Karlsruhe Institute of Technology

Computational Geometry • Lecture

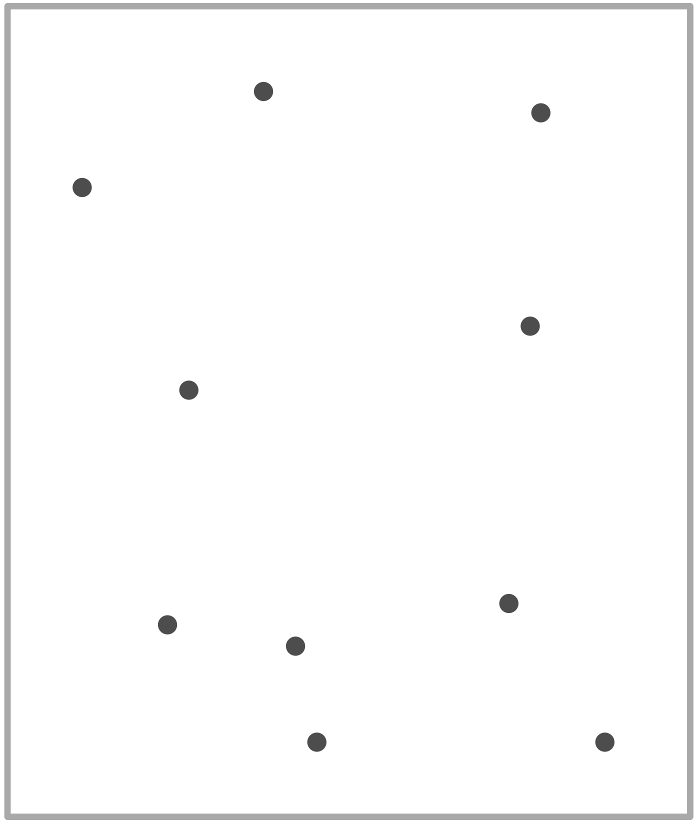
Well-Separated Pair Decompositions

INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

Tamara Mchedlidze · Darren Strash

18.1.2016

Motivation: Spanners



Task:

A set of cities shall be connected by a new road network.

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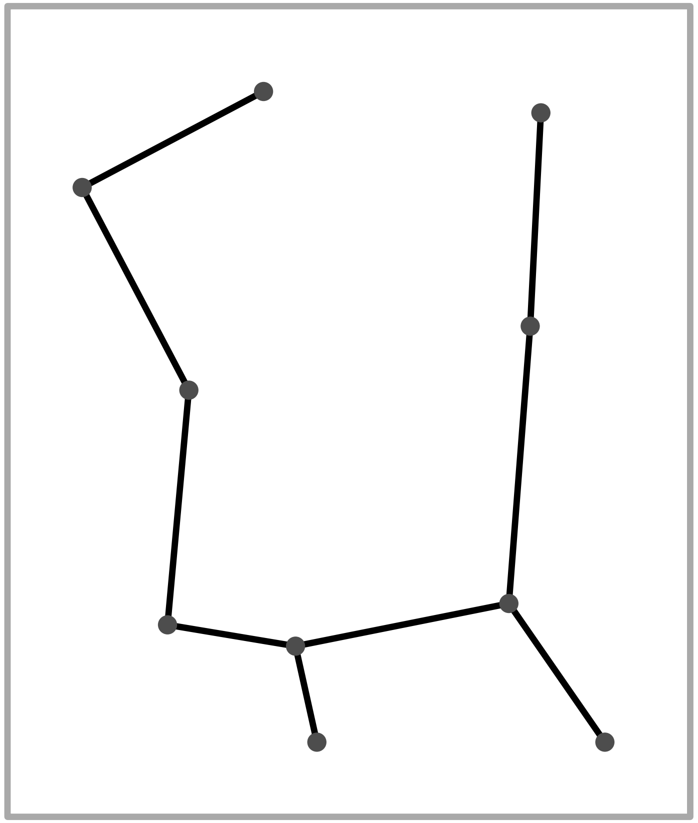

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Idea 1: Euclidean minimum spanning tree

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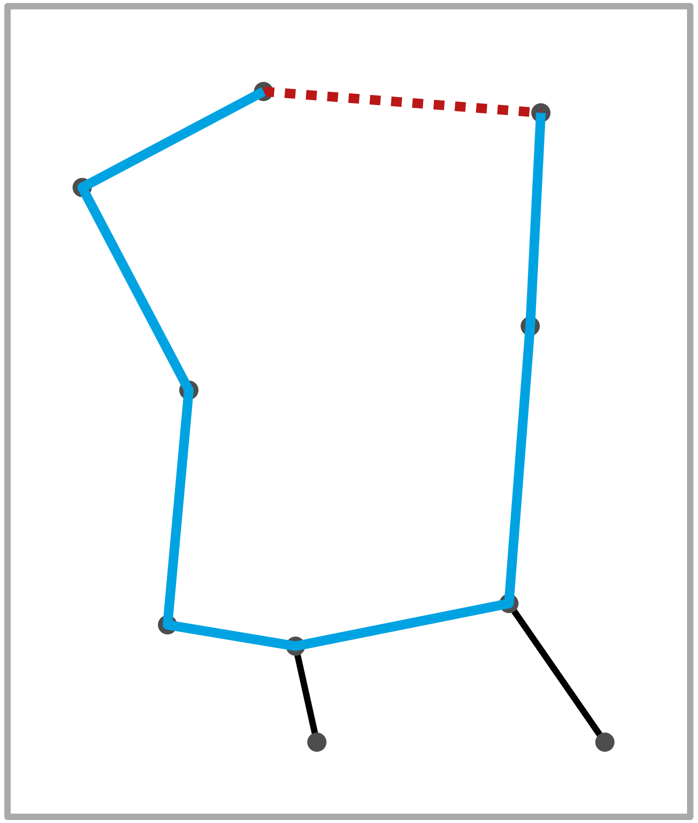

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But for no pair (x, y) the path length in the road network should be much larger than the distance $||xy||$.

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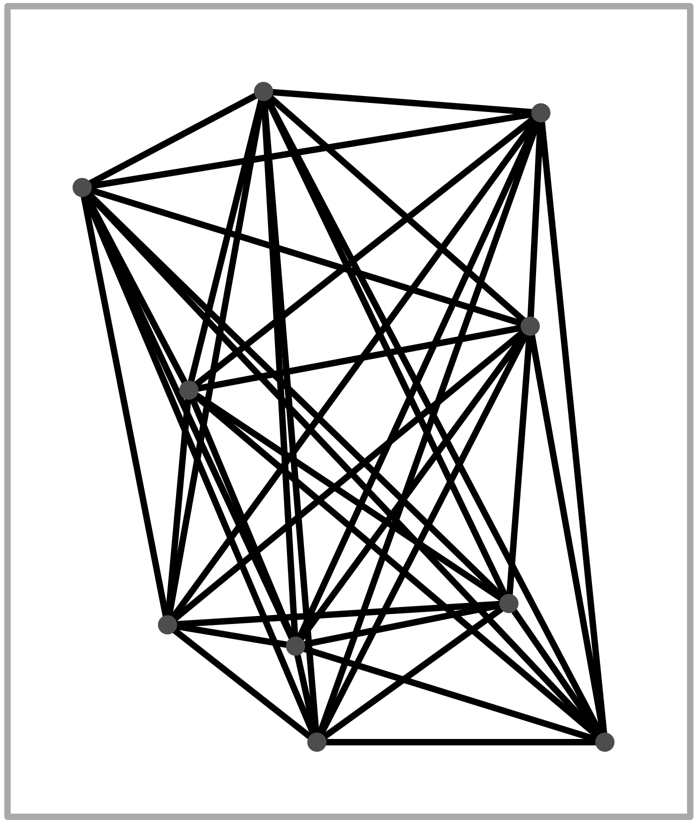

Idea 1: Euclidean minimum spanning tree

Idea 2: complete graph

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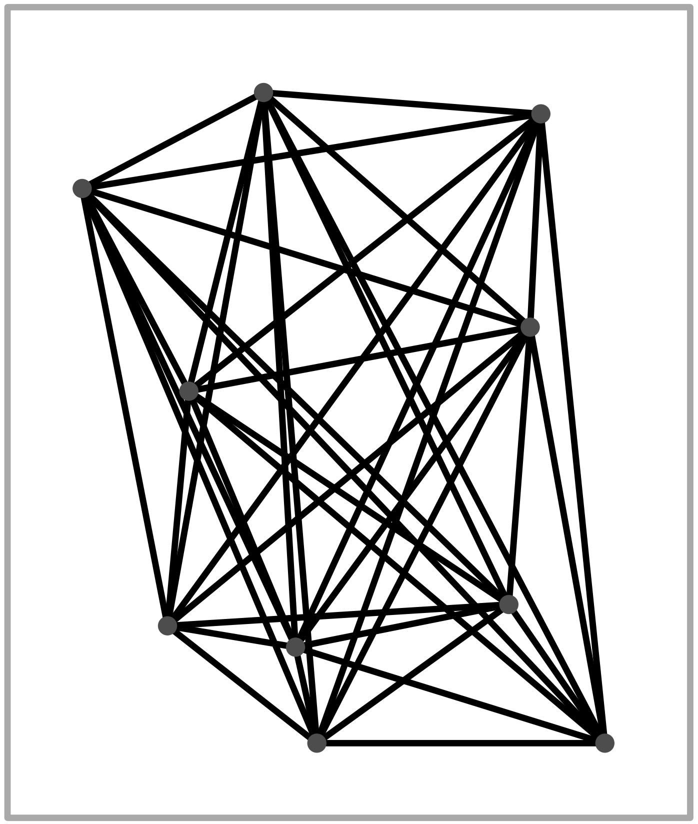

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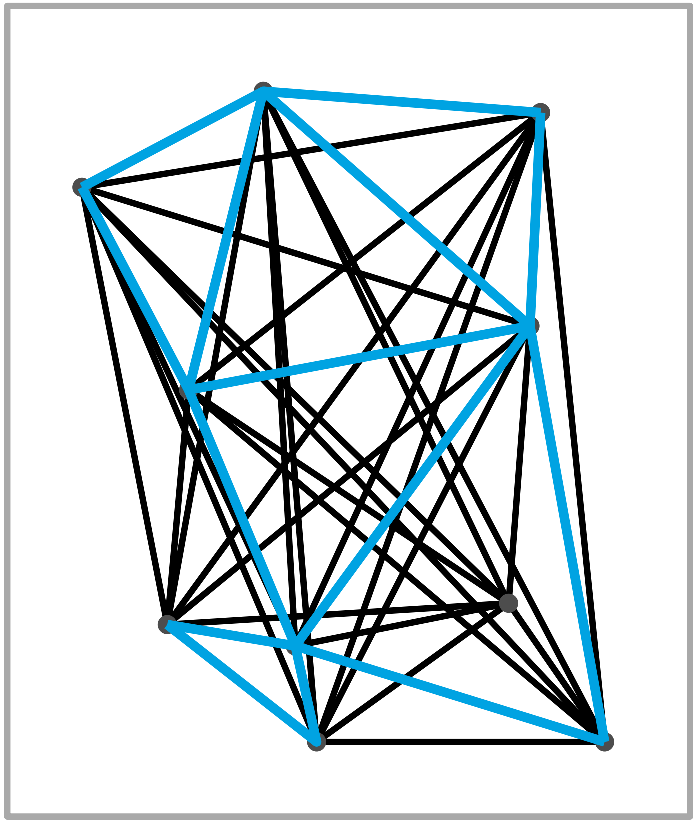

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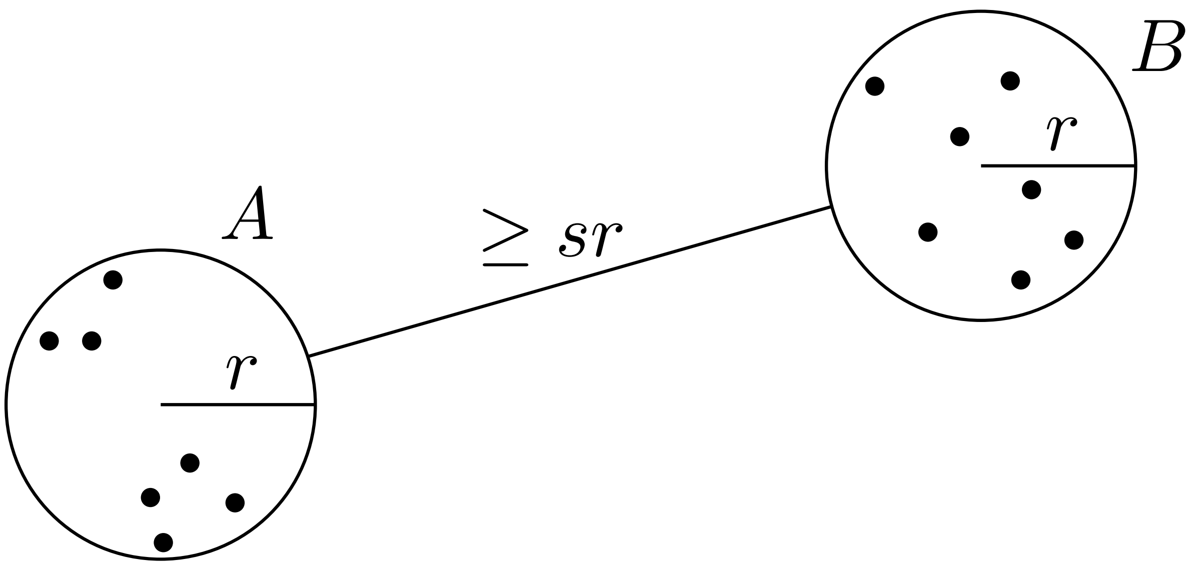
Idea 1: Euclidean minimum spanning tree
Idea 2: complete graph
Idea 3: sparse t-spanner

Well-Separated Pairs

Def: A pair of disjoint point sets A and B in $\mathbb{R}^d$ is called s-**well separated** for some $s > 0$, if A and B can each be covered by a ball of radius r whose distance is at least sr.



