Computational Geometry • Lecture

Delaunay Triangulation

INSTITUTE FOR THEORETICAL INFORMATICS · FACULTY OF INFORMATICS

Tamara Mchedlidze · Darren Strash

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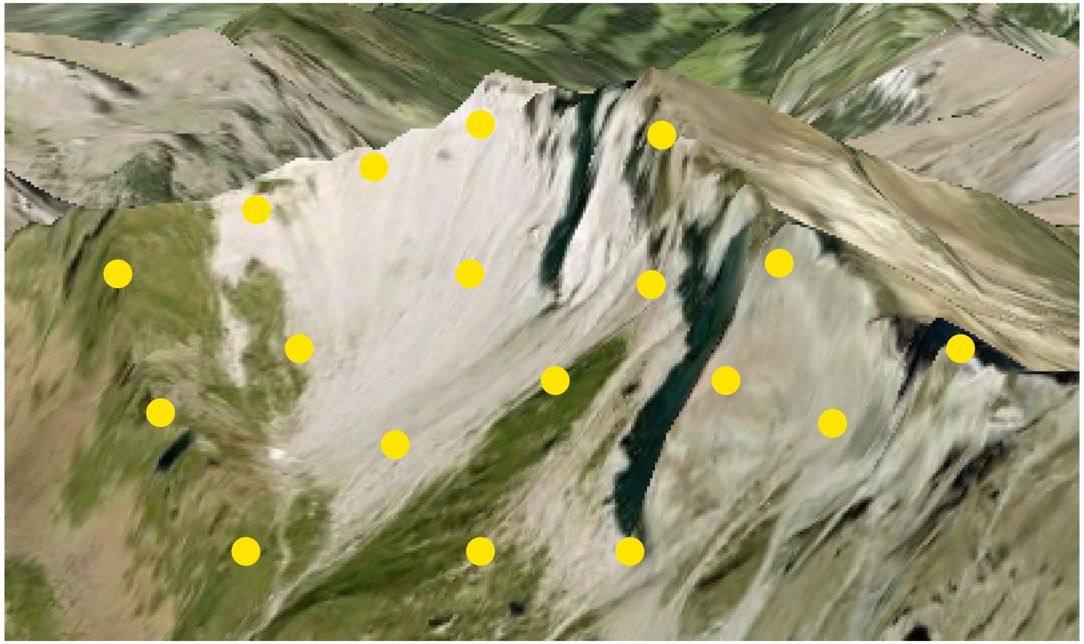
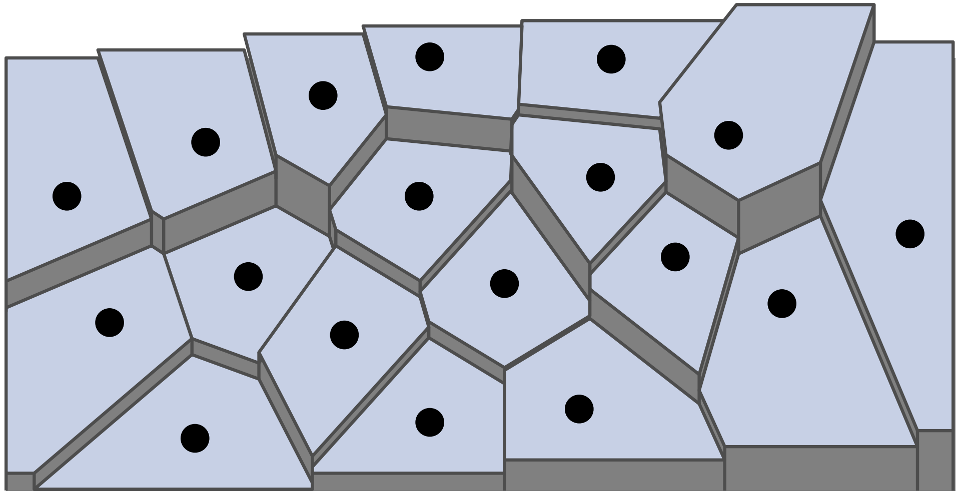
Modelling a Terrain



