Computational Geometry Lecture
Voronoi Diagrams

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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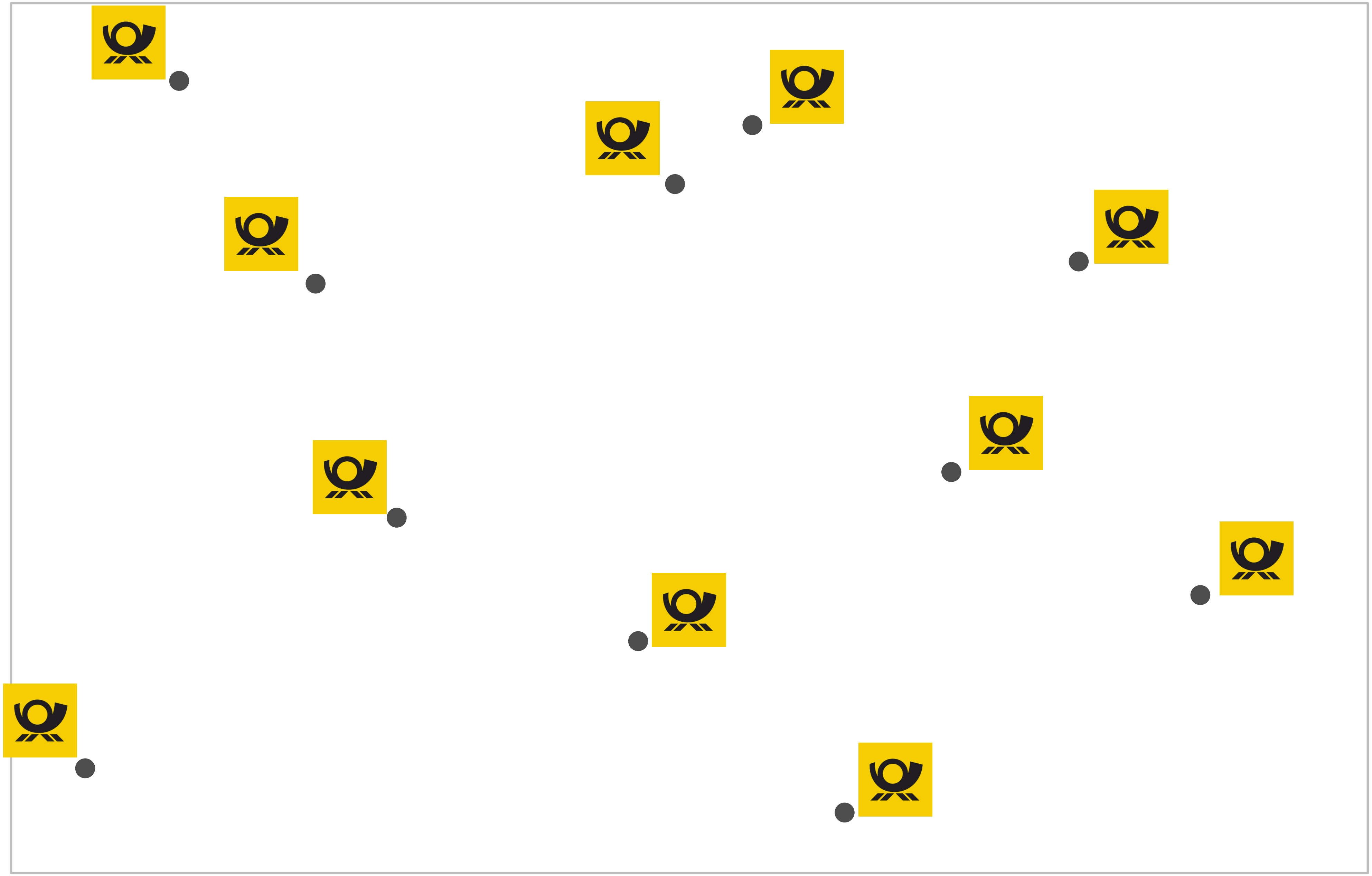
03.06.2014

The Post Office Problem



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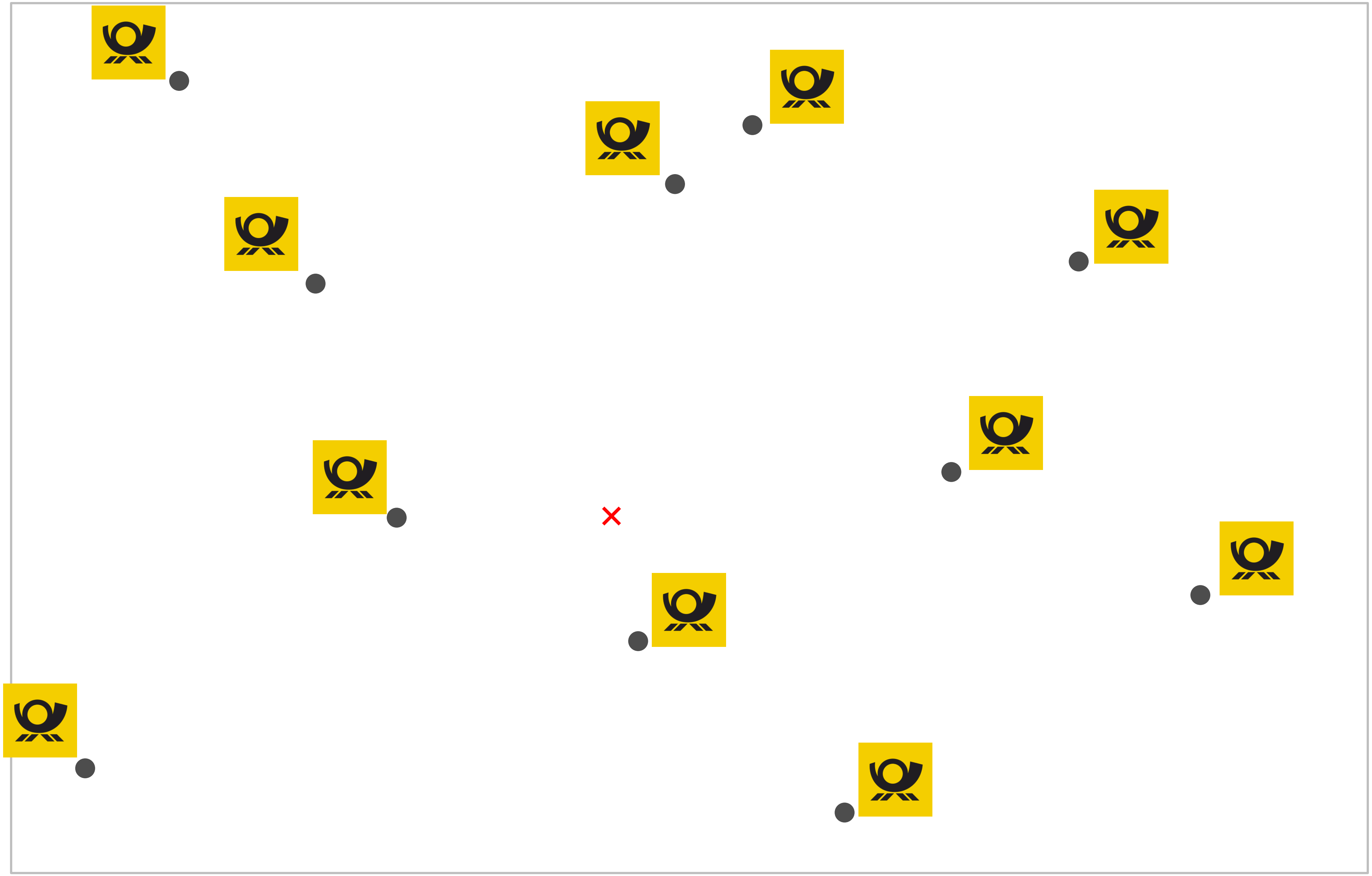

