Algorithms for Graph Visualization

Layered Layout

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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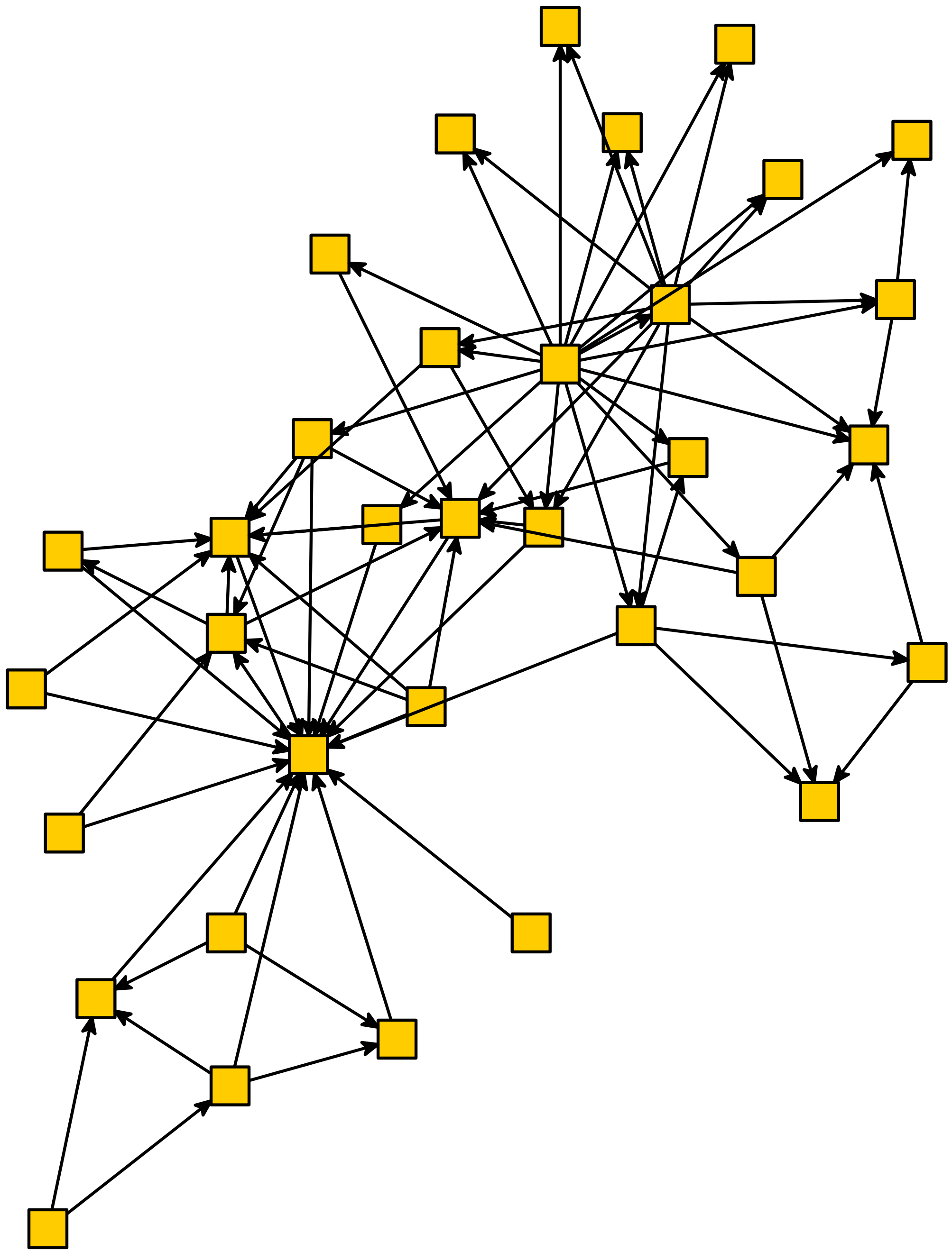
5.12.2016

Example

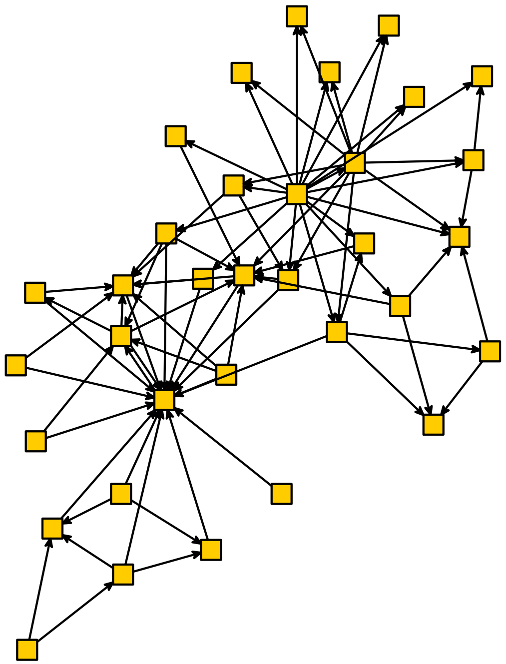


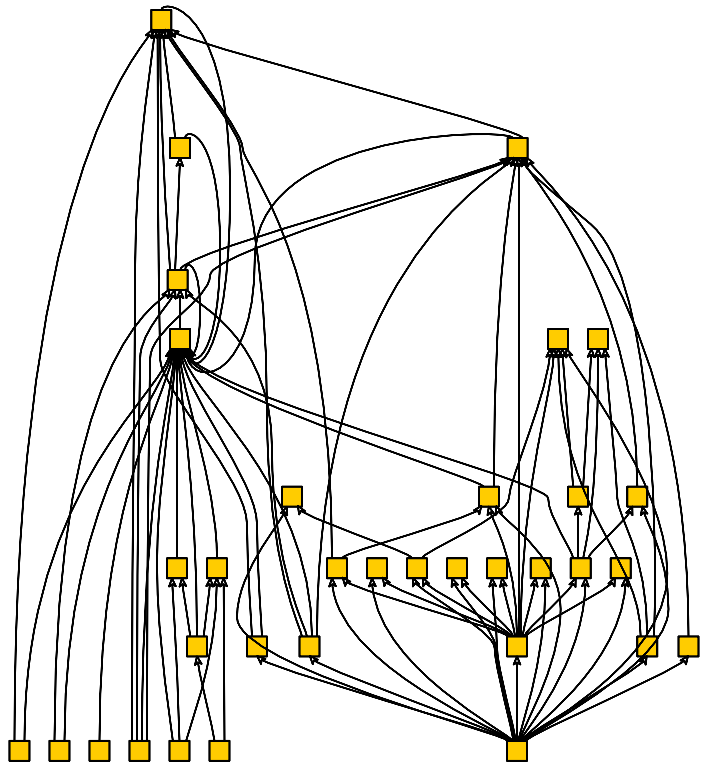
- Which are the properties?
- Which aesthetic ctireria are usefull?

Layered Layout

Given: directed graph $D = (V, A)$

Find: drawing of D that emphasized the hierarchy





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Criteria:

- many edges pointing to the same direction
- edges preferably straght and short
- position nodes on (few) horizontal lines
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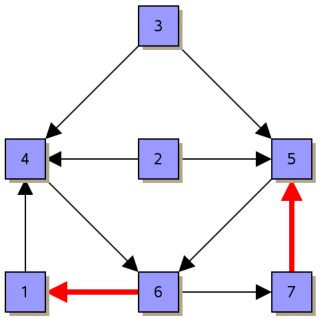
⚠ Optimization criteria partially overlap

Sugiyama Framework (Sugiyama, Tagawa, Toda 1981)



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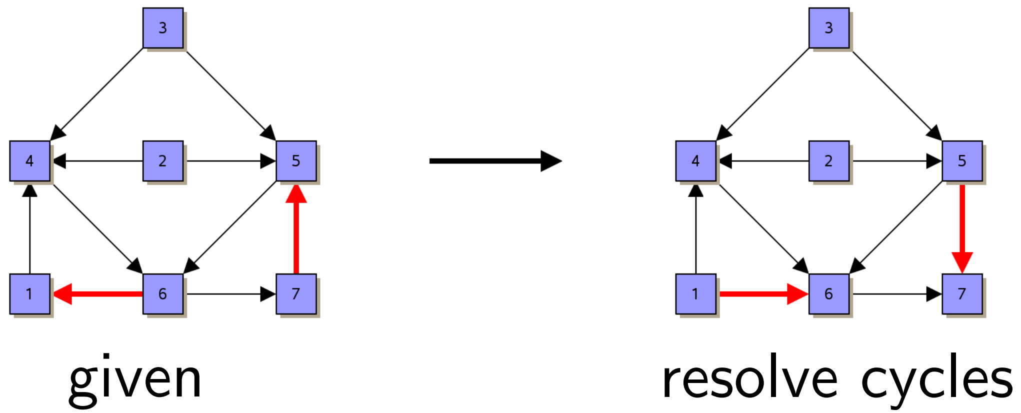

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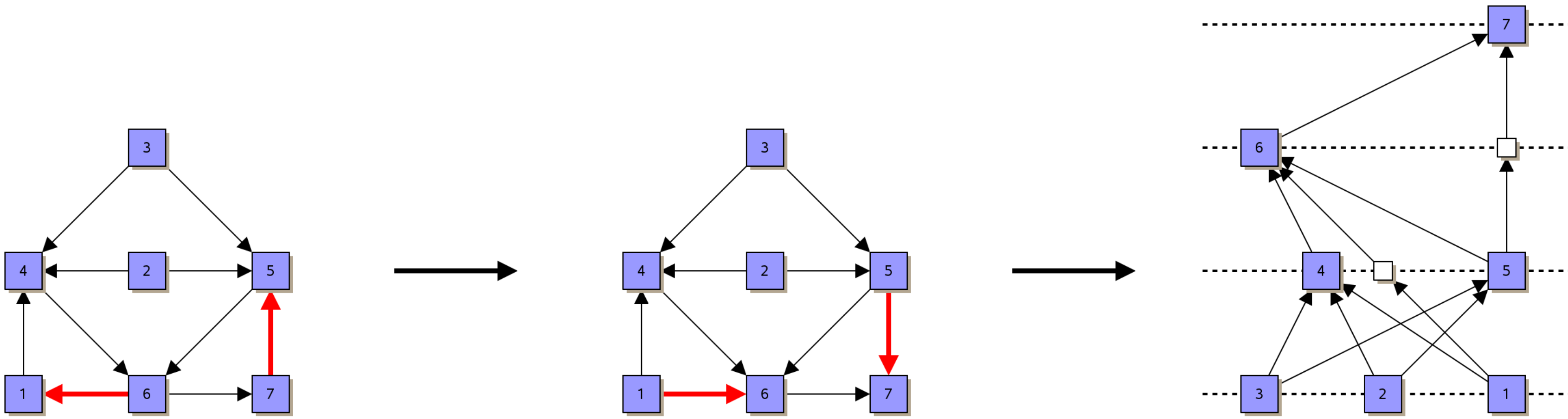