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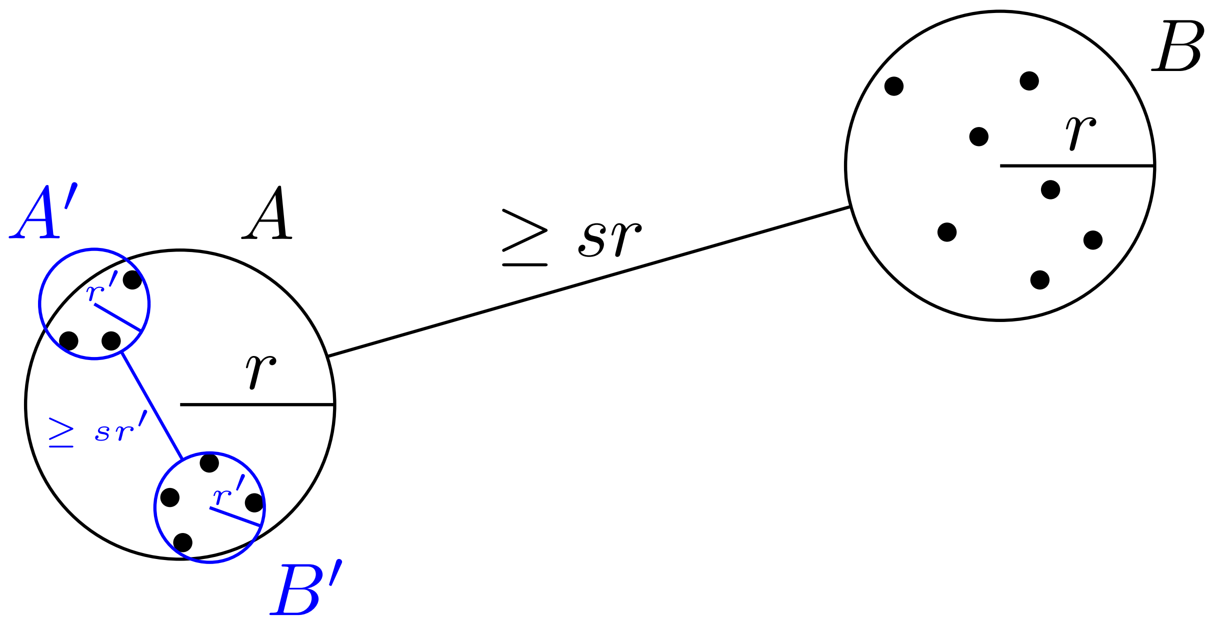

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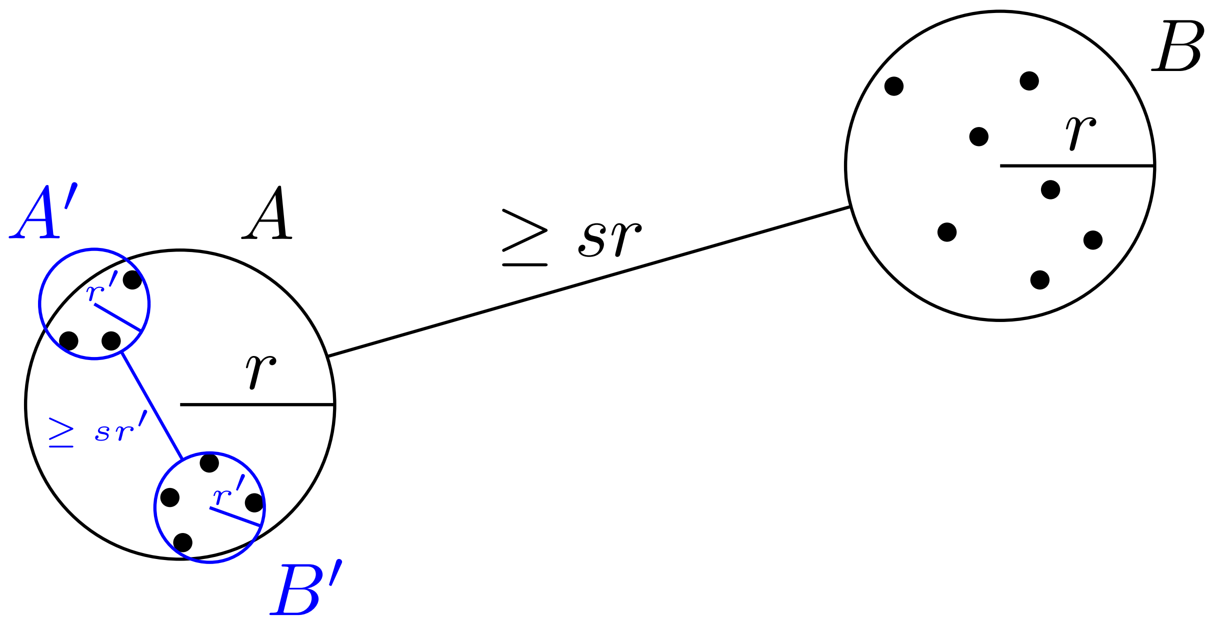

Obs:
- s-well separated $\Rightarrow$ s'-well separated for all $s' \leq s$
- singletons $\{a\}$ and $\{b\}$ are s-well separated for all $s > 0$

Well-Separated Pair Decomposition (WSPD)

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For well-separated pair $\{A, B\}$ we know that the distance for all point pairs in $A \otimes B = \{\{a, b\} \mid a \in A, b \in B, a \neq b\}$ is similar.

Goal: $o(n^2)$-sized data structure that approximates the distances of all $\binom{n}{2}$ pairs of points in a set $P = \{p_1, \ldots, p_n\}$.

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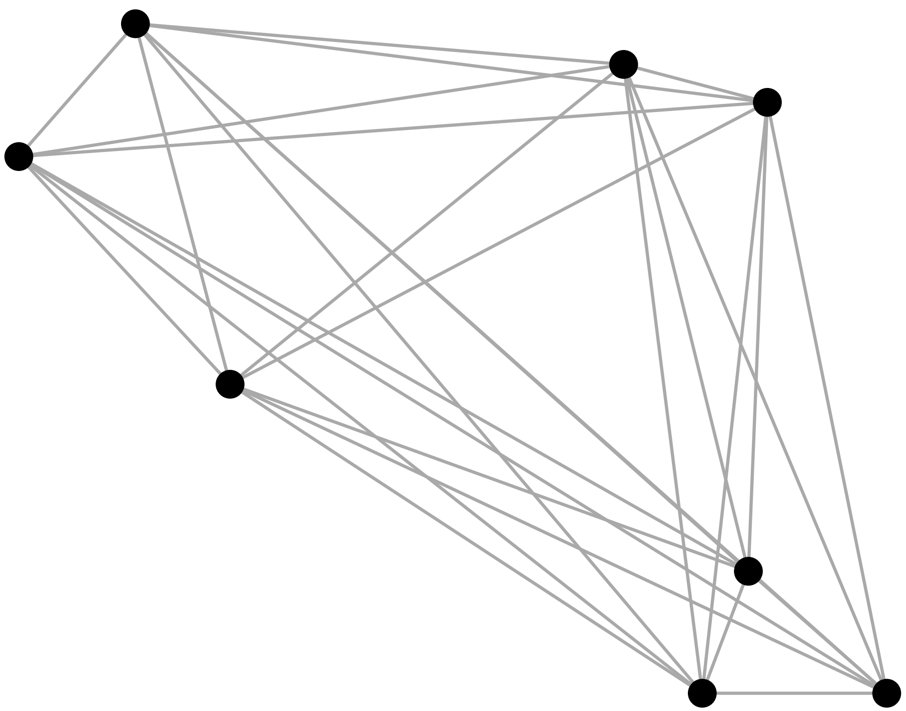
Def: For a point set P and some $s > 0$ an s-**well separated pair decomposition** (s-WSPD) is a set of pairs $\{\{A_1, B_1\}, \ldots, \{A_m, B_m\}\}$ with

- $A_i, B_i \subset P$ for all i
- $A_i \cap B_i = \emptyset$ for all i
- $\bigcup_{i=1}^{m} A_i \otimes B_i = P \otimes P$
- $\{A_i, B_i\}$ s-well separated for all i

Example



28 point pairs

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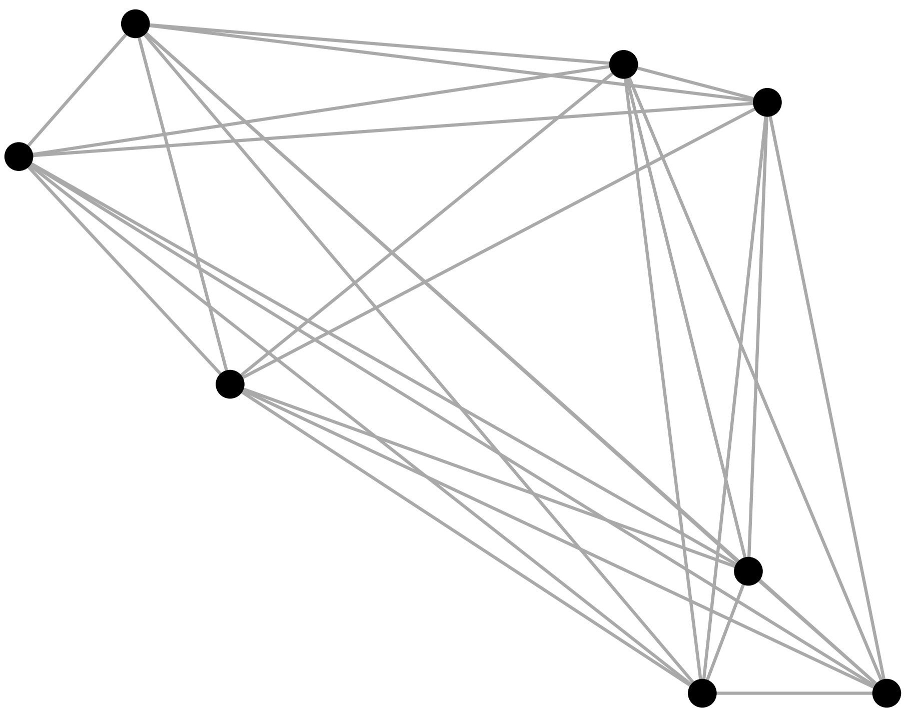

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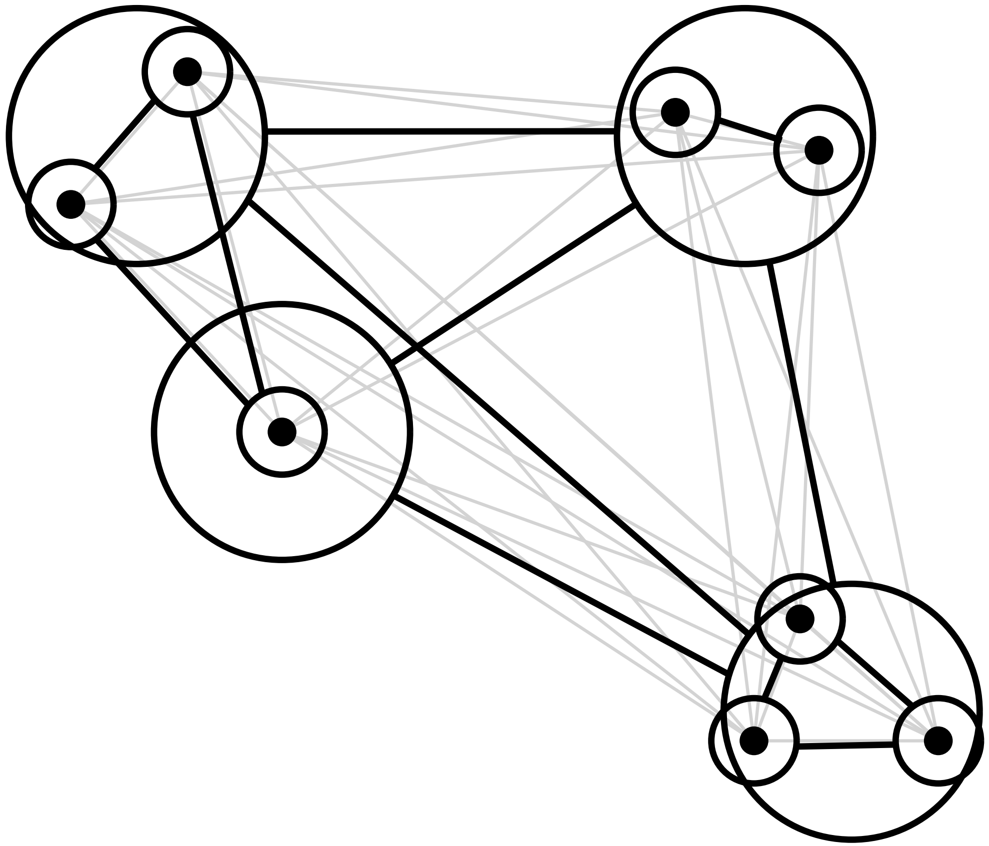