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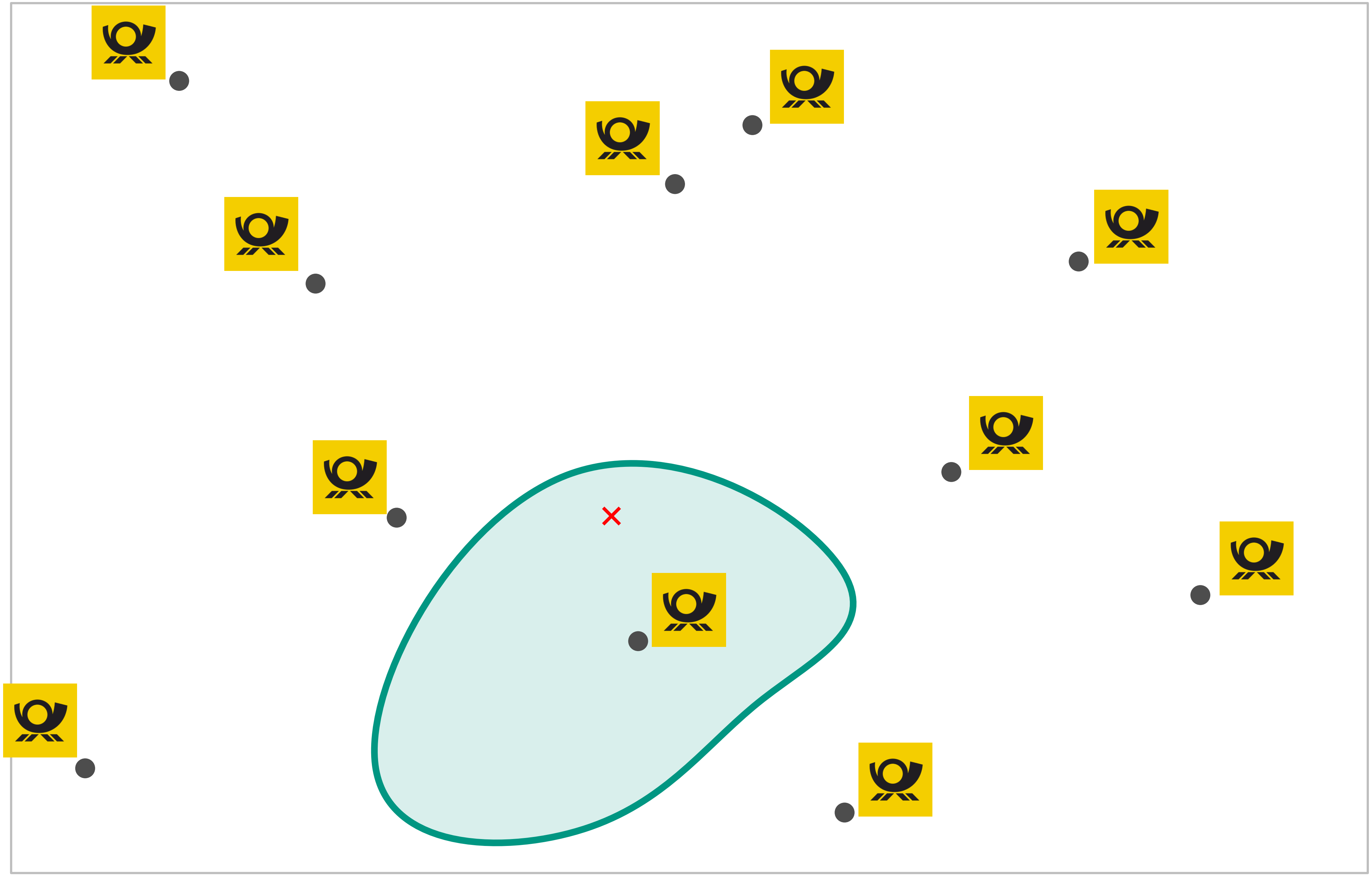

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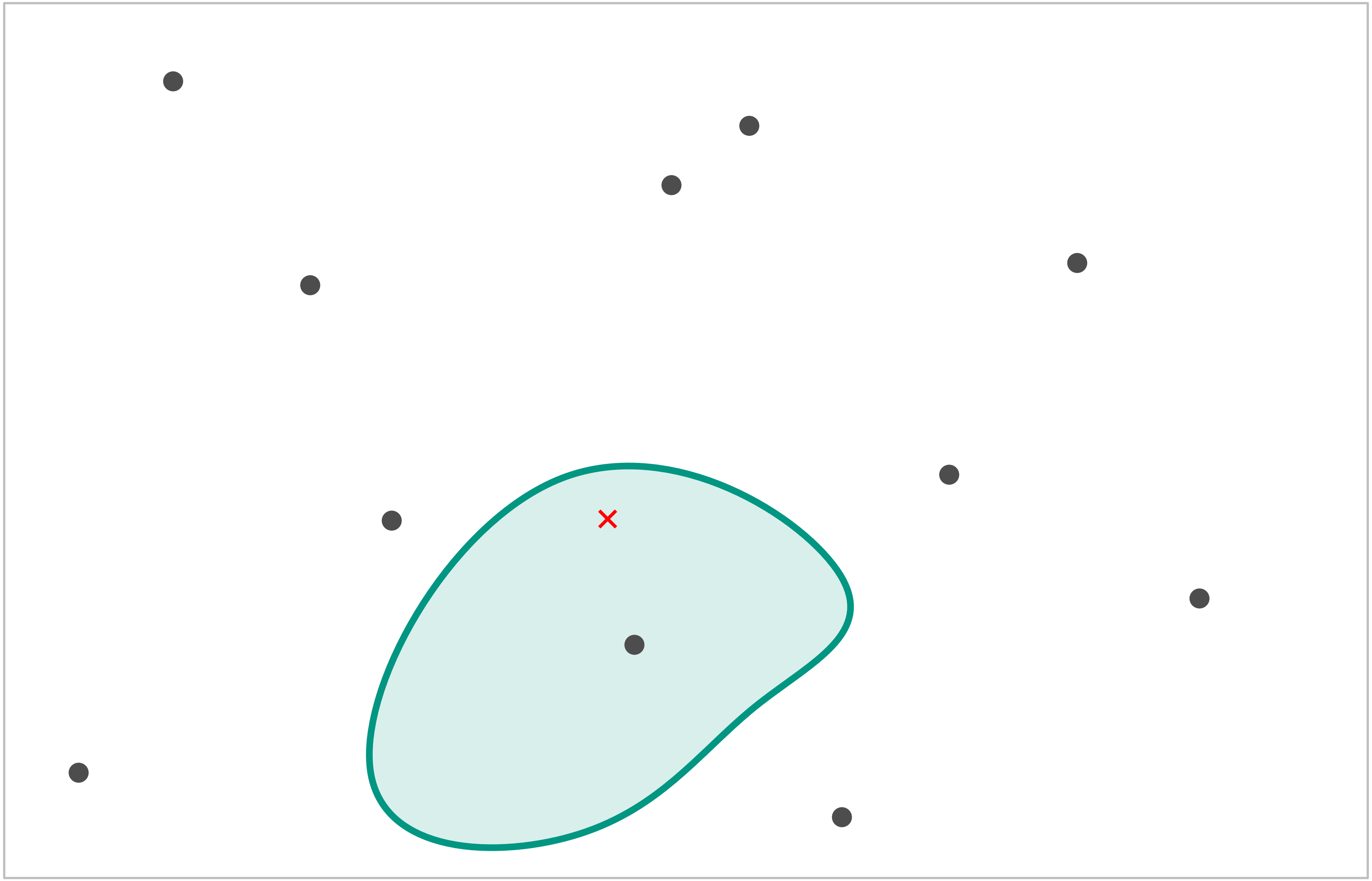