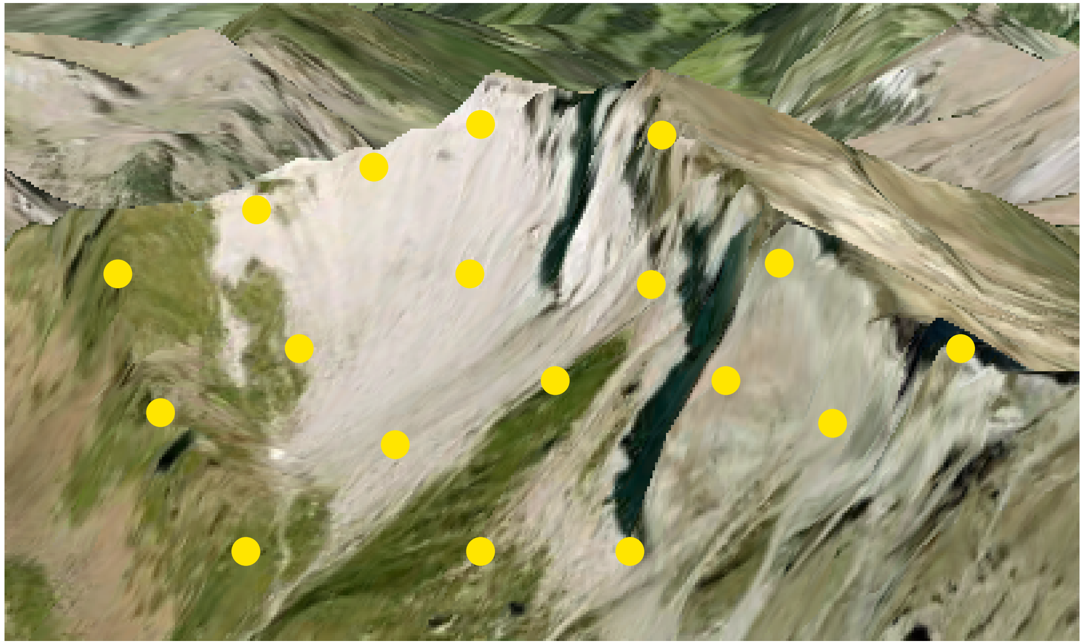
Sample points
$p = (x_p, y_p, z_p)$



Projection
$\pi(p) = (p_x, p_y, 0)$

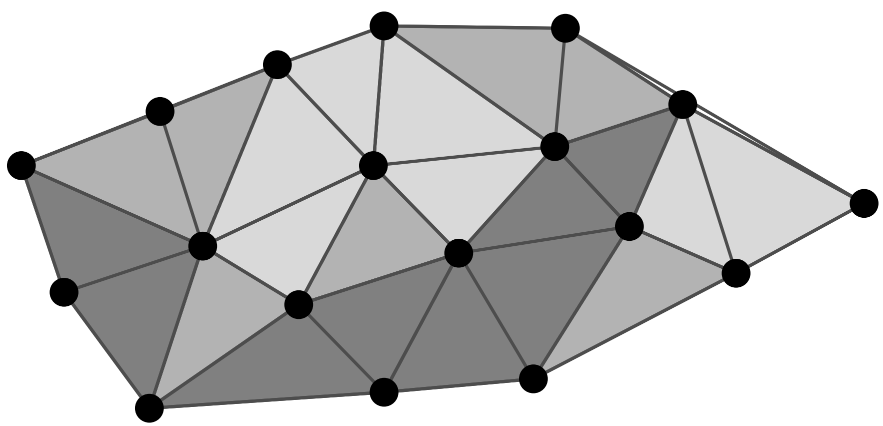
Interpolation 1: each point gets the height of the nearest sample point