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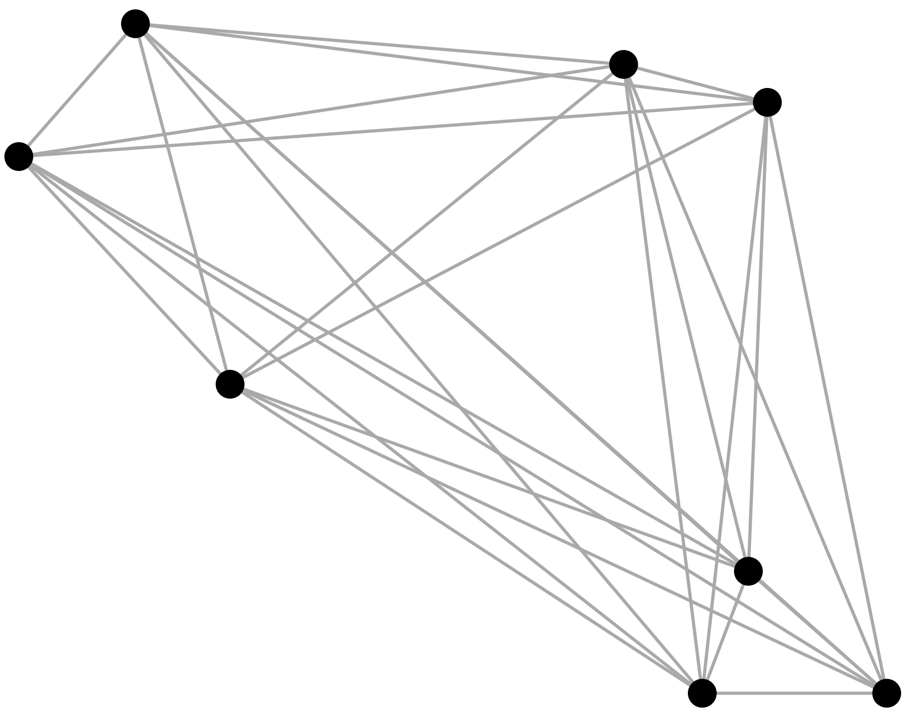

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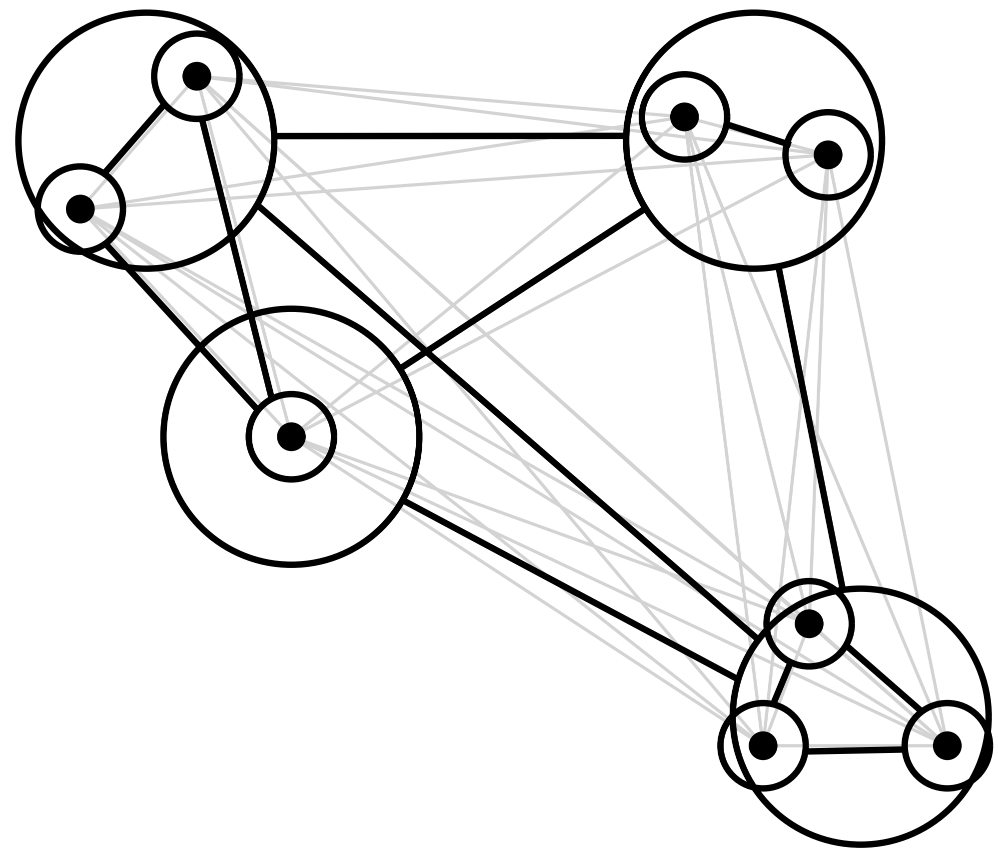
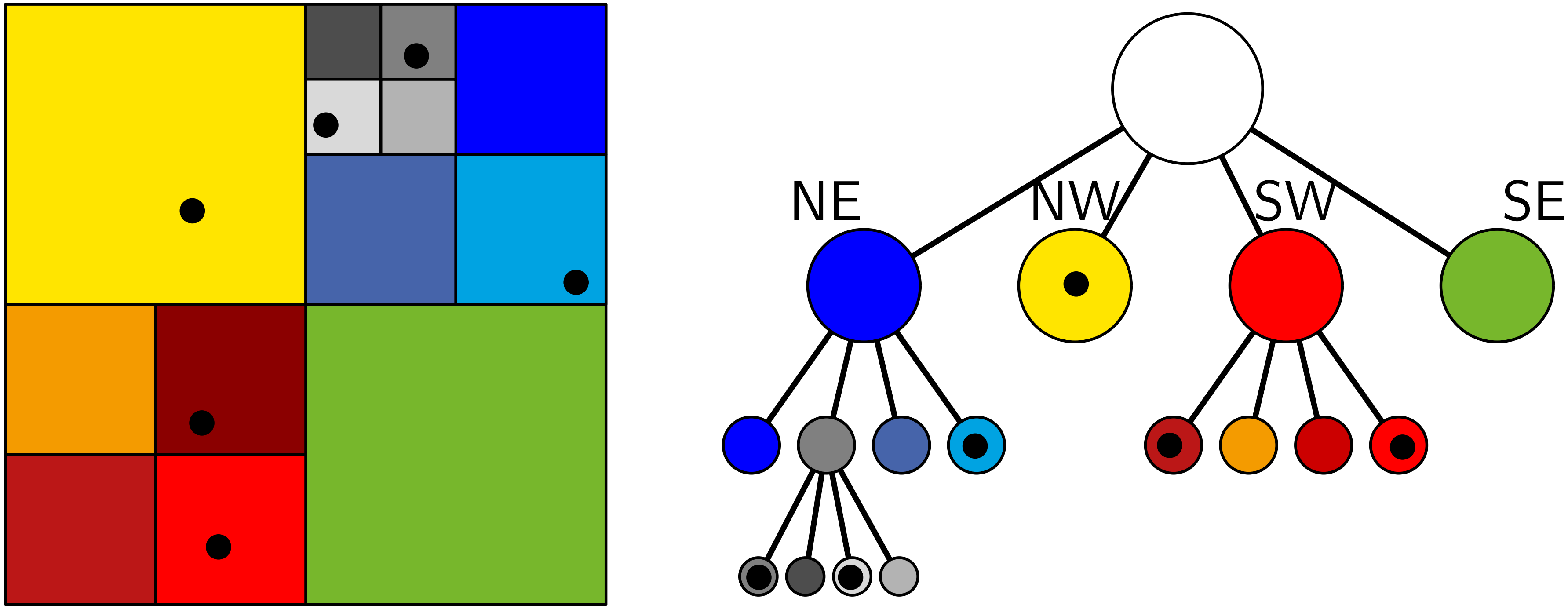


12 s-well separated pairs

WSPD of size $O(n^2)$ is trivial. Can we do it in $O(n)$?

Recall: Quadtrees

Def: A **quadtree** $\mathcal{T}(P)$ for a point set P is a rooted tree, where each internal node has four children. Each node corresponds to a square, and the squares of the leaves form a partition of the root square.



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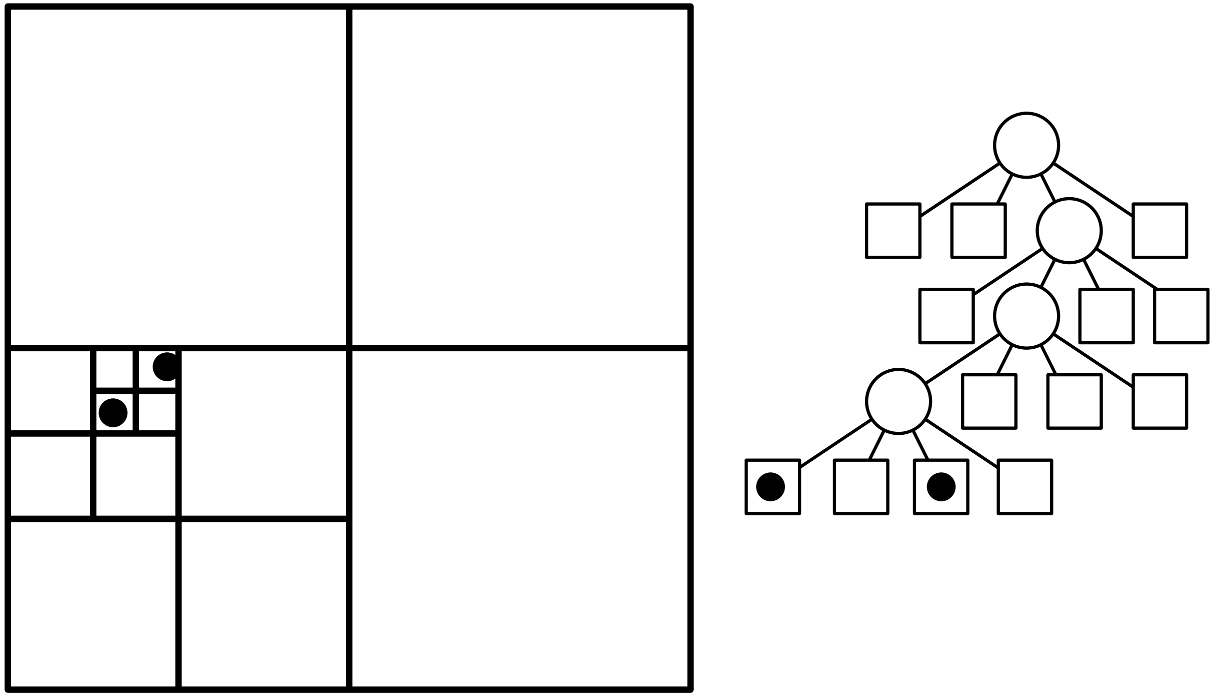
Lemma 1: The height of $\mathcal{T}(P)$ is at most $\log(s/c) + 3/2$, where c is the smallest distance in P and s is the side length of the root square Q.

Thm 1: A quadtree $\mathcal{T}(P)$ on n points with height h has $O(hn)$ nodes and can be constructed in $O(hn)$ time.

Compressed Quadtrees

Def: A **compressed** quadtree is a quadtree, in which each path of non-separating inner nodes is contracted into a single edge. Each such edge has a label to reconstruct the path structure.



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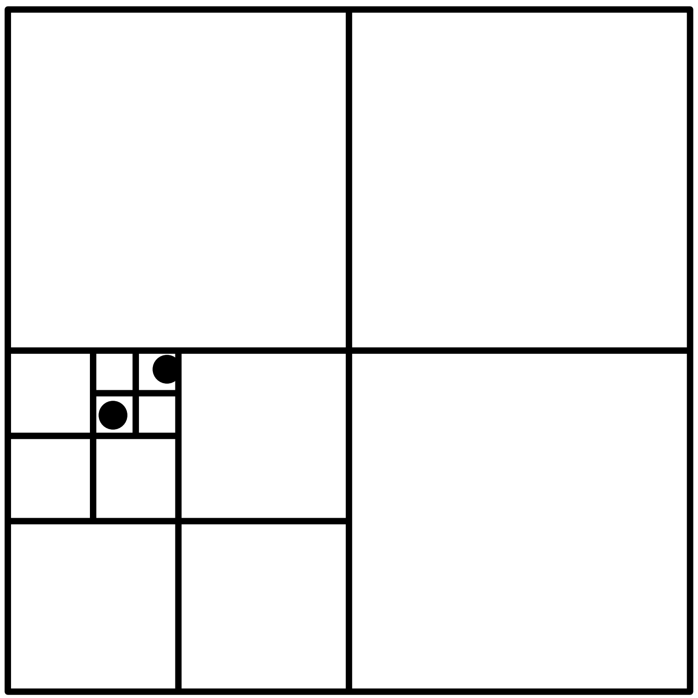

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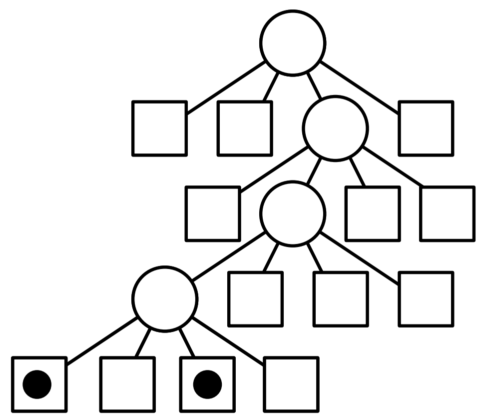