given

resolve cycles

layer assignement

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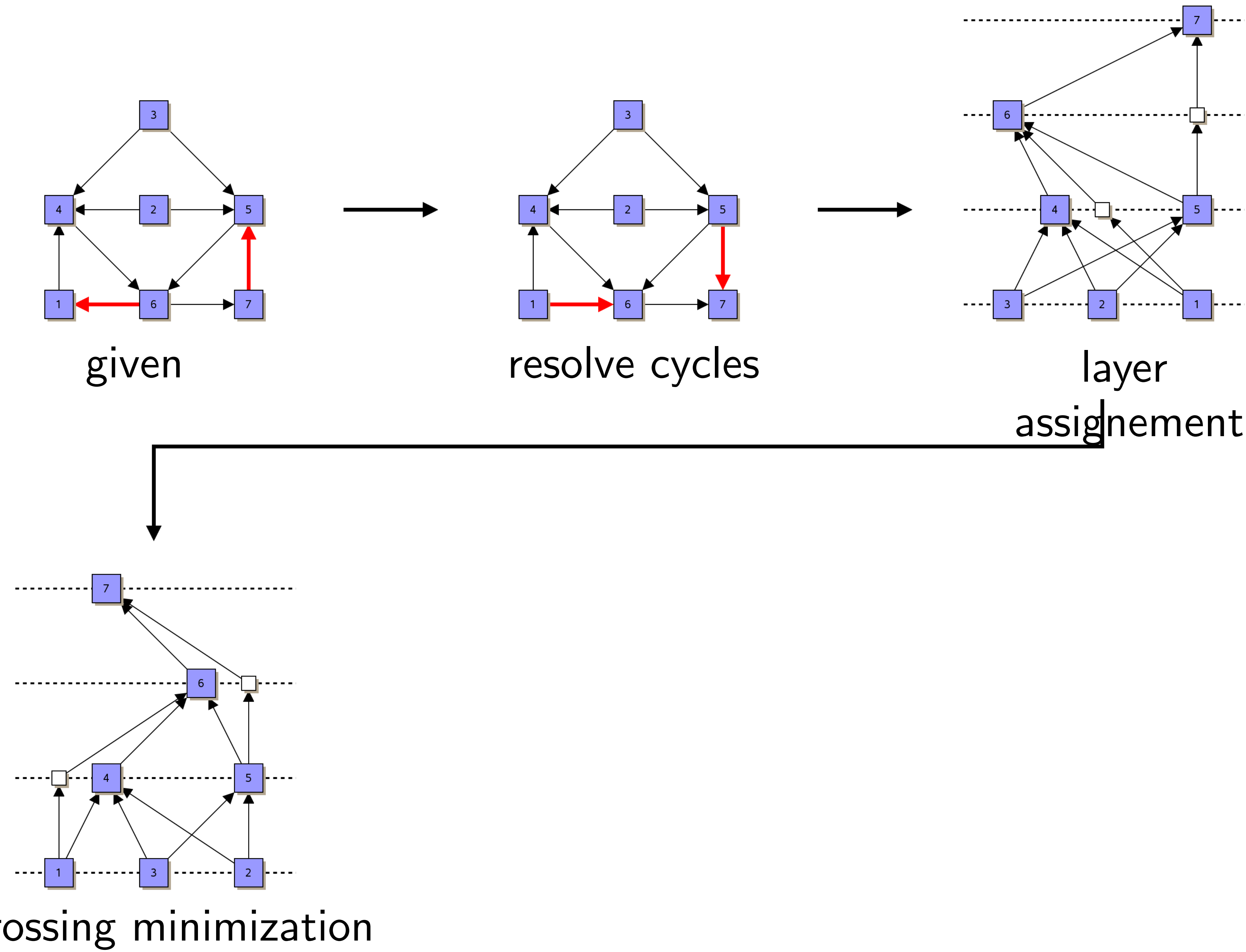

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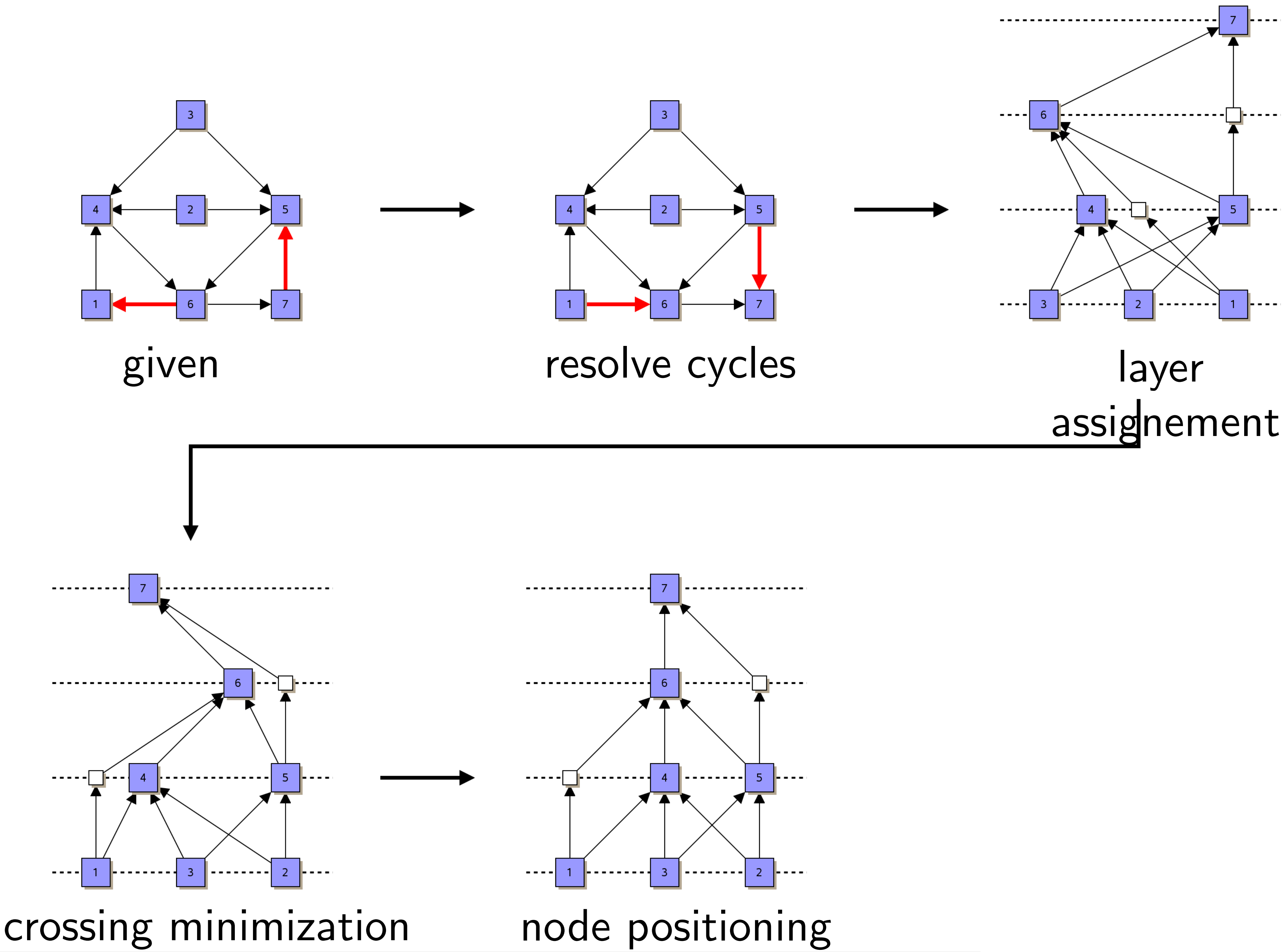

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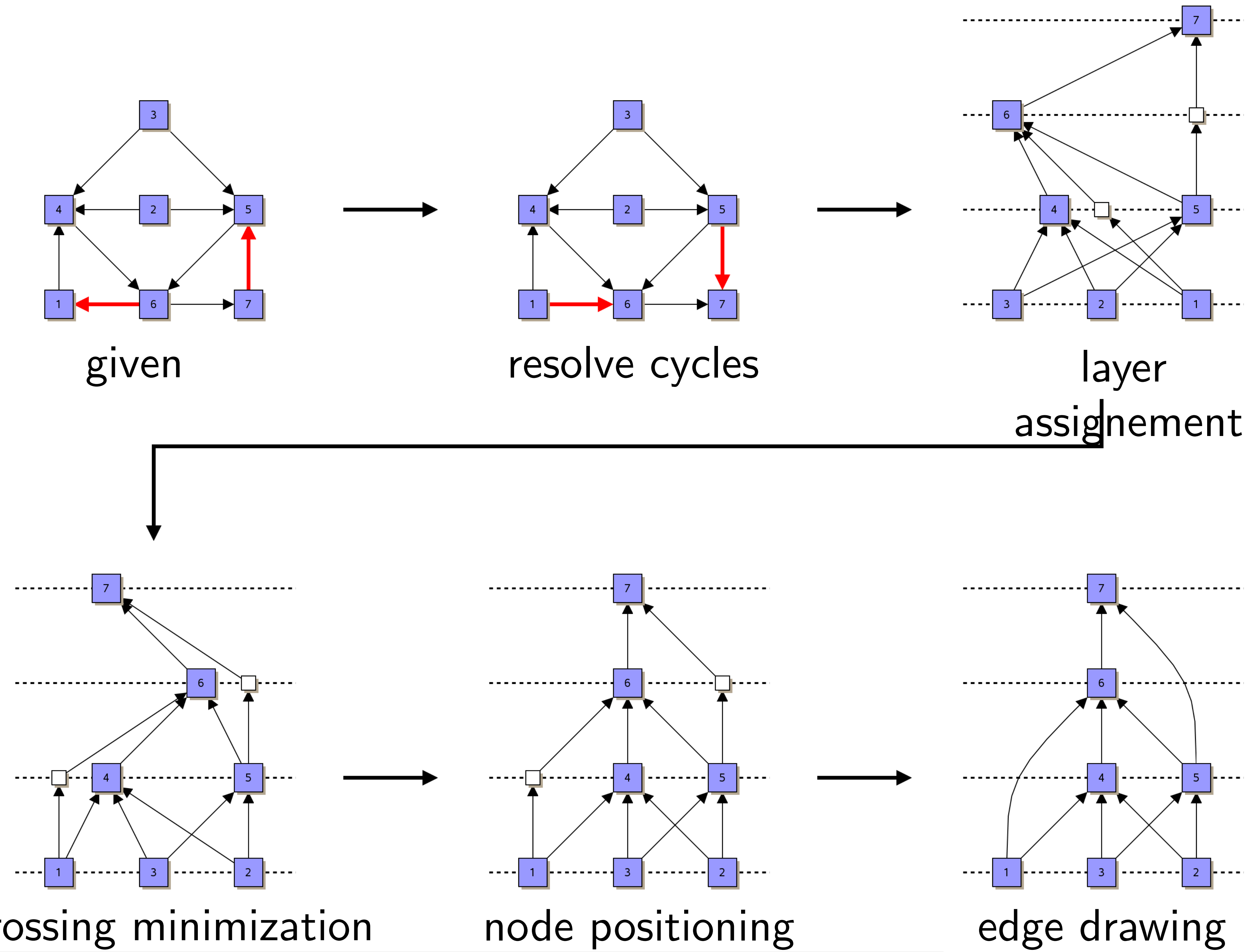

