Karlsruhe Institute of Technology

# Computational Geometry • Lecture

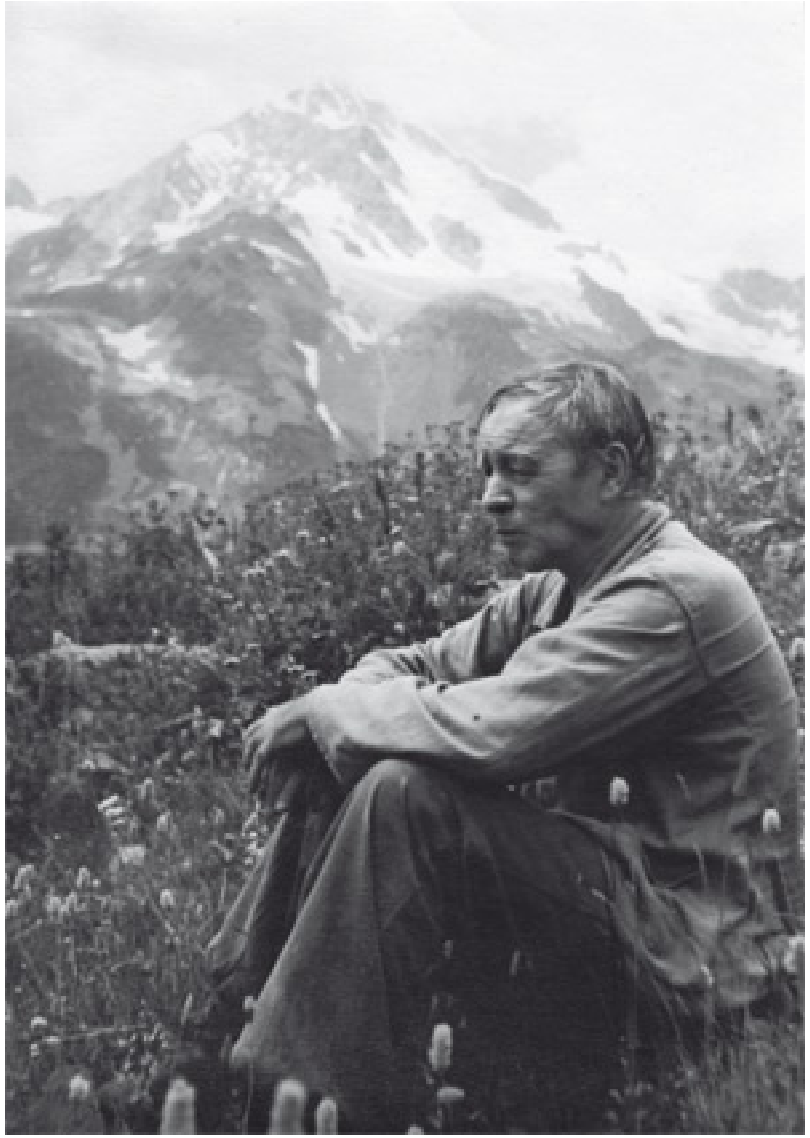
## Delaunay Triangulation

INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

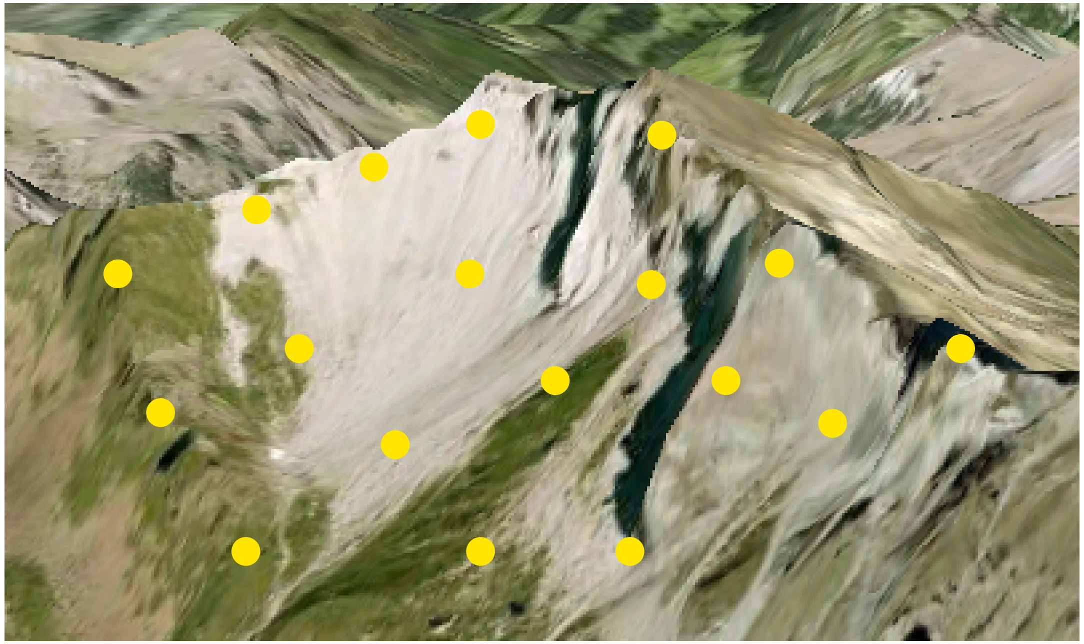
Tamara Mchedlidze · Darren Strash

7.12.2015

# Delaunay



# Modelling a Terrain



Sample points
$p = (x_p, y_p, z_p)$

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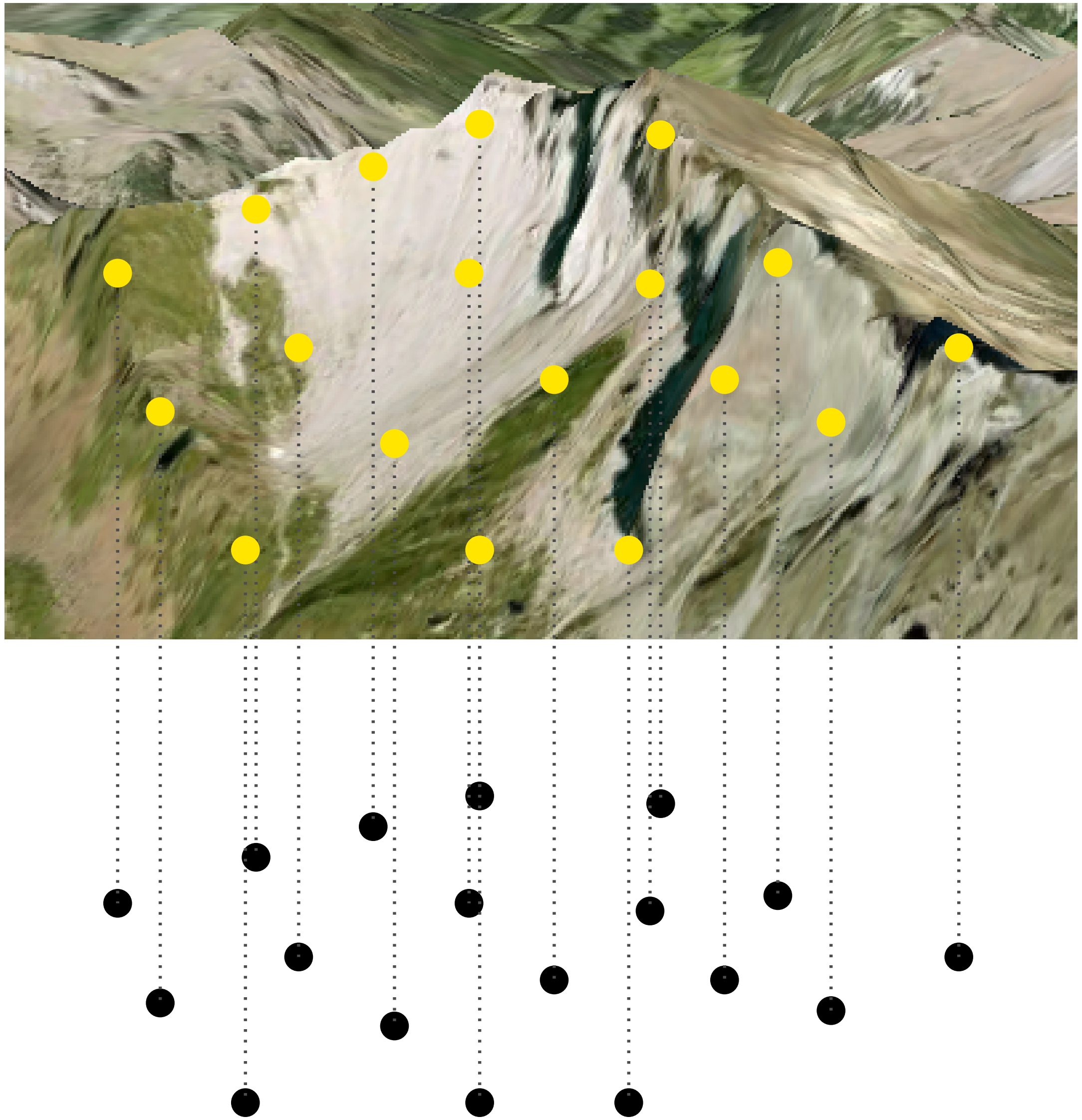

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↓

Projection
$\pi(p) = (p_x, p_y, 0)$

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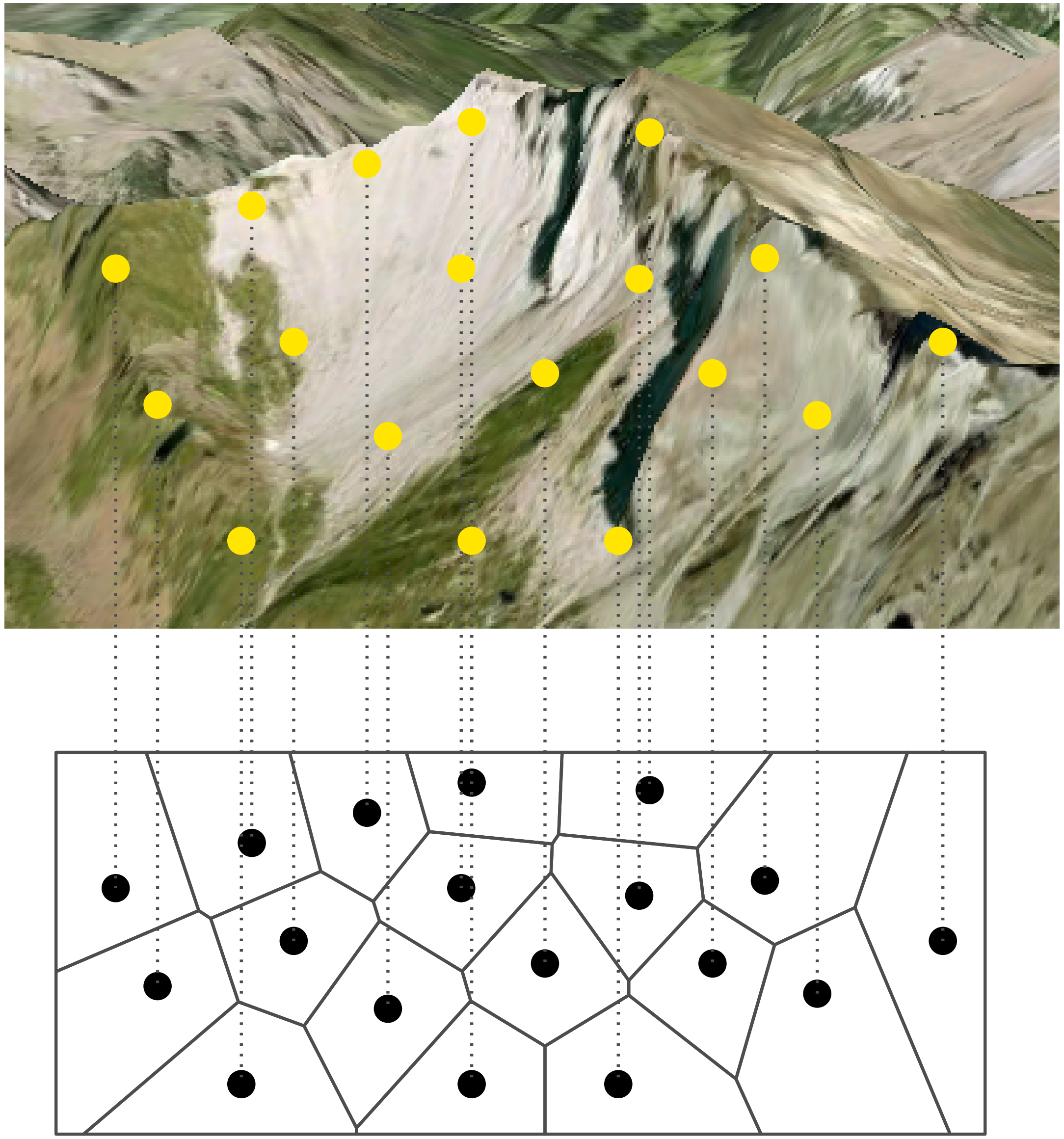

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Interpolation 1: each point gets the height of the nearest sample point

