylines



Source: ABC news, Australia

Applications: Text-Variant Graphs



Source: Visualization of Text-Variant Graphs. Jänicke et al.

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- Will be provided as XML format on the lecture's web-page during this week

Hiwi

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WHAT??? THEN USE SOMEBODY ELSE'S SQL, BUT I WANT THE PRODUCT IN TIME.
Brainstuck.com

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