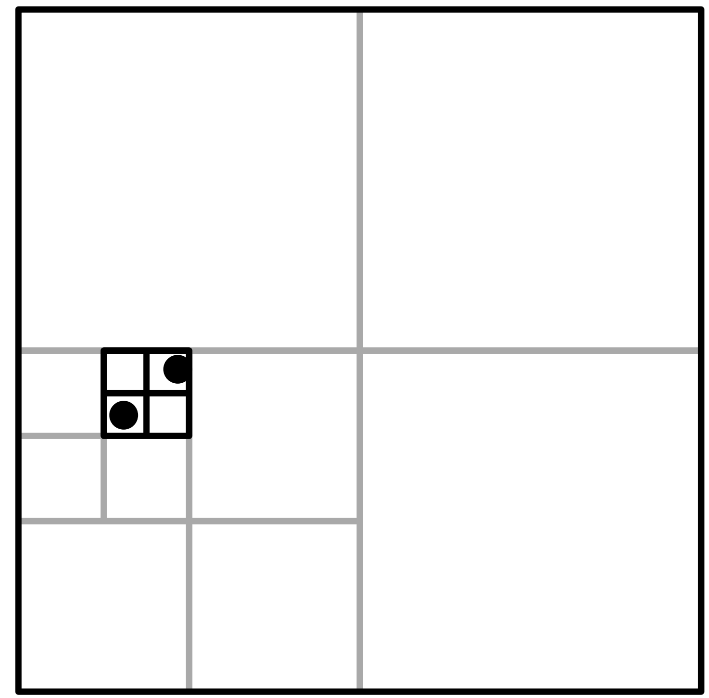

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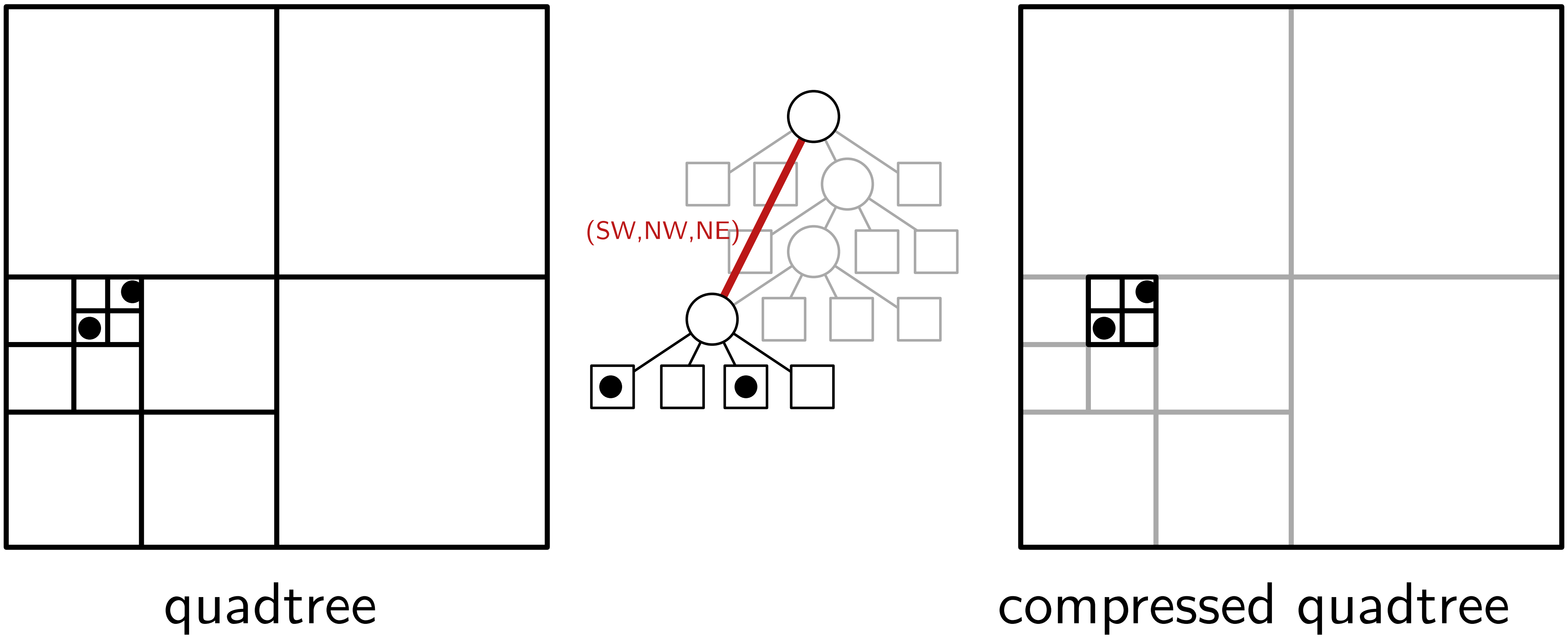
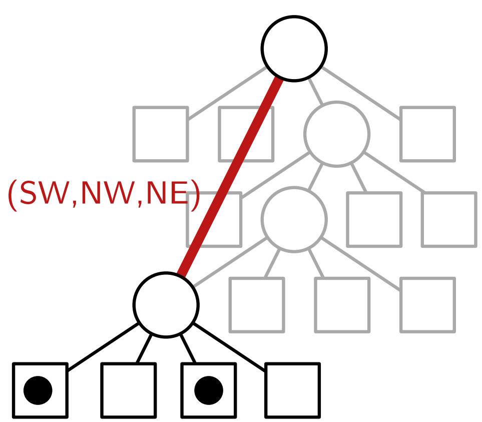
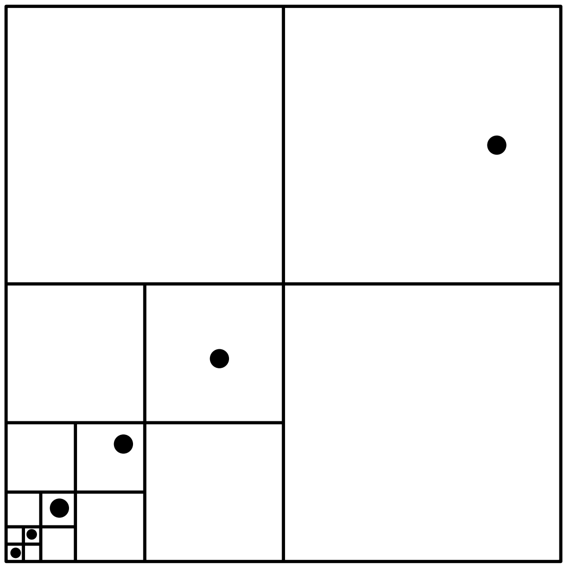


Properties of Compressed Quadtrees

Obs:

- inner nodes split their point set into ≥ 2 non-empty parts $\Rightarrow$ max. $n-1$ inner nodes
- depth can be $d = n$, so the algorithm to construct quadtrees takes $O(n^2)$ time





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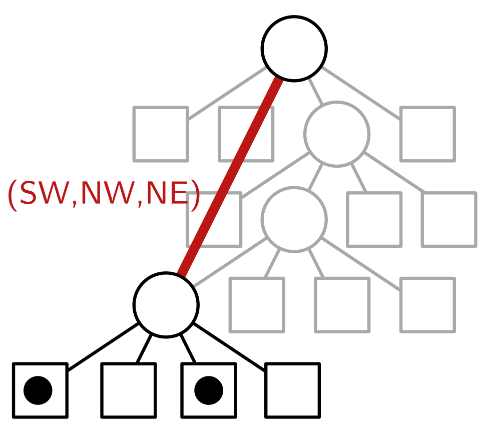
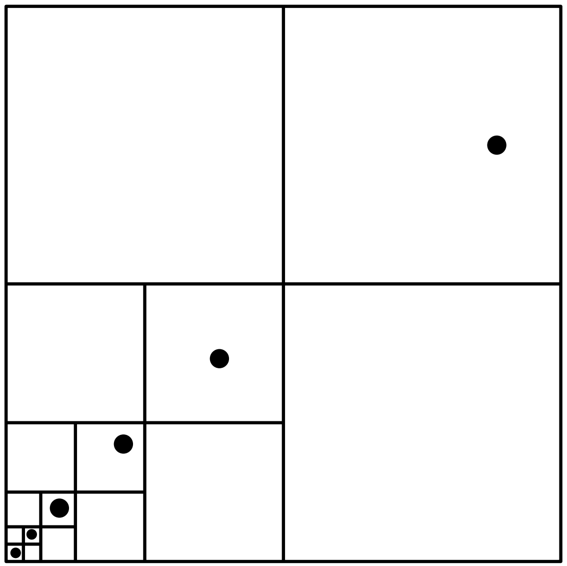
Thm 2: A compressed quadtree for n points in $\mathbb{R}^d$ with a fixed dimension d can be constructed in $O(n \log n)$ time.

e.g. skip-quadtree [Eppstein et al. 2005] (without proof)





Packing Lemma

Lemma 2: Let K be a ball with radius r in $\mathbb{R}^d$ and let X be a set of pairwise disjoint quadtree cells with side length $\geq x$ that intersect K. Then it holds

$$|X| \leq (1 + \lceil 2r/x \rceil)^d.$$

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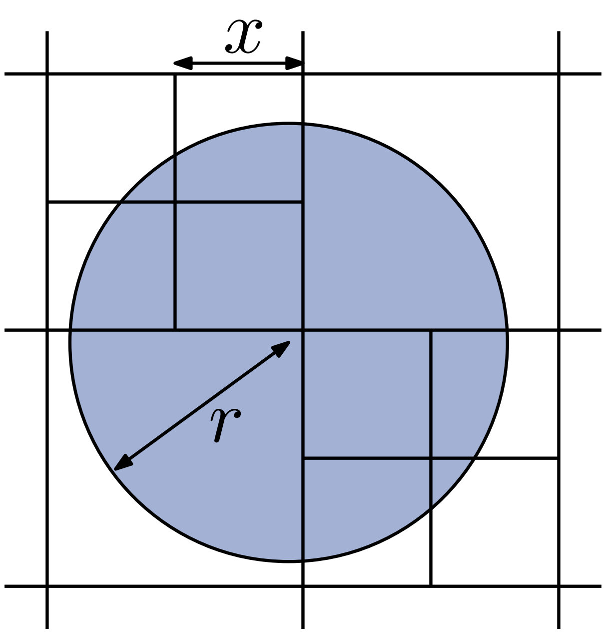

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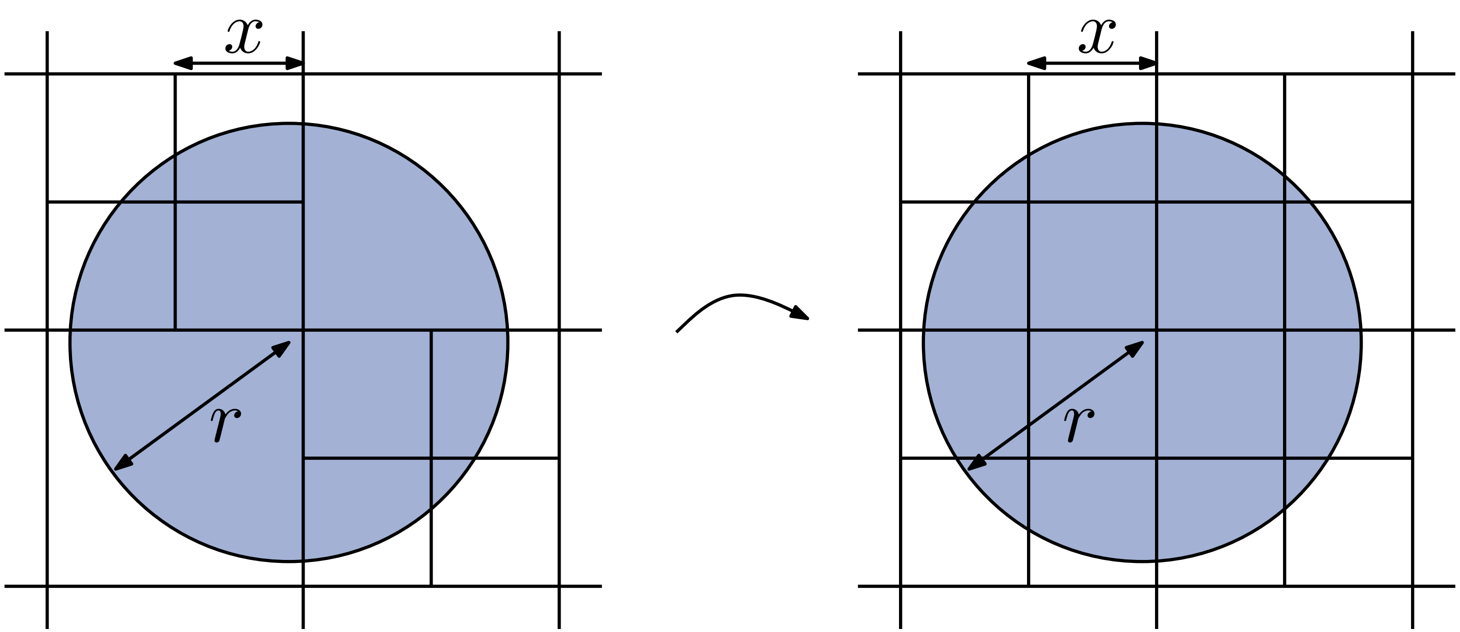

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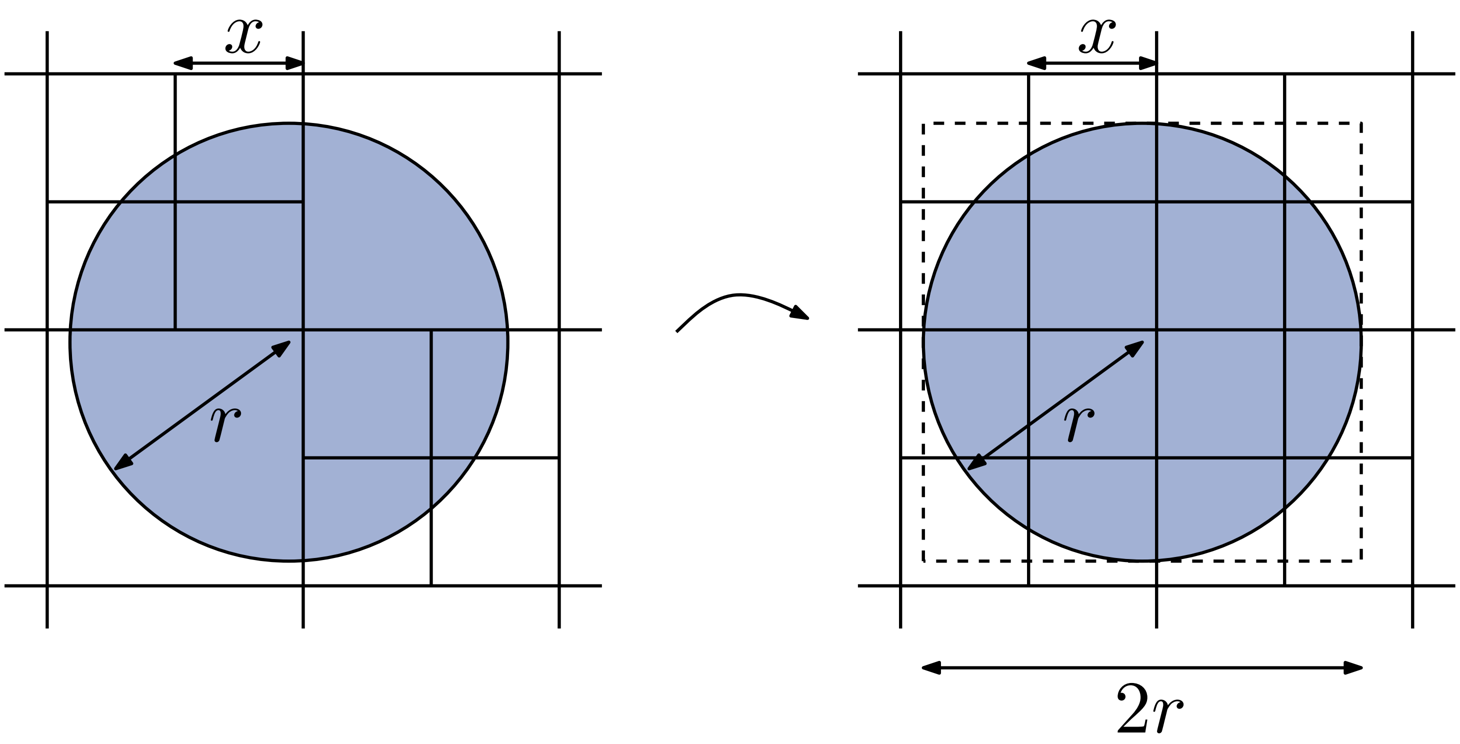

Representatives and Level

Def: For each node u of a quadtree $\mathcal{T}(P)$ for point set P let $P_u = Q_u \cap P$ be the set of points in the corresponding square Q_u. In each leaf u define the representative

$$\text{rep}(u) = \begin{cases} p & \text{falls } P_u = \{p\} \ (u \text{ is leaf}) \\ \emptyset & \text{otherwise.} \end{cases}$$

For an inner node v assign $\text{rep}(v) = \text{rep}(u)$ for a non-empty child u of v.

