Computational Geometry Lecture
Applications of WSPD & Visibility Graphs

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25.01.2016
Recall: Well-Separated Pair Decomposition

**Def:** A pair of disjoint point sets $A$ and $B$ in $\mathbb{R}^d$ is called $s$-well separated for some $s > 0$, if $A$ and $B$ can each be covered by a ball of radius $r$ whose distance is at least $sr$. 

![Diagram showing well-separated pair decomposition](image_url)
Recall: Well-Separated Pair Decomposition

**Def:** A pair of disjoint point sets $A$ and $B$ in $\mathbb{R}^d$ is called **$s$-well separated** for some $s > 0$, if $A$ and $B$ can each be covered by a ball of radius $r$ whose distance is at least $sr$.

**Def:** For a point set $P$ and some $s > 0$ an **$s$-well separated pair decomposition** ($s$-WSPD) is a set of pairs $\left\{ \{A_1, B_1\}, \ldots, \{A_m, B_m\} \right\}$ with
- $A_i, B_i \subset P$ for all $i$
- $A_i \cap B_i = \emptyset$ for all $i$
- $\bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P$
- $\{A_i, B_i\}$ $s$-well separated for all $i$
Recall: Well-Separated Pair Decomposition

**Def:** A pair of disjoint point sets \( A \) and \( B \) in \( \mathbb{R}^d \) is called \textit{s-well separated} for some \( s > 0 \), if \( A \) and \( B \) can each be covered by a ball of radius \( r \) whose distance is at least \( sr \).

**Def:** For a point set \( P \) and some \( s > 0 \) an \textit{s-well separated pair decomposition} (\( s \)-WSPD) is a set of pairs \( \{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\} \) with

- \( A_i, B_i \subset P \) for all \( i \)
- \( A_i \cap B_i = \emptyset \) for all \( i \)
- \( \bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P \)
- \( \{A_i, B_i\} \) \( s \)-well separated for all \( i \)

**Thm 3:** Given a point set \( P \) in \( \mathbb{R}^d \) and \( s \geq 1 \) we can construct an \( s \)-WSPD with \( O(s^d n) \) pairs in time \( O(n \log n + s^d n) \).
Further Applications of WSPD
Euclidean MST

**Problem:** Given a point set $P$ find a minimum spanning tree (MST) in the Euclidean graph $\mathcal{E}G(P)$. 
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- $\mathcal{E}G(P)$ has $\Theta(n^2)$ edges $\Rightarrow$ running time $O(n^2)$
- $(1 + \varepsilon)$-spanner for $P$ has $O(n/\varepsilon^d)$ edges $\Rightarrow$ running time $O(n \log n + n/\varepsilon^d)$
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How good is the MST of a $(1 + \varepsilon)$-spanner?
Euclidean MST

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- \((1 + \varepsilon)\)-spanner for \( P \) has \( O(n/\varepsilon^d) \) edges
  \( \Rightarrow \) running time \( O(n \log n + n/\varepsilon^d) \)

**Thm 5:** The MST obtained from a \((1 + \varepsilon)\)-spanner of \( P \) is a \((1 + \varepsilon)\)-approximation of the EMST of \( P \).
Diameter of $P$

**Problem:** Find the diameter of a point set $P$ (i.e., the pair $\{x, y\} \subset P$ with maximum distance).
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- test distances $\|\text{rep}(u)\|\text{rep}(v)\|$ of all ws-pairs $\{P_u, P_v\}$
  $\Rightarrow$ running time $O(n \log n + s^d n)$ :-)


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How good is the computed diameter?
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How good is the computed diameter?

**Thm 6:** The diameter obtained from an $s$-WSPD of $P$ for $s = 4/\varepsilon$ is a $(1 + \varepsilon)$-approximation of the diameter of $P$. 
Closest Pair of Points

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**Exercise:** For \( s > 2 \) this actually yields the closest pair.
Discussion

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WSPD is useful whenever one can do without knowing all $\Theta(n^2)$ exact distances in a point set and approximate them instead. One example are force-based layout algorithms in graph drawing, where pairwise repulsive forces of $n$ points need to be calculated.
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On the one hand, this replaces slow computations by faster (but less precise) ones; on the other hand, often the input data are imprecise so that approximate solutions can be sufficient depending on the application.
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Can we achieve the same time bounds with exact computations?