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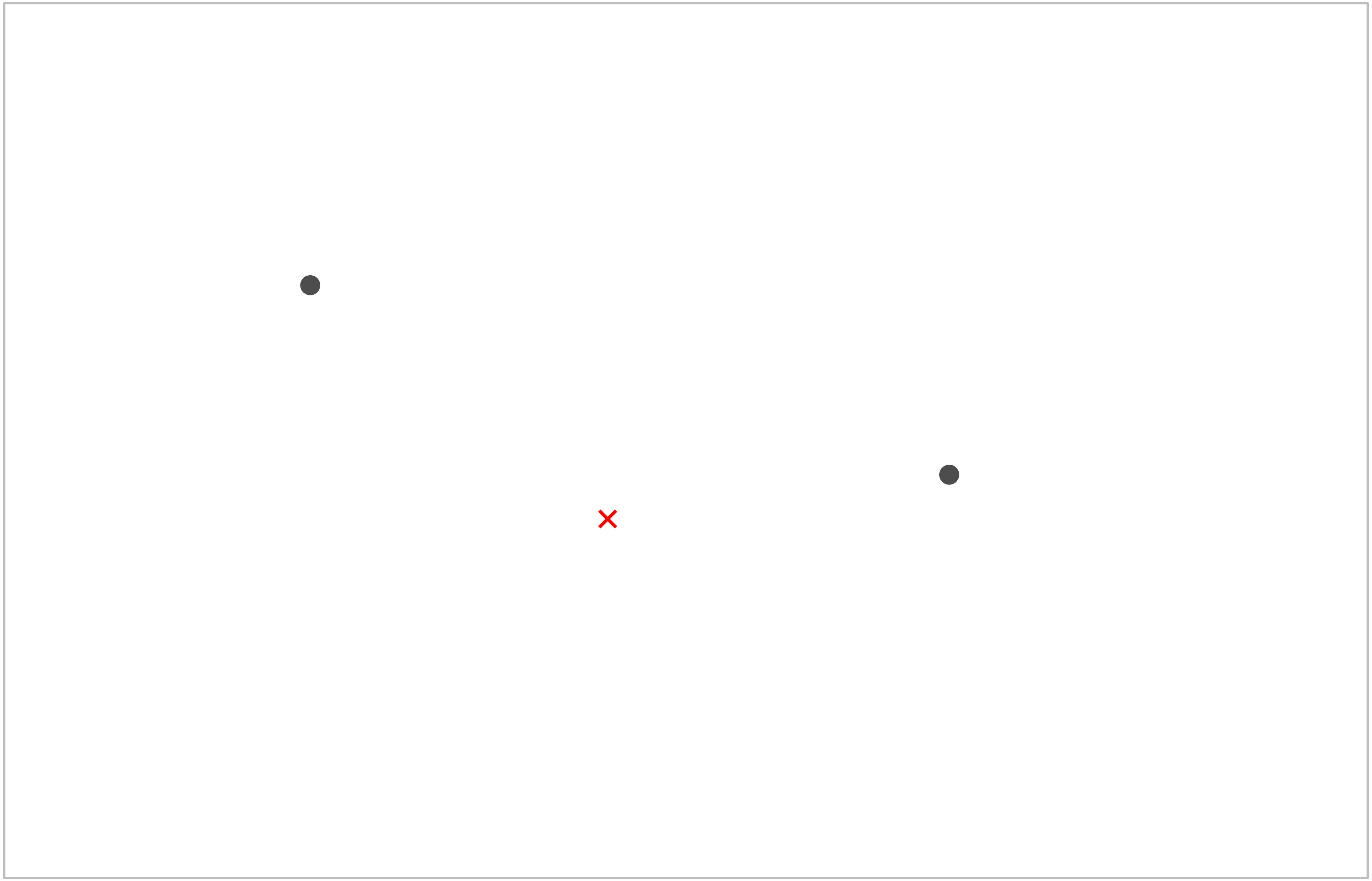

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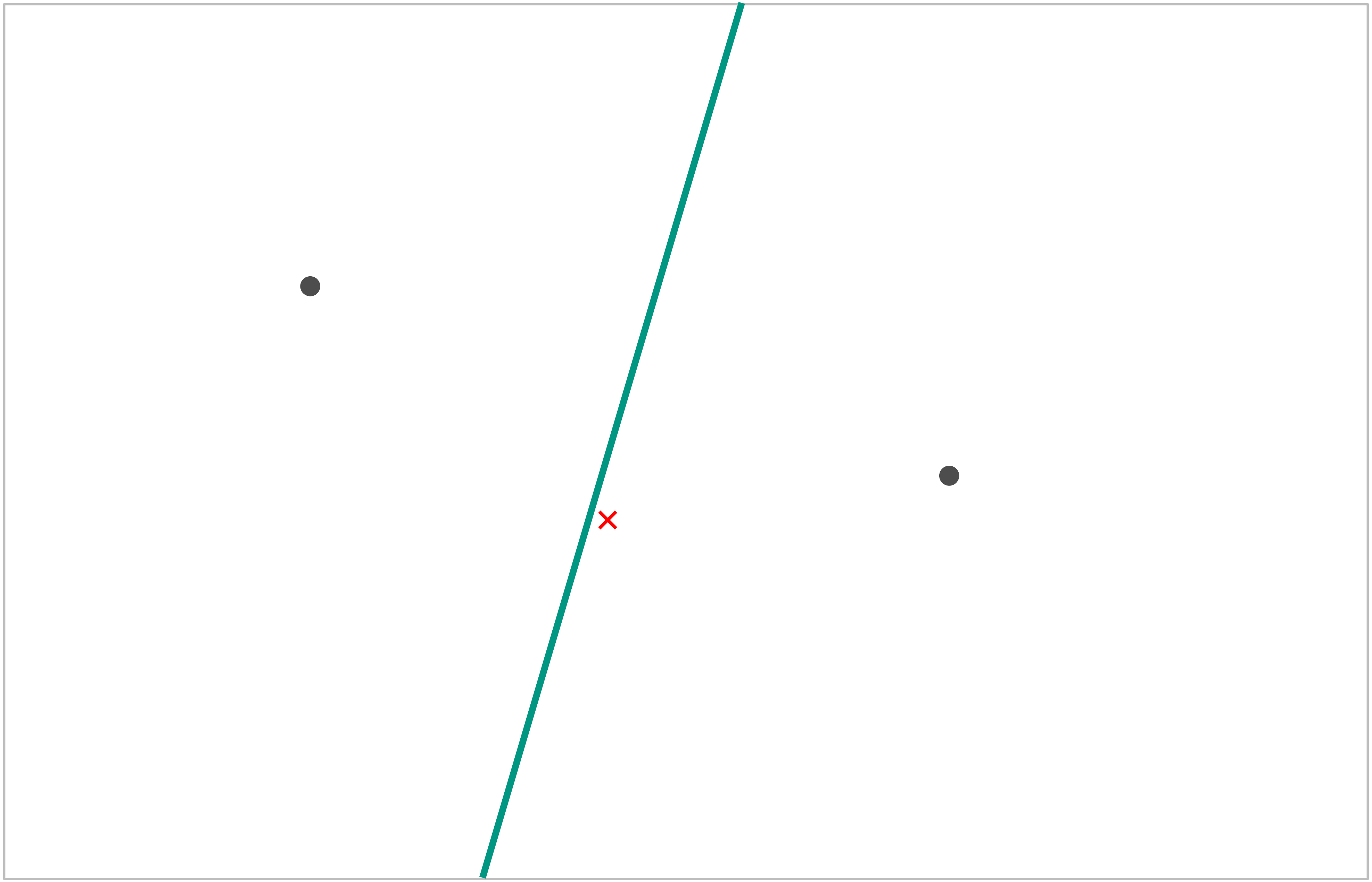