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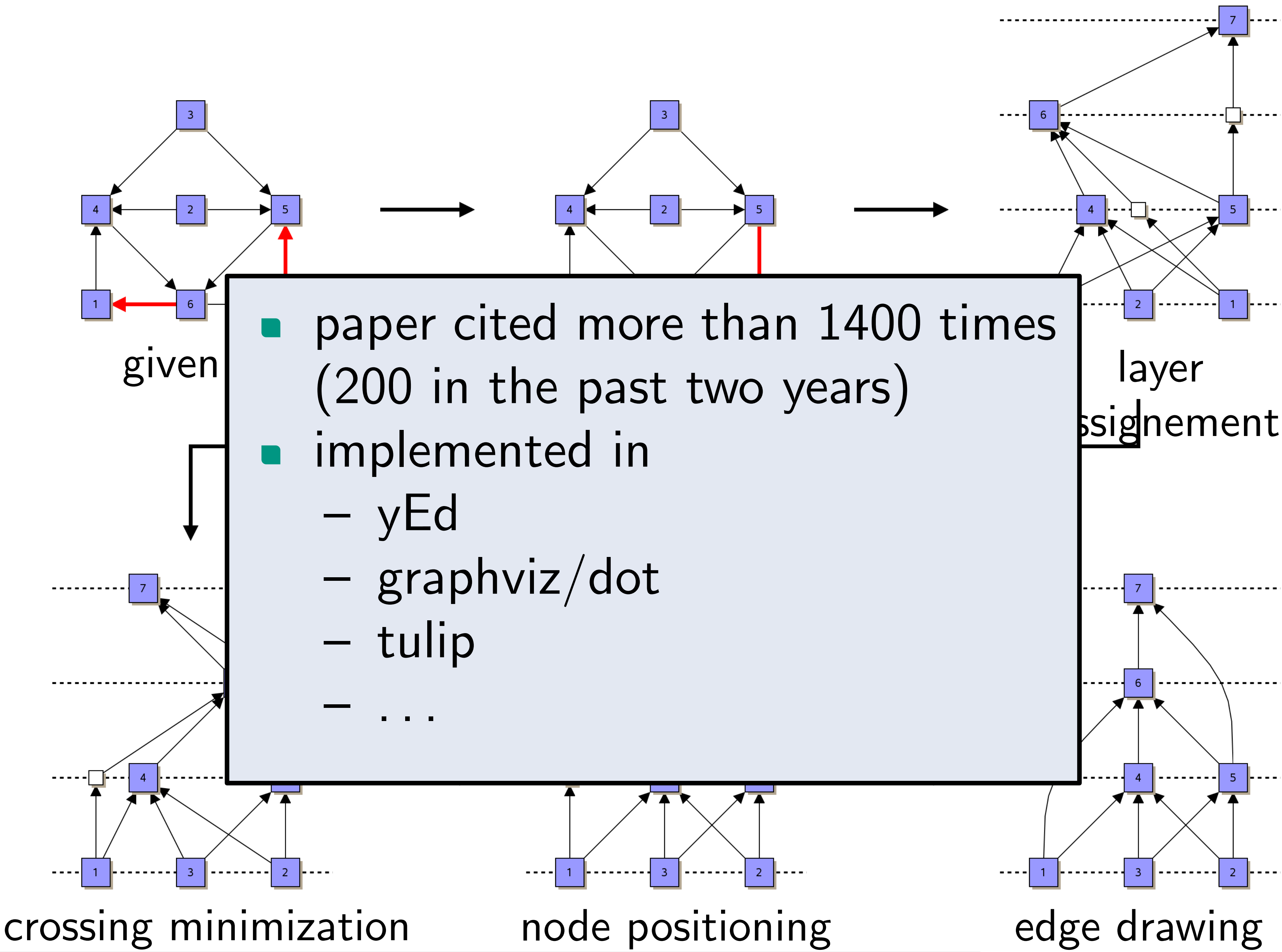

- paper cited more than 1400 times (200 in the past two years)
- implemented in
  - yEd
  - graphviz/dot
  - tulip
  - . . .

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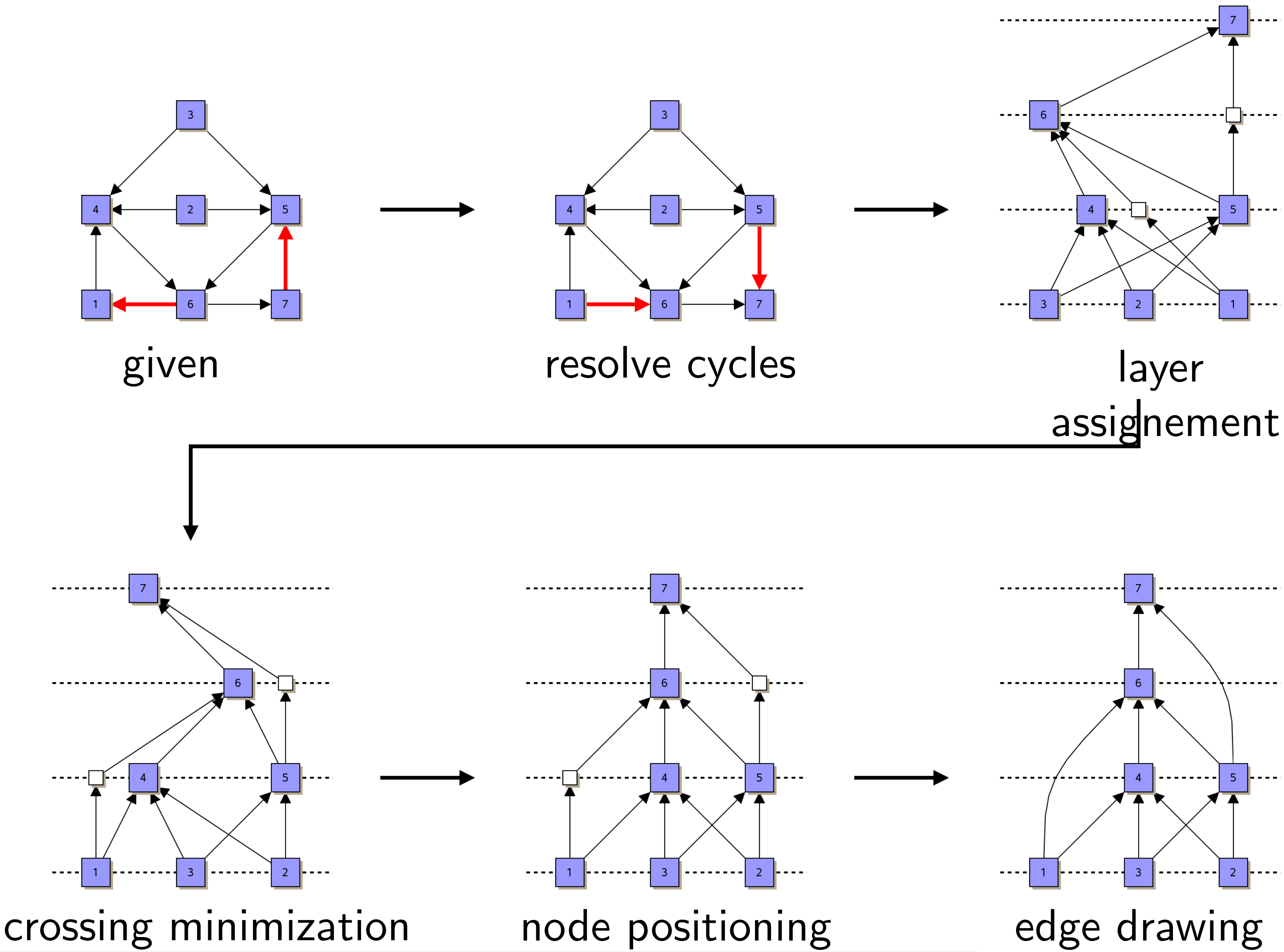

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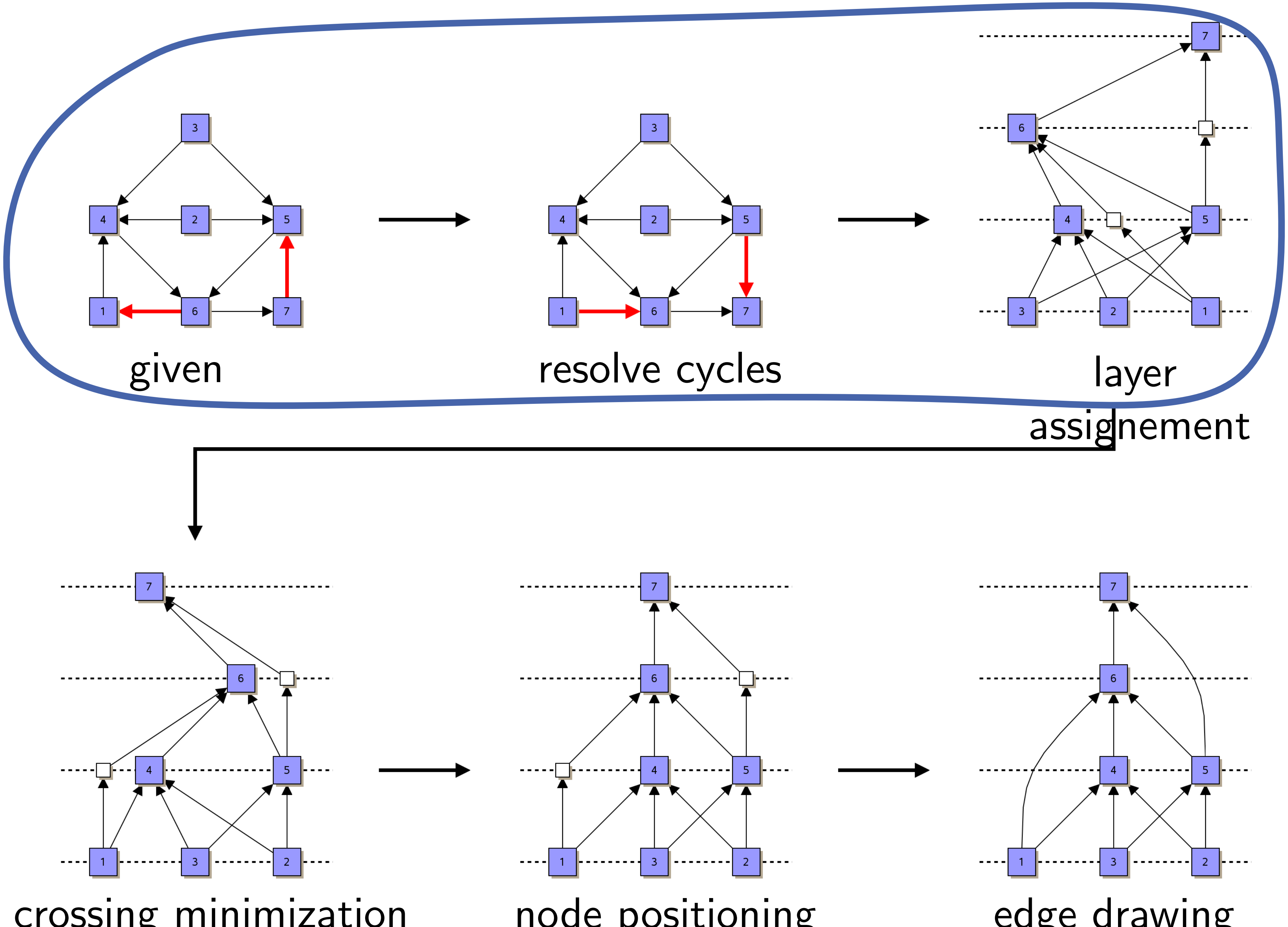
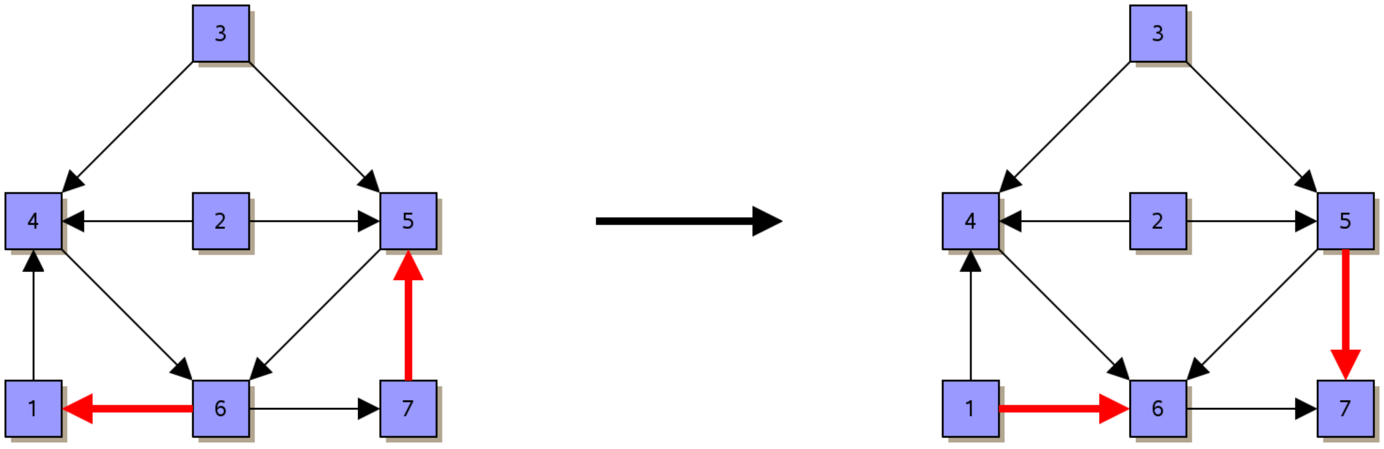


Step 1: Resolve Cycles



How would you proceed?