In $\mathbb{R}^2$ this is often true, but not in $\mathbb{R}^d$ for $d > 2$. (e.g. EMST, diameter)
Organizational Information

Oral Exams:

**Length:** 30 minutes  
**Dates:** Feb. 23, 24, 25; April 12, 13, 14  
**Times:** 9, 9:30, 10, 10:30, 11  
**Doodle:** Select all time slots that you have available!

Please come to our offices and ask questions!

Project presentations:

Next week on Feb. 1 and 3.
Motion planning and Visibility Graphs
Robot Motion Planning

Problem: Given a (point) robot at position $p_{\text{start}}$ in a area with polygonal obstacles, find a shortest path to $p_{\text{goal}}$ avoiding obstacles.
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First Idea: Shortest Paths in Graphs

\[ p_{\text{start}} \quad \times \quad p_{\text{goal}} \]
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- First compute trapezoidal map
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- Remove segments in obstacles
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- Nodes in trapezoids and vertical line segments
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- Locate start and goal
- Shortest path with Dijkstra in $G$
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*not the shortest path!*
First Idea: Shortest Paths in Graphs

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not the shortest path!
Lemma 1: For a set $S$ of disjoint open polygons in $\mathbb{R}^2$ and two points $s$ and $t$ not in $S$. 
Shortest Paths in Polygonal Areas

**Lemma 1:** For a set $S$ of disjoint open polygons in $\mathbb{R}^2$ and two points $s$ and $t$ not in $S$, each shortest $st$-path in $\mathbb{R}^2 \setminus \bigcup S$ is a polygonal path whose internal vertices are vertices of $S$. 
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**Proof sketch:**

![Diagram](image)
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Visibility Graph

Given a set $S$ of disjoint open polygons...
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...with point set $V(S)$.
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Def.: Then $G_{vis}(S) = (V(S), E_{vis}(S))$ is the **visibility graph** of $S$ with $E_{vis}(S) = \{uv \mid u, v \in V(S) \text{ and } u \text{ sees } v\}$ und $w(uv) = |uv|$.
Visibility Graph

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Define $S^* = S \cup \{s, t\}$ and $G_{vis}(S^*)$ analogously.
Visibility Graph

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**Def.:** Then $G_{\text{vis}}(S) = (V(S), E_{\text{vis}}(S))$ is the visibility graph of $S$ with $E_{\text{vis}}(S) = \{uv \mid u, v \in V(S) \text{ and } u \text{ sees } v\}$ and $w(uv) = |uv|$. Where $u$ sees $v :\Leftrightarrow \overline{uv} \cap \bigcup S = \emptyset$.

Define $S^* = S \cup \{s, t\}$ and $G_{\text{vis}}(S^*)$ analogously.

**Lemma 1**

$\Rightarrow$ A shortest $st$-path in $\mathbb{R}^2$ avoiding obstacles in $S$ is equivalent to a shortest $st$-path in $G_{\text{vis}}(S^*)$. 

Algorithm

ShortestPath($S, s, t$)

**Input:** Obstacles $S$, points $s, t \in \mathbb{R}^2 \setminus \bigcup S$

**Output:** Shortest collision-free $st$-path in $S$

1. $G_{vis} \leftarrow$ VisibilityGraph($S \cup \{s, t\}$)
2. foreach $uv \in E_{vis}$ do $w(uv) \leftarrow |uv|$
3. return Dijkstra($G_{vis}, w, s, t$)
Algorithm

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$n = |V(S)|$, $m = |E_{\text{vis}}(S)|$
Algorithm

ShortestPath($S, s, t$)

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3. **return** Dijkstra($G_{\text{vis}}, w, s, t$)

\[ n = |V(S)|, m = |E_{\text{vis}}(S)| \]

\[ O(n^2 \log n) \quad O(m) \quad O(n \log n + m) \quad O(n^2 \log n) \]