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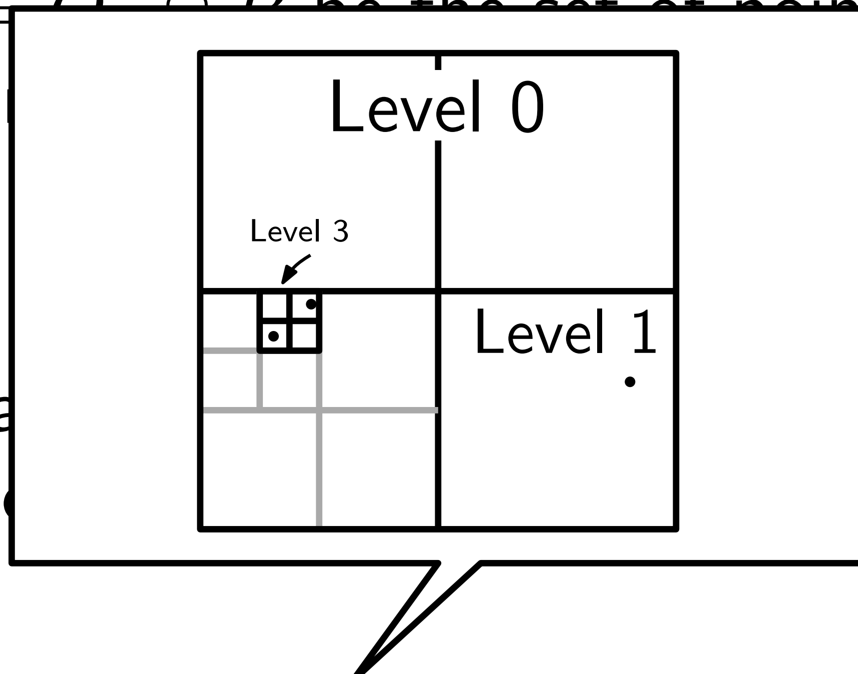

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Constructing a WSPD

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Input: quadtree nodes u, v, quadtree $\mathcal{T}$, $s > 0$
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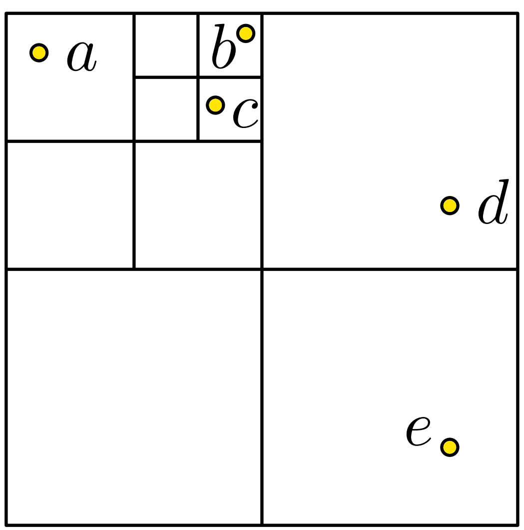

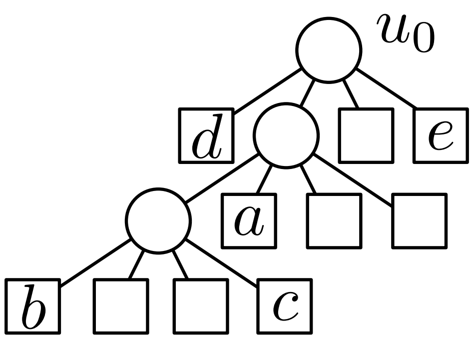

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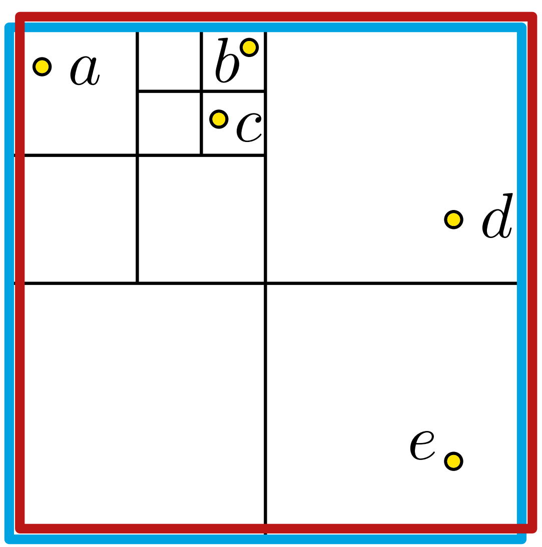

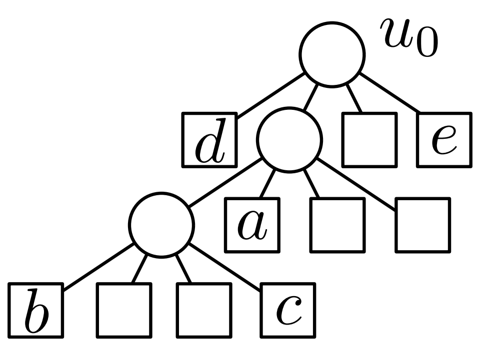

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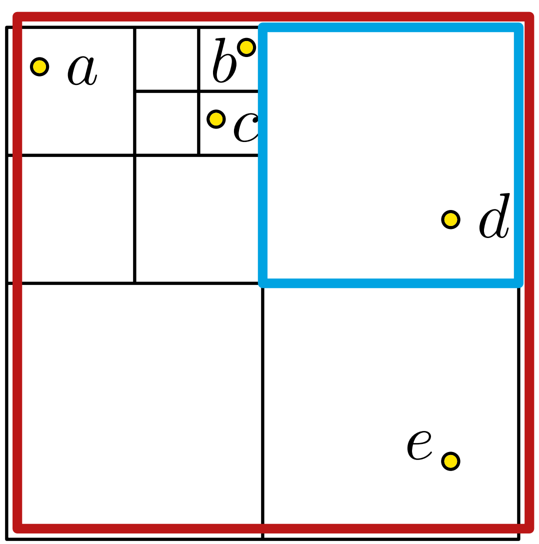

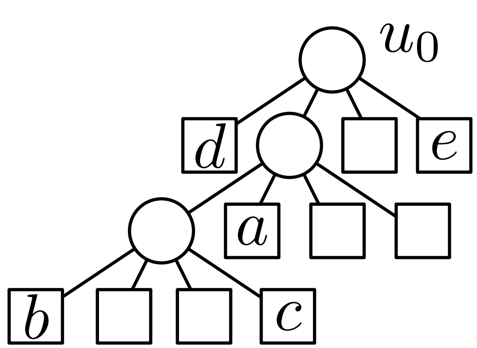

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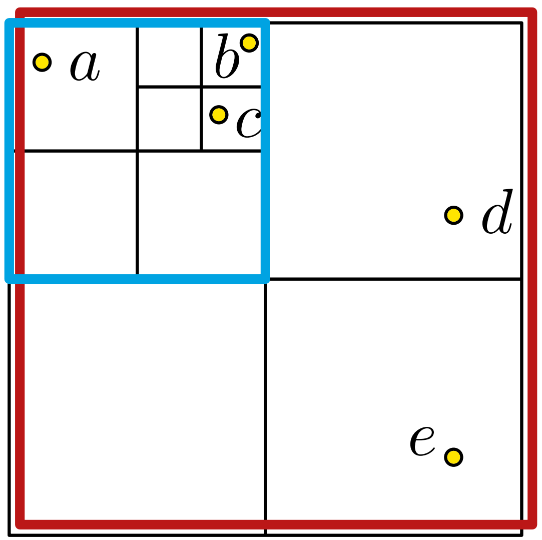

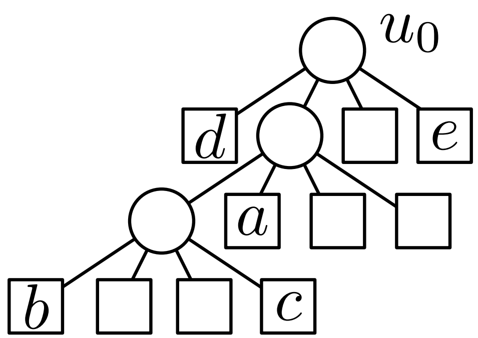

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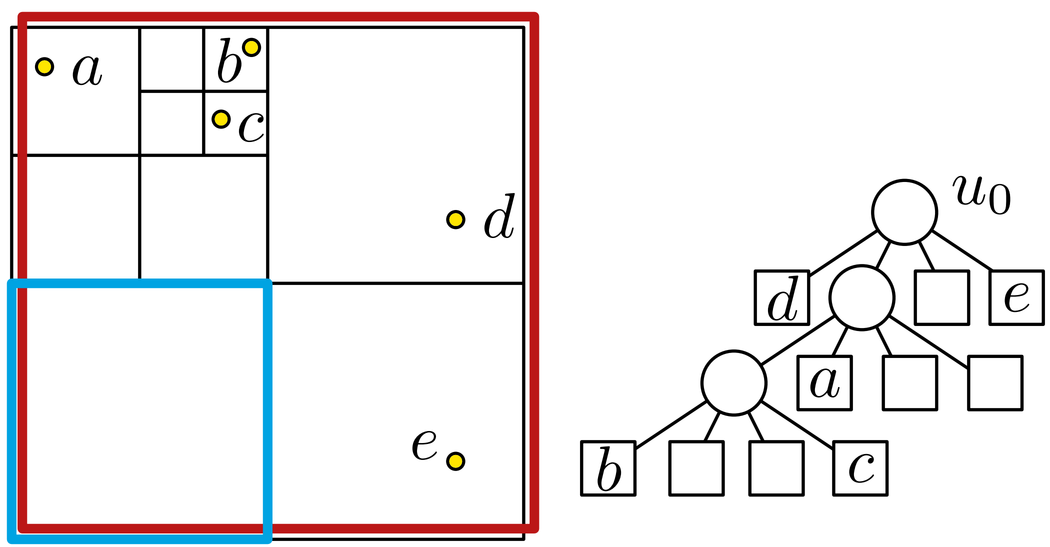

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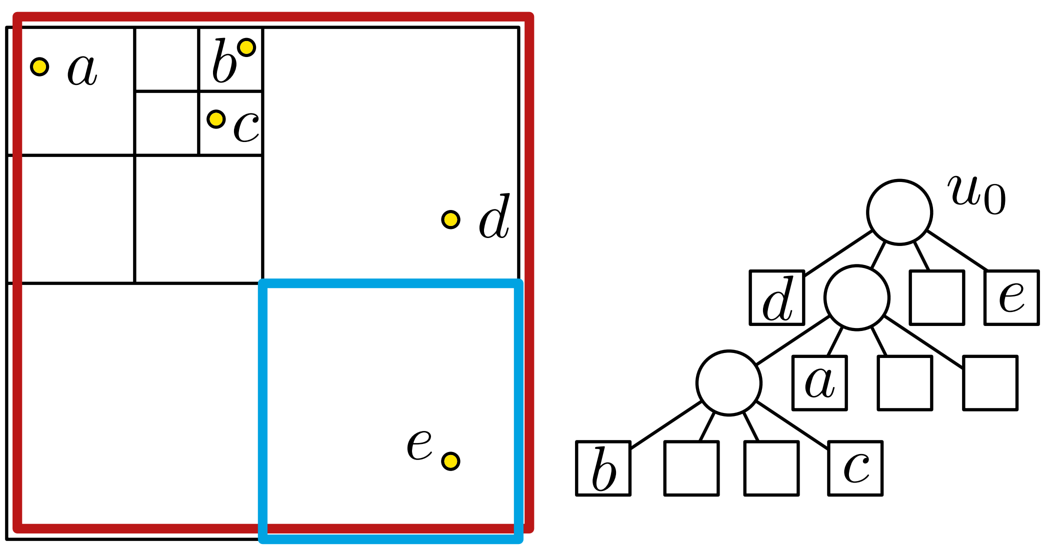