Modelling a Terrain



Sample points
$p = (x_p, y_p, z_p)$

↓

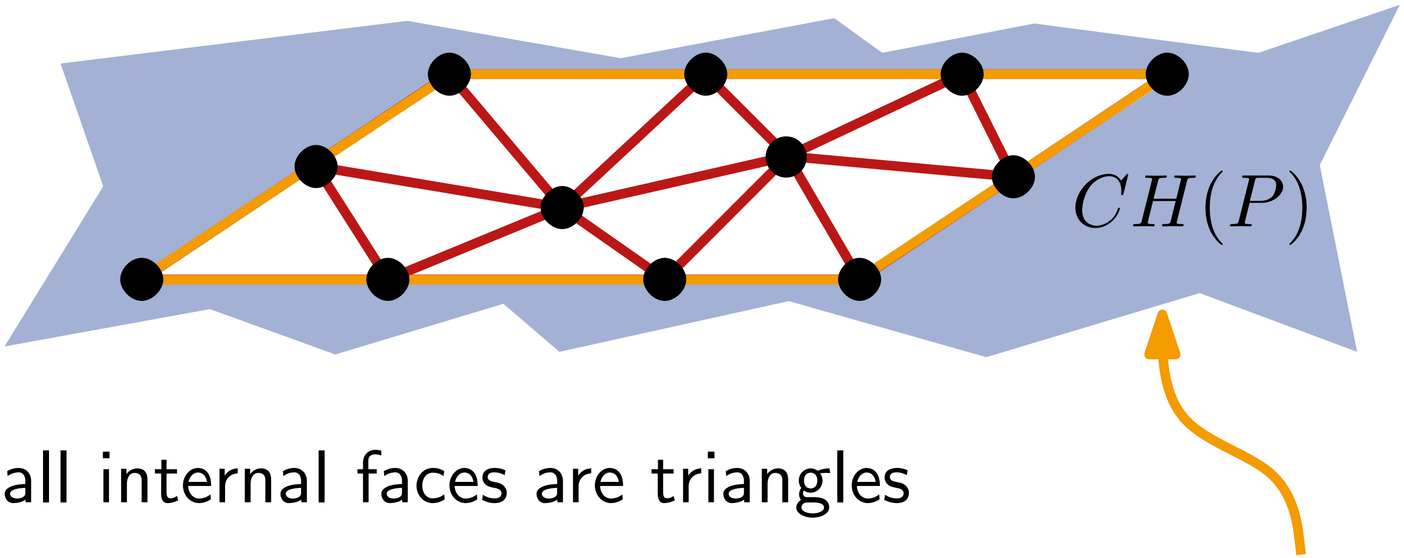


Projection
$\pi(p) = (p_x, p_y, 0)$

Interpolation 2: triangulate the set of sample points and interpolate on the triangles

Triangulation of a Point Set

Def.: A **triangulation** of a point set $P \subset \mathbb{R}^2$ is a maximal planar subdivision with a vertex set P.