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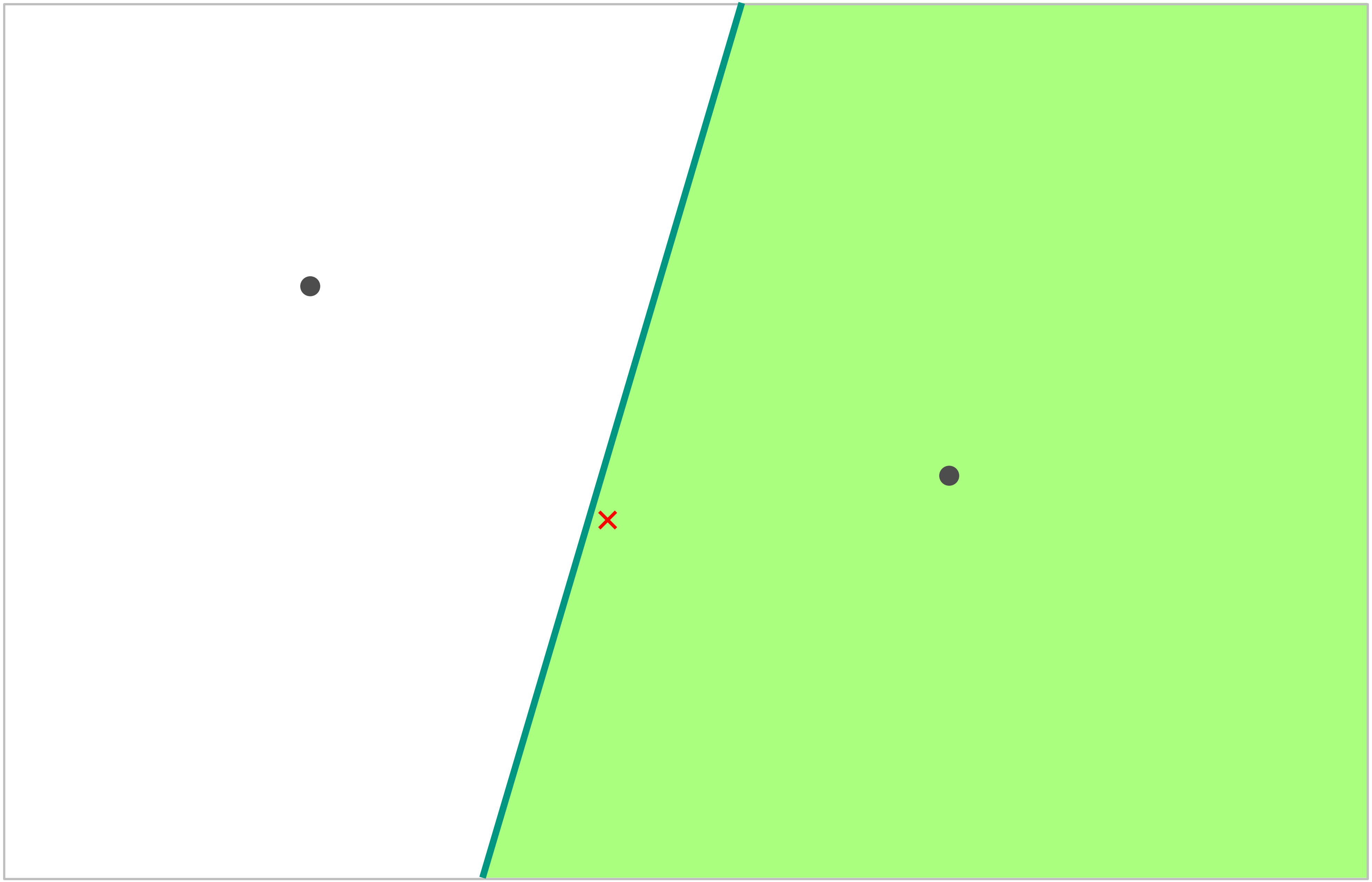

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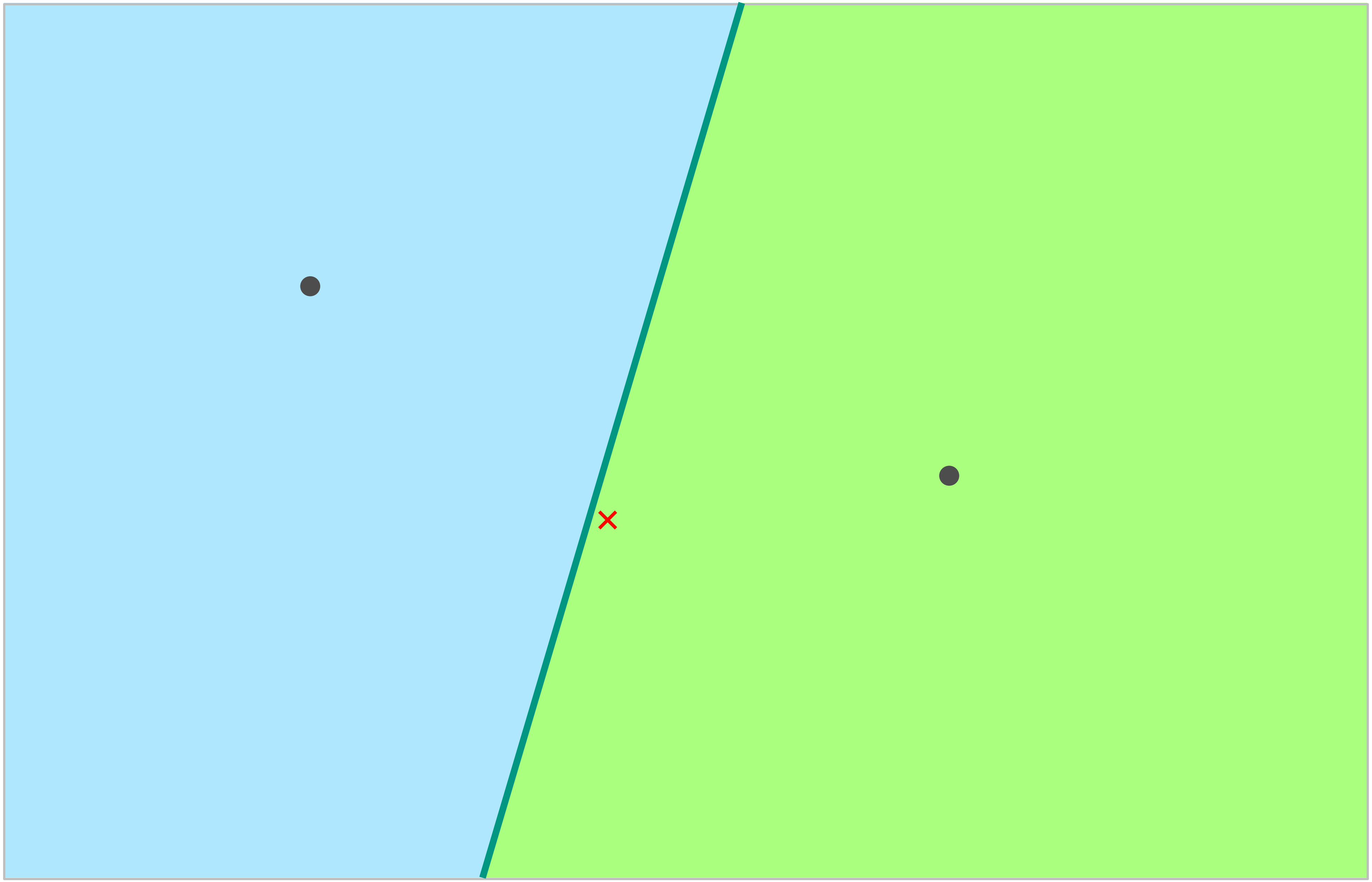