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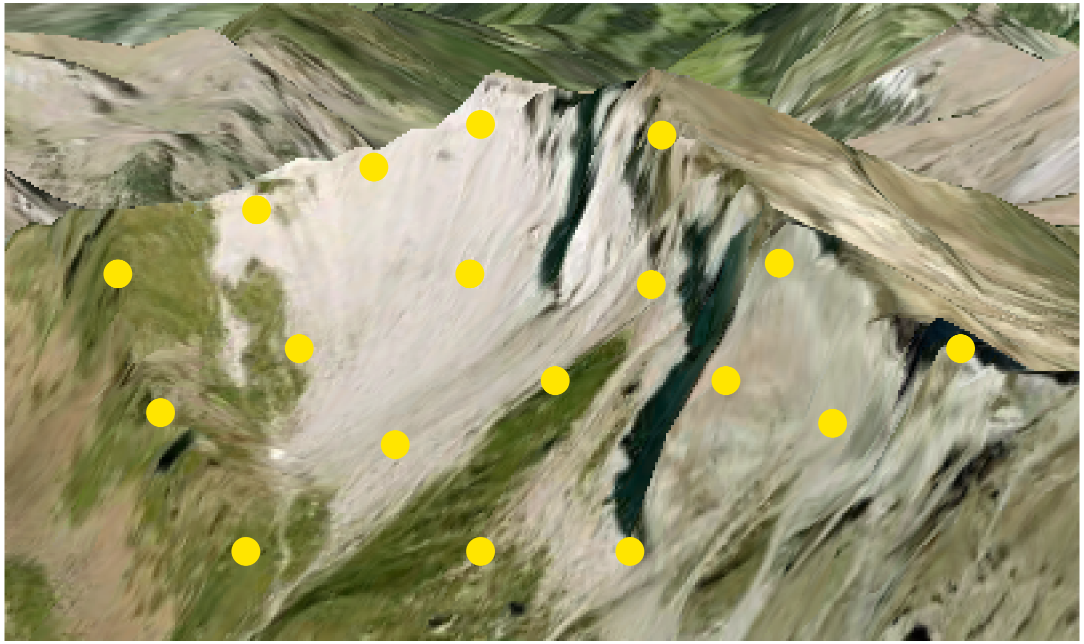

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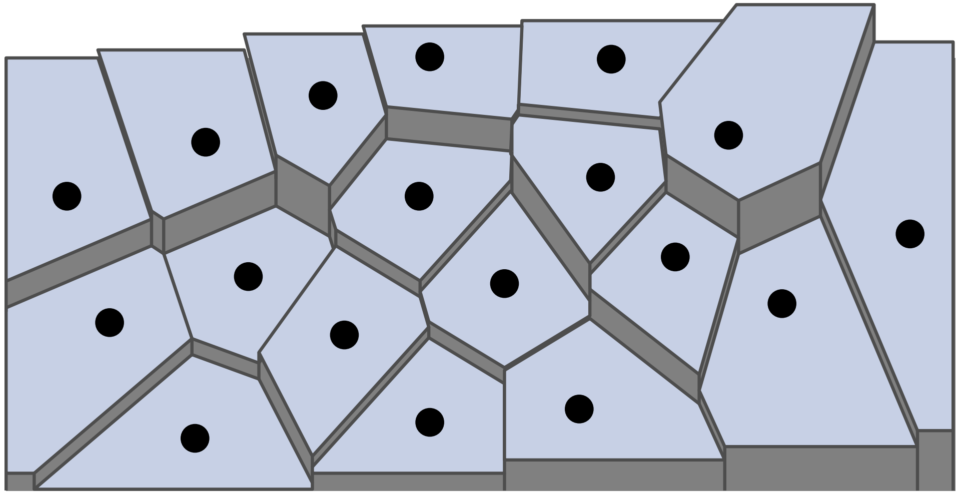

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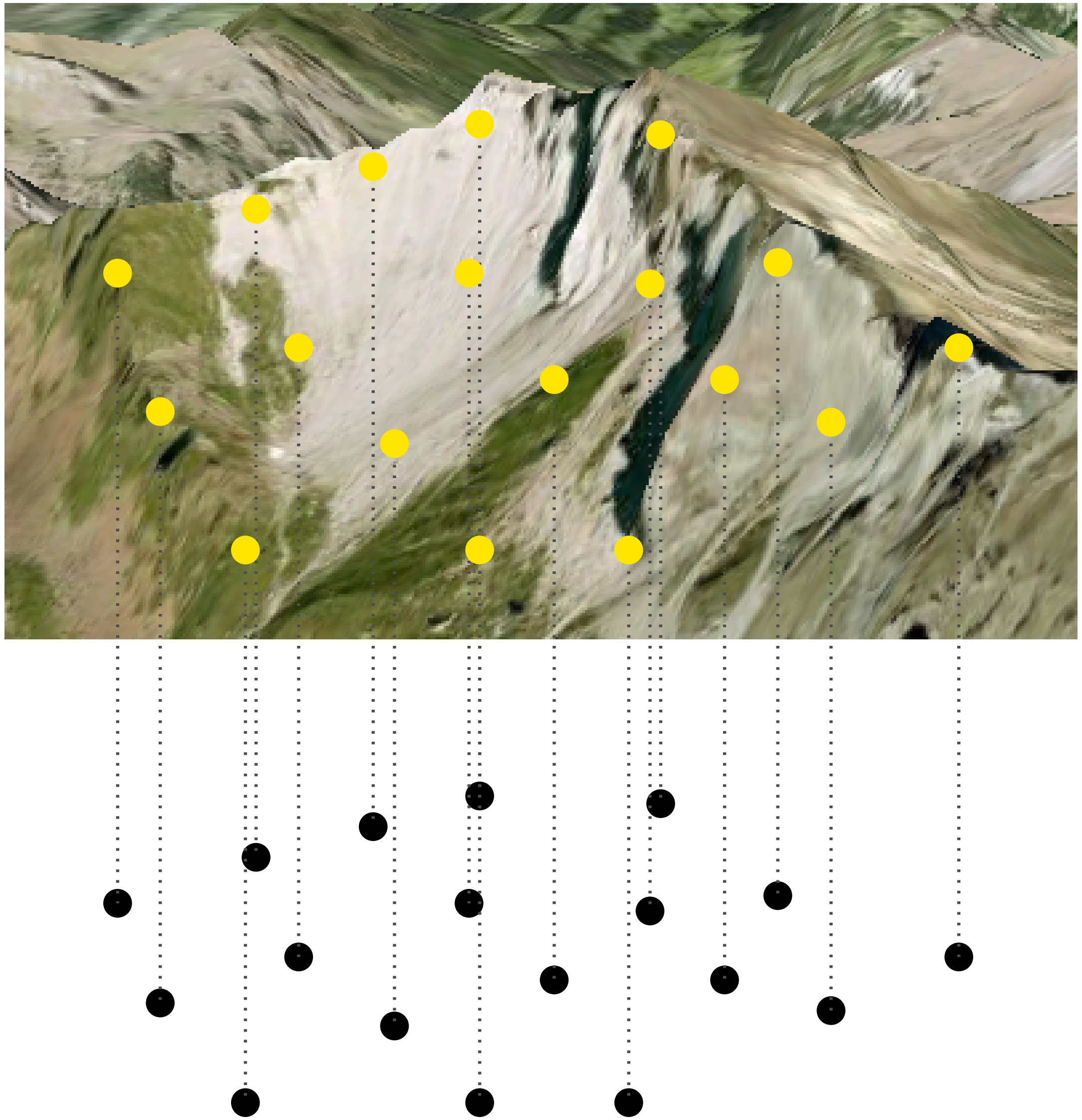

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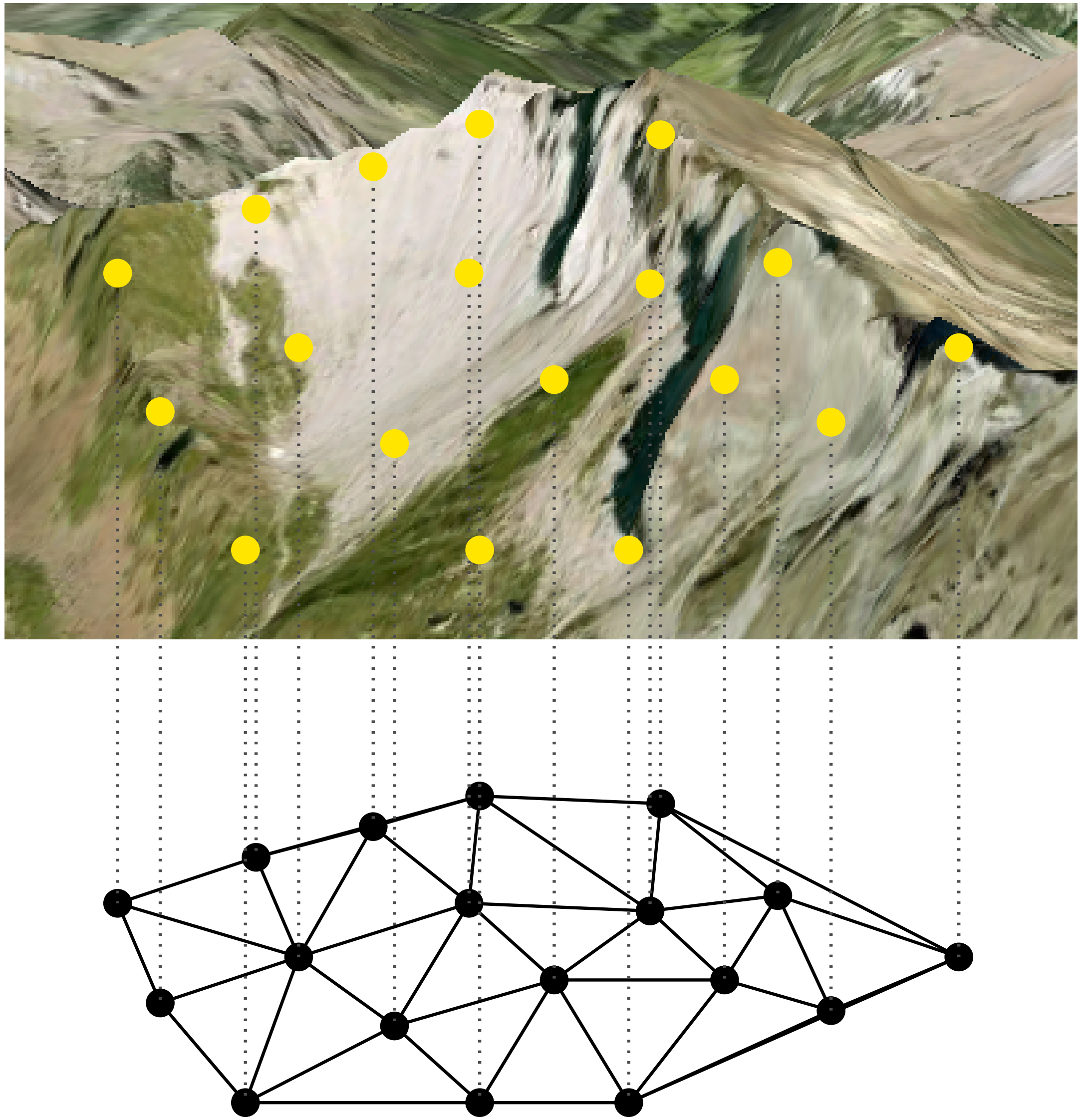

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Interpolation 2: triangulate the set of sample points and interpolate on the triangles

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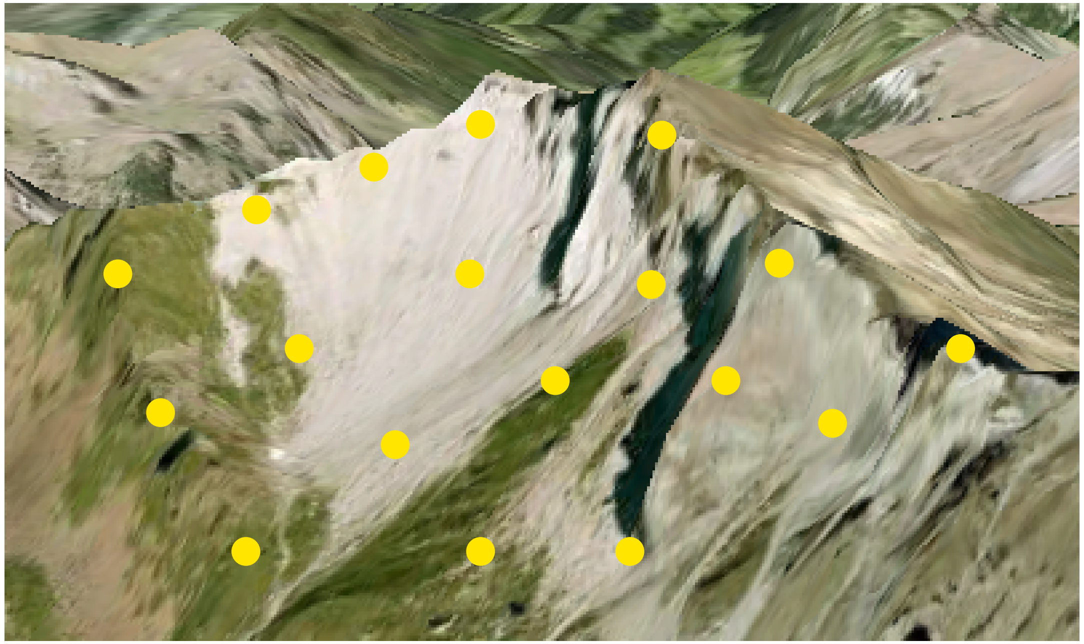

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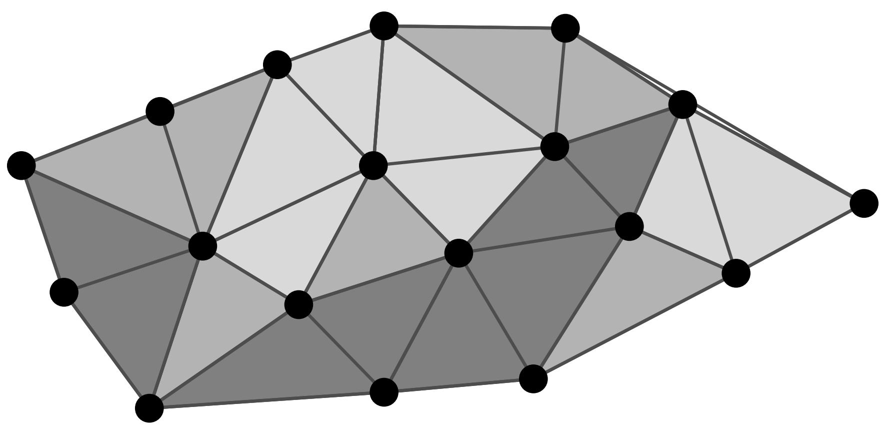