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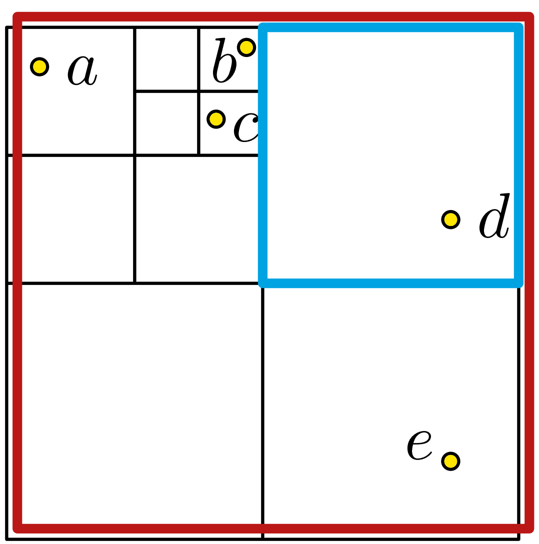

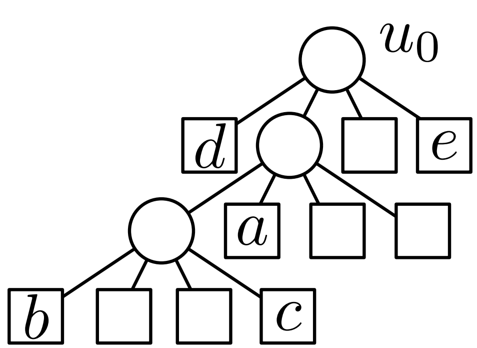

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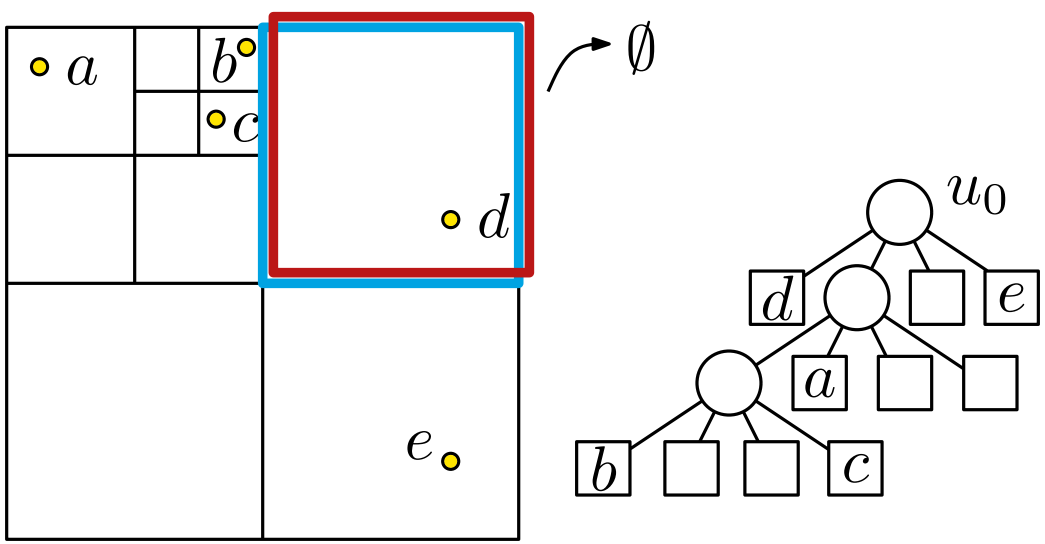

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Output: WS

circles around Q_u and Q_v (or radius 0 for point in a leaf)

increase smaller circle and check if distance $\geq sr$

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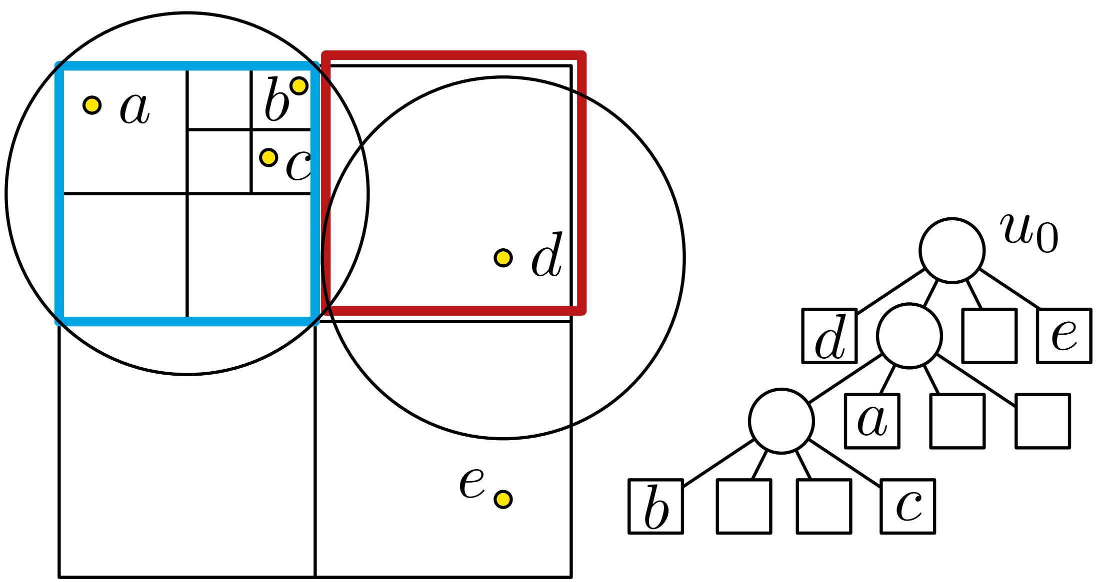

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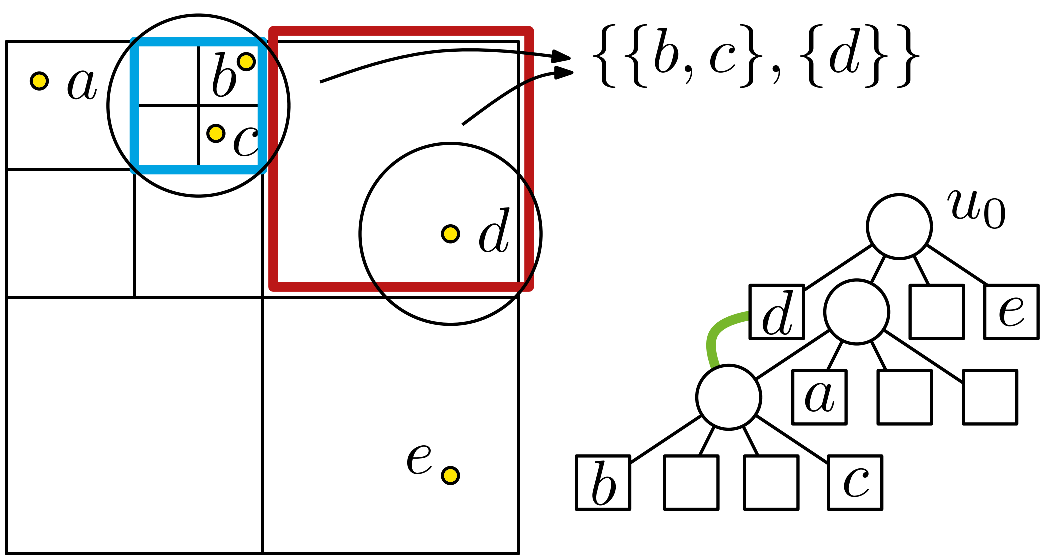

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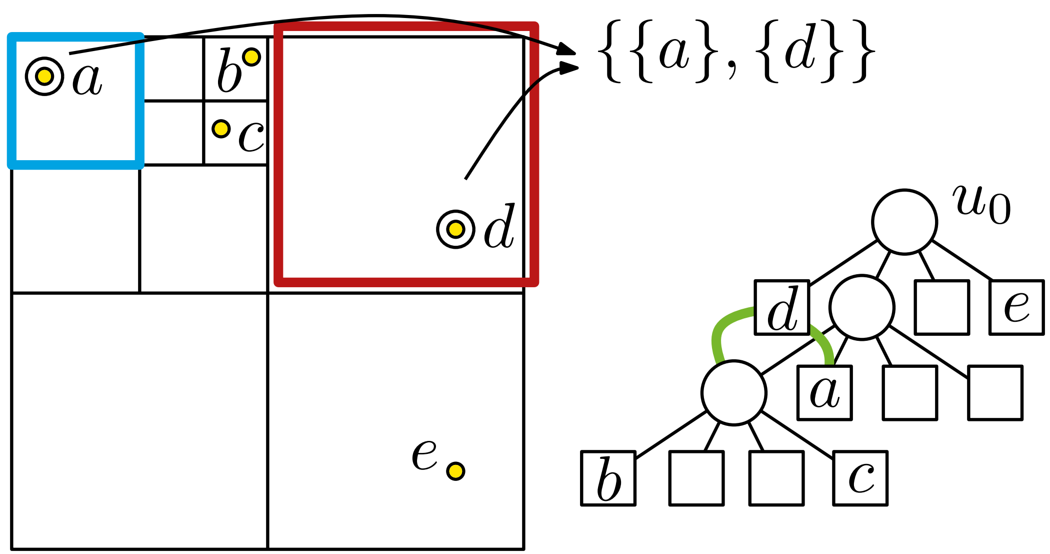

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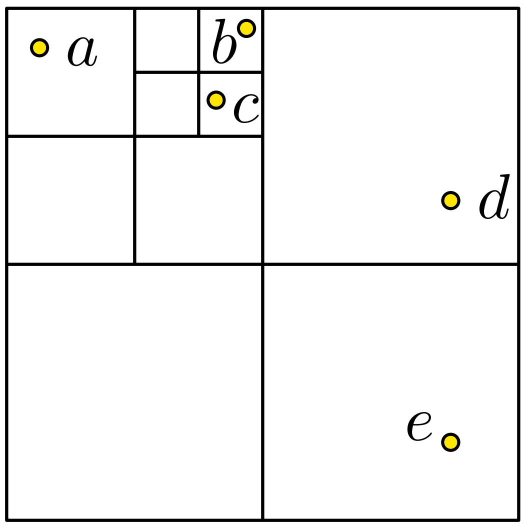

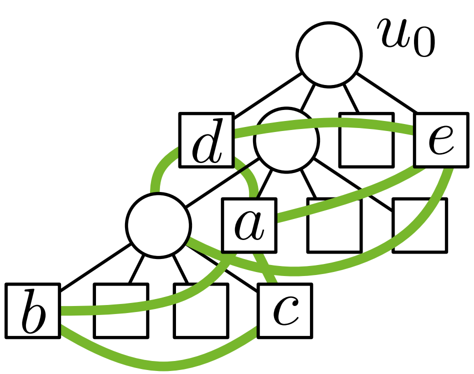

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Karlsruhe Institute of Technology

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Question: How many pairs does the algorithm create?

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Thm 3: Given a point set P in $\mathbb{R}^d$ and $s \geq 1$ we can construct an s-WSPD with $O(s^d n)$ pairs in time $O(n \log n + s^d n)$.

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goal: each quadtree node has cost $O(s^d)$

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- u, v not ws $\Rightarrow$ ball distance $\leq s \max\{r_u, r_v\} \leq 2sr_v = sx\sqrt{d}$

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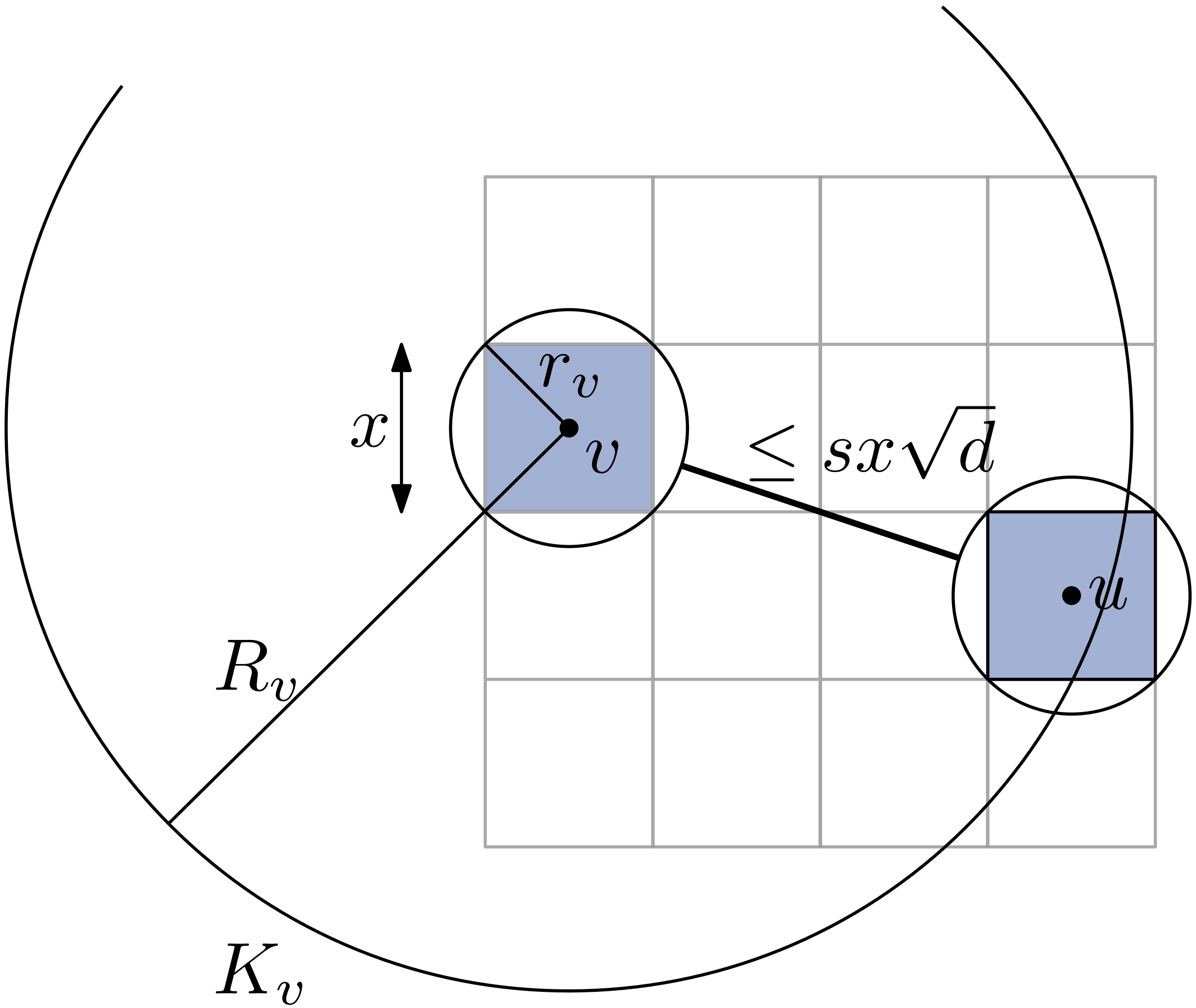