**Thm 1:** A shortest $st$-path in an area with polygonal obstacles with $n$ edges can be computed in $O(n^2 \log n)$ time.
Computing a Visibility Graph

VisibilityGraph($S$)

**Input:** Set of disjoint polygons $S$

**Output:** Visibility graph $G_{\text{vis}}(S)$

1. $E \leftarrow \emptyset$
2. **foreach** $v \in V(S)$ **do**
3.   $W \leftarrow \text{VisibleVertices}(v, S)$
4.   $E \leftarrow E \cup \{vw \mid w \in W\}$
5. **return** $(V(S), E)$
Computing Visible Nodes

$$\text{VisibleVertices}(p, S)$$
Computing Visible Nodes

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Computing Visible Nodes

VisibleVertices\( (p, S) \)
Computing Visible Nodes

VisibleVertices\((p, S)\)

**Problem:** Given \(p\) and \(S\), find in \(O(n \log n)\) time all nodes that \(p\) sees in \(V(S)\)!
Computing Visible Nodes

VisibleVertices\((p, S)\)

\[ r \leftarrow \{p + (k, 0) \mid k \in \mathbb{R}_0^+\} \]
Computing Visible Nodes

VisibleVertices\((p, S)\)

\[ r \leftarrow \{ p + (k, 0) \mid k \in \mathbb{R}_0^+ \} \]

\[ I \leftarrow \{ e \in E(S) \mid e \cap r \neq \emptyset \} \]
Computing Visible Nodes

VisibleVertices\((p, S)\)

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\[ \mathcal{T} \leftarrow \text{balancedBinaryTree}(I) \]
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$w_1, \ldots, w_n \leftarrow \text{sort } V(S) \text{ in cyclic order around } p$
Computing Visible Nodes

VisibleVertices\((p, S)\)

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\begin{align*}
    r & \leftarrow \{p + (k, 0) \mid k \in \mathbb{R}_0^+\} \\
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\(v \prec v' \iff\)

\[
\begin{align*}
\angle v < \angle v' \text{ or } \\
(\angle v = \angle v' \text{ and } |pv| < |pv'|)
\end{align*}
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Computing Visible Nodes

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\textbf{Sweep method with rotation}
Computing Visible Nodes

**VisibleVertices**(\(p, S\))

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W \leftarrow \emptyset
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\begin{algorithm}
\textbf{for} \: i = 1 \: \textbf{to} \: n \: \textbf{do}
\begin{algorithm}
\textbf{if} \: \text{Visible}\(p, w_i)\: \textbf{then}
\begin{algorithm}
W \leftarrow W \cup \{w_i\}
\end{algorithm}
\end{algorithm}
\text{Add to } \mathcal{T} \text{ edges incident to } w_i: \text{CW from } \overrightarrow{pw_i}^+ \n\end{algorithm}
\text{Remove from } \mathcal{T} \text{ edges incident to } w_i: \text{CCW from } \overrightarrow{pw_i}^-
\end{algorithm}

\textbf{return} \: W
Computing Visible Nodes

VisibleVertices((p, S))

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2. \( I \leftarrow \{e \in E(S) \mid e \cap r \neq \emptyset\} \)
3. \( \mathcal{T} \leftarrow \text{balancedBinaryTree}(I) \)
4. \( w_1, \ldots, w_n \leftarrow \text{sort } V(S) \text{ in cyclic order around } p \)
5. \( W \leftarrow \emptyset \)
6. \( \text{for } i = 1 \text{ to } n \text{ do} \)
   - \( \text{if } \text{Visible}(p, w_i) \text{ then} \)
     - \( W \leftarrow W \cup \{w_i\} \)
     - Add to \( \mathcal{T} \) edges incident to \( w_i \): CW from \( \overrightarrow{pw_i} \)
     - Remove from \( \mathcal{T} \) edges incident to \( w_i \): CCW from \( \overrightarrow{pw_i} \)
7. \( \text{return } W \)
Computing Visible Nodes