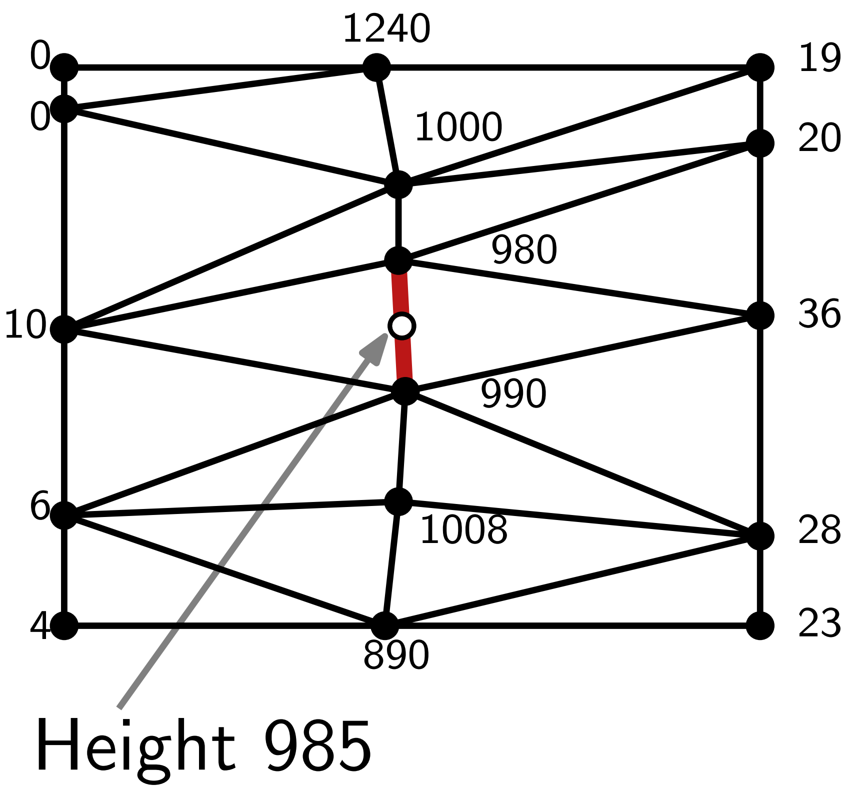


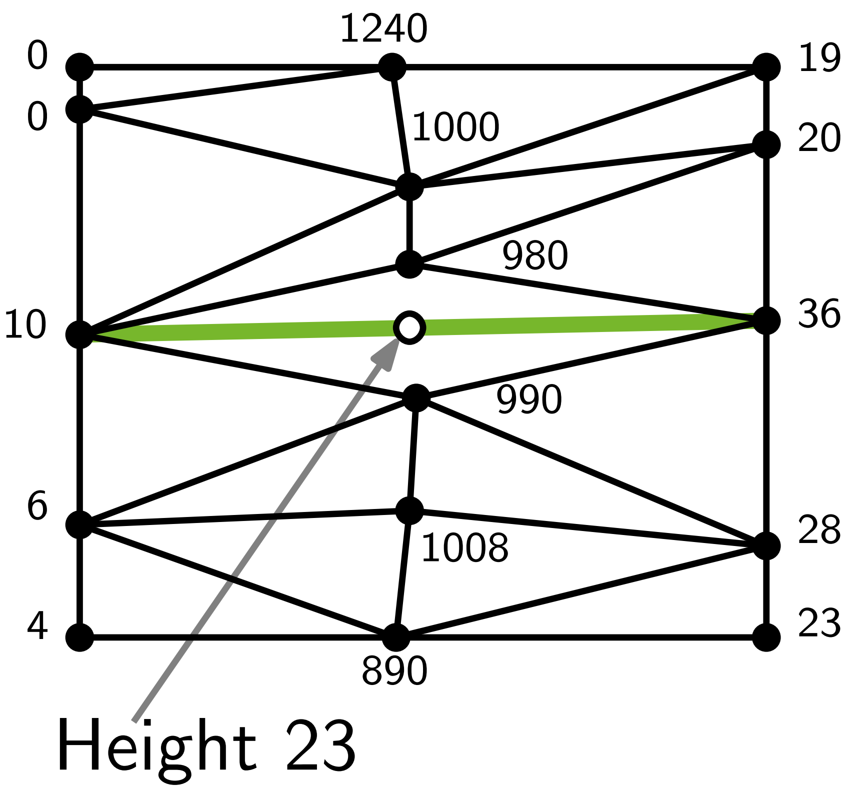
Obs.:
- all internal faces are triangles
- outer face is the complement of the convex hull

Theorem 1: Let P be a set of n points, not all collinear. Let h be the number of points in $CH(P)$.

Then any triangulation of P has $(2n - 2 - h)$ triangles and $(3n - 3 - h)$ edges.

Back to Height Interpolation





Intuition: Avoid narrow triangles!

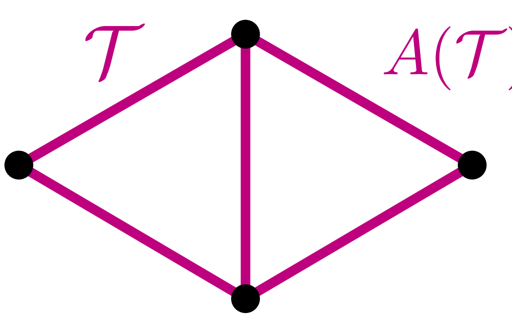
Or: maximize the smallest angle!

Angle-optimal Triangulations

Def.: Let $P \subset \mathbb{R}^2$ be a set of points, $\mathcal{T}$ be a triangulation of P and m be the number of the triangles.
$A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ is called **angle-vector** of $\mathcal{T}$ where $\alpha_1 \leq \cdots \leq \alpha_{3m}$.

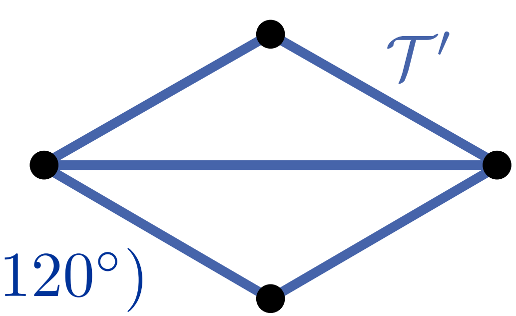
For two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of P we define the **order** of the angle-vectors $A(\mathcal{T}) > A(\mathcal{T}')$ as lexicographic order of corresponding angle sequences.

$\mathcal{T}$ is called **angle-optimal**, if $A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of P.