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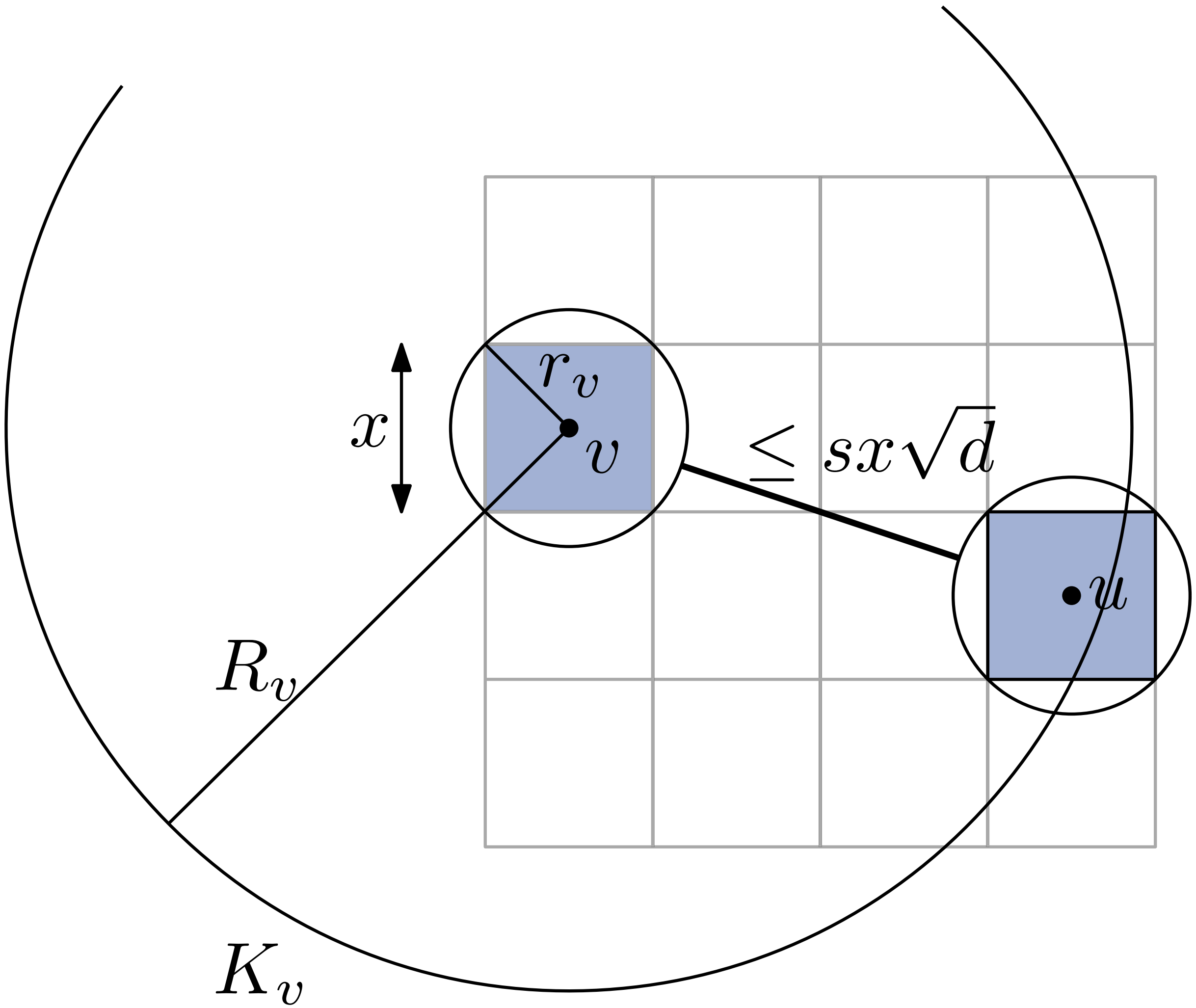

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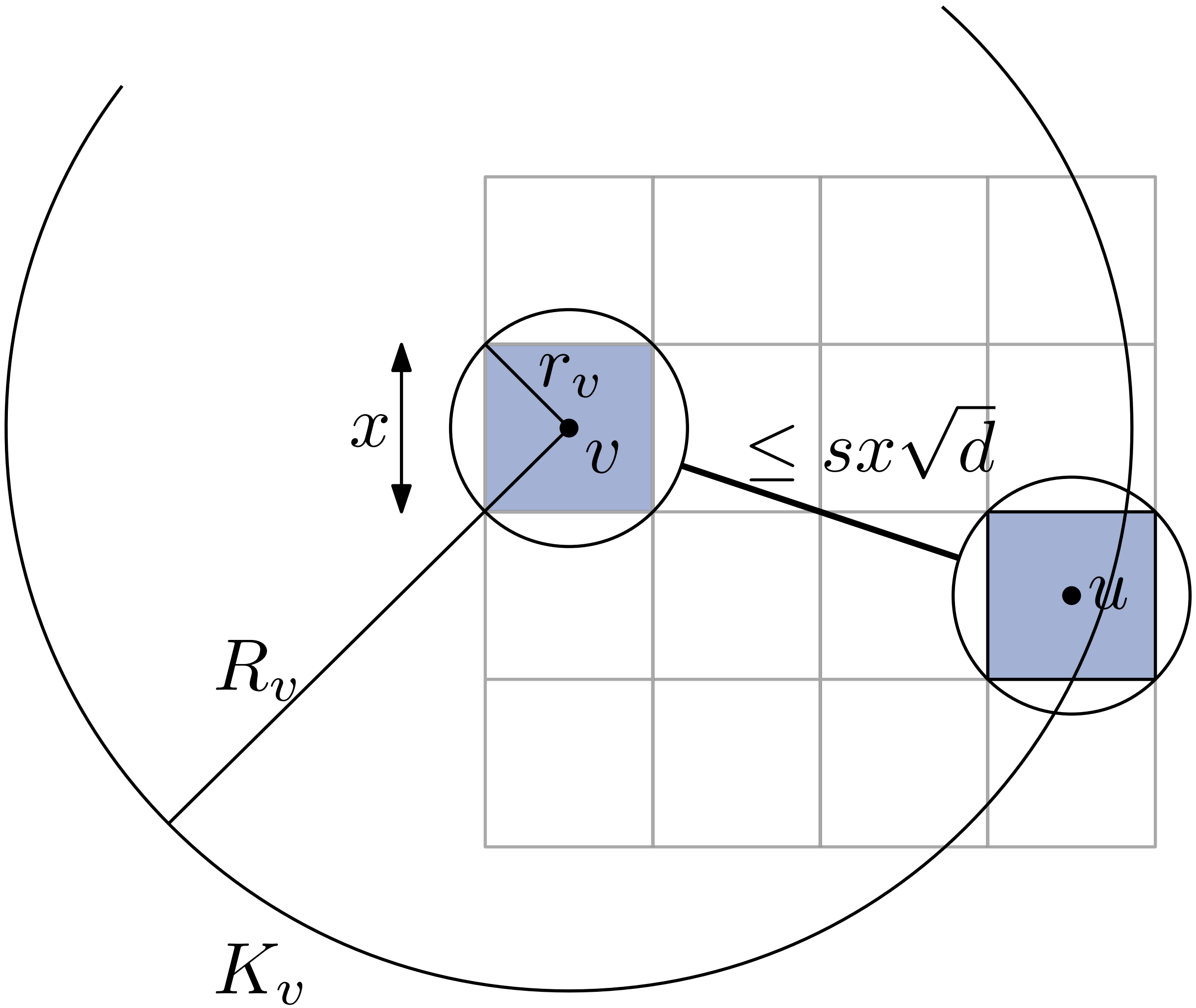

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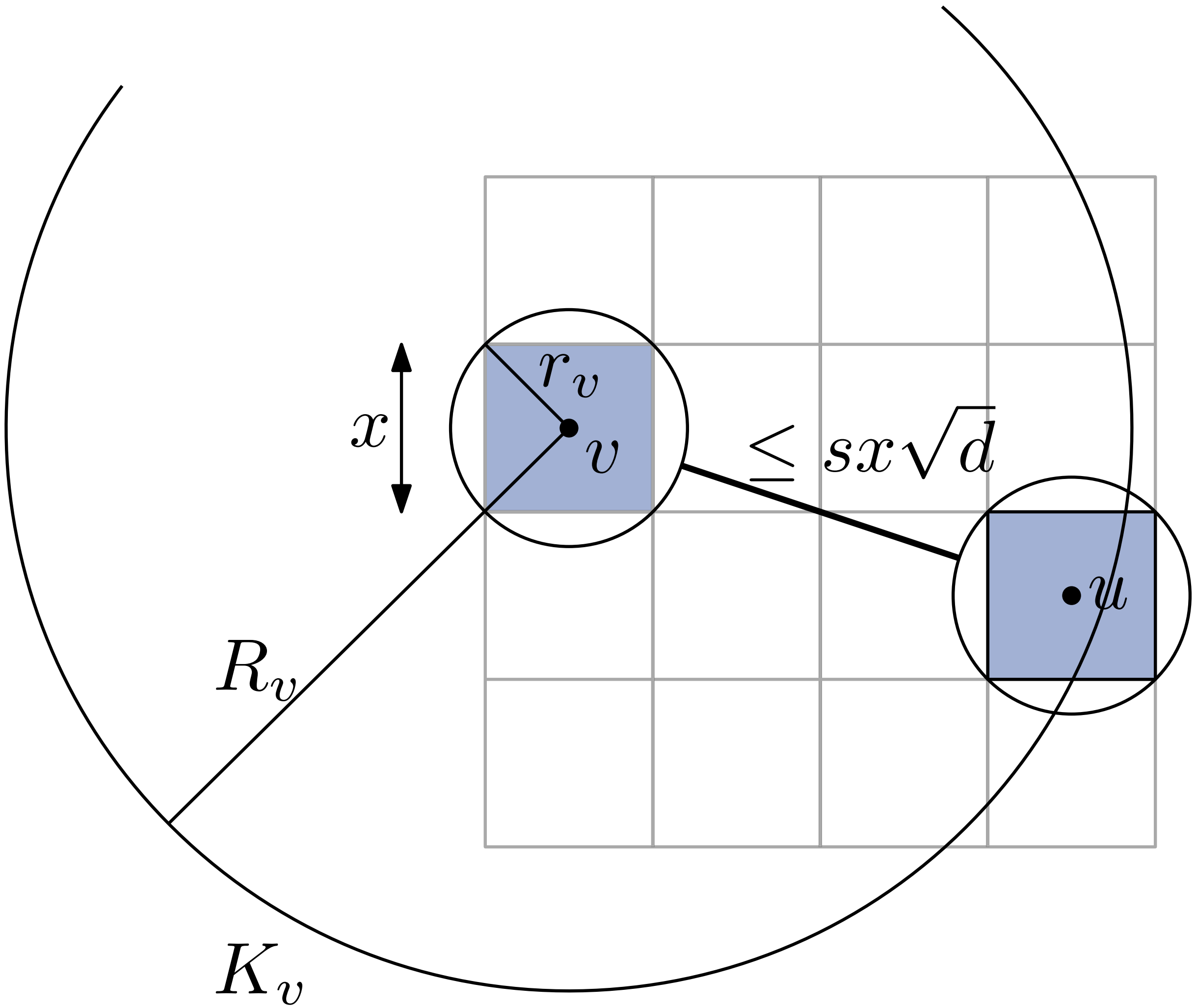