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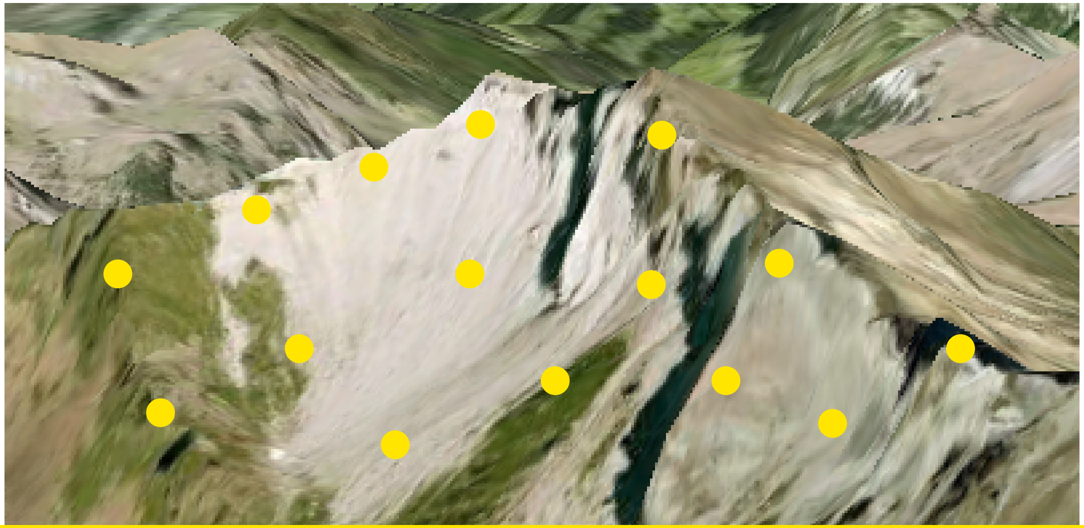

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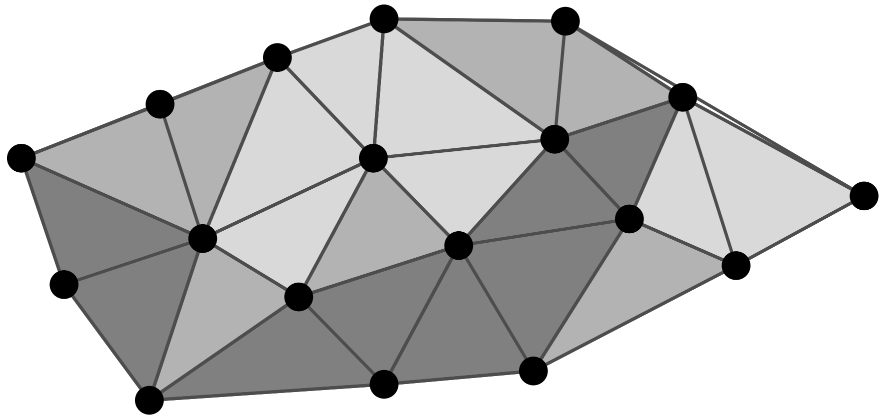
What a good triangulation looks like?



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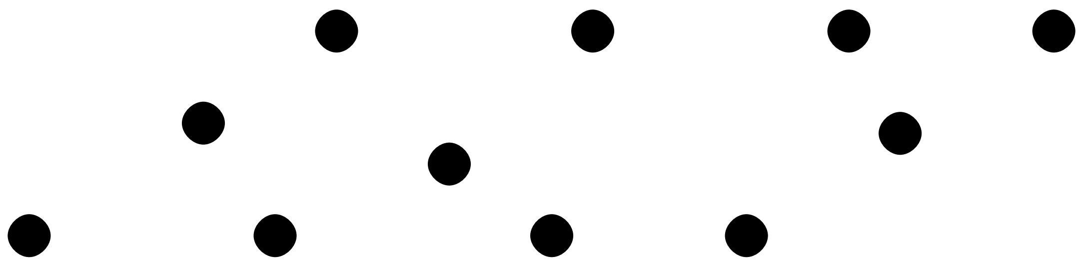
# Triangulation of a Point Set

**Def.:** A **triangulation** of a point set $P \subset \mathbb{R}^2$ is a maximal planar subdivision with a vertex set $P$.



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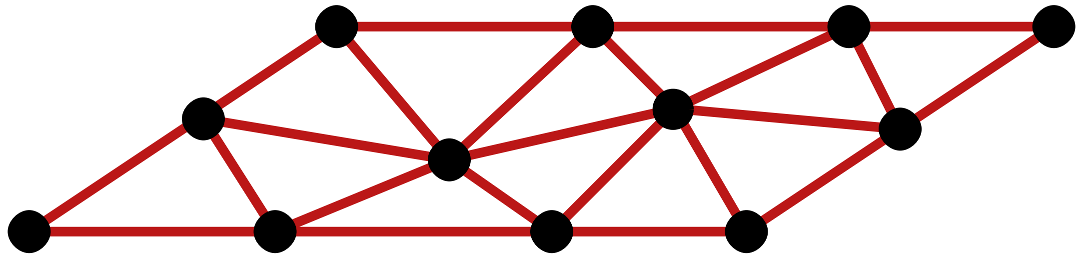

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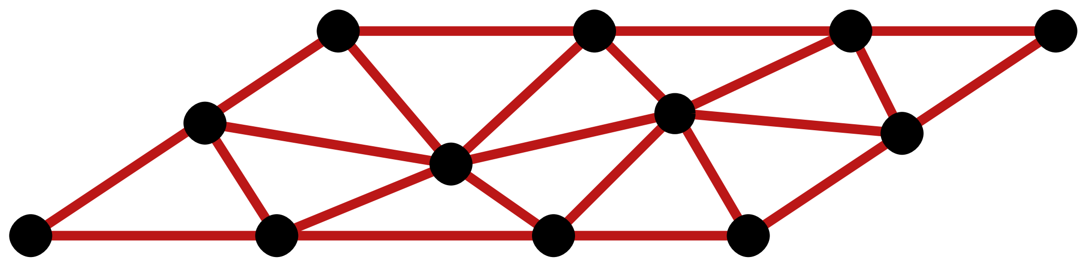

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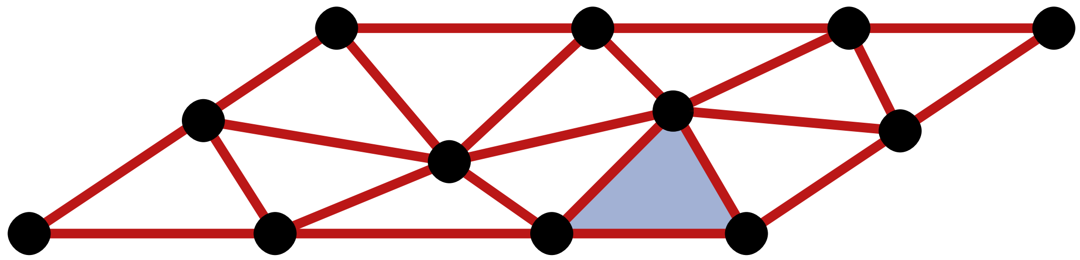

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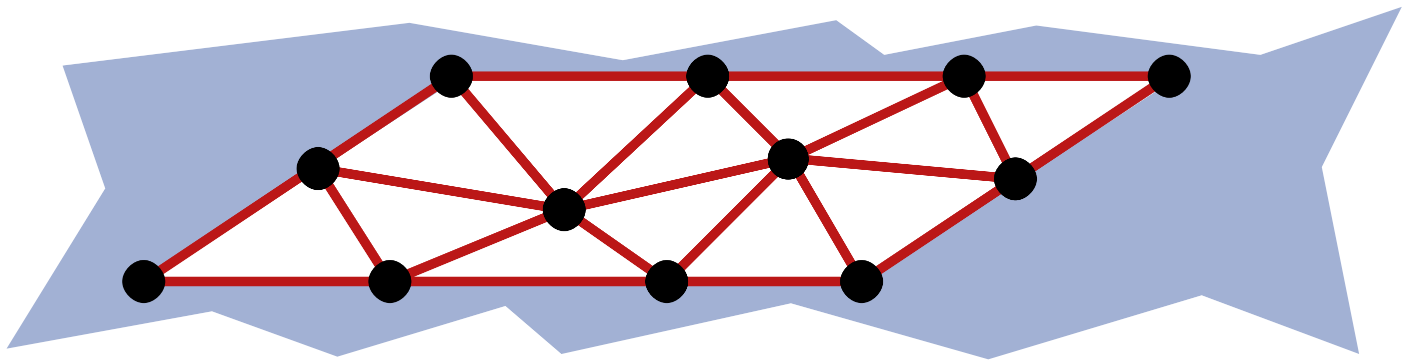

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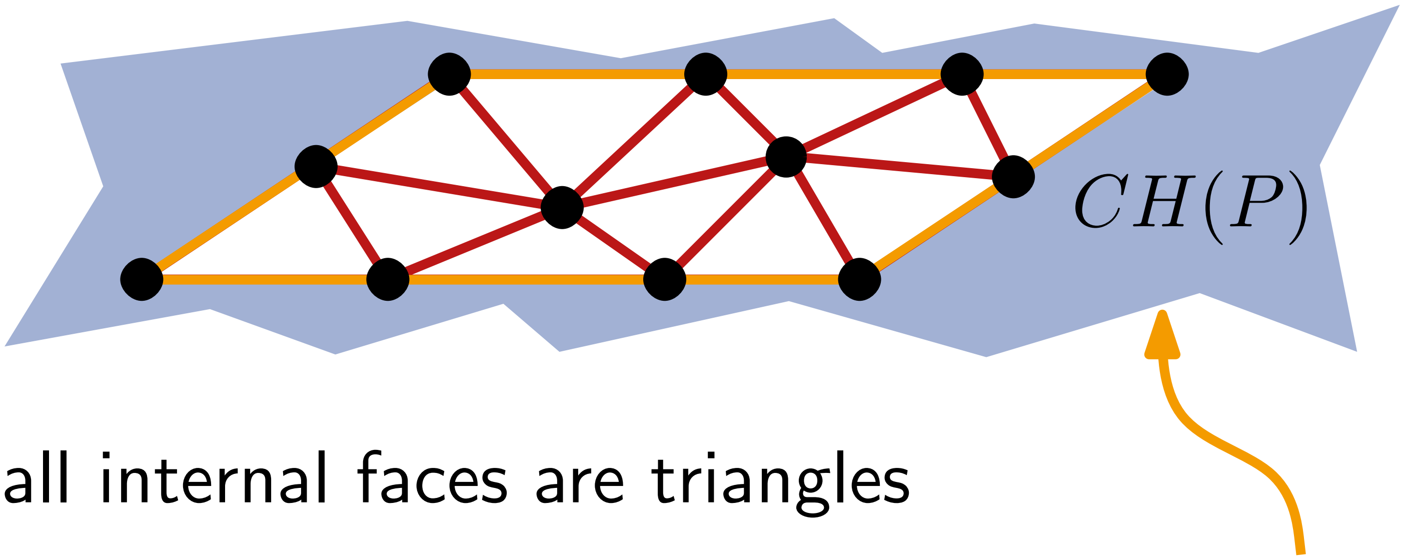


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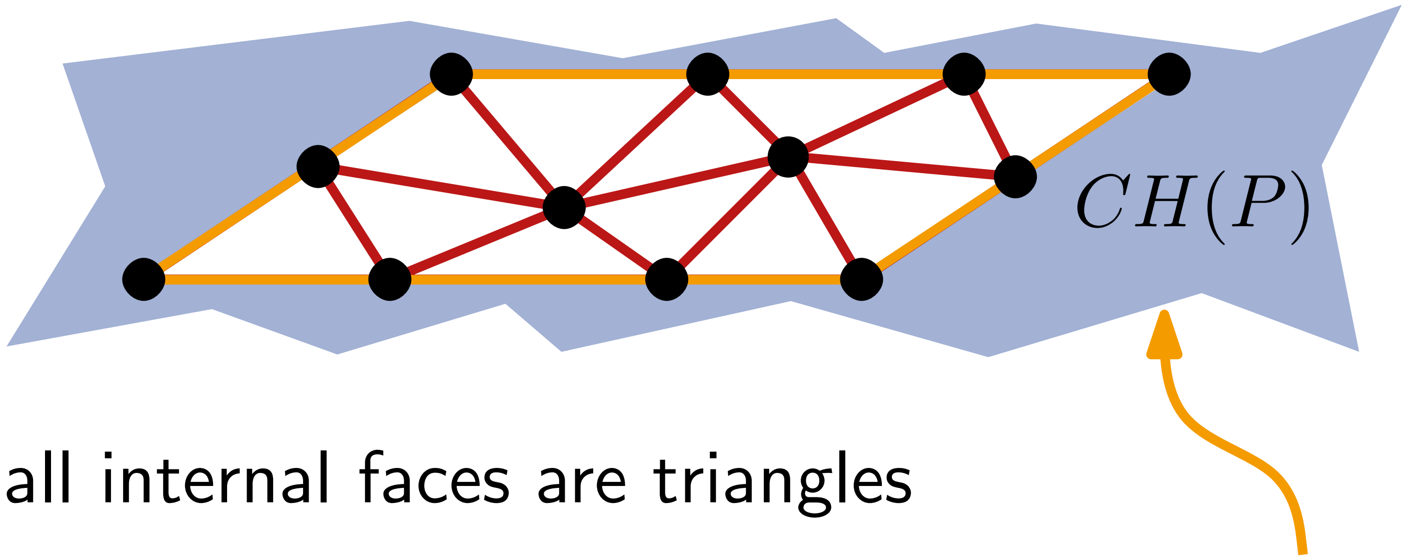


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**Theorem 1:** Let $P$ be a set of $n$ points, not all collinear. Let $h$ be the number of points in $CH(P)$.

Then any triangulation of $P$ has $t(n, h)$ triangles and $e(n, h)$ edges.