$A(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$

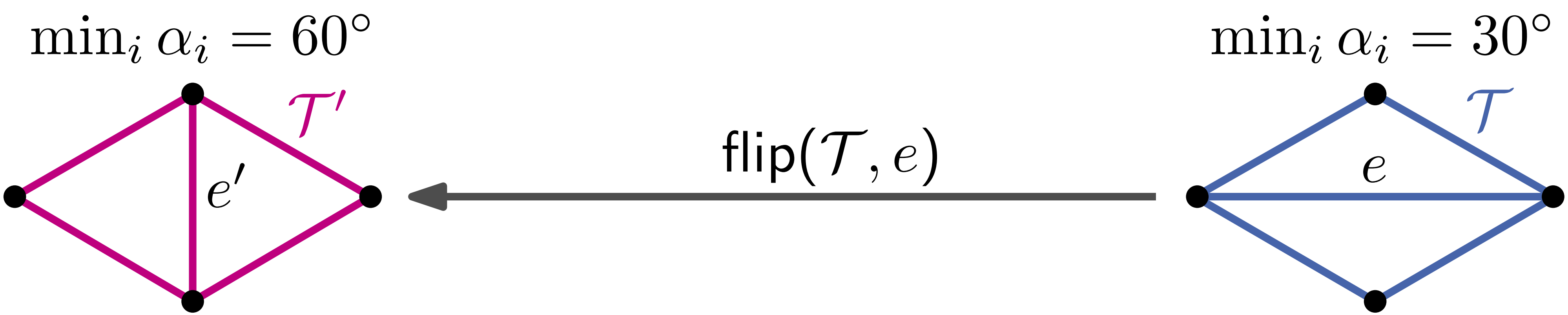
$A(\mathcal{T}') = (30°, 30°, 30°, 30°, 120°, 120°)$



Edge Flips

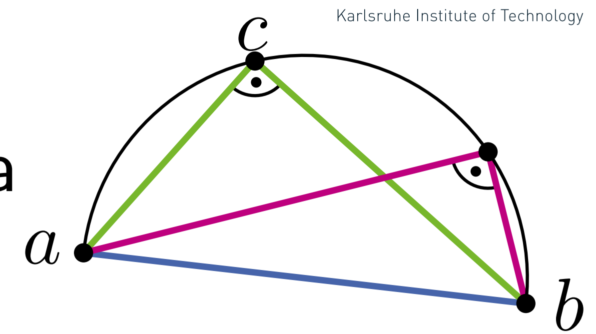
Def.: Let $\mathcal{T}$ be a triangulation. An edge e of $\mathcal{T}$ is called **illegal**, when the smallest angle incident to e increases after the flip of e.

Obs.: Let e be an illegal edge of $\mathcal{T}$ and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.

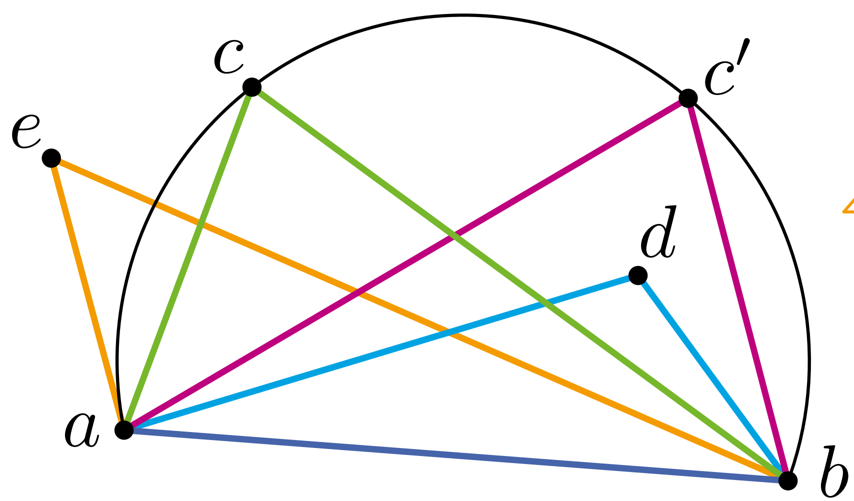


Thales's Theorem

Theorem 2: If a, b and c are points on a circle where the segment ab is a diameter of the circle, then the angle $\angle bca$ is a right angle.



Theorem 2′: Consider a circle C through a, b, c. For any point c' on C on the same side of ab as c , holds that $\angle acb = \angle ac'b$. For any point d inside C holds that $\angle adb > \angle acb$, and for point e outside C, holds that $\angle aeb < \angle acb$.