Feedback Arc Set

Idea:
- find maximum acyclic subgraph
- inverce the directions of the other edges

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Minimum Feedback Set (FS)

Given: directed graph $D = (V, A)$

Find: $A_f \subset A$, with $D_f = (V, A \setminus A_f \cup \text{rev}(A_f))$ acyclic with minimum $|A_f|$

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All three problems are NP-hard!

Heuristic 1 (Berger, Shor 1990)

$A' := \emptyset$;

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $A' := A' \cup N^{\rightarrow}(v)$;

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  remove v and $N(v)$ from D.

return (V, A')

$$N^{\rightarrow}(v) := \{(v,u) : (v,u) \in A\}$$
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- $D' = (V, A')$ is a DAG
- $A \setminus A'$ is a feedback arc set

- Why D' does not contain cycles?
- Is $D'' = (V, A' \cup \text{rev}(A \setminus A'))$ acyclic?
- What is the running time?
- What one can say about $|A'|$?

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- $D' = (V, A')$ is a DAG
- $A \setminus A'$ is a feedback arc set
- Running time $O(|V| + |A|)$
- $|A'| \geq |A|/2$

Heuristic 1 (Berger, Shor 1990)

Lemma 1: Let $D = (V, A)$ be a connected, directed digraph. Heuristic 1 produces an acyclic digraph $D' = (V, A')$.

proof:

For the sake of contradiction assume there is a cycle C. Let u be the first visited vertex of C. Either incomming or outgoing edges of u are not in A', i.e. D' can not contain a cycle.

Heuristic 1 (Berger, Shor 1990)

Lemma 2: The digraph $D'' = (V, A' \cup \text{rev}(A \setminus A'))$, where A' is produced by Heuristic 1, is acyclic.

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- For the sake of contr. assume there is a cycle C in D''.
- Let u be the first visited vertex of C. Cycle C contains a reversed edge incident to u, otherwise u can not have both incomming and outgoing edges in C.

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- W.l.o.g. assume (u, v) is the reversed edge. I.e. the original edge was (v, u), i.e. $(v, u) \in A \setminus A'$. Therefore, no other incomming edge to u is in A'. I.e. u has no incomming edges in C that are in A'.

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- Therefore the incomming edge to u in C is also a reversed edge. I.e. both incomming and outgoing edges of u in C are in $A \setminus A'$, which is impossible, as u is the first vertex visited by the algorithm in C.

Heuristic 2 (Eades, Lin, Smyth 1993)

$A' := \emptyset$;

while $V \neq \emptyset$ **do**

  while in V exists a sink v **do**

    $A' \leftarrow A' \cup N^{\leftarrow}(v)$

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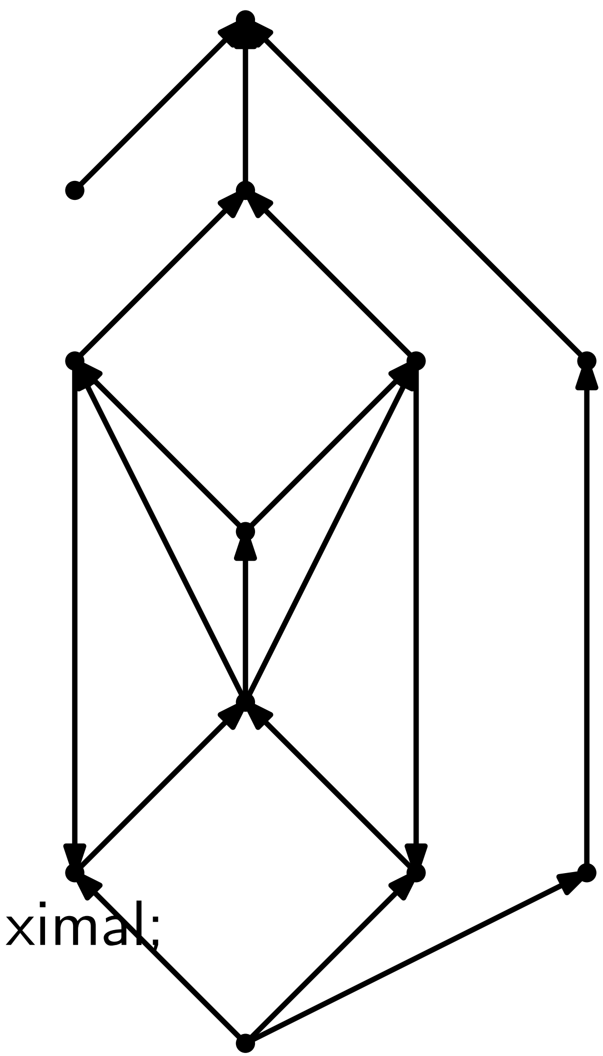

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```
A' := ∅;
while V ≠ ∅ do
    while in V exists a sink v do
        A' ← A' ∪ N←(v)
        remove v and N←(v): {V, n, m}_sink
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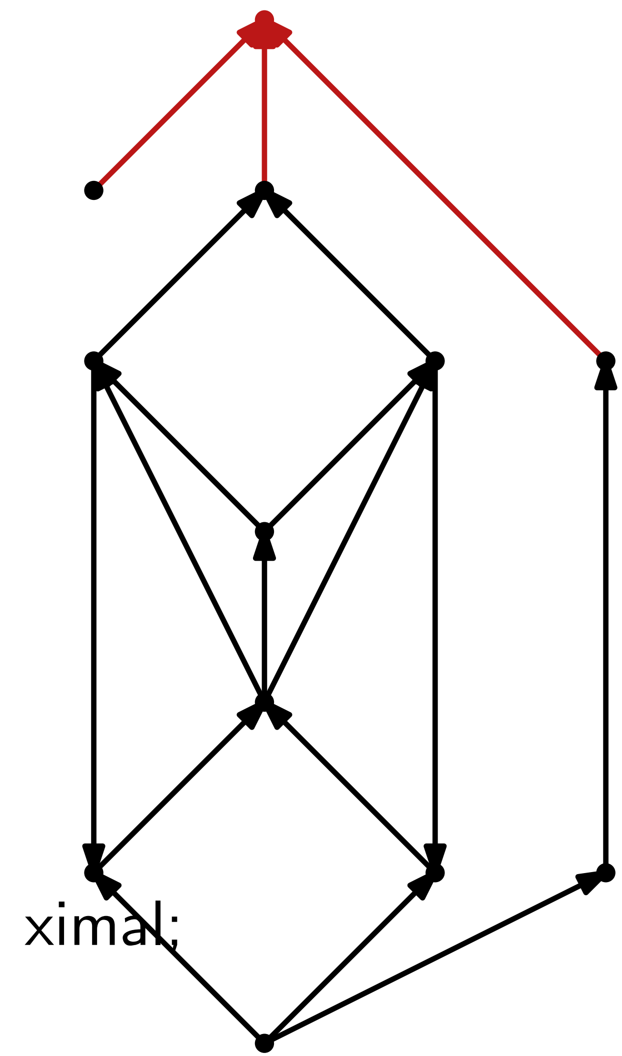

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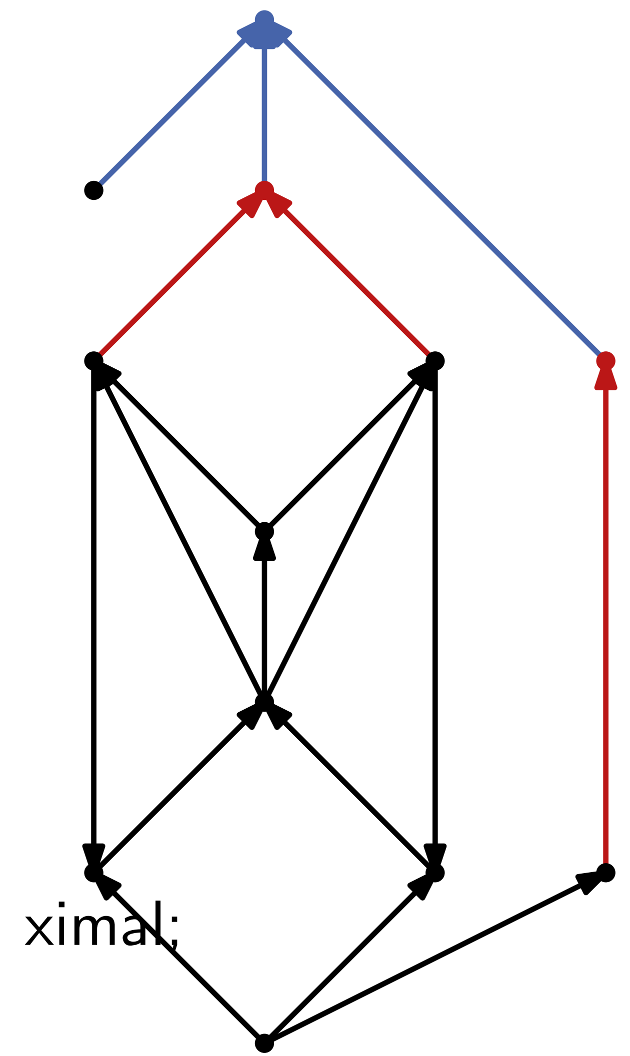

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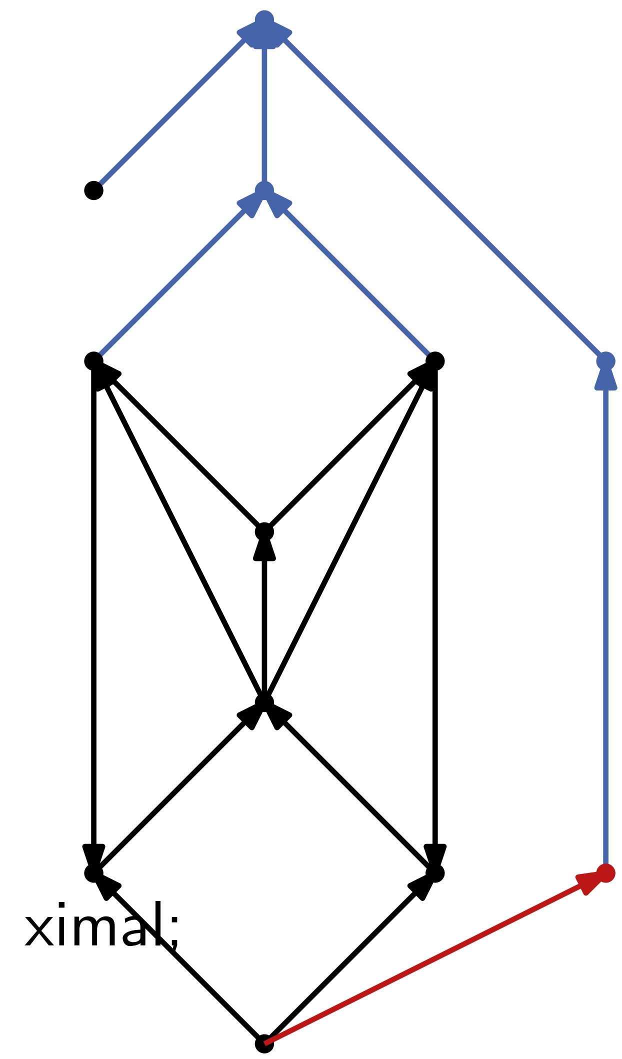