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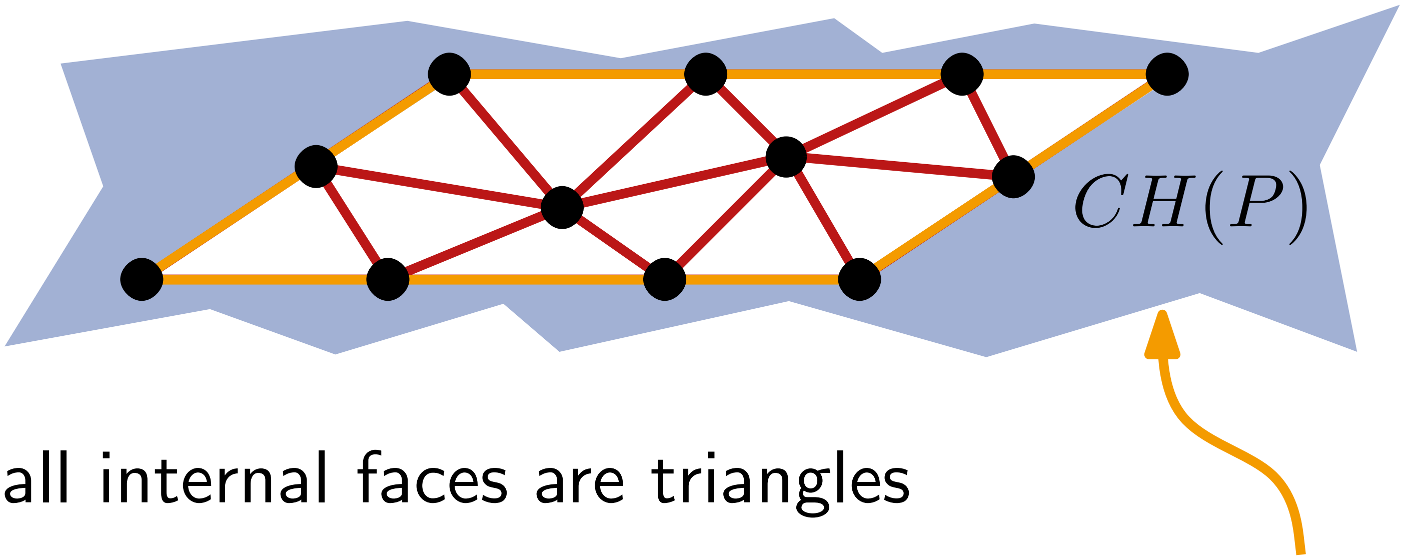


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Compute $t$ and $e$!

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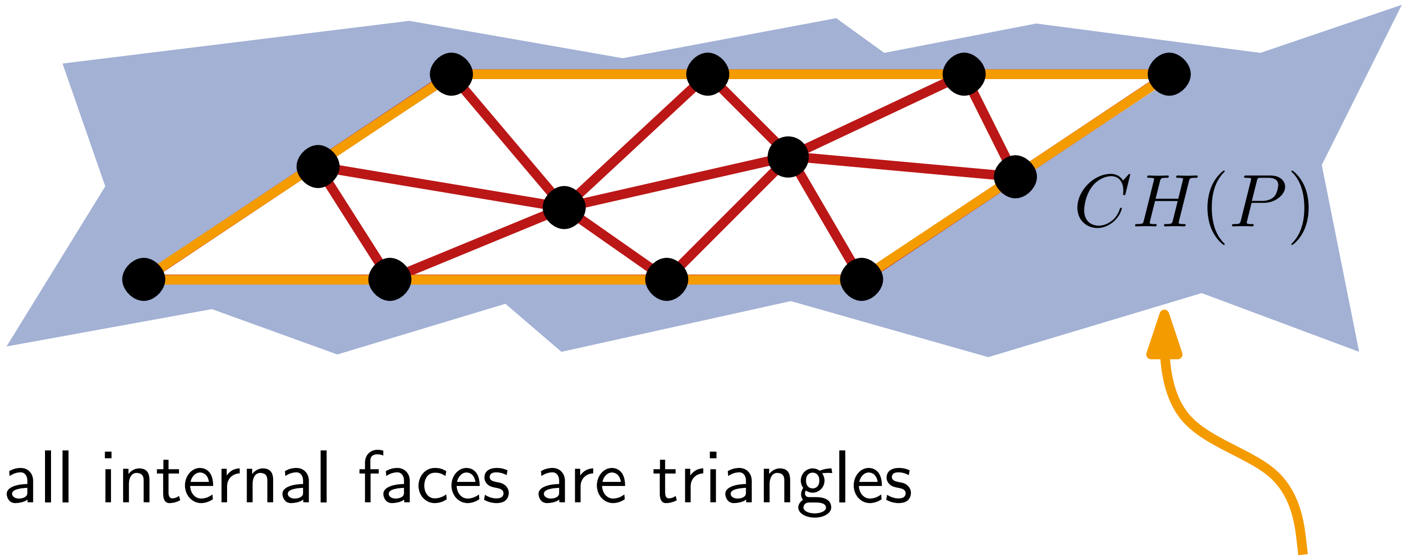


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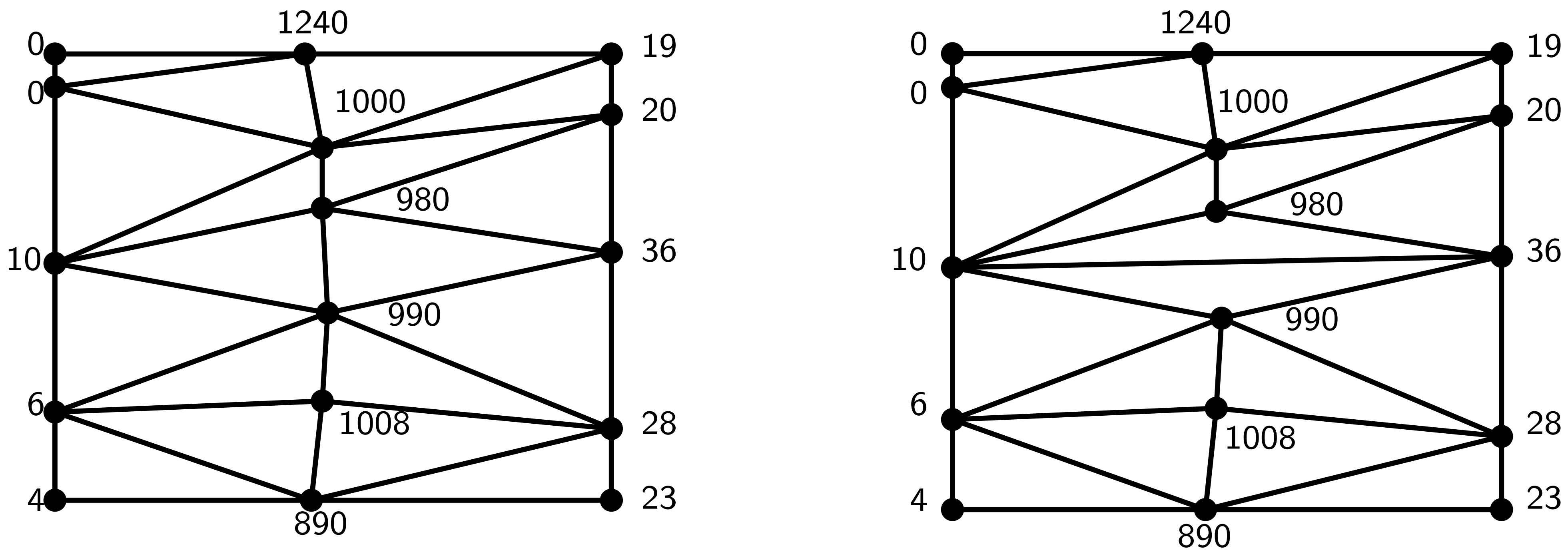
**Theorem 1:** Let $P$ be a set of $n$ points, not all collinear. Let $h$ be the number of points in $CH(P)$.

Then any triangulation of $P$ has $(2n - 2 - h)$ triangles and $(3n - 3 - h)$ edges.

# Back to Height Interpolation





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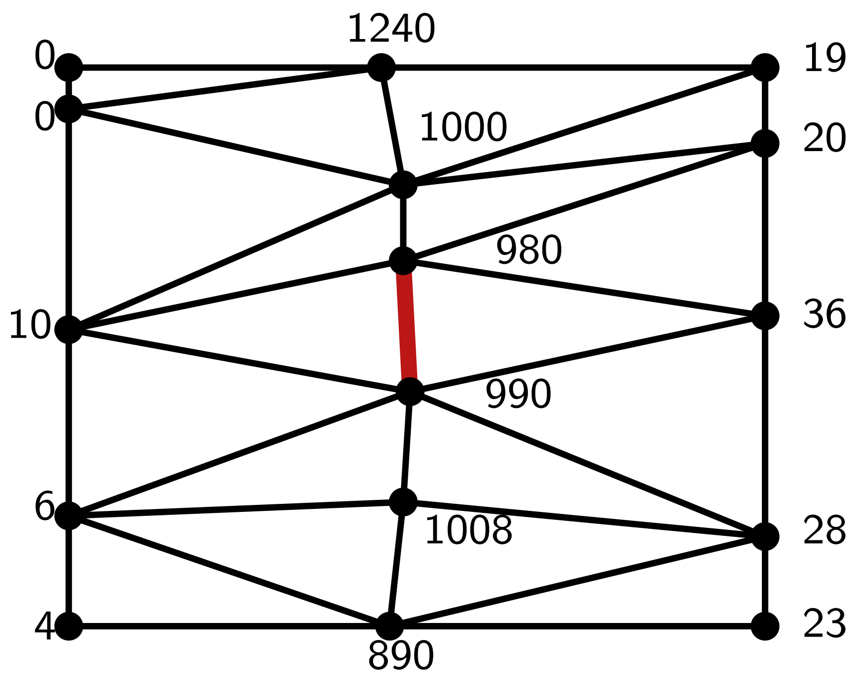


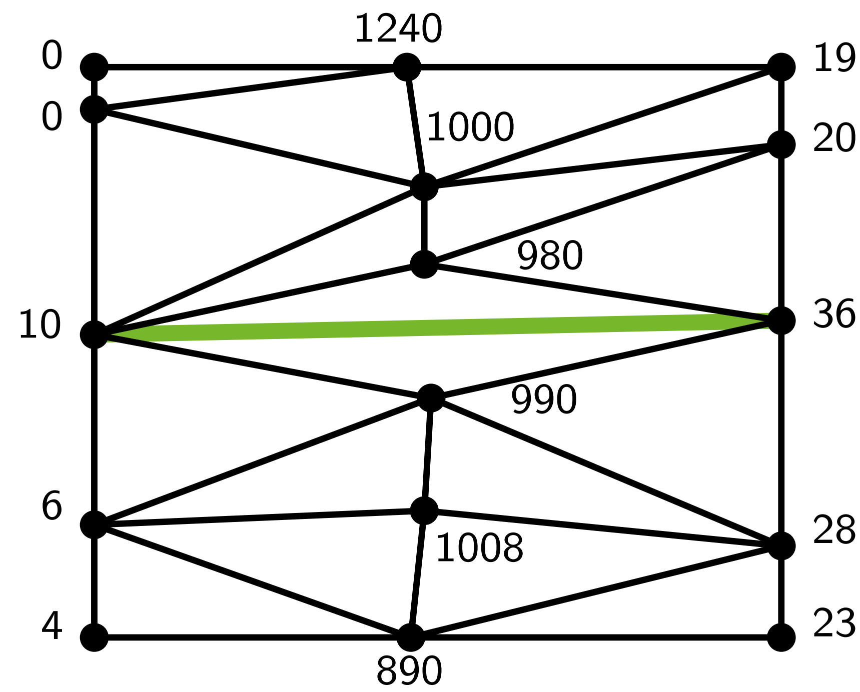

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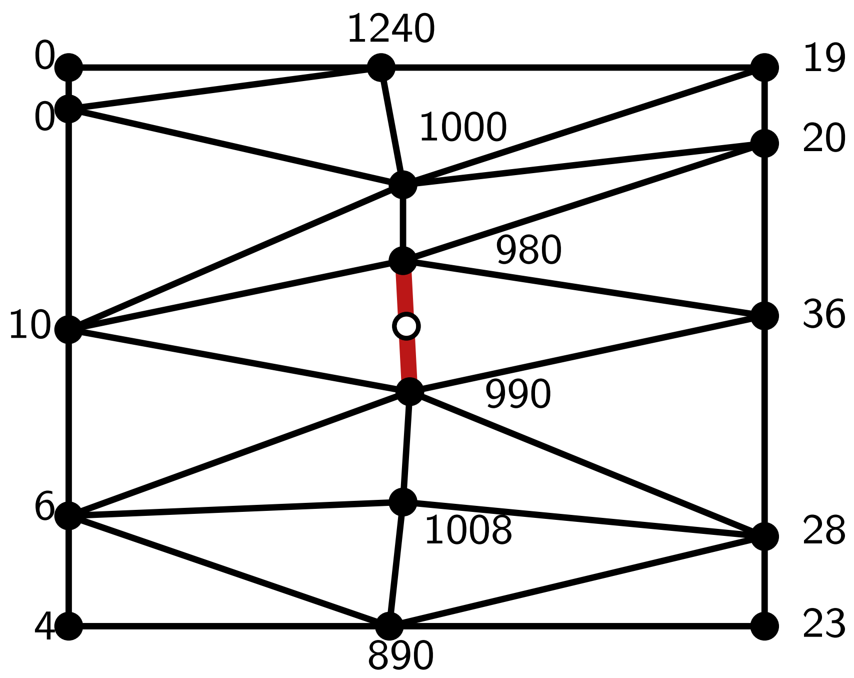