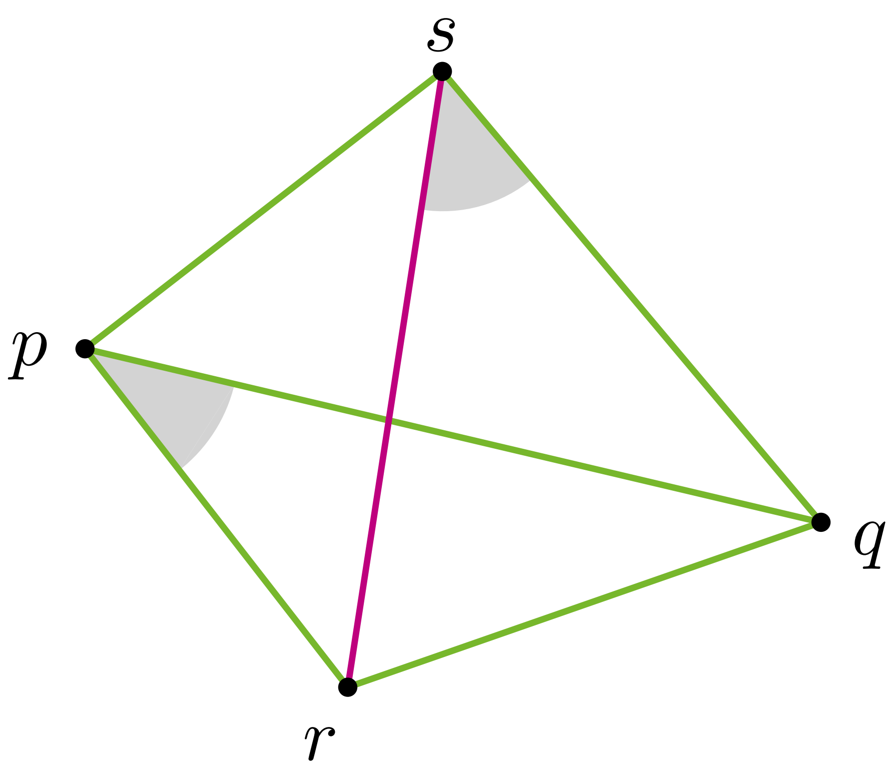


$\angle aeb < \angle acb = \angle ac'b < \angle adb$

Legal Triangulation

Lemma 1: Let Δpqr and Δpqs be two triangles in $\mathcal{T}$ and let C be the circle through Δpqr. Then $\overline{pq}$ is illegal iff $s \in \text{int}(C)$. If p, q, r, s form a convex quadrilateral and do not lie on a common circle ($s \notin \partial C$) then exactly one of $\overline{pq}$ and $\overline{rs}$ is an illegal edge.

Sketch of proof:



$\angle rsq > \angle rpq$
$\angle psr > \angle pqr$

$\angle prs > \angle pqs$
$\angle srq > \angle spq$

Legal Triangulation

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Obs.:
- The characterization is symmetric w.r.t. r and s
- $s \in \partial C \Rightarrow \overline{pq}$ and $\overline{rs}$ are legal
- an illegal edge $\Rightarrow$ quadrilateral is convex

Def.: A triangulation without illegal edges in called **legal**.

Is there always a legal triangulation?

Legal Triangulation

Lemma 1: Let Δpqr and Δpqs be two triangles in $\mathcal{T}$ and let C be the circle through Δpqr. Then $\overline{pq}$ is illegal iff $s \in \text{int}(C)$. If p, q, r, s form a convex quadrilateral and do not lie on a common circle ($s \notin \partial C$) then exactly one of $\overline{pq}$ and $\overline{rs}$ is an illegal edge.

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Def.: A triangulation without illegal edges in called **legal**.

while $\mathcal{T}$ has an illegal edge e **do**
  flip$(\mathcal{T}, e)$
return $\mathcal{T}$

terminates, since $A(\mathcal{T})$ increases and #Triangulations is finite ($< 30^n$, [Sharir, Sheffer 2011])

Reverse statement?

It holds that: each angle-optimal triangulation is legal.

But is each legal triangulation also angle-optimal?