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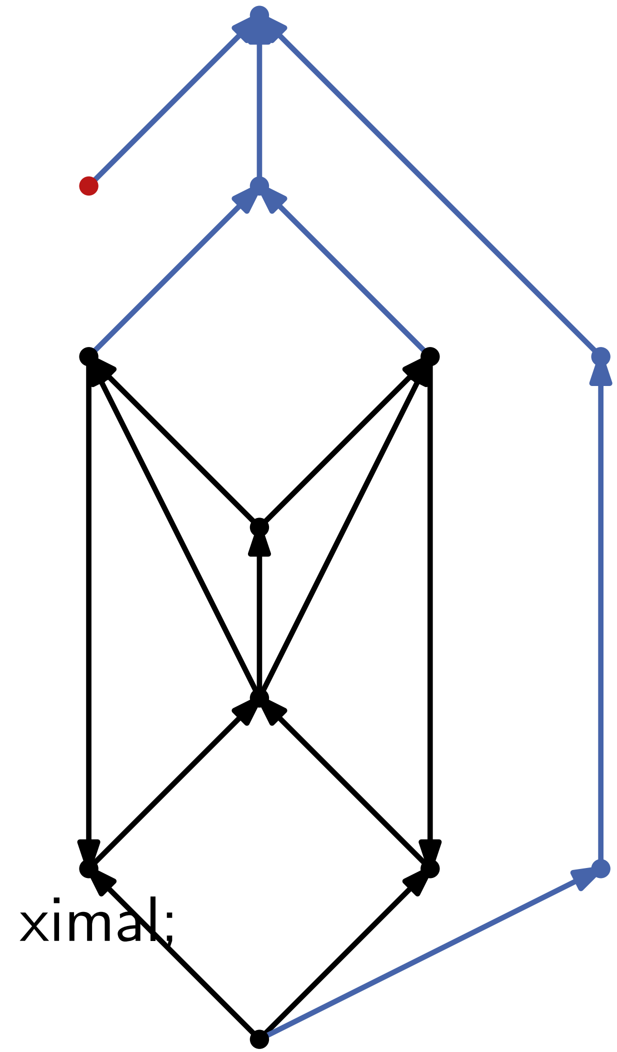

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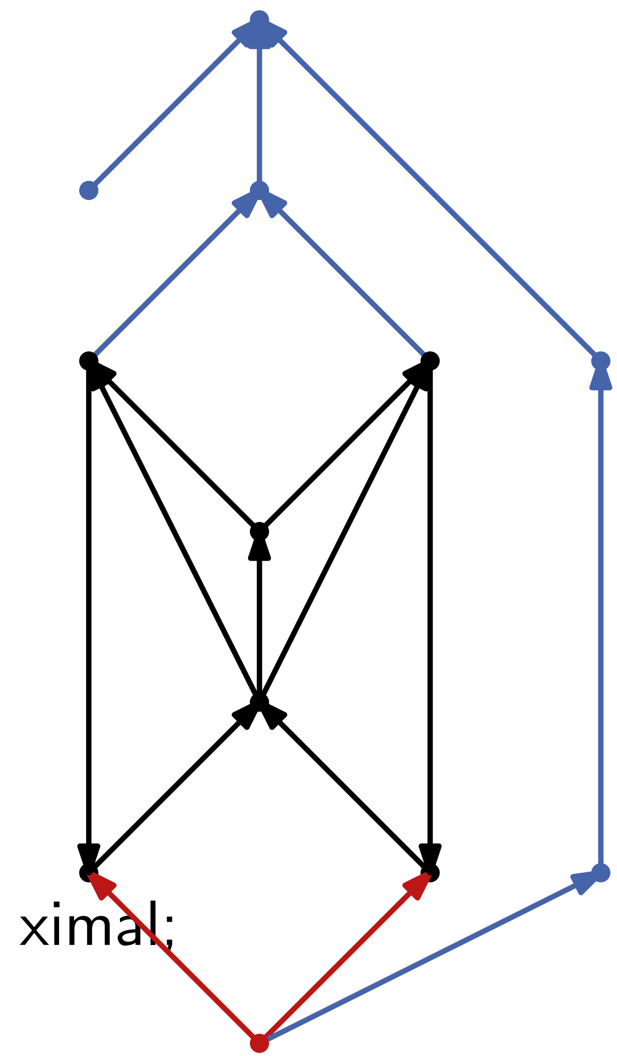

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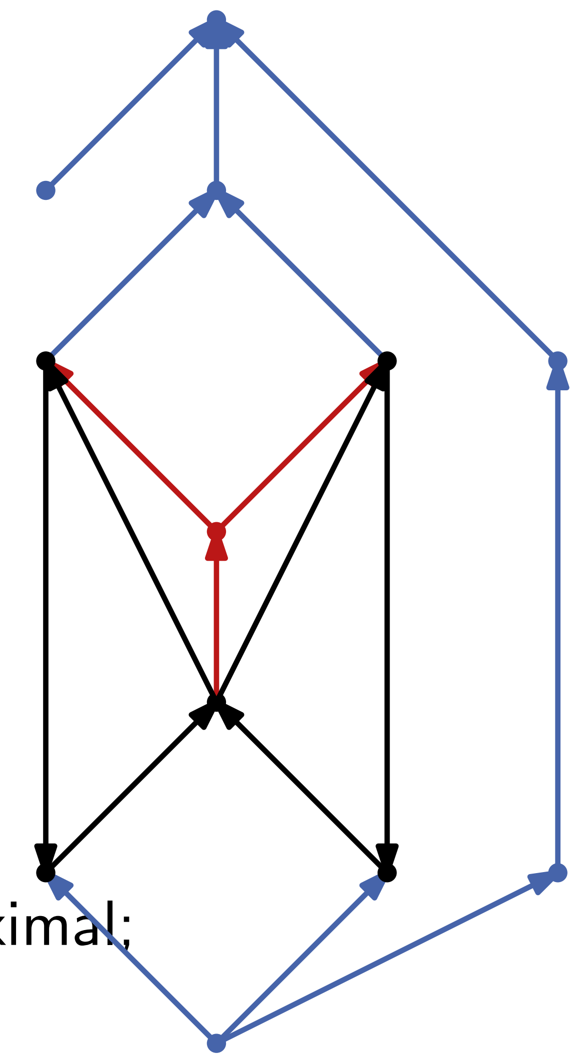

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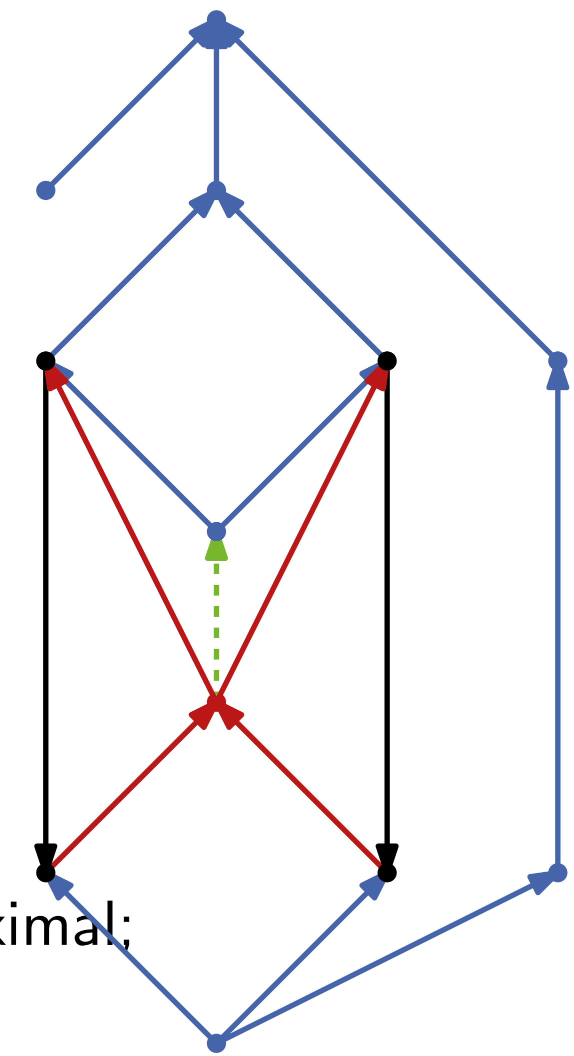

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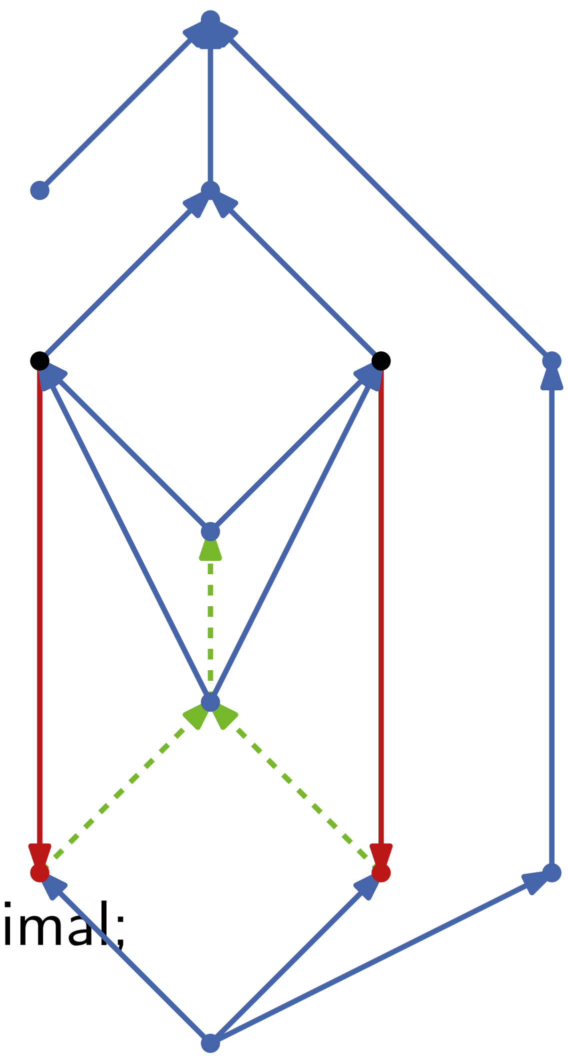

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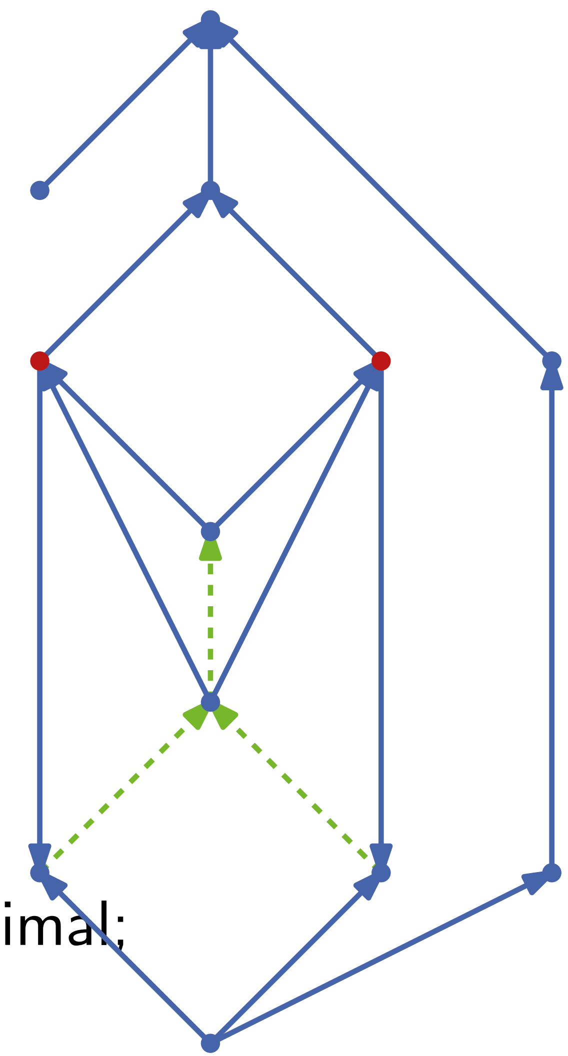