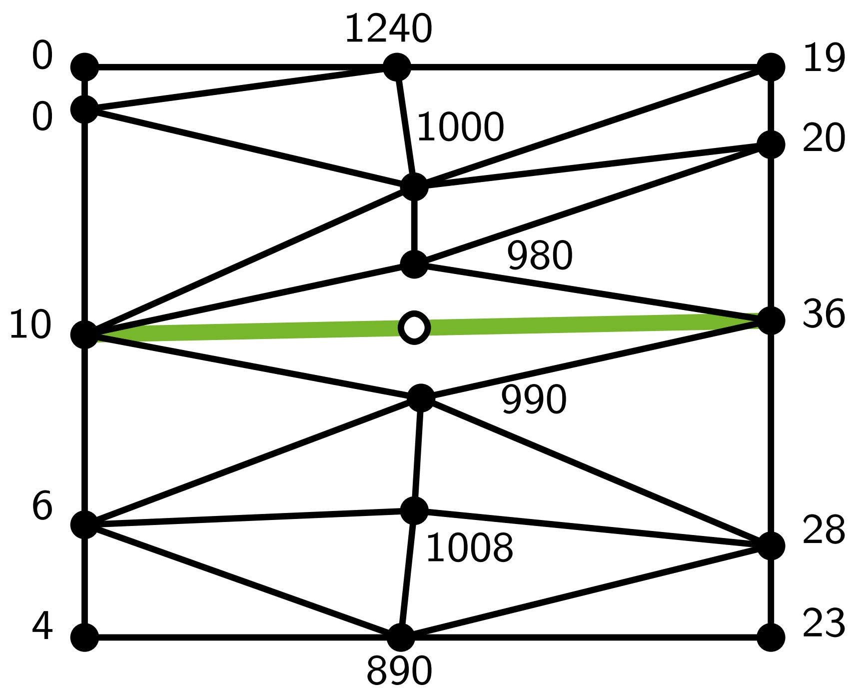

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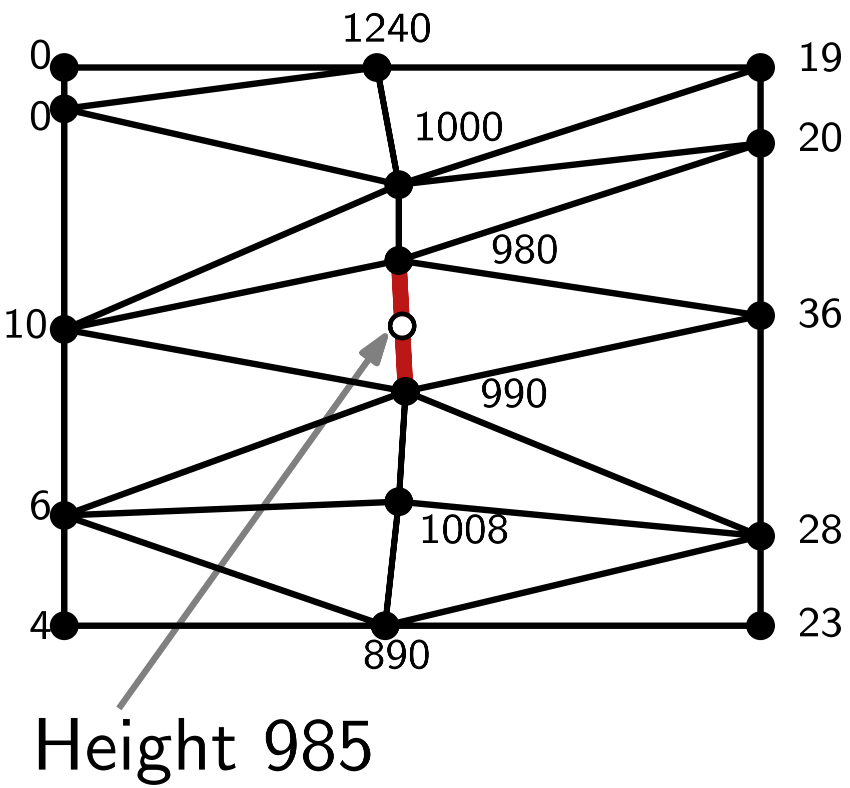


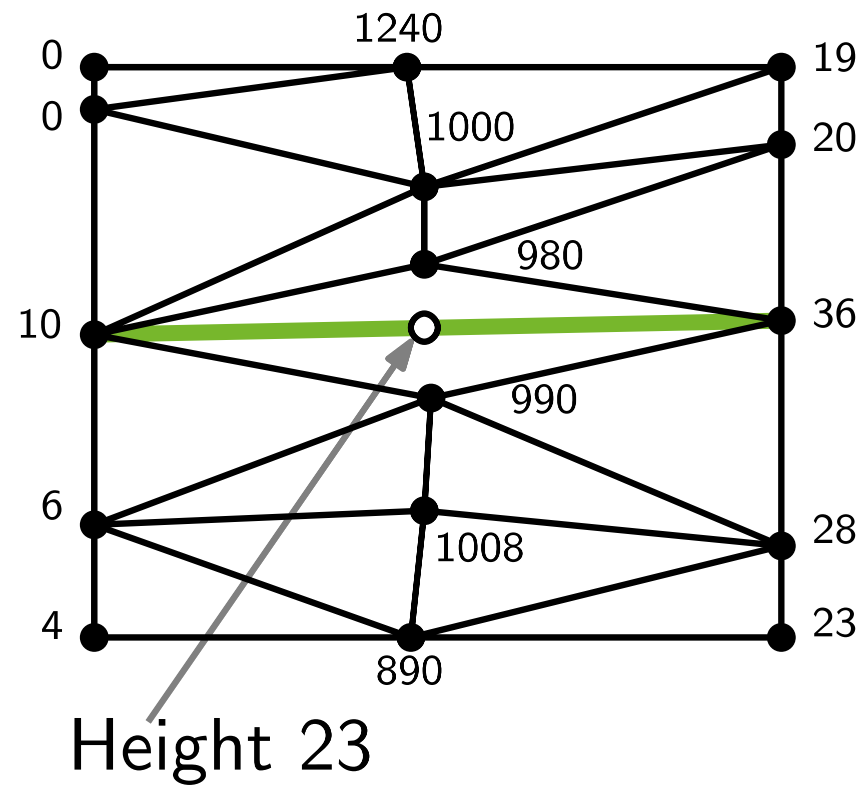

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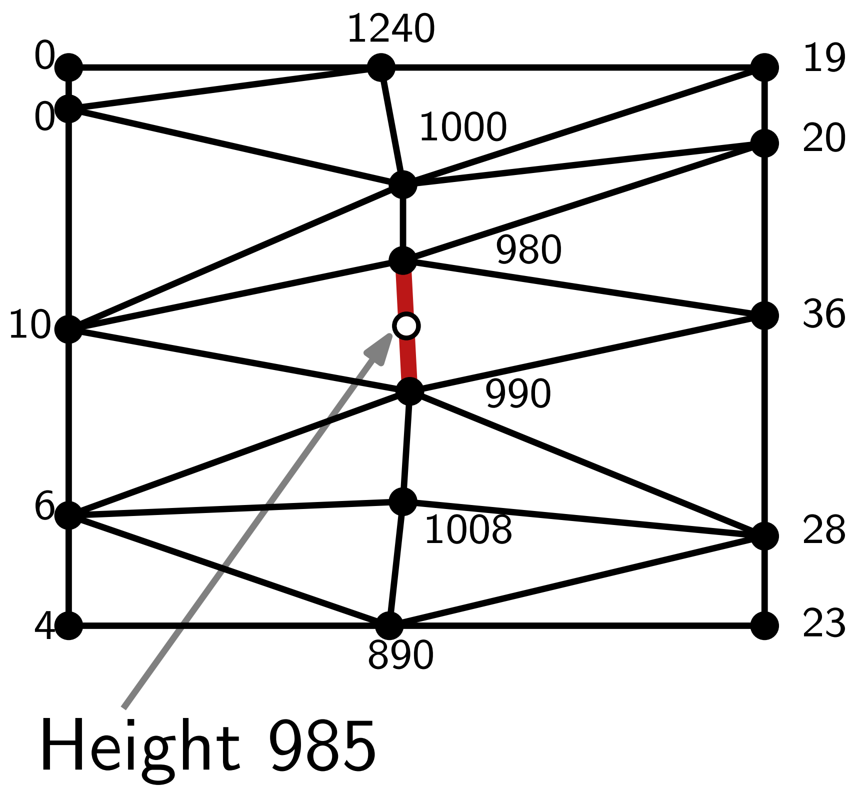


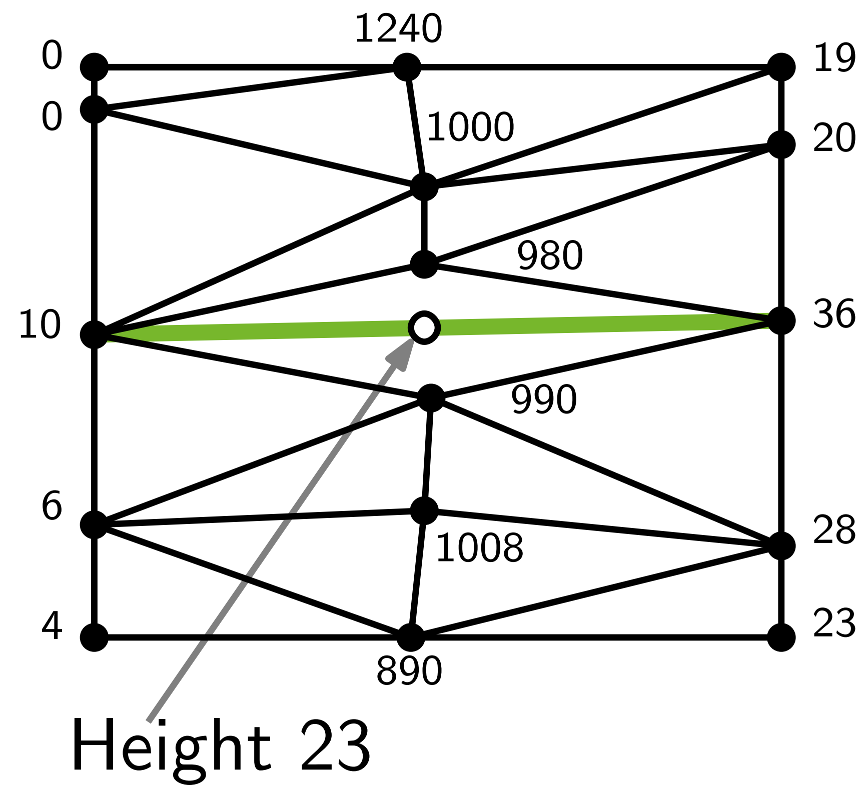


**Intuition:** Avoid narrow triangles!

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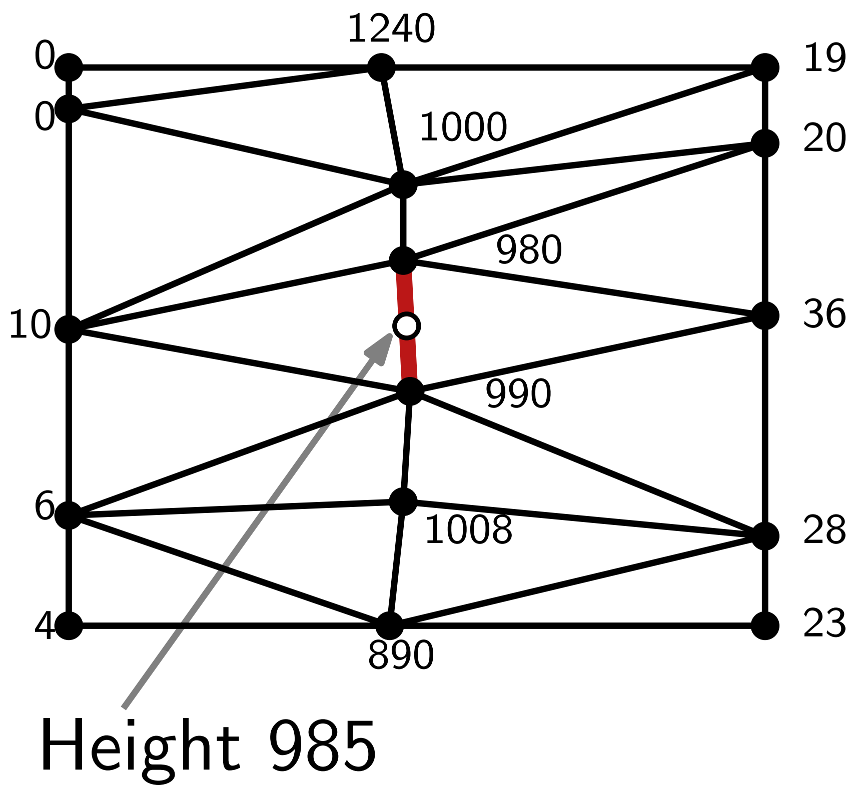
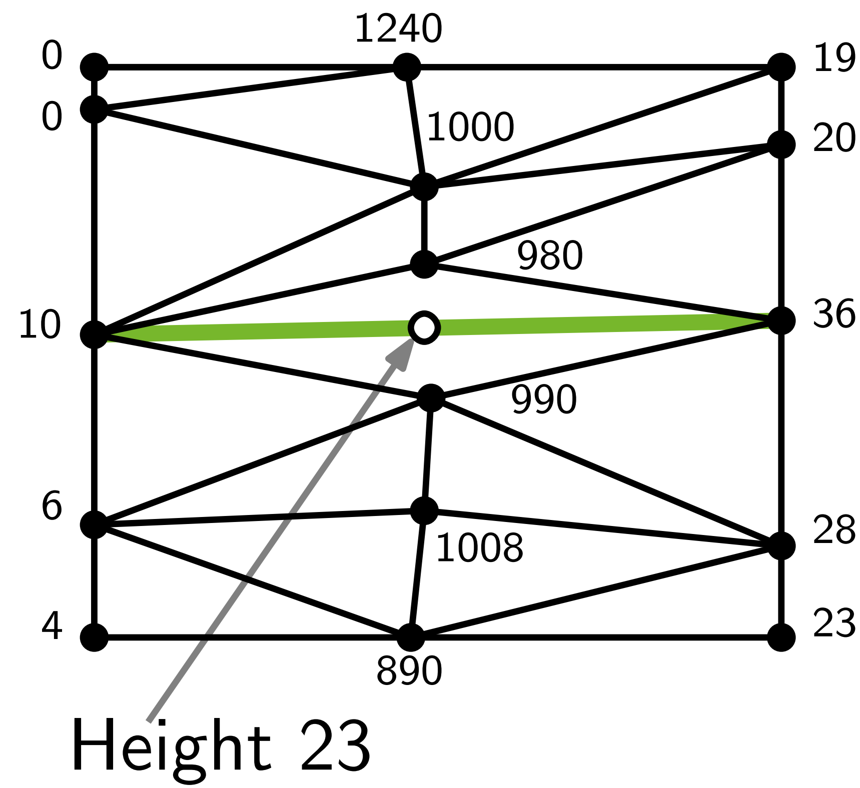


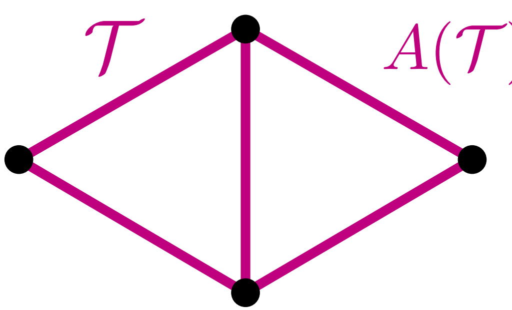




**Intuition:** Avoid narrow triangles!

Or: maximize the smallest angle!