Delaunay-Triangulation

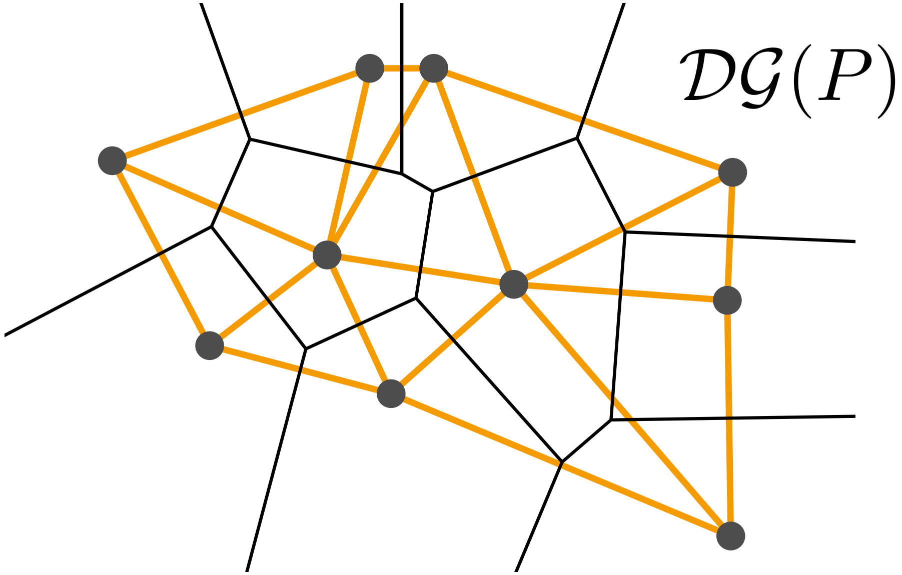
Karlsruhe Institute of Technology

Let $\mathrm{Vor}(P)$ be the Voronoi-Diagram of a point set P.

Def.: The graph $\mathcal{G} = (P, E)$ with
$E = \{pq \mid \mathcal{V}(p)$ and $\mathcal{V}(q)$ are adjacent$\}$
is called **dual graph** of $\mathrm{Vor}(P)$.

Def.: The straight-line drawing of $\mathcal{G}$ is called
Delaunay-Graph $\mathcal{DG}(P)$.

Georgy Voronoi
(1868–1908)



Boris Delone
(1890–1980)

Properties

Theorem. 3: $\mathcal{DG}(P)$ is crossing-free.

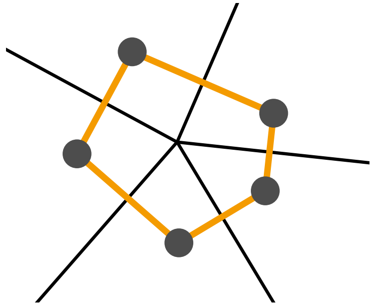
Sketch of proof:

The bisector $b(p, q)$ defines a Voronoi-edge
$\Leftrightarrow \exists r \in b(p, q)$ with $C_P(r) \cap P = \{p, q\}$.

or

The edge pq is in $\mathcal{DG}(P)$
$\Leftrightarrow$ there is an empty circle $C_{p,q}$ with p and q on the boundary.

Obs.: A Voronoi-vertex v in $\mathrm{Vor}(P)$ with degree k corresponds to a convex k-gon in $\mathcal{DG}(P)$.

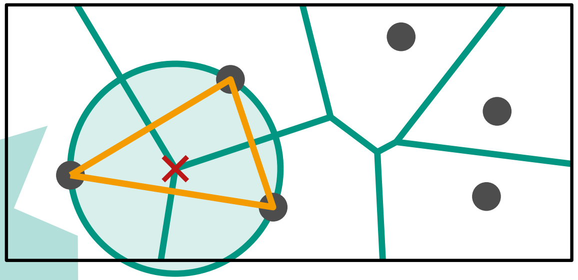


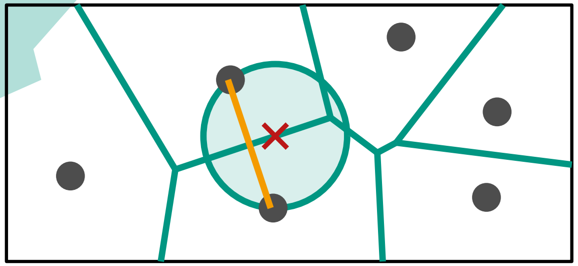
If P is in *general position* (no four points on a circle), then all faces of $\mathcal{DG}(P)$ are triangles $\rightarrow$ **Delaunay-triangulation**

Characterization

Theorem about Voronoi-Diagram:

- point q is a Voronoy-vertex
  $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
- bisector $b(p_i, p_j)$ defines a Voronoi-edge
  $\Leftrightarrow \exists q \in b(p_i, p_j)$ with $C_P(q) \cap P = \{p_i, p_j\}$.





Theorem 4: Let P be a set of points.

- Points p, q, r are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow$ circle through p, q, r is empty
- Edge pq is in $\mathcal{DG}(P) \Leftrightarrow$ there is an empty circle $C_{p,q}$ through p and q

Theorem 5: Let P be a set of points and let $\mathcal{T}$ be a triangulation of P. $\mathcal{T}$ is Delaunay-Triangulation $\Leftrightarrow$ the circumcircle of each triangle has an empty interior.