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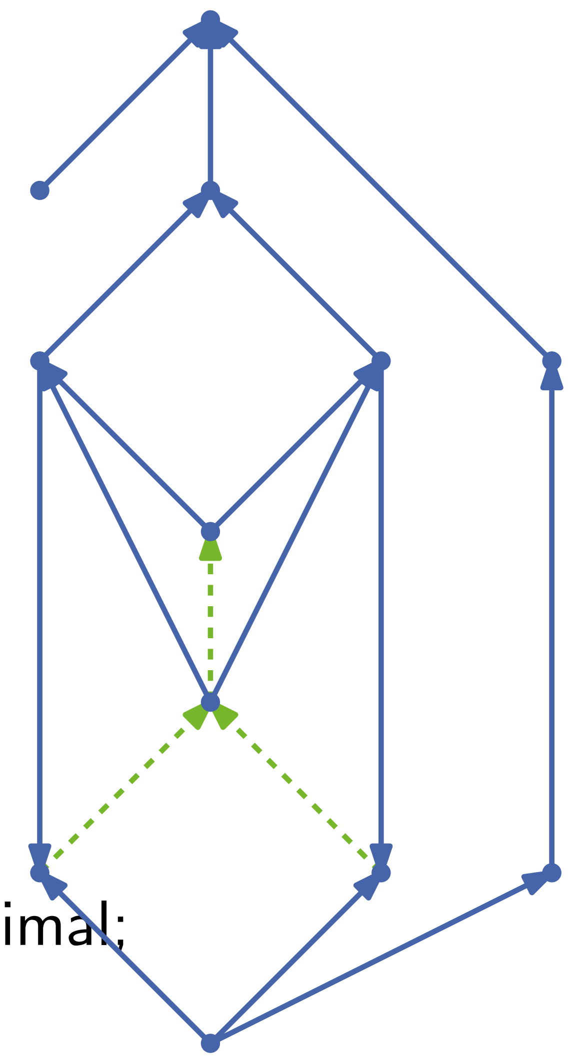

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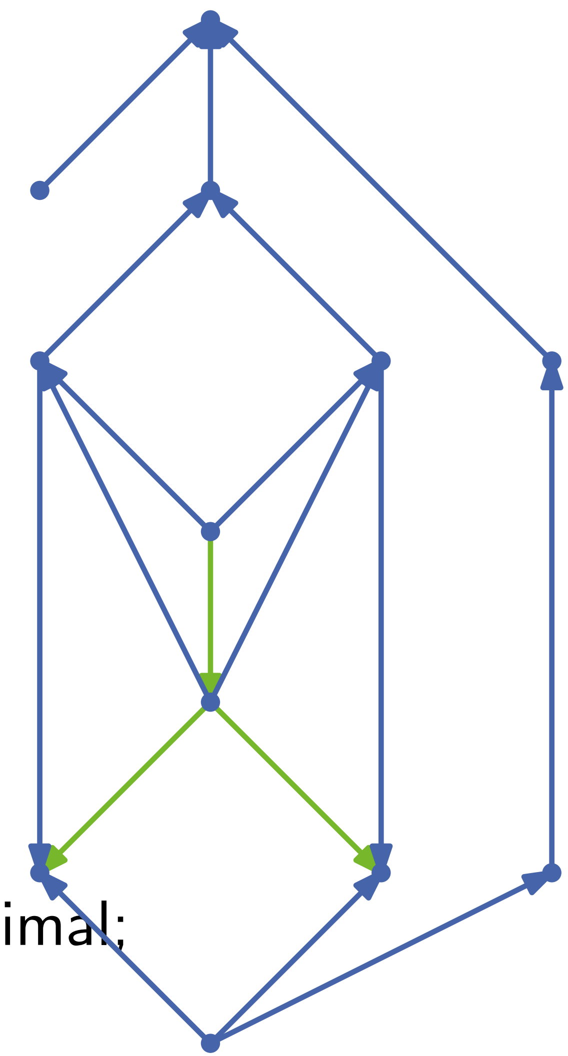

Heuristic 2 – Analysis

Theorem 1: Let $D = (V, A)$ be a connected, direted graph without 2-cycles. Heuristic 2 computes a set of edges A' with $|A'| \geq |A|/2 + |V|/6$.
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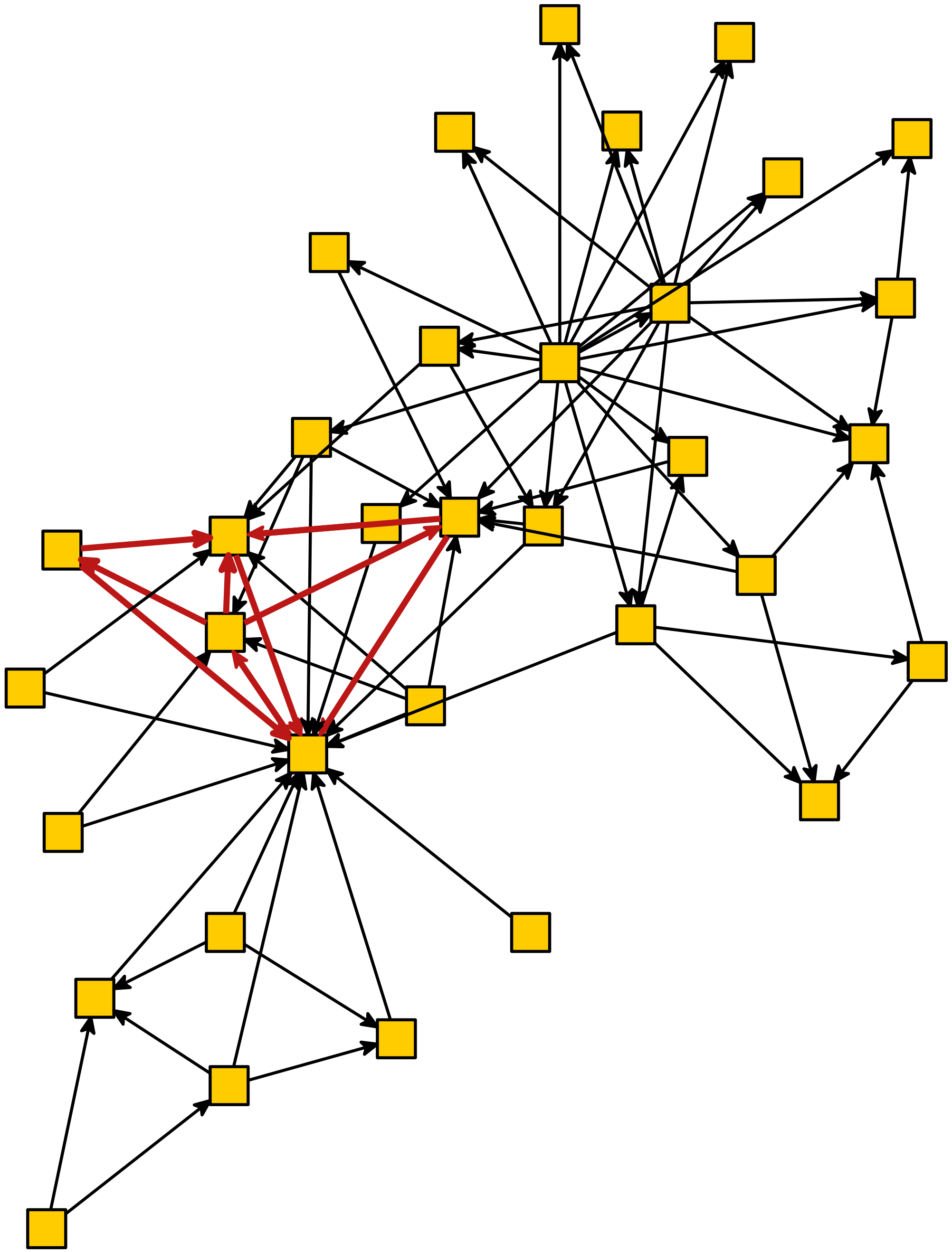
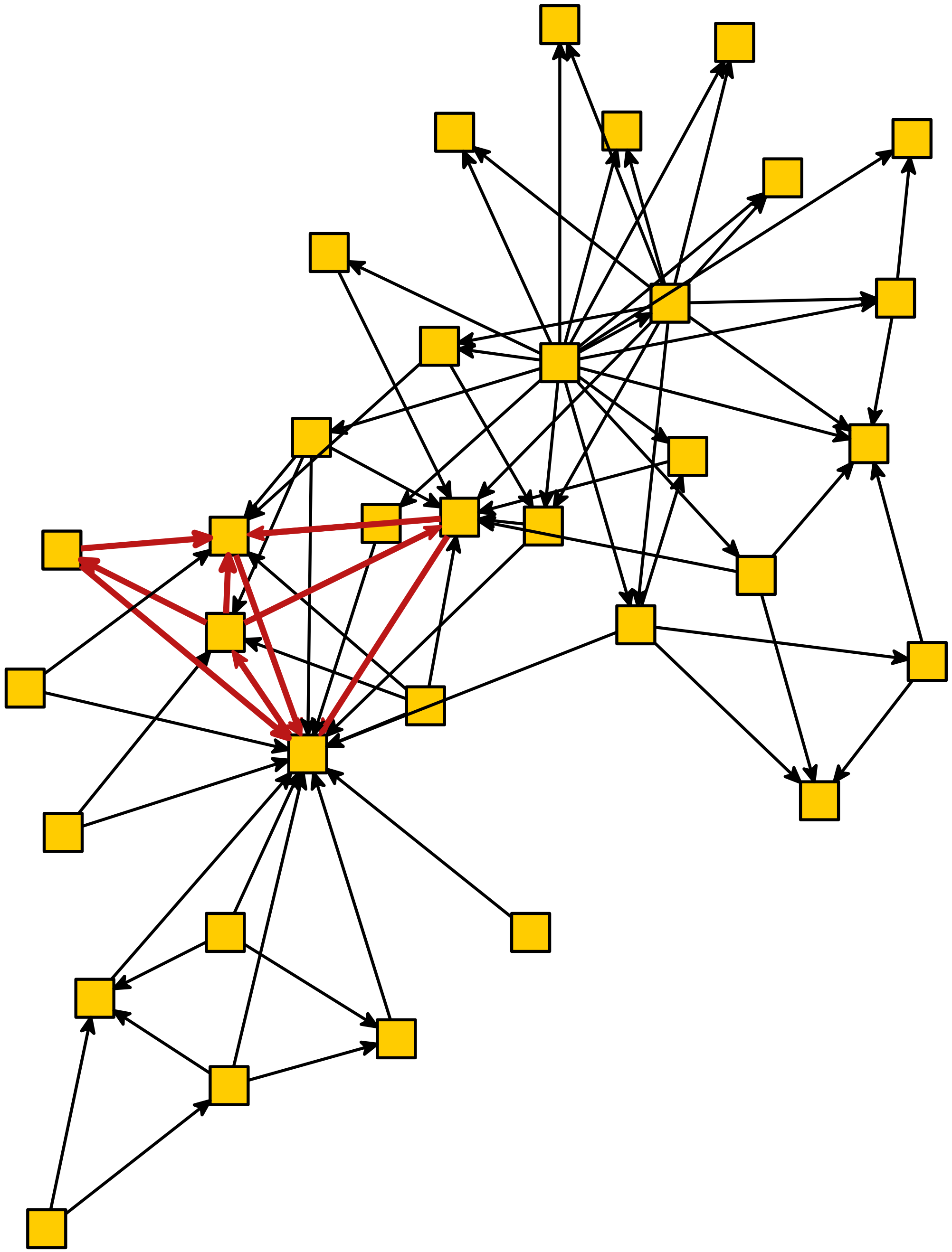
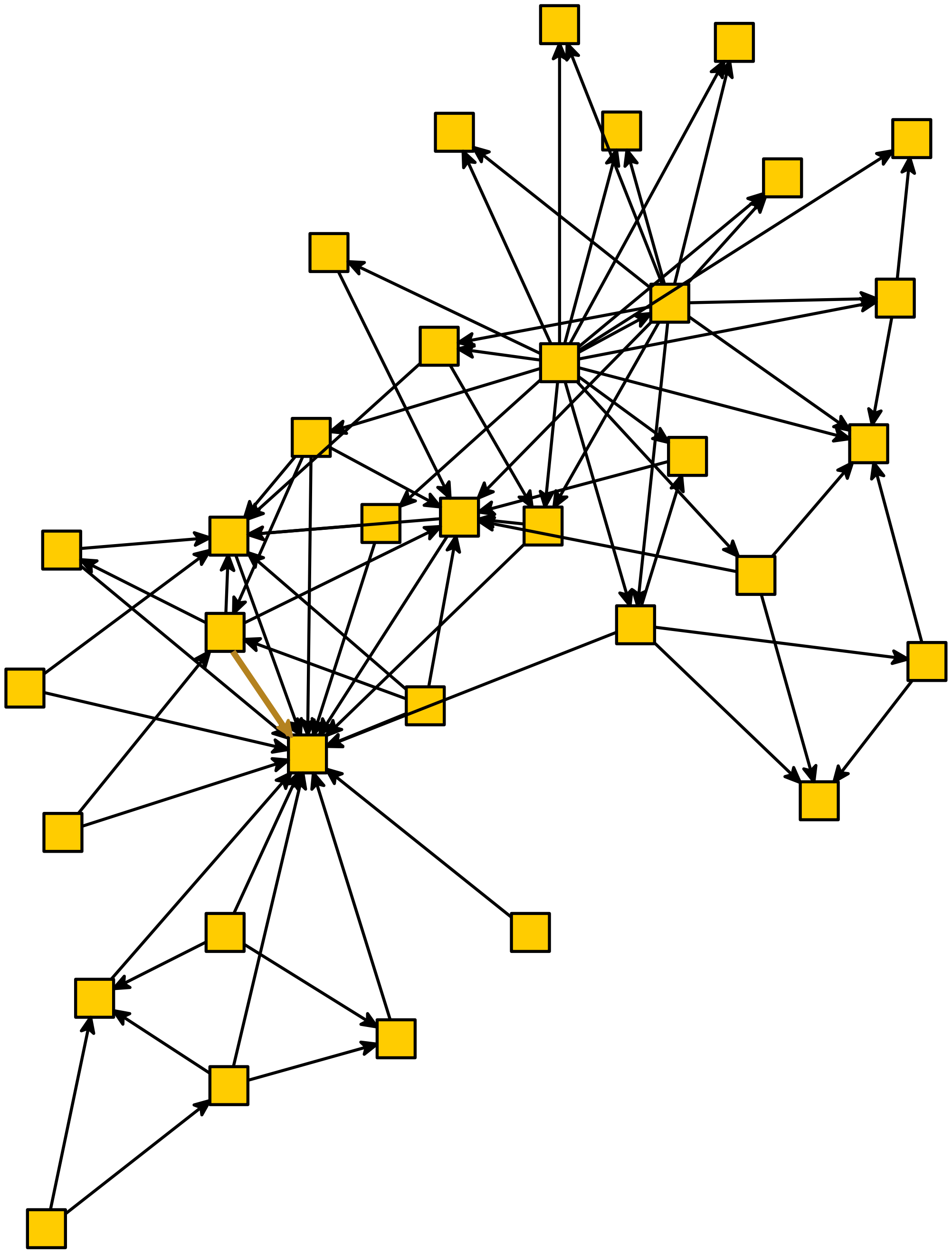
For $|A| \in O(|V|)$ Heuristic 2 performs similarly.