# Angle-optimal Triangulations

**Def.:** Let $P \subset \mathbb{R}^2$ be a set of points, $\mathcal{T}$ be a triangulation of $P$ and $m$ be the number of the triangles. $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ is called **angle-vector** of $\mathcal{T}$ where $\alpha_1 \leq \cdots \leq \alpha_{3m}$.




$A(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$

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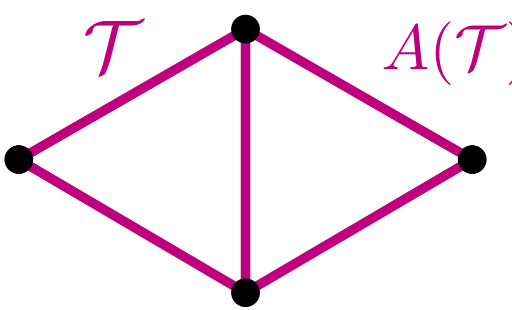
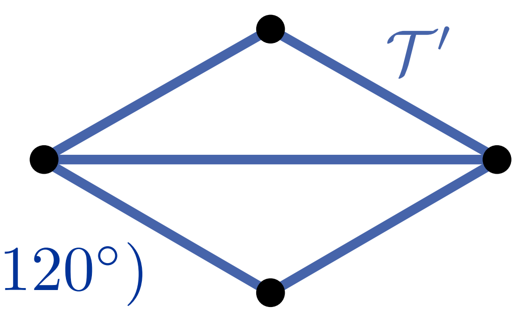
For two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $P$ we define the **order** of the angle-vectors $A(\mathcal{T}) > A(\mathcal{T}')$ as lexicographic order of corresponding angle sequences.




$$A(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$$




$$A(\mathcal{T}') = (30°, 30°, 30°, 30°, 120°, 120°)$$

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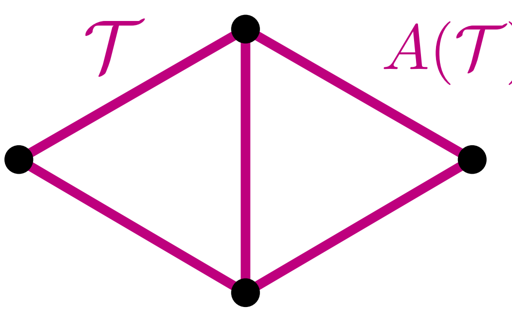
For two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $P$ we define the **order** of the angle-vectors $A(\mathcal{T}) > A(\mathcal{T}')$ as lexicographic order of corresponding angle sequences.

$\mathcal{T}$ is called **angle-optimal**, if $A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of $P$.

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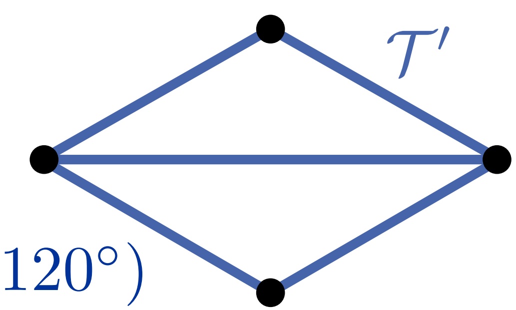
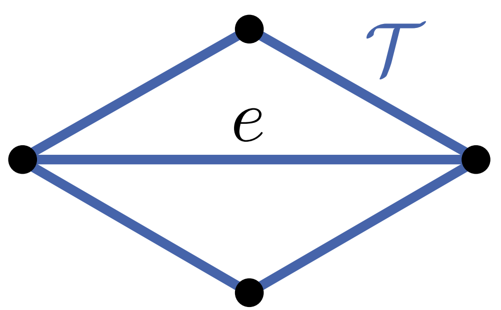


# Edge Flips

**Def.:** Let $\mathcal{T}$ be a triangulation. An edge $e$ of $\mathcal{T}$ is called **illegal**, when the smallest angle incident to $e$ increases after the flip of $e$.



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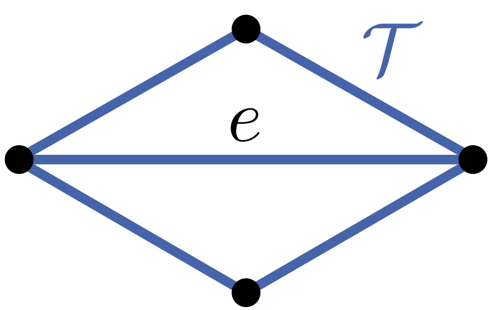
$\min_i \alpha_i = 30°$



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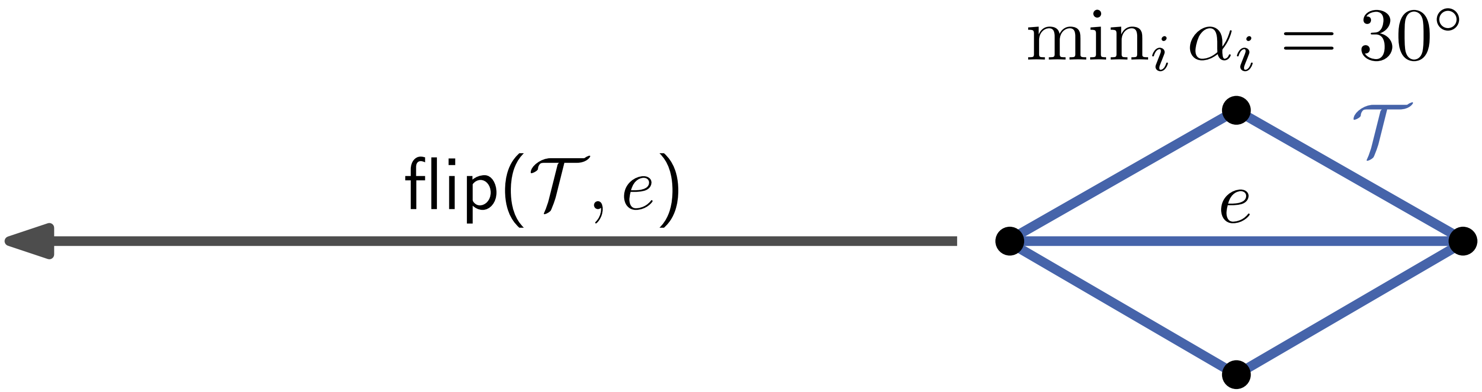

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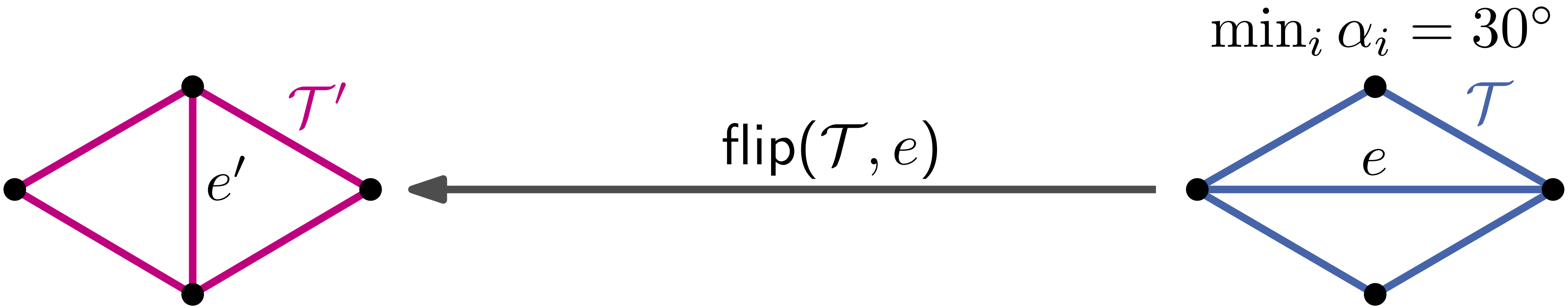

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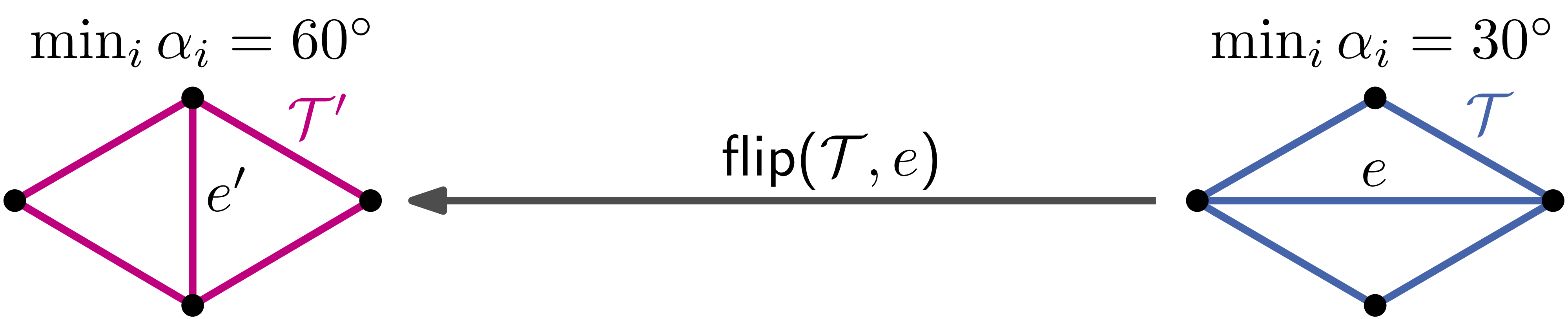

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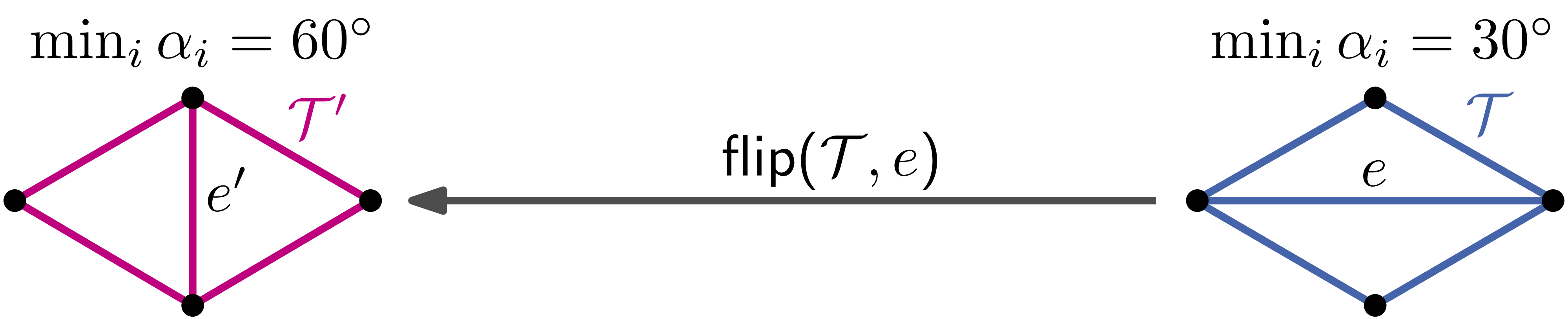
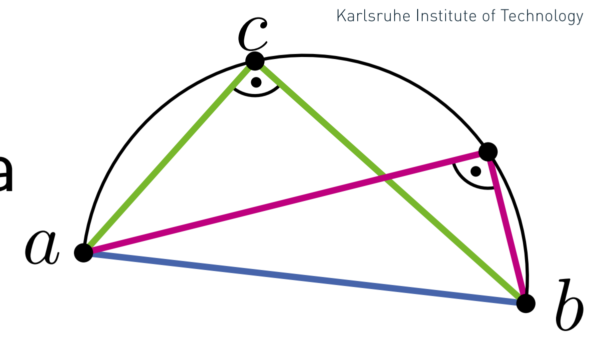
**Obs.:** Let $e$ be an illegal edge of $\mathcal{T}$ and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$.
Then $A(\mathcal{T}') > A(\mathcal{T})$.



# Thales's Theorem

**Theorem 2:** If $a$, $b$ and $c$ are points on a circle where the segment $ab$ is a diameter of the circle, then the angle $\angle bca$ is a right angle.



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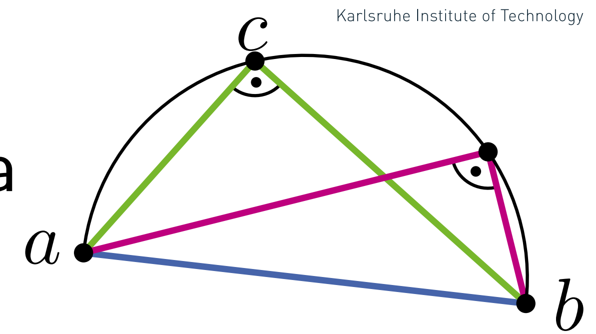
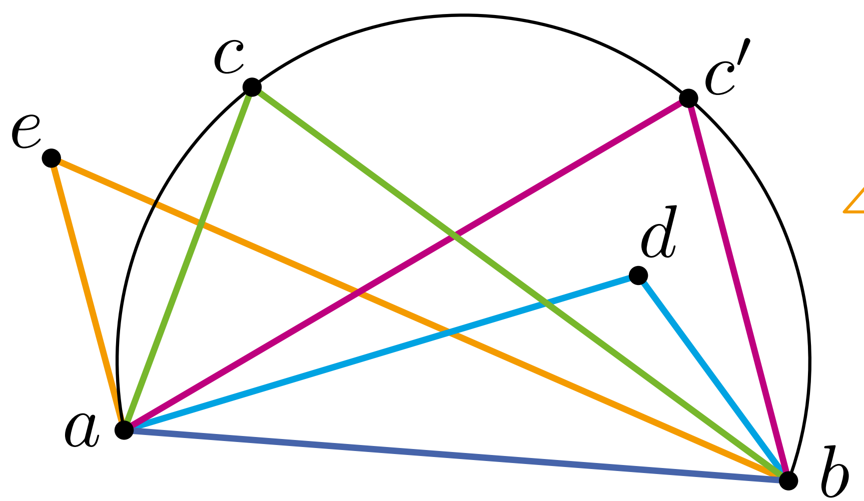



**Theorem 2:** If $a$, $b$ and $c$ are points on a circle where the segment $ab$ is a diameter of the circle, then the angle $\angle bca$ is a right angle.