Recall Lemma 2:
Given ball K with radius r in $\mathbb{R}^d$ and set X of pairwise disjoint quadtree cells with side length $\geq x$ that intersect K. Then

$$|X| \leq (1 + \lceil 2r/x \rceil)^d.$$

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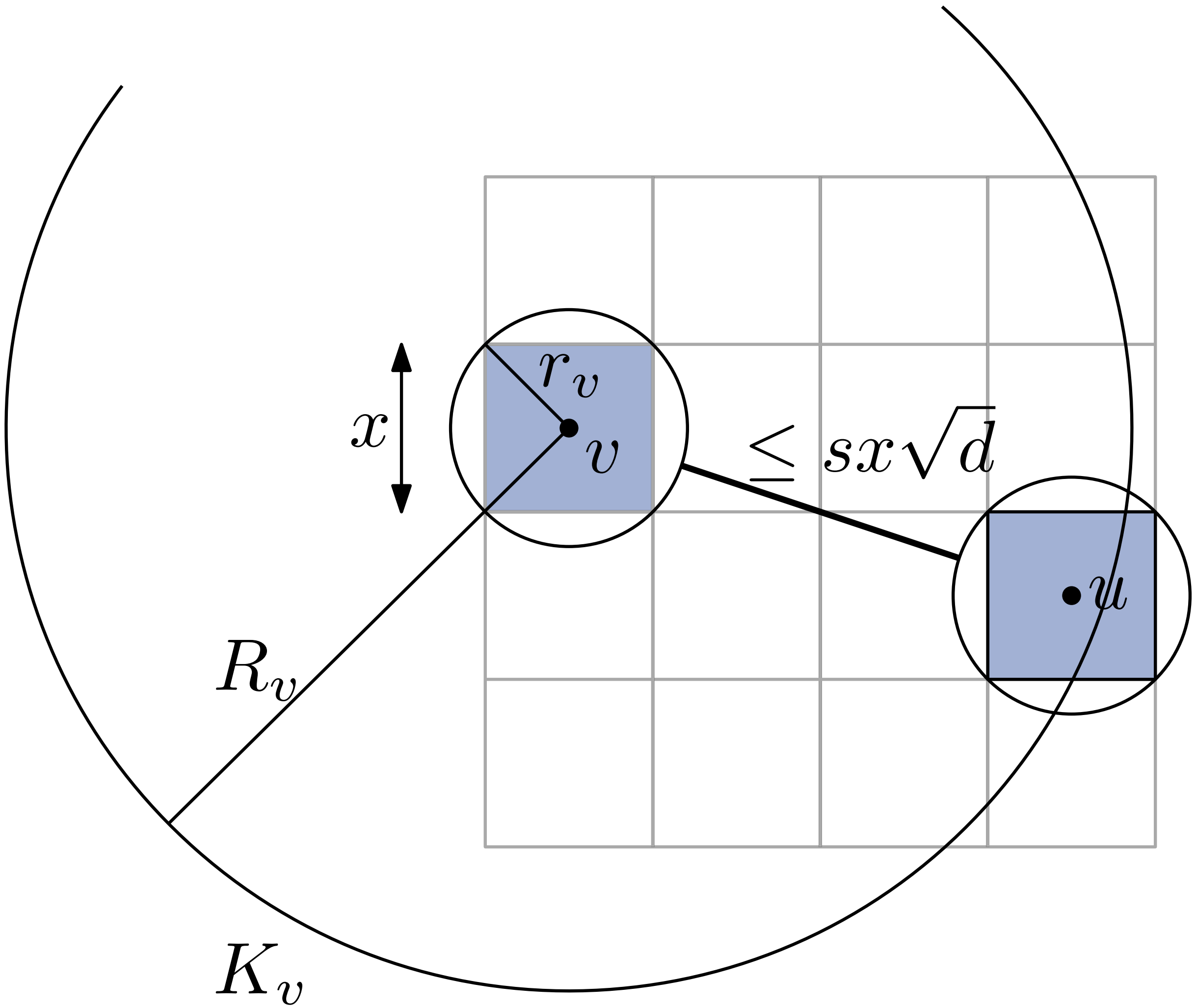

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- in total $O(s^d n)$ ws-pairs
- time: $O(n \log n)$ for quadtree and $O(s^d n)$ for the s-WSPD □

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Obs: each point pair $\{u, v\}$ is represented by exactly one ws-pair $\{A_i, B_i\}$ in this WSPD

t-Spanner

For a set P of n points in $\mathbb{R}^d$ the **Euclidean graph** $\mathcal{EG}(P) = (P, \binom{P}{2})$ is the complete weighted graph, whose edge weights correspond to the Euclidean distances of the edges' endpoints.

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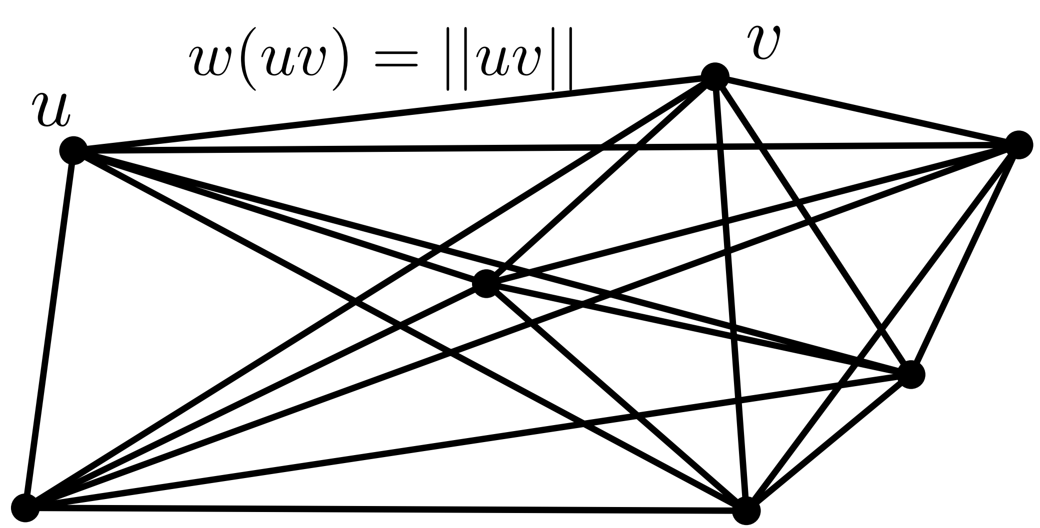

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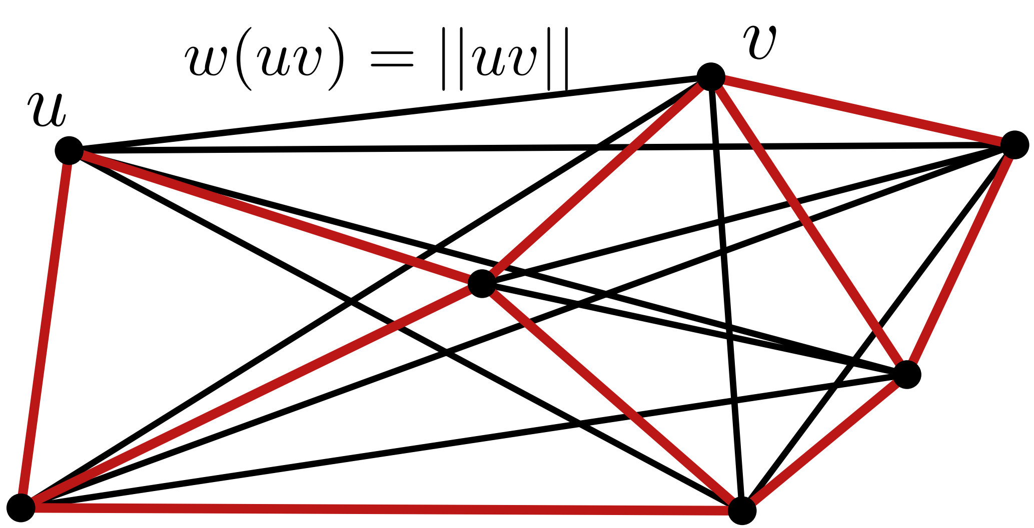

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Def: A weighted graph G with vertex set P is called t-**spanner** for P and a stretch factor $t \geq 1$, if for all pairs $x, y \in P$ it holds

$$||xy|| \leq \delta_G(x, y) \leq t \cdot ||xy||,$$

where $\delta_G(x, y) =$ length of shortest x-y-path in G.

WSPD und t-Spanner

Def: For n points P in $\mathbb{R}^d$ and a WSPD W of P define the graph $G = (P, E)$, where
$E = \{\{x, y\} \mid \exists\{u, v\} \in W$ with $\text{rep}(u) = x, \text{rep}(v) = y\}$.

Recall: For each node u of a quadtree $\mathcal{T}(P)$ for point set P let $P_u = Q_u \cap P$ be the set of points in the corresponding square Q_u.
In each leaf u define the representative

$$\text{rep}(u) = \begin{cases} p & \text{falls } P_u = \{p\} \ (u \text{ is leaf}) \\ \emptyset & \text{otherwise.} \end{cases}$$

For inner node v assign $\text{rep}(v) = \text{rep}(u)$ for non-empty child u of v.

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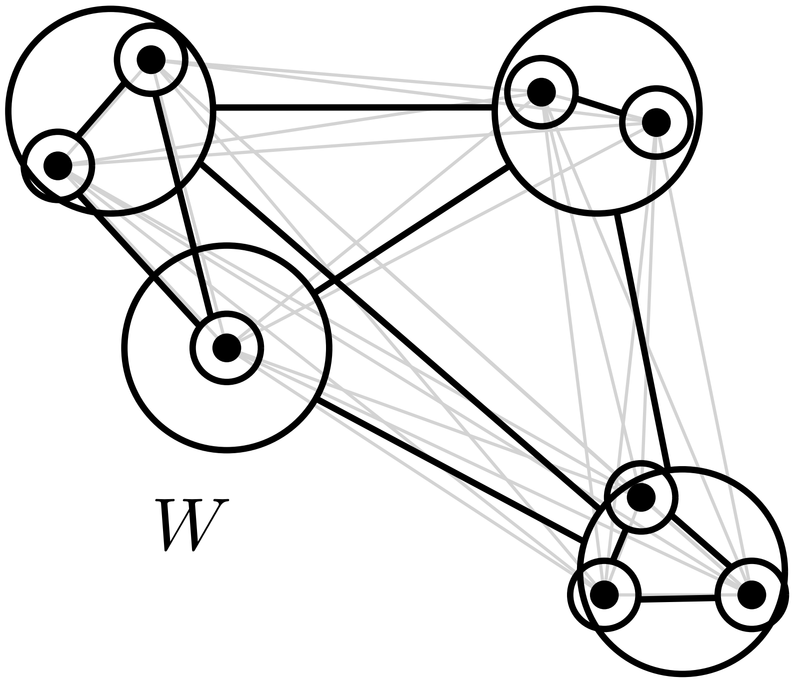
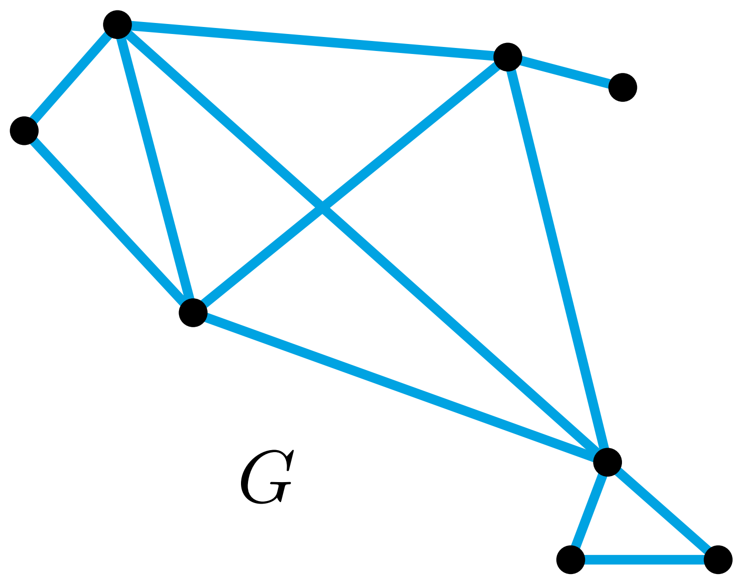


W



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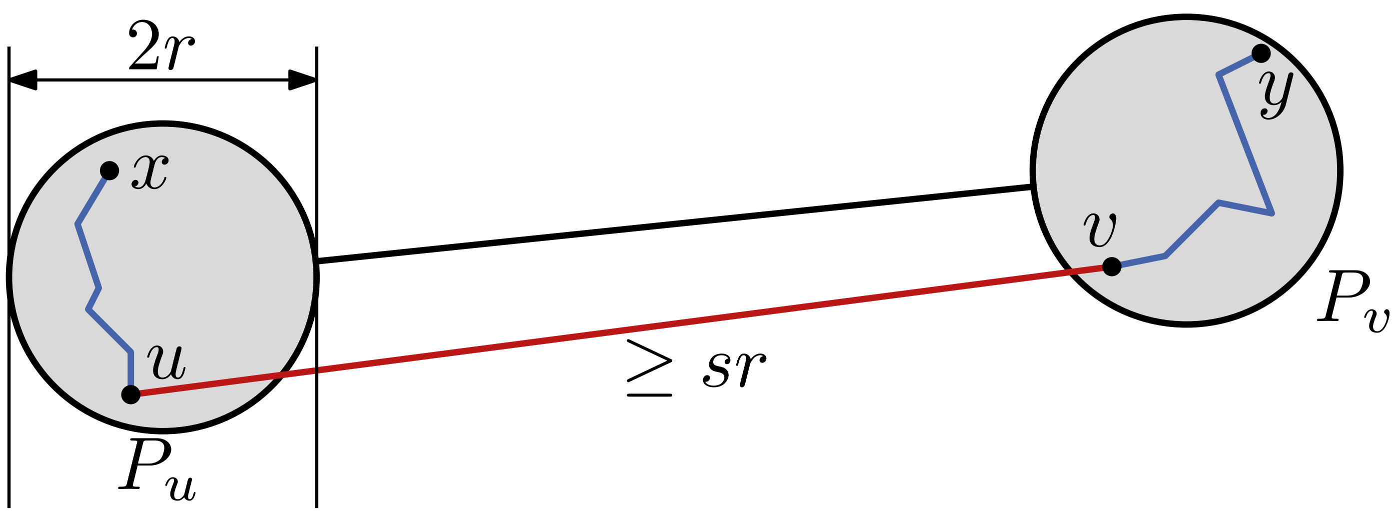
Proof: (blackboard)



Summary

Thm 4: For a set P of n points in $\mathbb{R}^d$ and some $\varepsilon \in (0,1]$ we can compute an $(1+\varepsilon)$-spanner for P with $O(n/\varepsilon^d)$ edges in $O(n \log n + n/\varepsilon^d)$ time.

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Proof: For $t = (1+\varepsilon)$ we have with $s = 4 \cdot \frac{t+1}{t-1}$ that

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P

$\downarrow$ $\quad O(n \log n)$

compressed quadtree

$\downarrow$ $\quad O(n/\varepsilon^d)$

WSPD

$\downarrow$ $\quad O(n/\varepsilon^d)$

$(1+\varepsilon)$-spanner

$\square$