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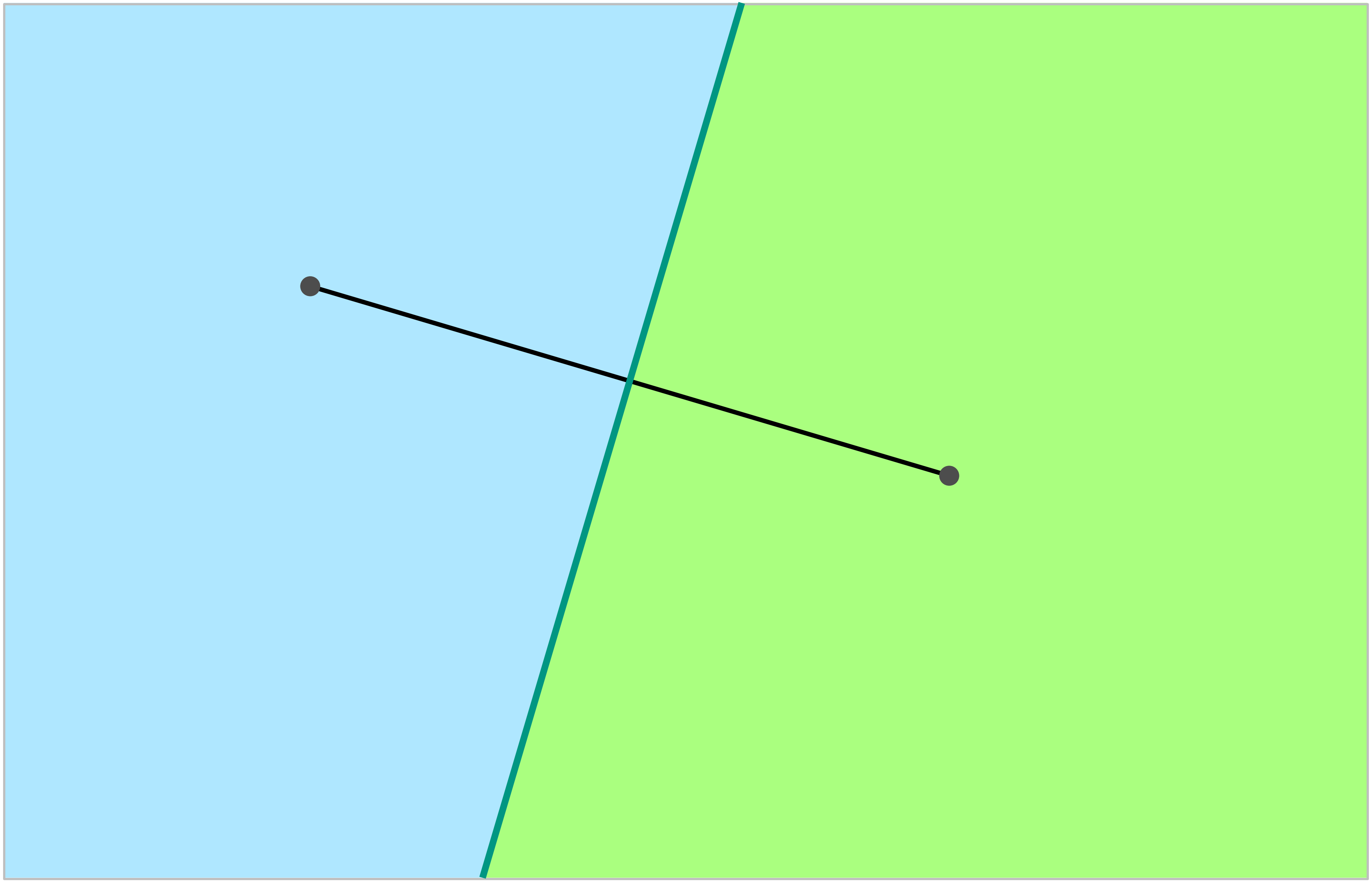

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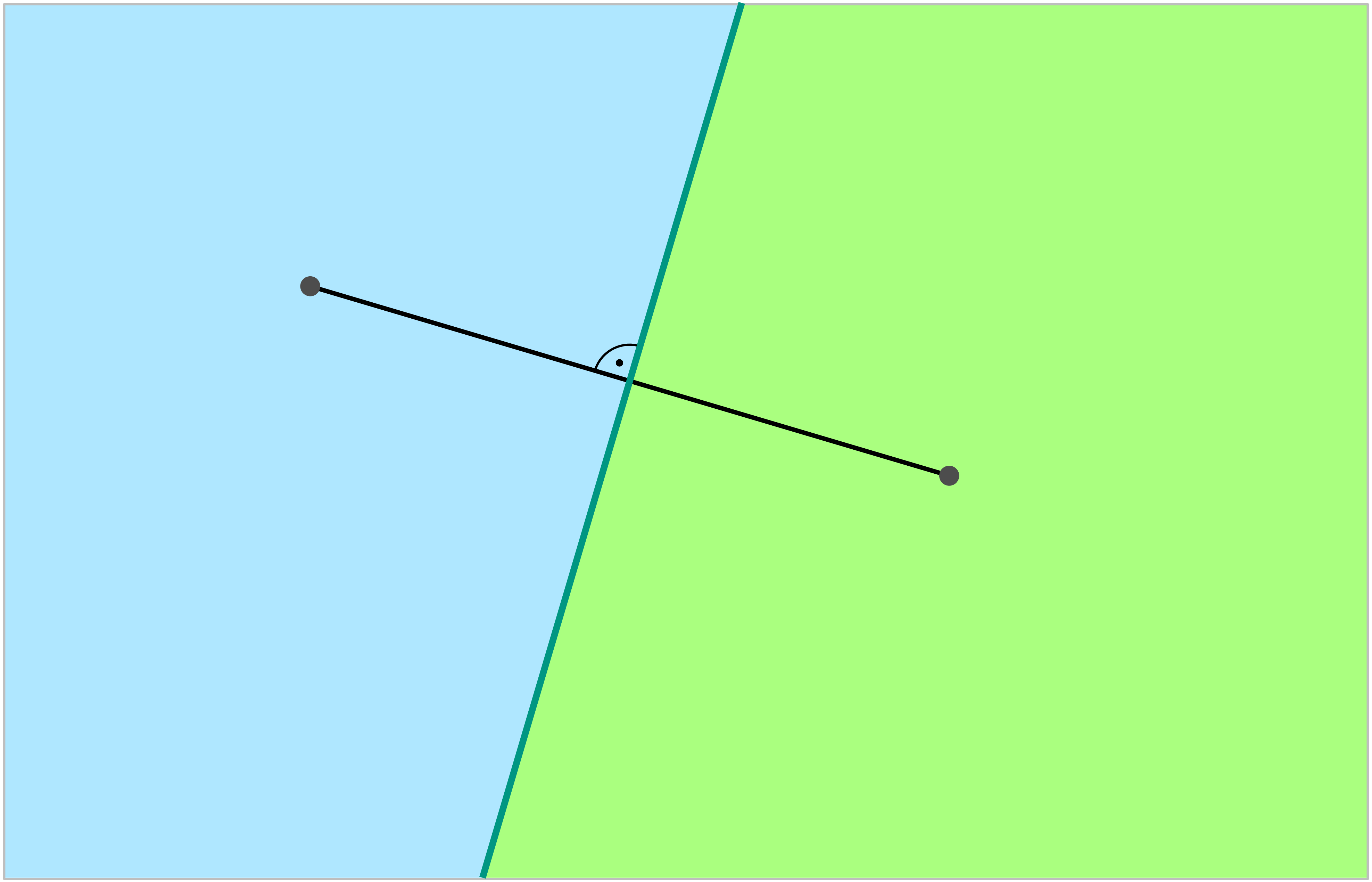