**Theorem 2′:** Consider a circle $C$ through $a, b, c$. For any point $c'$ on $C$ on the same side of $ab$ as $c$ , holds that $\angle acb = \angle ac'b$. For any point $d$ inside $C$ holds that $\angle adb > \angle acb$, and for point $e$ outside $C$, holds that $\angle aeb < \angle acb$.

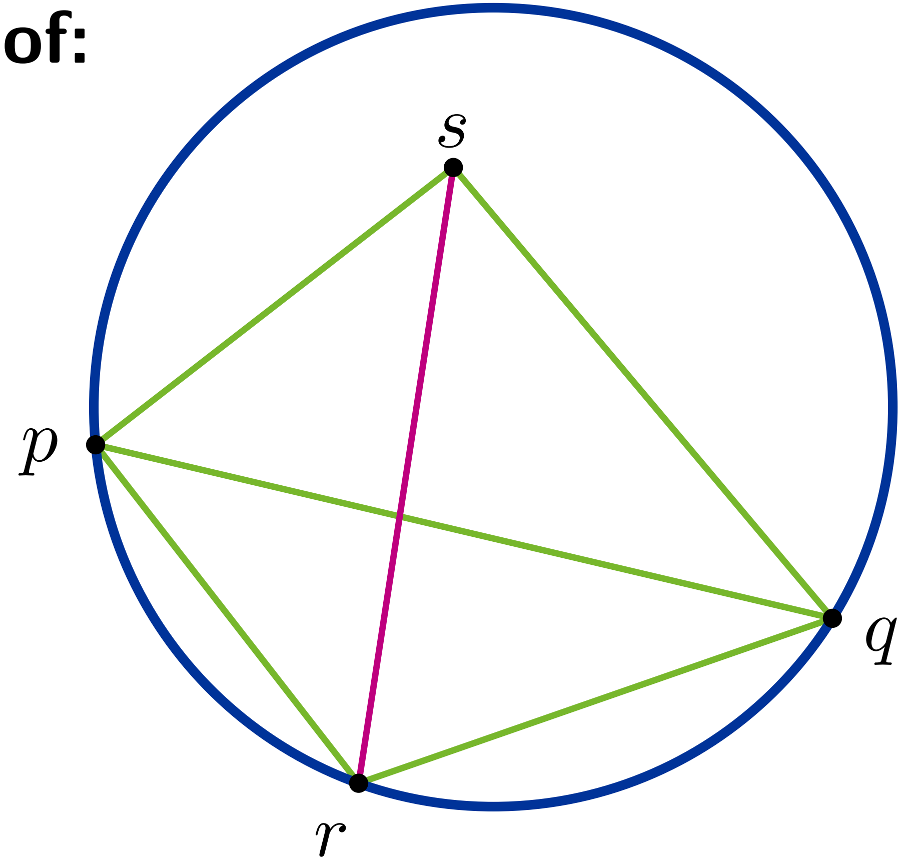



$$\angle aeb < \angle acb = \angle ac'b < \angle adb$$

# Legal Triangulation

**Lemma 1:** Let $\Delta pqr$ and $\Delta pqs$ be two triangles in $\mathcal{T}$ and let $C$ be the circle through $\Delta pqr$. Then $\overline{pq}$ is illegal iff $s \in \text{int}(C)$. If $p, q, r, s$ form a convex quadrilateral and do not lie on a common circle ($s \notin \partial C$) then exactly one of $\overline{pq}$ and $\overline{rs}$ is an illegal edge.

**Sketch of proof:**




$\angle rsq > \angle rpq$
$\angle psr > \angle pqr$

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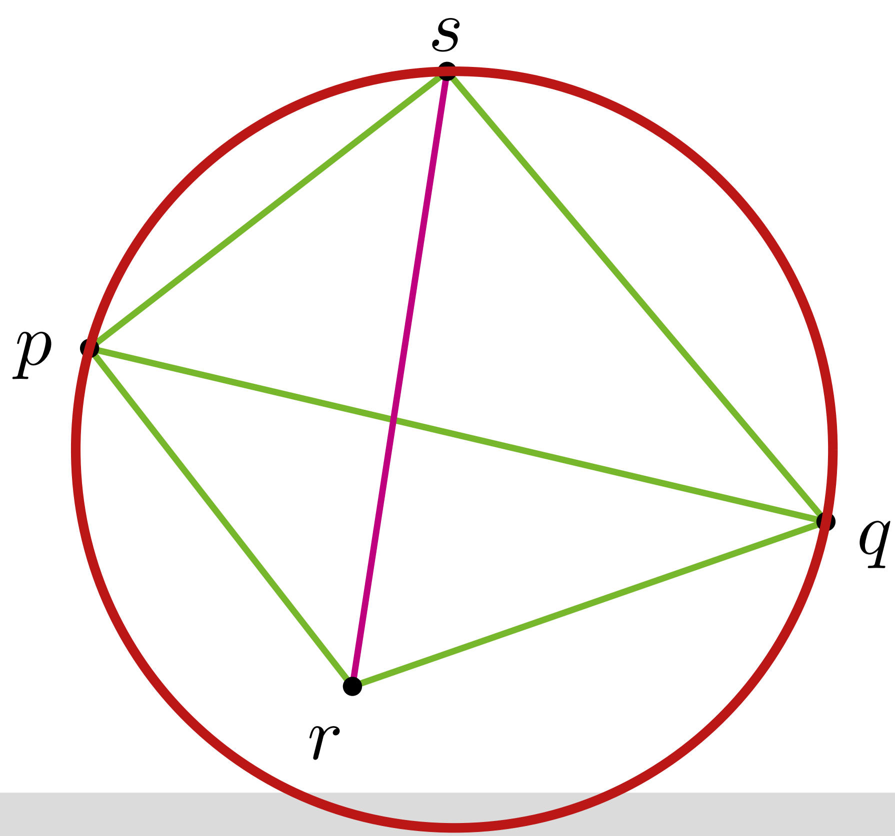


$\angle rsq > \angle rpq$
$\angle psr > \angle pqr$

$\angle prs > \angle pqs$
$\angle srq > \angle spq$

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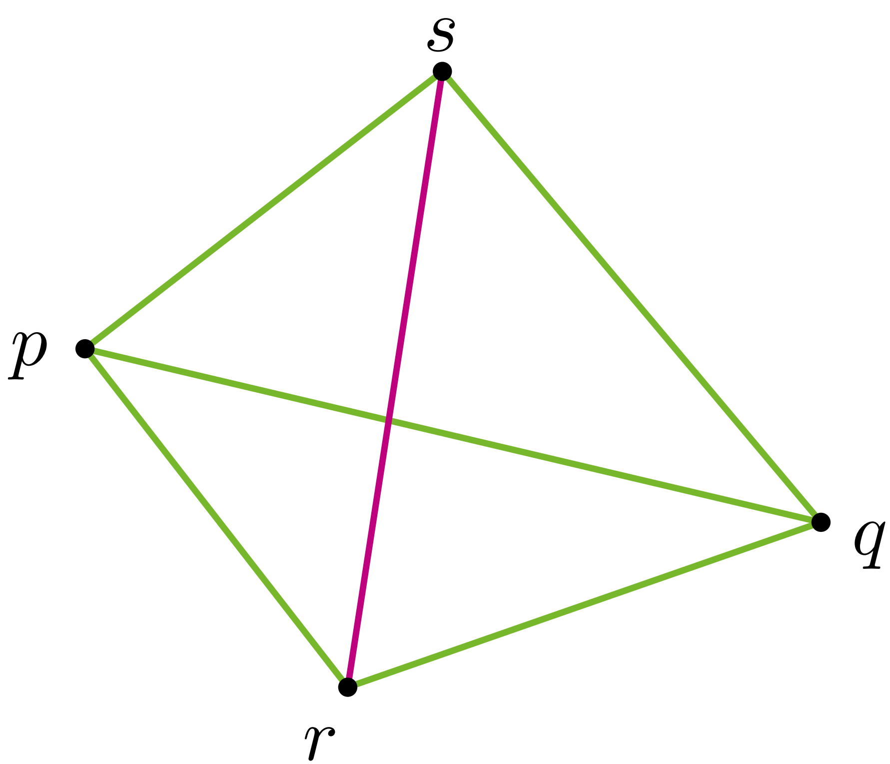


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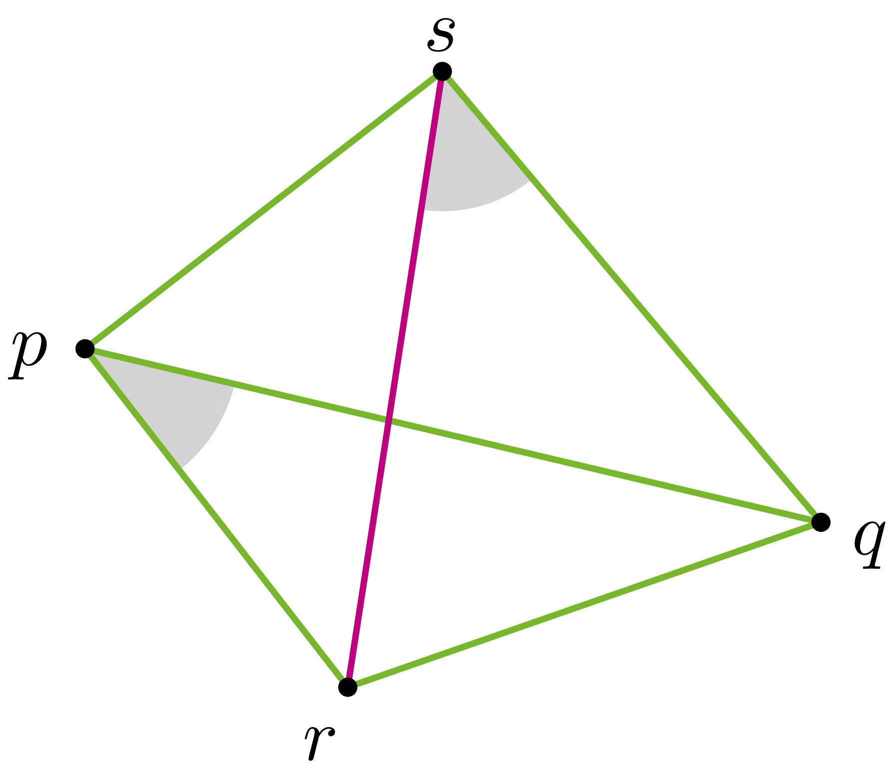


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Is there always a legal triangulation?

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terminates, since $A(\mathcal{T})$ increases and #Triangulations is finite ($< 30^n$, [Sharir, Sheffer 2011])

# Reverse statement?

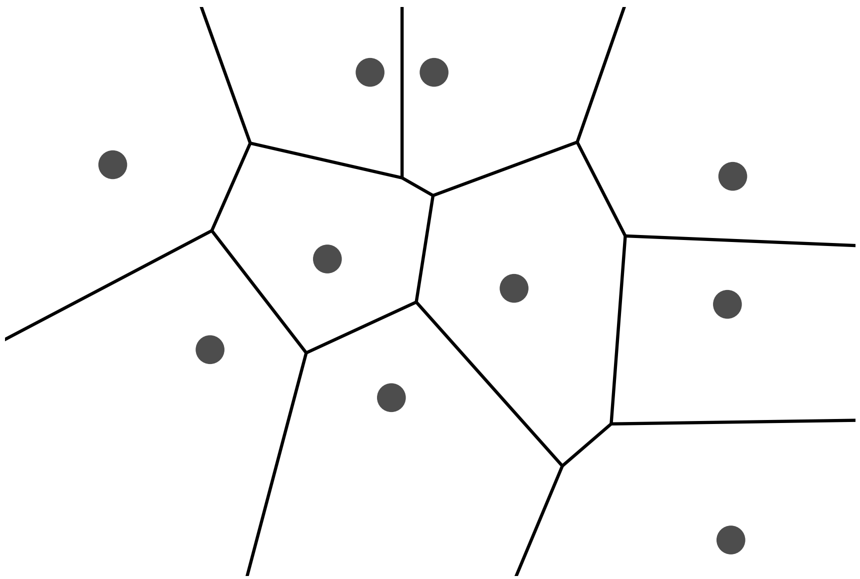
**It holds that:** each angle-optimal triangulation is legal.

But is each legal triangulation also angle-optimal?

# Delaunay-Triangulation

Let $\mathrm{Vor}(P)$ be the Voronoi-Diagram of a point set $P$.

**Def.:** The graph $\mathcal{G} = (P, E)$ with
$E = \{pq \mid \mathcal{V}(p) \text{ and } \mathcal{V}(q) \text{ are adjacent}\}$
is called **dual graph** of $\mathrm{Vor}(P)$.