Example

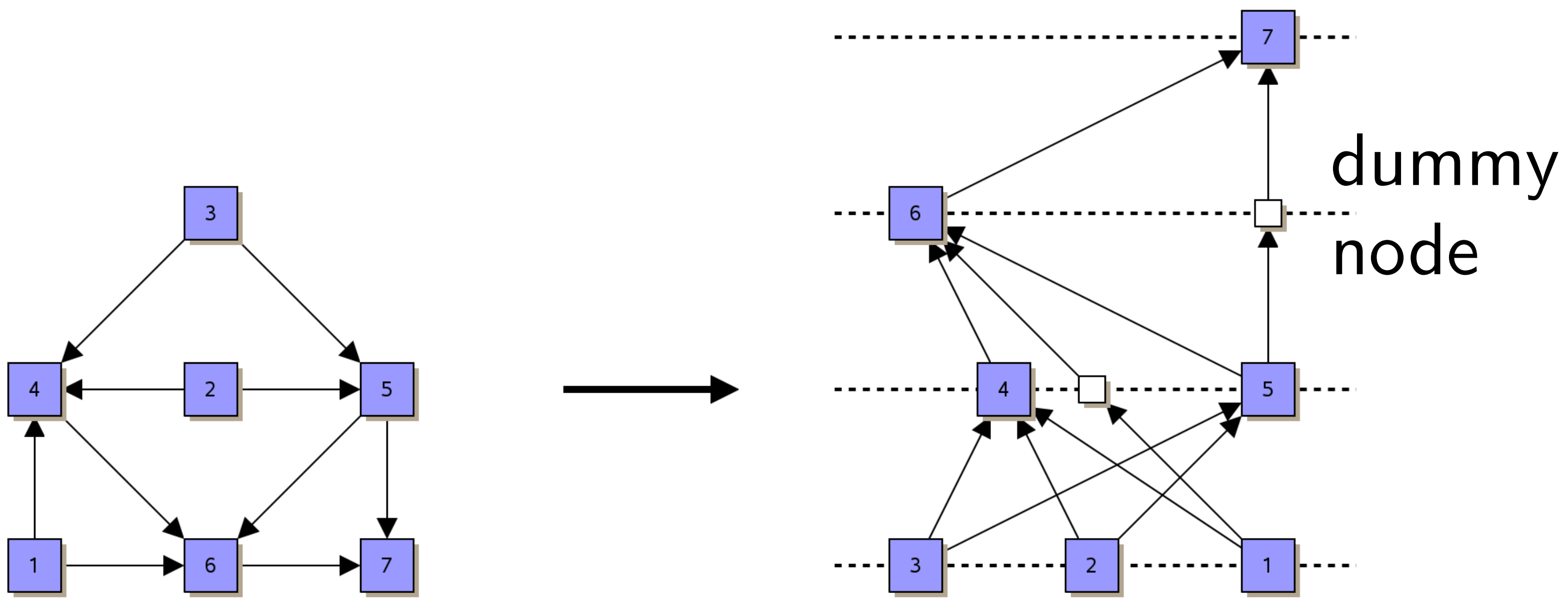


Example





Step 2: Layer Assignement



How would you proceed?

Step 2: Layer Assignement

Given.: directed acyclic graph (DAG) $D = (V, A)$

Find: Partition the vertex set V into disjoint subsets (**layers**) $L_1, \dots, L_h$ s.t. $(u, v) \in A$, $u \in L_i$, $v \in L_j \Rightarrow i < j$

Def: y-Coordinate $y(u) = i \Leftrightarrow u \in L_i$

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Criteria

- minimize the number of layers h ($=$ height of the layouts)
- minimize width, e.g. $\max\{|L_i| \mid 1 \leq i \leq h\}$
- minimize lengs of the longest edge, d.h. $\max\{j - i \mid (u, v) \in A,\, u \in L_i,\, v \in L_j\}$
- minimize the total length of edges ($\approx$ number of dummy nodes)

Height Optimization

Idea: assign each node v to the layer L_i, where i is the length of the longest simple path from a source to v

- all incomming neighbourse lie below v
- the resulting height h is minimized

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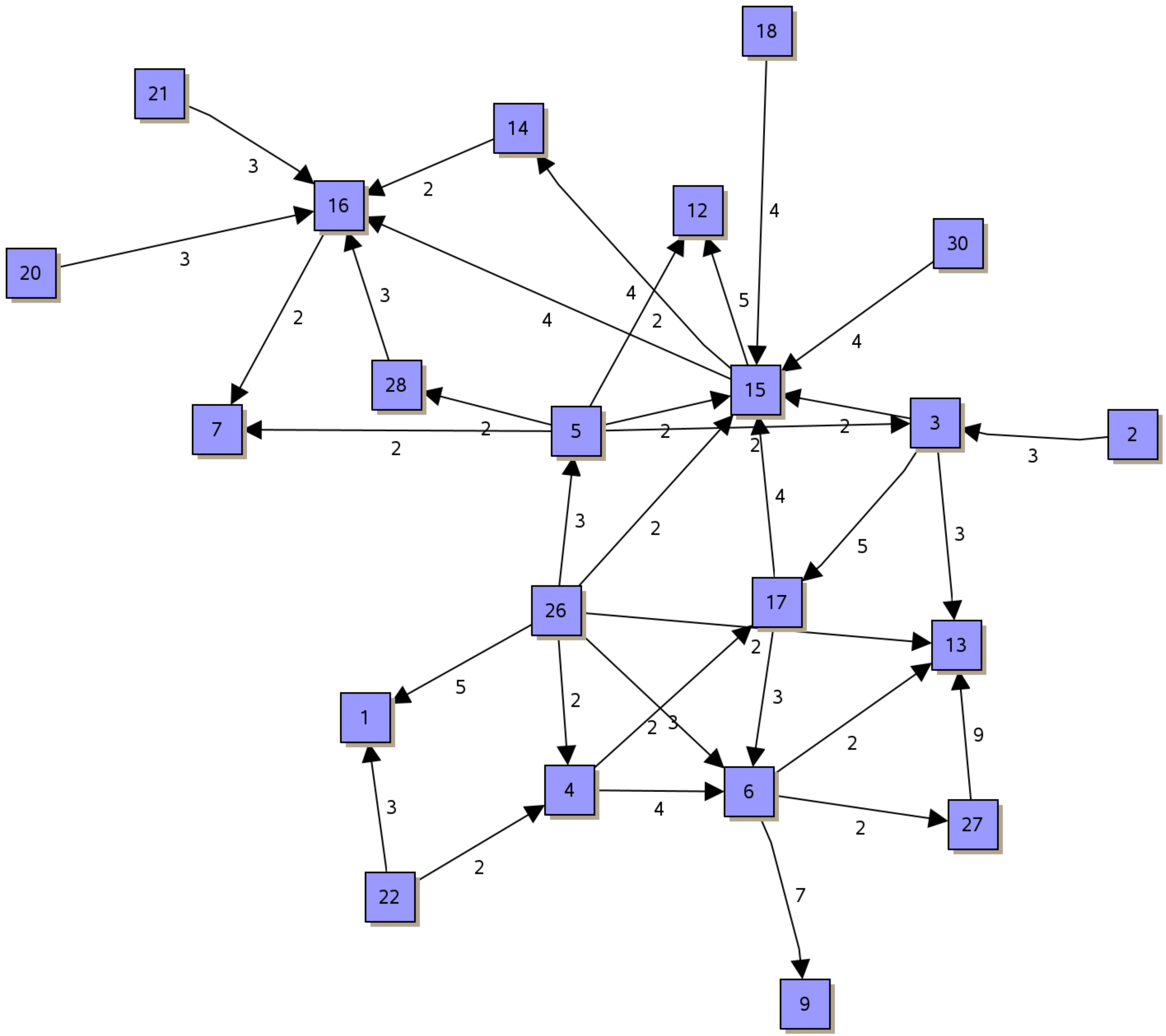
How can we implement the algorithm in $O(|V| + |A|)$ time?