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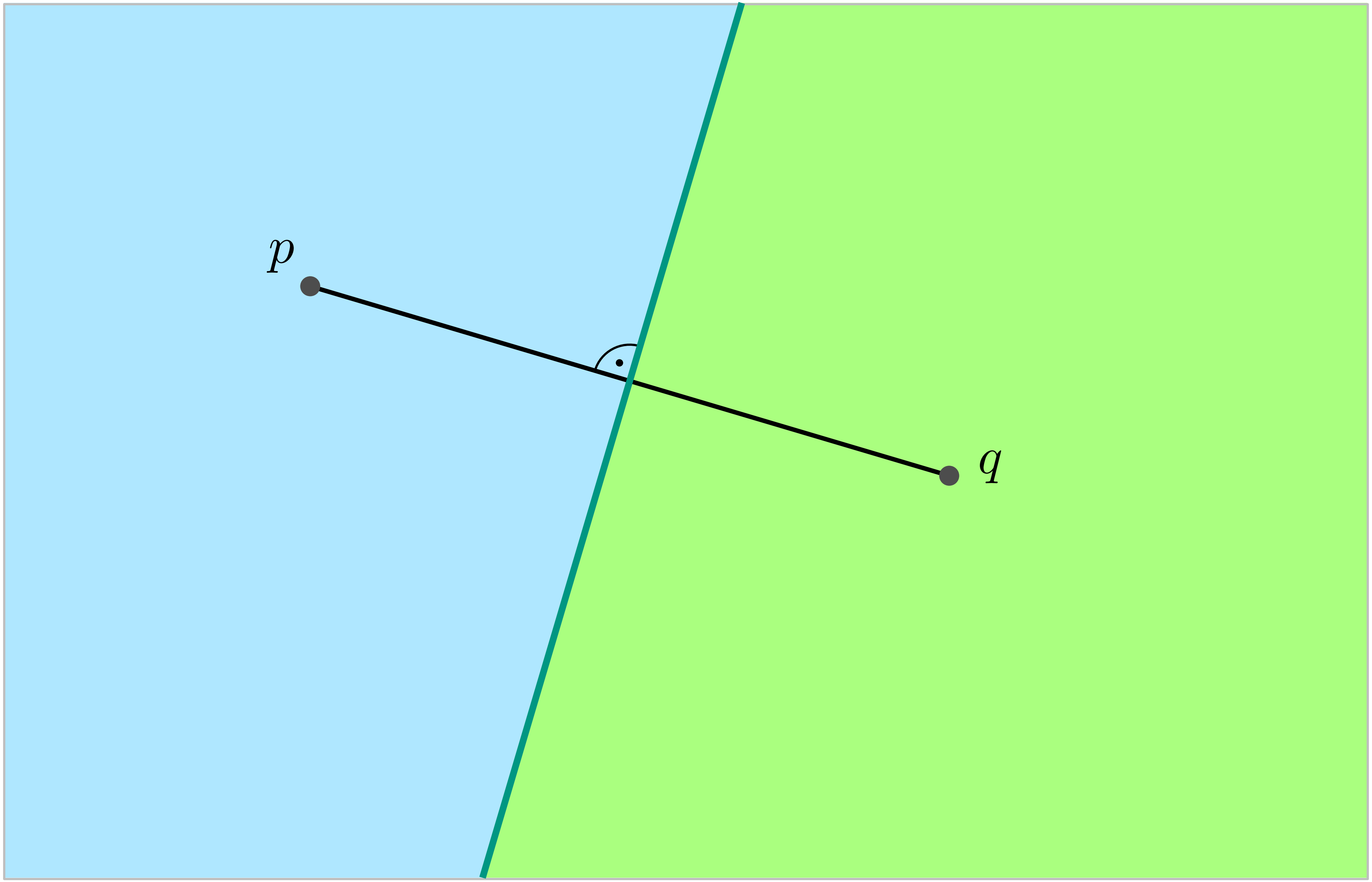

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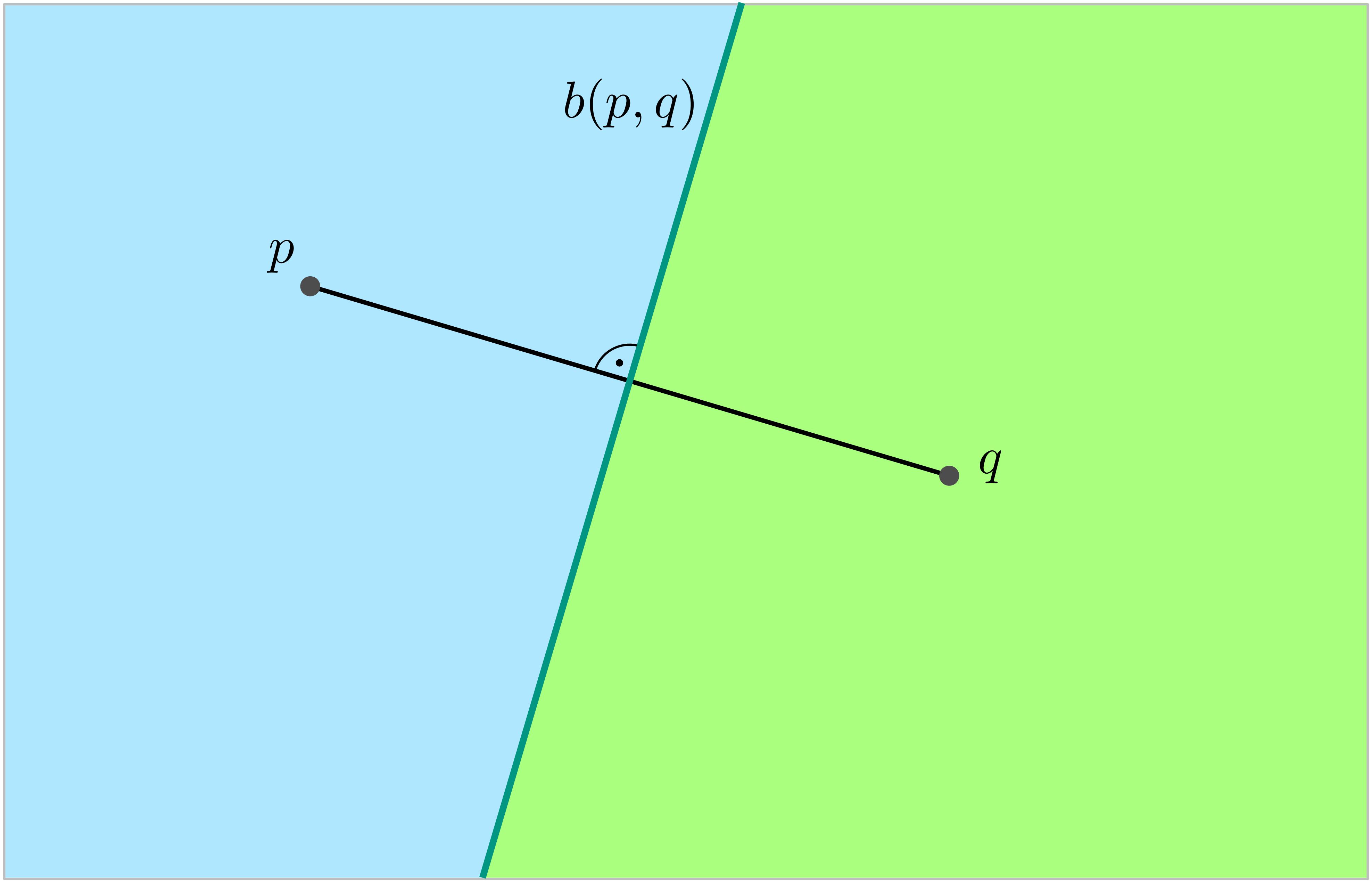

