Exercise Sheet 2

Assignment:	November 14, 2016
Delivery:	None, Discussion on November 23, 2016

1 Canonical Orderings for Triconnected Planar Graphs

Let $G = (V, E)$ be a triconnected plane graph with a vertex v_1 on the exterior face. Further, let $\pi = (V_1.....V_K)$ be an ordered partition of V, that is, $V_1 \cup \cdots \cup V_K = V$ and $V_i \cap V_j = 0$ for $i \neq j$.

We define G_k to be the subgraph of G induced by $V_1 \cup \cdots \cup V_K$ and denote by C_k the exterior face of G_k.

The sequence π is a *canonical ordering* of G if:

- V_1 consists of $\{v_1, v_2\}$, where v_2 lies on the outerface and $(v_1, v_2) \in E$.
- $V_K = \{v_n\}$ is a singleton, where v_n lies on the outerface, $\{v_1, v_n\} \in E$, and $v_n \neq v_2$.
- Each C_k $(k > 1)$ is a cycle containing $\{v_1, v_2\}$.
- Each G_k is biconnected and internally triconnected, that is, removing two interior vertices of G_k does not disconnect it.
- For each k with $2 \leq k \leq K - 1$, one of the following conditions holds:
  1. $V_k = \{z\}$, where z belongs to C_k and has at least one neighbor in $G - G_k$.
  2. $V_k = \{z_1, \ldots, z_\ell\}$ is a chain, where each z_i has at least one neighbor in $G - G_k$ and where z_1 and z_ℓ each have one neighbor on C_{k-1}, and these are the only two neighbors of V_k in G_{k-1}.

Prove that every triconnected planar graph admits a canonical ordering.

Hint: Use reverse induction. For the induction step, consider the two cases that G_k *is triconnected* and G_k *is not triconnected.*

2 Visibility Representation of Maximal Planar Graphs

Recall the definition of *visibility representation* from the previous exersize set. Prove that every maximal planar graph admits a visibility representation.

Hint: Use canonical ordering.

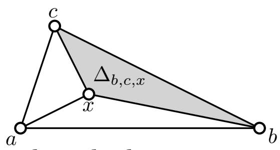
3 Barycentric Coordinates

Let $\Delta_{a,b,c}$ be a triangle on the plane on vertices a, b and c. For each point x laying inside triangle $\Delta_{a,b,c}$ there exists a triple (x_a, x_b, x_c) such that $x_a \cdot a + x_b \cdot b + x_c \cdot c = x$ and $x_a + x_b + x_c = 1$. The triple (x_a, x_b, x_c) is called *barycentric coordinates* of x with respect to $\Delta_{a,b,c}$.

Prove that:

(a) If $A(\Delta)$ denotes the area of the triangle A, then

$$x_a = \frac{A(\Delta_{b,c,x})}{A(\Delta_{a,b,c})},\ x_b = \frac{A(\Delta_{a,c,x})}{A(\Delta_{a,b,c})},\ x_c = \frac{A(\Delta_{a,b,x})}{A(\Delta_{a,b,c})}$$



(b) Equations $x_a = 0$, $x_b = 0$, $x_c = 0$ represent lines through bc, ab and ab, respectively.

(c) Let (x_a, x_b, x_c) be barycentric coordinates of point x in triagnle Δ_{abc}. The set of points $\{(x_a, x'_b, x'_c) : x'_b, x'_c \in \mathbb{R}\}$ represents a line parallel to edge bc passing through point x. Similarly, sets of points $\{(x'_a, x_b, x'_c) : x'_a, x'_c \in \mathbb{R}\}$, $\{(x'_a, x'_b, x_c) : x'_a, x'_b \in \mathbb{R}\}$ represent lines parallel to edges ac, ab, respectively, passing through point x.

4 Baricentric representation

A *Barycentric Representation* of a graph $G = (V, E)$ is an assigment of barycentric coordinates to the vertices of G, i.e. it is an *injective* function $v \in V \mapsto (v_a, v_b, v_c) \in \mathbb{R}^3$, such that:

- $v_a + v_b + v_c = 1$ for all $v \in V$
- for each $(x, y) \in E$ and each $z \in V \setminus \{x, y\}$, $\exists k \in \{a, b, c\}$ with $x_k < z_k$ and $y_k < z_k$.

Prove the following lemma.

Lemma 1. *A barycentric representation f of a graph G is a planar straight line drawing of G in the plane spanned by three points* $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.