Example



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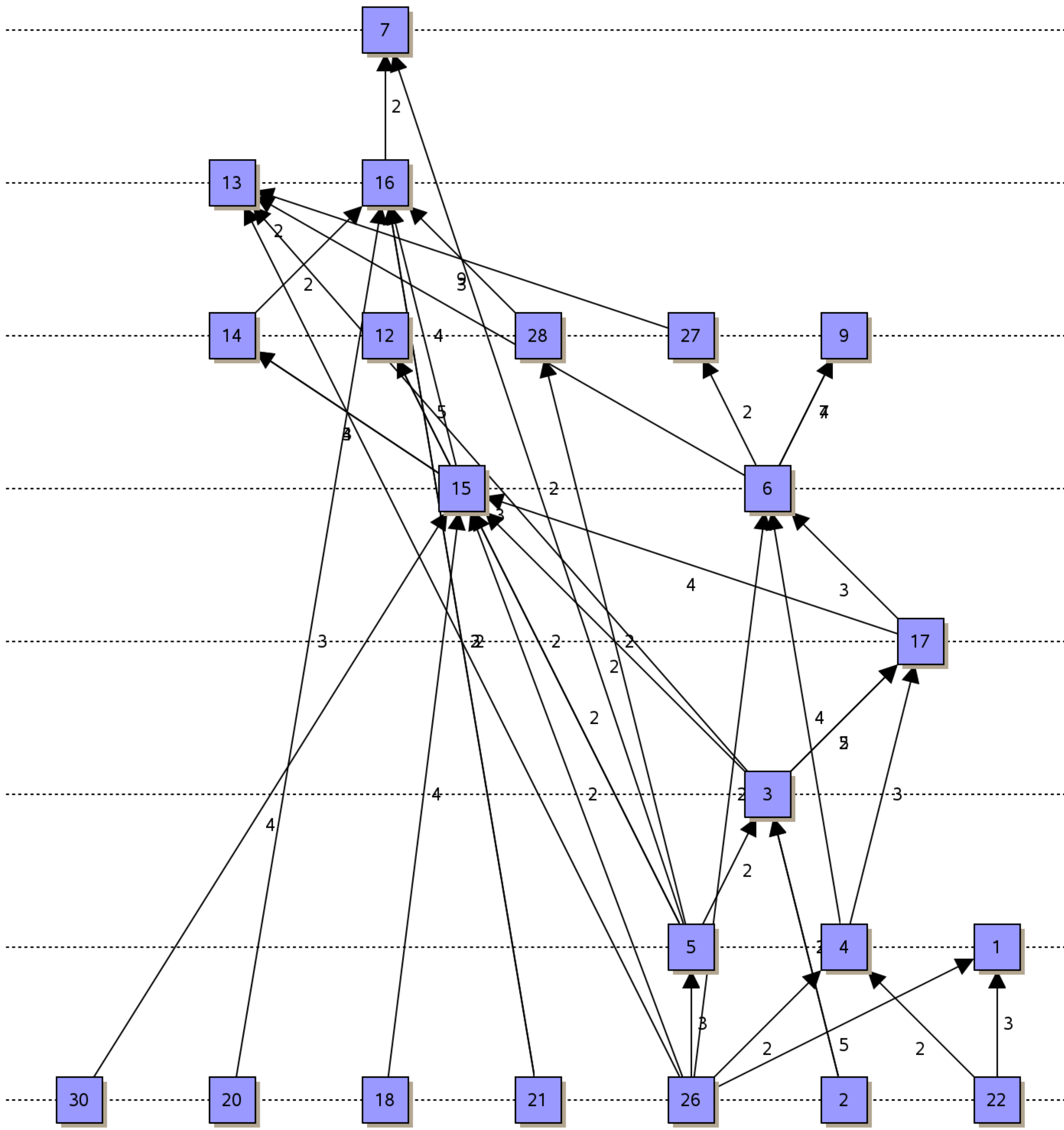

Total Edge Length

Can be formulated as an integer linear program:

$$
\begin{array}{rll}
\text{min} & \sum_{(u,v)\in A} (y(v) - y(u)) & \\
\text{subject to} & y(v) - y(u) \geq 1 & \forall (u,v) \in A \\
 & y(v) \geq 1 & \forall v \in V \\
 & y(v) \in \mathbb{Z} & \forall v \in V
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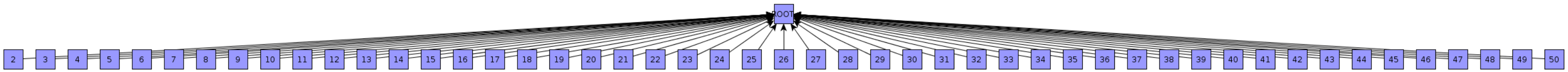
$$

One can show that:

- Constraint-Matrix is **totally unimodular**
- $\Rightarrow$ Solution of the relaxed linear program is integer
- The total edge length can be minimized in a polynomial time

Width of the Layout



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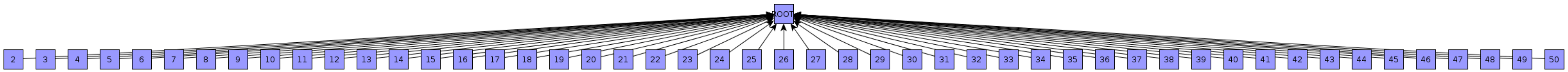


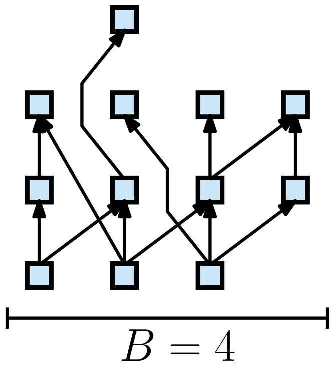
$\rightarrow$ bound the width!

Layer Assignement with Fixed Width

Fixed-Width Layer Assignment

Given: directed acyclic graph $D = (V, A)$, width B

Find: layer assignement $\mathcal{L}$ of minumum height with at most B nodes per layer



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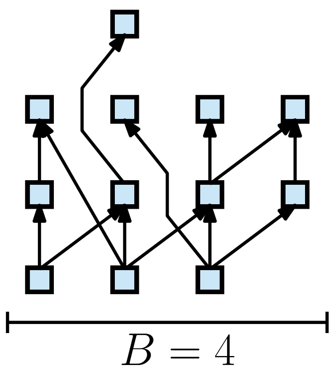

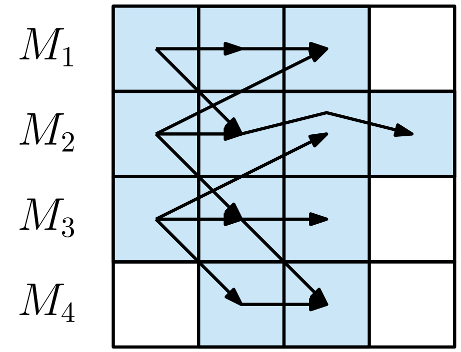

$\rightarrow$ equivalent to the following scheduling problem:

Minimum Precedence Constrained Scheduling (MPCS)

Given: n Jobs $J_1, \ldots, J_n$ with identical unit processing time, precedence constraints $J_i < J_k$, and B identical machines

Find: Schedule of minimum length, that satisfies all the precendence constraints

Complexity

Theorem 2: It is NP-hard to decide, whether for n jobs $J_1, \ldots, J_n$ of identical length, given partial ordering constraints, and number of machinces B, there exists a schedule of height at most T, even if $T = 3$.

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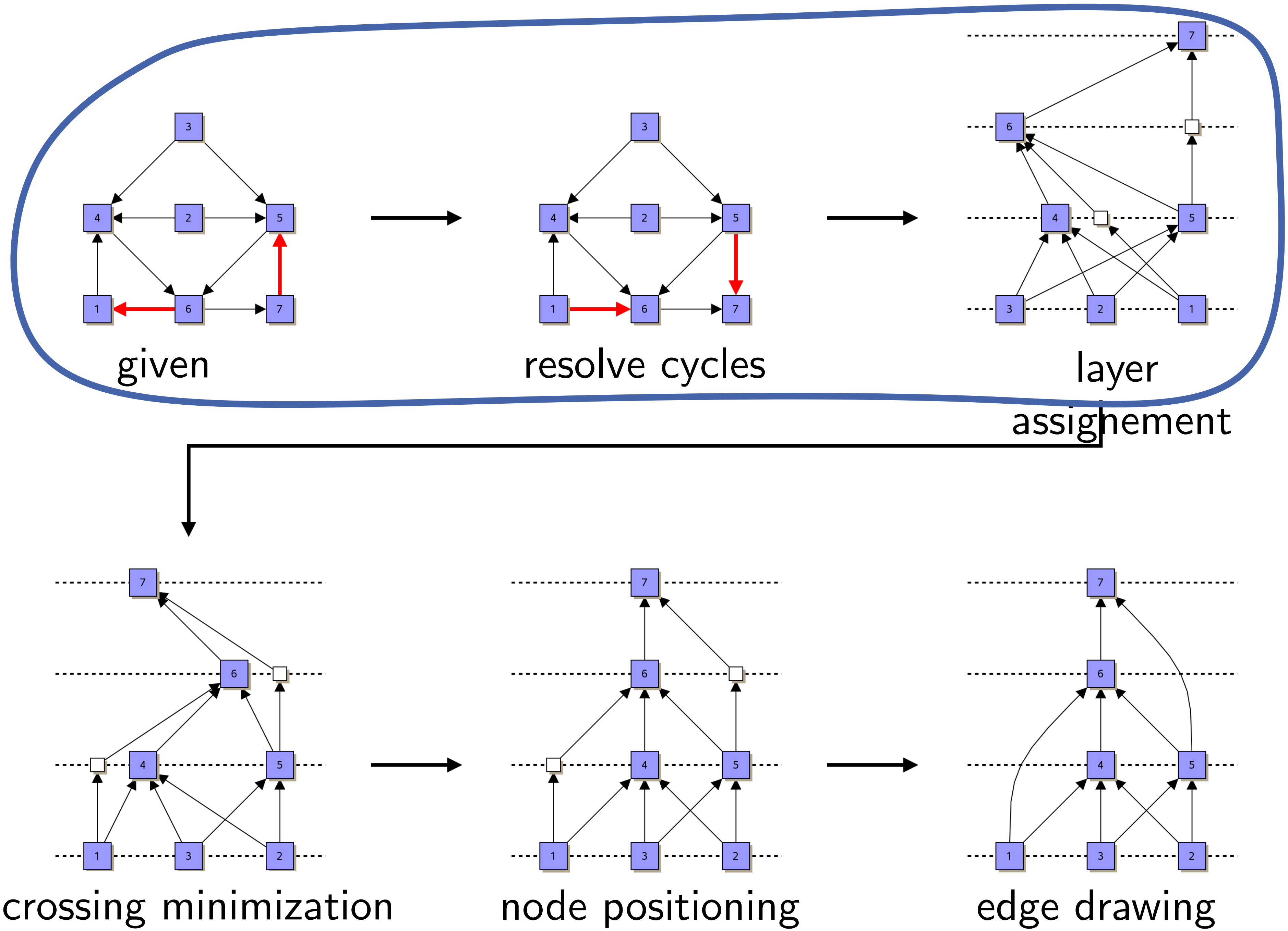
List-Scheduling-Algorithm:

- order jobs arbitrarily as a list $\mathcal{L}$
- when a machine is free, select an allowed job from $\mathcal{L}$; Machine is idle of there is no such job

Summary


given
resolve cycles
layer assignement
crossing minimization
node positioning
edge drawing