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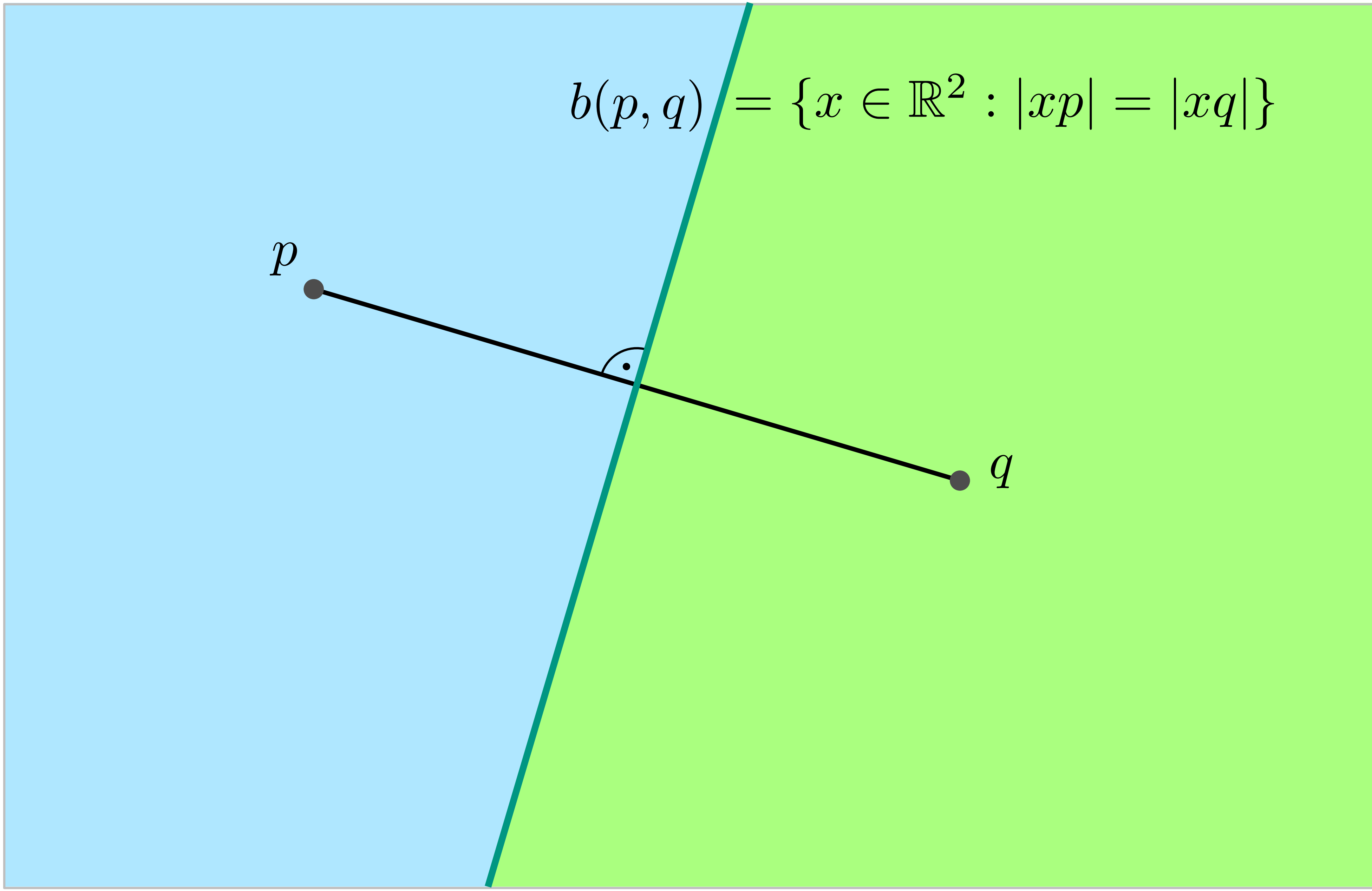

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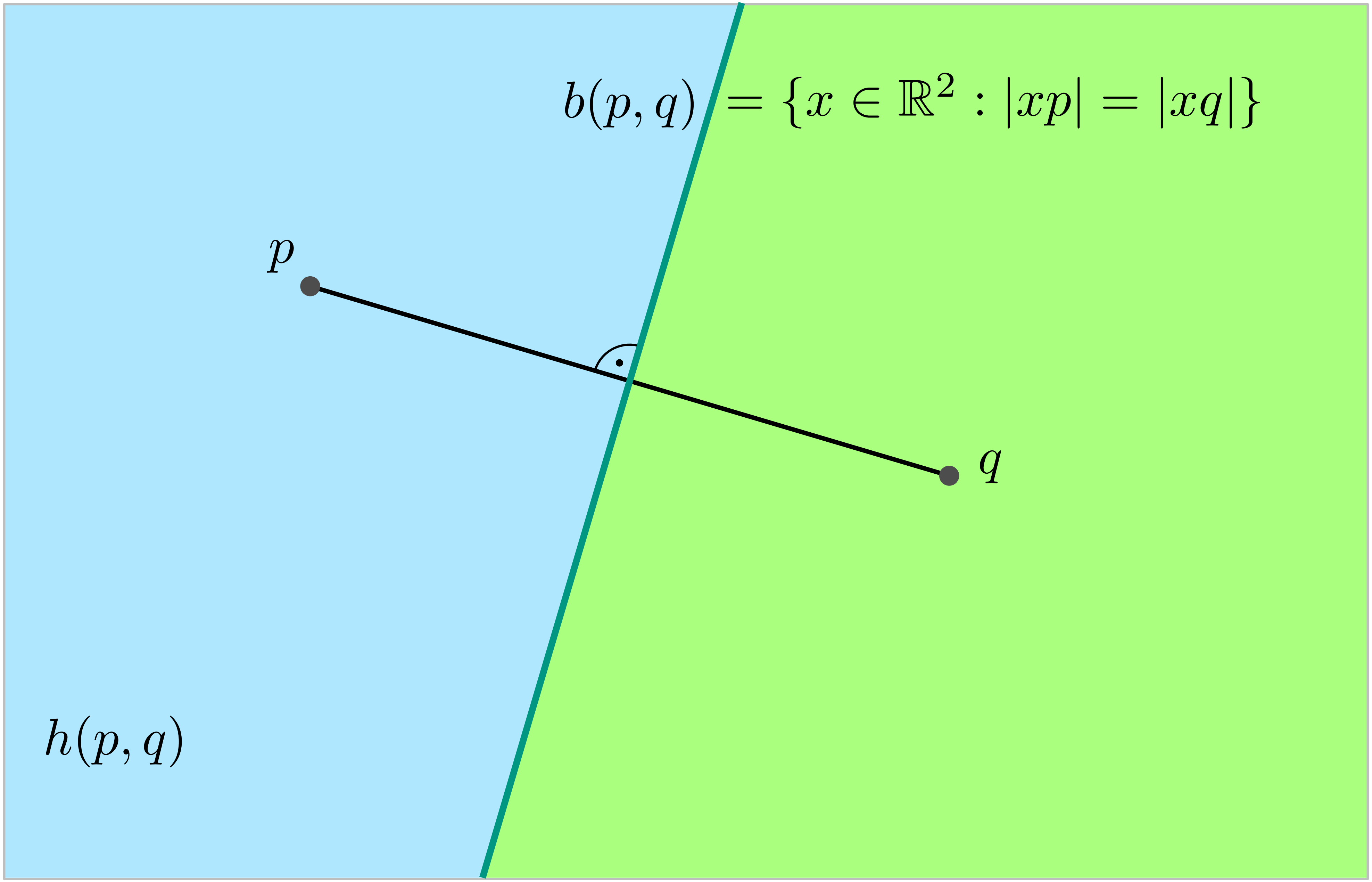