Legality and Delaunay-Triangulation

Theorem 6: Let P be a set of points and $\mathcal{T}$ a triangulation of P. $\mathcal{T}$ is legal $\Leftrightarrow$ $\mathcal{T}$ is Delaunay-Triangulation.

Sketch of proof:

„$\Leftarrow$“ clear; use

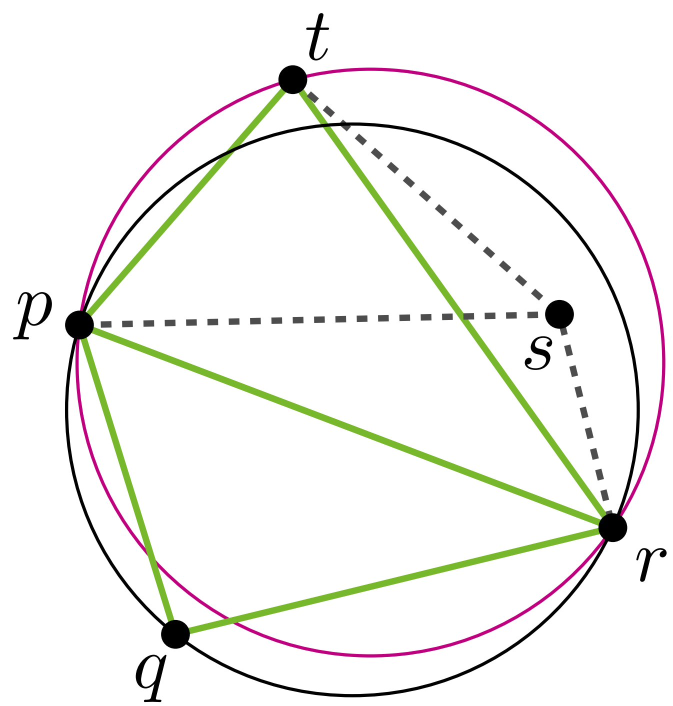
Lemma 1: Let Δprq and Δpqs be two triangles of $\mathcal{T}$ and C the circumcircle of Δprq. Edge $\overline{pq}$ is illegal iff $s \in \text{int}(C)$.

Legality and Delaunay-Triangulation

Theorem 6: Let P be a set of points and $\mathcal{T}$ a triangulation of P. $\mathcal{T}$ is legal $\Leftrightarrow$ $\mathcal{T}$ is Delaunay-Triangulation.

Sketch of proof:

„$\Rightarrow$“



Legality and Delaunay-Triangulation

Theorem 6: Let P be a set of points and $\mathcal{T}$ a triangulation of P. $\mathcal{T}$ is legal $\Leftrightarrow$ $\mathcal{T}$ is Delaunay-Triangulation.

Obs.: When P is in general position $\mathcal{DG}(P)$ is unique
$\Rightarrow$ legal triangulation is unique
I know that: $\mathcal{T}$ is angle-optimal $\Rightarrow$ $\mathcal{T}$ is legal
$\Rightarrow$ $\mathcal{DG}(P)$ is angle-optimal!

If P *is not* in general position, then for *any* triangulation of a „bigger“ face of $\mathcal{DG}(P)$ the *minimal* angles are equal (exercise!).

Summary

Theorem 7: For n points on the plane a Delaunay-triangulation can be computed in $O(n \log n)$ time
(Voronoi-Diag. + Triangulation of „big“ faces)

Corollary: For n points in general position an angle-optimal triangulation can be computed in $O(n \log n)$ time.
If the points are not in general position, a triangulation with maximal smallest angle can be computed in the same $O(n \log n)$ time.

Outlook: In the general case the angle-optimal triangulation can be computed in $O(n \log n)$ time.

[Mount, Saalfeld '88]

Discussion

Are there alternative approaches for the height interpolation using triangulations?

The *data-independent* Delaunay-triangulations is an initial step of *data-dependent* triangulations, which start from $\mathcal{DG}(P)$ and perform edge-flips. Rippa (1990) showed that $\mathcal{DG}(P)$ minimizes *roughness* independently from the height information.

Has $\mathcal{DG}(P)$ other interesting properties?

Yes, Delaunay-Graph contains the edges of other interesting graphs on P (see exersices). For example it holds that

$$\text{EMST}(P) \subseteq \text{Gabriel-Graph}(P) \subseteq \mathcal{DG}(P)$$

Where to find further information on Voronoi-Diagrams und Delaunay-Triangulations?

Relatively new book (2013) of Aurenhammer, Klein, Lee!