\textbf{VisibleVertices}(\(p, S\))

\[ r \leftarrow \{p + (k, 0) \mid k \in \mathbb{R}_0^+\} \]

\[ I \leftarrow \{e \in E(S) \mid e \cap r \neq \emptyset\} \]

\[ \mathcal{T} \leftarrow \text{balancedBinaryTree}(I) \]

\[ w_1, \ldots, w_n \leftarrow \text{sort } V(S) \text{ in cyclic order around } p \]

\[ W \leftarrow \emptyset \]

\textbf{for} \( i = 1 \) \textbf{to} \( n \) \textbf{do}

\[ \text{if Visible}(p, w_i) \text{ then} \]

\[ W \leftarrow W \cup \{w_i\} \]

Add to \( \mathcal{T} \) edges incident to \( w_i \): \text{CW from} \( pw_i^+ \)

Remove from \( \mathcal{T} \) edges incident to \( w_i \): \text{CCW from} \( pw_i^- \)

\textbf{return} \( W \)
Visibility Case Analysis

$\text{Visible}(p, w_i)$

\[
\text{if } \overline{pw_i} \text{ intersects polygon of } w_i \text{ then} \\
\quad \text{return false}
\]
Visibility Case Analysis

Visible\((p, w_i)\)

\[
\text{if } \overline{pw_i} \text{ intersects polygon of } w_i \text{ then} \\
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\]

\[
\text{if } i = 1 \text{ or } w_{i-1} \notin \overline{pw_i} \text{ then} \\
\quad e \leftarrow \text{edge of leftmost leaf of } T \\
\quad \text{if } e \neq \text{nil} \text{ and } \overline{pw_i} \cap e \neq \emptyset \text{ then} \\
\quad \quad \text{return false}
\]

\[
\text{else return true}
\]
Visibility Case Analysis

\textbf{Visible}(p, w_i)

\begin{itemize}
  \item \textbf{if} $\overline{pw_i}$ intersects polygon of $w_i$ \textbf{then} \textbf{return} false
  \item \textbf{if} $i = 1$ or $w_{i-1} \not\in \overline{pw_i}$ \textbf{then}
    \begin{itemize}
      \item $e \leftarrow$ edge of leftmost leaf of $\mathcal{T}$
      \item \textbf{if} $e \neq \text{nil}$ and $\overline{pw_i} \cap e \neq \emptyset$ \textbf{then}
        \begin{itemize}
          \item \textbf{return} false
        \end{itemize}
      \item \textbf{else} \textbf{return} true
    \end{itemize}
  \item \textbf{else}
    \begin{itemize}
      \item \textbf{if} $w_{i-1}$ is not visible \textbf{then} \textbf{return} false
      \item \textbf{else}
        \begin{itemize}
          \item $e \leftarrow$ find edge in $\mathcal{T}$, that $\overline{w_{i-1}w_i}$ cuts; \textbf{if} $e \neq \text{nil}$ \textbf{then} \textbf{return} false
          \item \textbf{else} \textbf{return} true
        \end{itemize}
    \end{itemize}
\end{itemize}
Summary

**Thm 1:** A shortest $st$-path in an area with polygonal obstacles with $n$ edges can be computed in $O(n^2 \log n)$ time.
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\( O(n^2) \) with duality
(see exercise or D. Mount [M12] Lect. 31)
Discussion

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For robots modelled by a convex polygon that cannot rotate, we can resize (grow) the polygons representing the obstacles (→ Minkowski Sums, Ch. 13 in [BCKO08]).
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Yes, by use duality and a simultaneous rotation sweep for all points in the dual. Computing the arrangement, is also in $O(n^2)$. Even though $G_{\text{vis}}$ can have $\Omega(n^2)$ edges, the visibility graph can be constructed even faster with an output sensitive $O(n \log n + m)$-time algorithm.

[Ghosh, Mount 1987]
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[Ghosh, Mount 1987]

If you search only for one shortest Euclidean $st$-path, there is an algorithm with optimal $O(n \log n)$ time.  

[Hershberger, Suri 1999]