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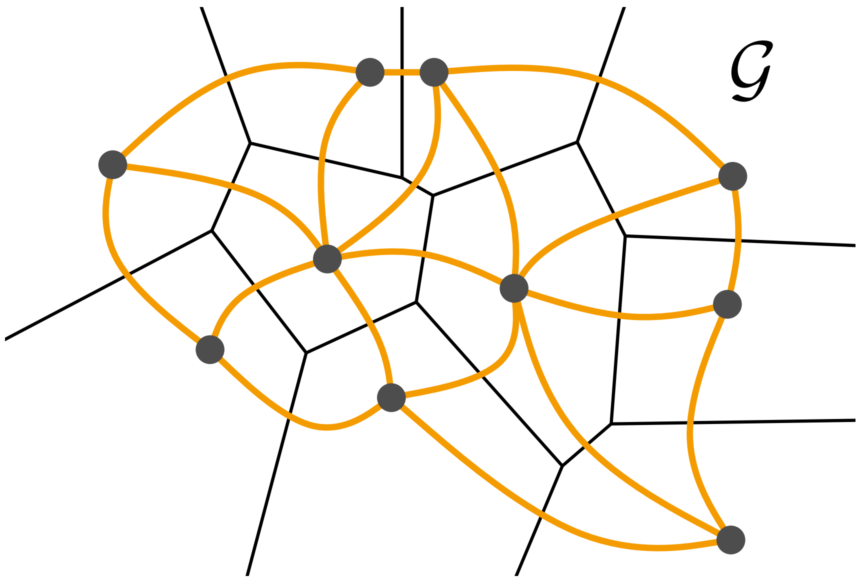

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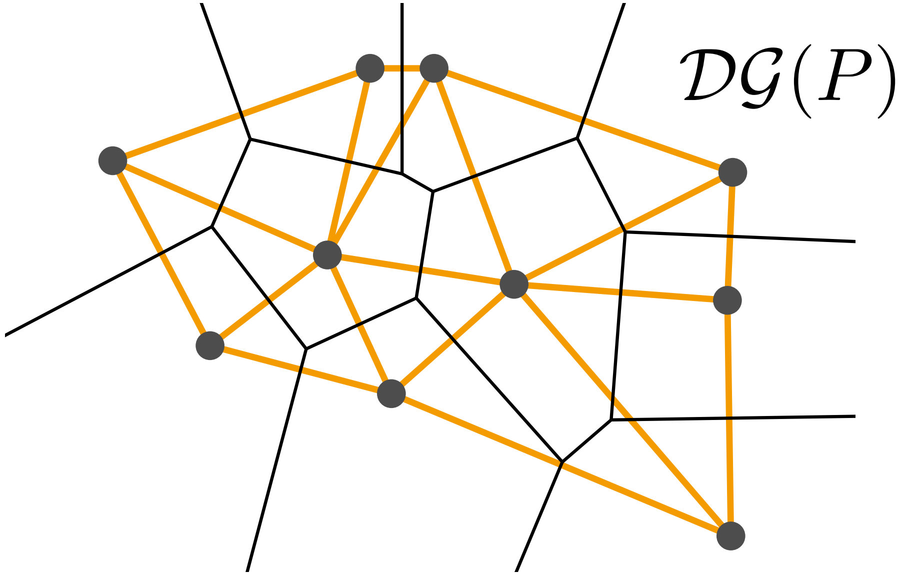

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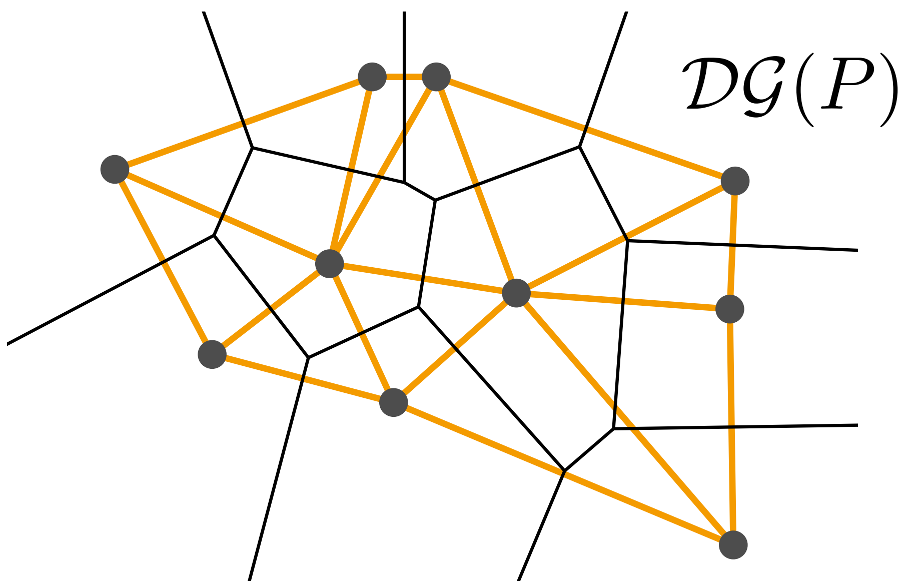


Boris Delone
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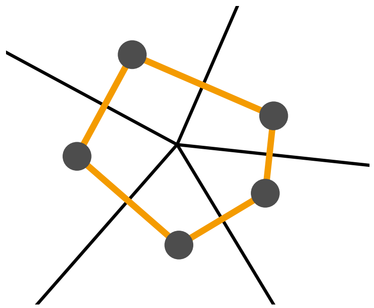

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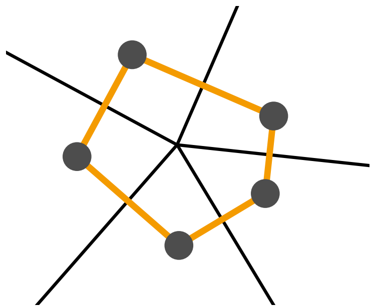

If $P$ is in *general position* (no four points on a circle), then all faces of $\mathcal{DG}(P)$ are triangles $\rightarrow$ **Delaunay-triangulation**

# Characterization

Theorem about Voronoi-Diagram:

- point $q$ is a Voronoy-vertex
  $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
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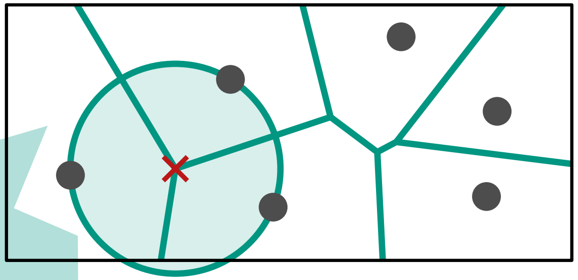

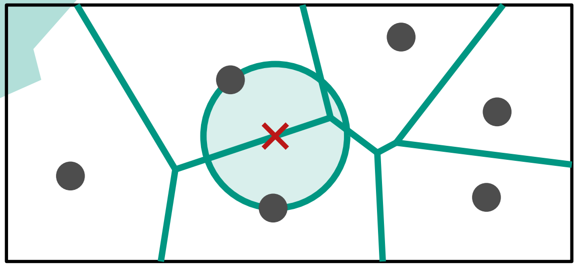

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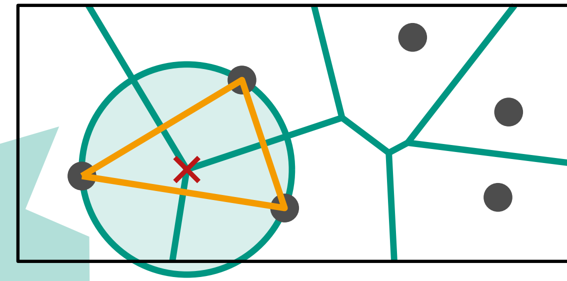

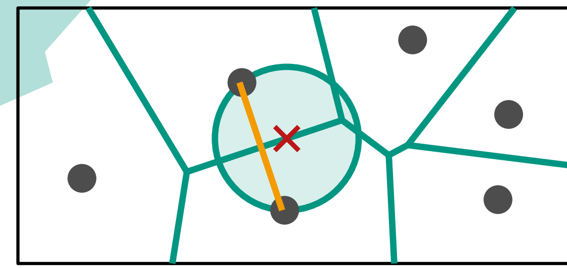

**Theorem 4:** Let $P$ be a set of points.

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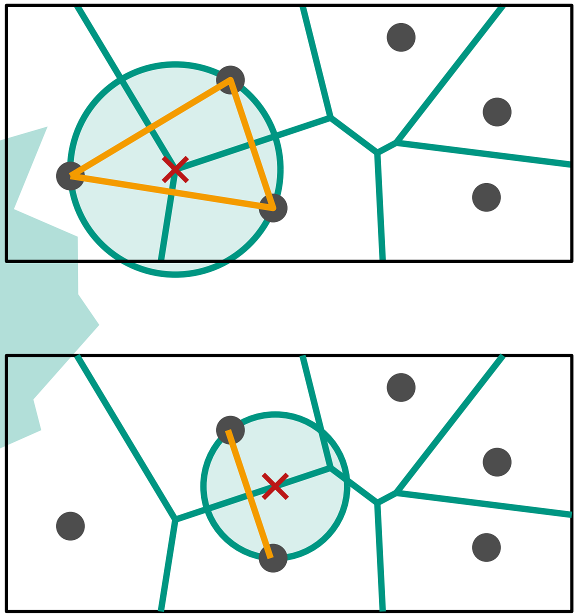

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**Theorem 5:** Let $P$ be a set of points and let $\mathcal{T}$ be a triangulation of $P$. $\mathcal{T}$ is Delaunay-Triangulation $\Leftrightarrow$ the circumcircle of each triangle has an empty interior.

# Legality and Delaunay-Triangulation

**Theorem 6:** Let $P$ be a set of points and $\mathcal{T}$ a triangulation of $P$. $\mathcal{T}$ is legal $\Leftrightarrow$ $\mathcal{T}$ is Delaunay-Triangulation.

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„$\Leftarrow$“ clear; use

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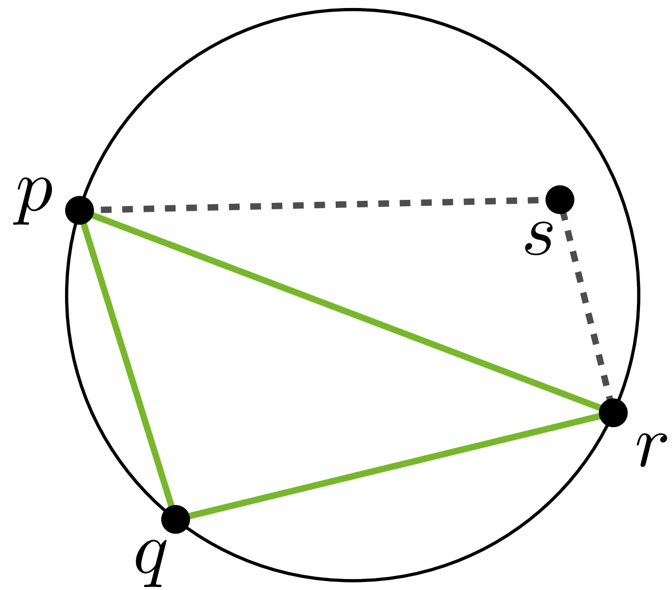

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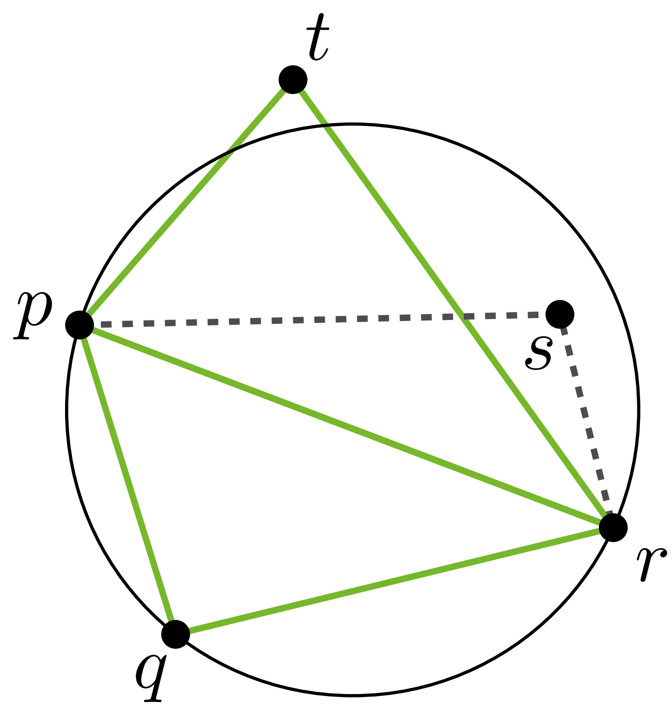

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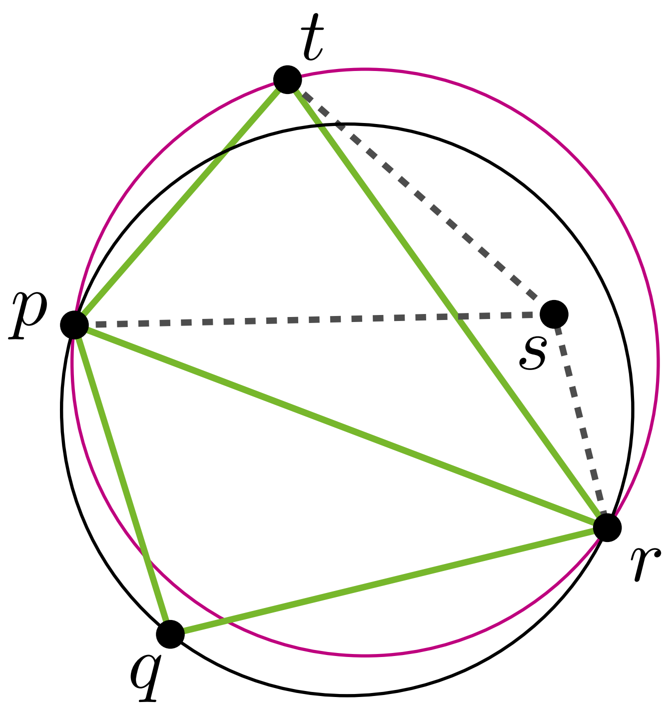

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$\Rightarrow$ $\mathcal{DG}(P)$ is angle-optimal!

If $P$ *is not* in general position, then for *any* triangulation of a „bigger“ face of $\mathcal{DG}(P)$ the *minimal* angles are equal (exercise!).

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**Theorem 7:** For $n$ points on the plane a Delaunay-triangulation can be computed in $O(n \log n)$ time
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**Outlook:** In the general case the angle-optimal triangulation can be computed in $O(n \log n)$ time.
[Mount, Saalfeld '88]

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Karlsruhe Institute of Technology

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## Where to find further information on Voronoi-Diagrams und Delaunay-Triangulations?

Relatively new book (2013) of Aurenhammer, Klein, Lee!