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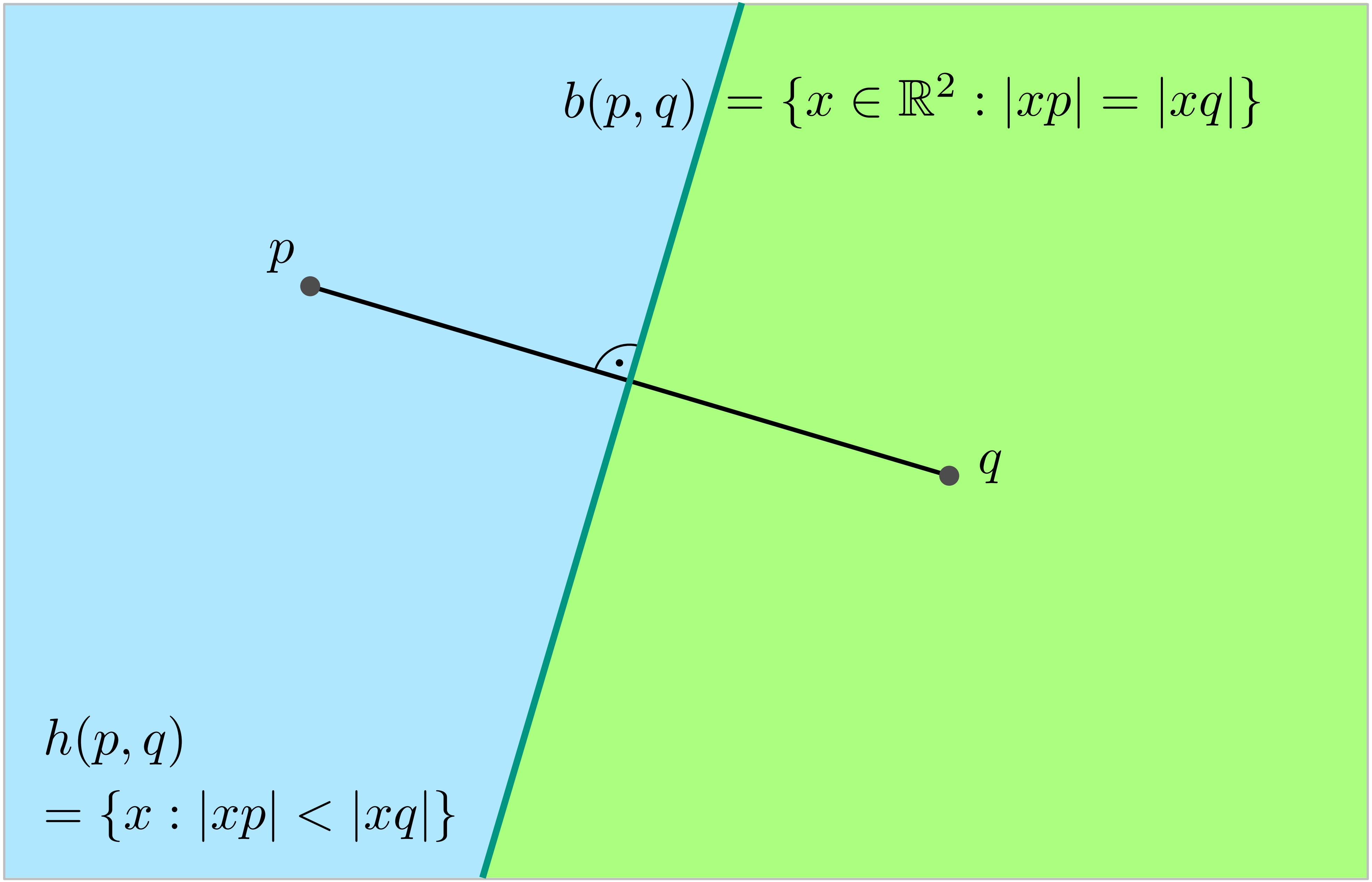

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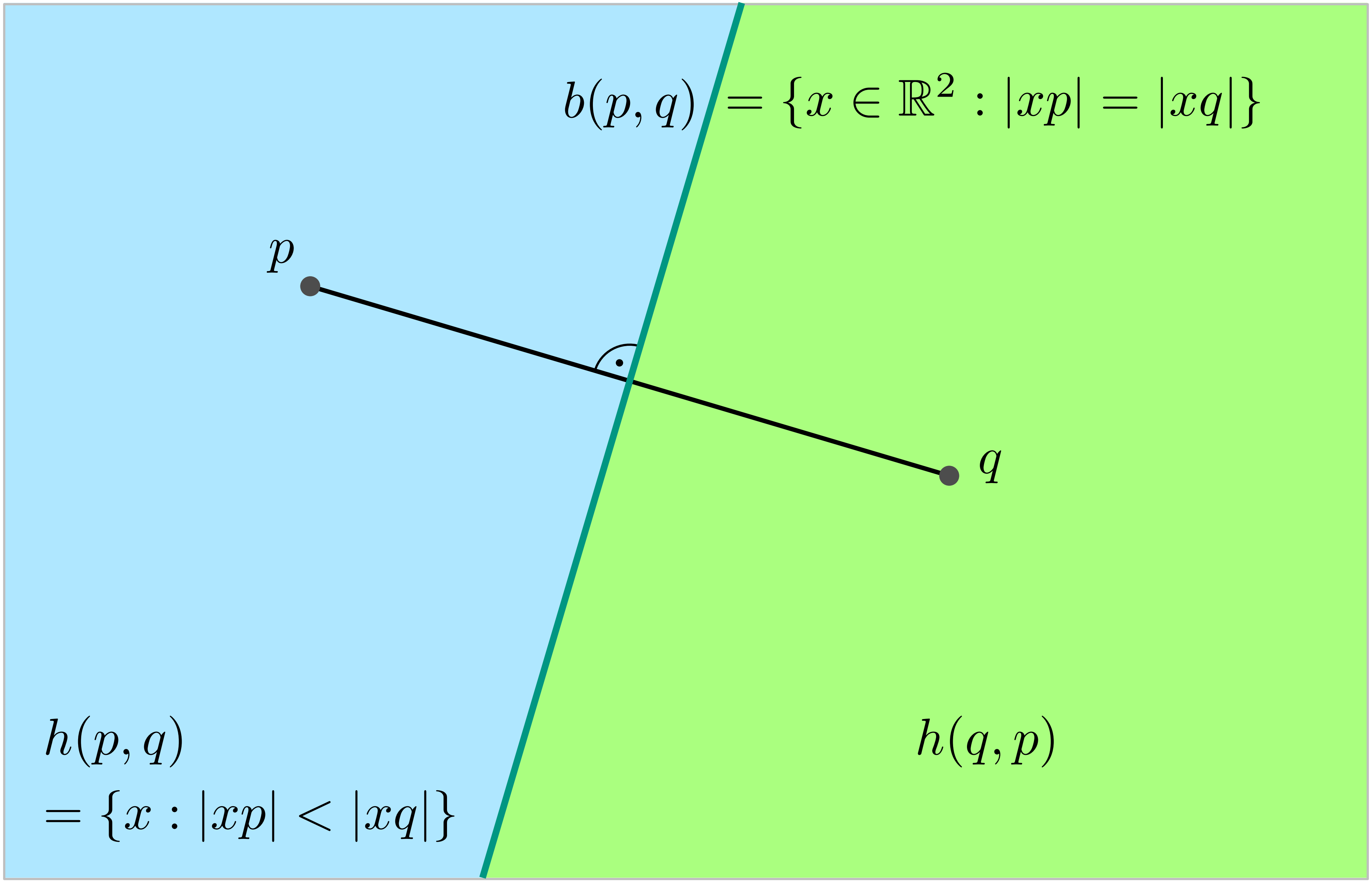

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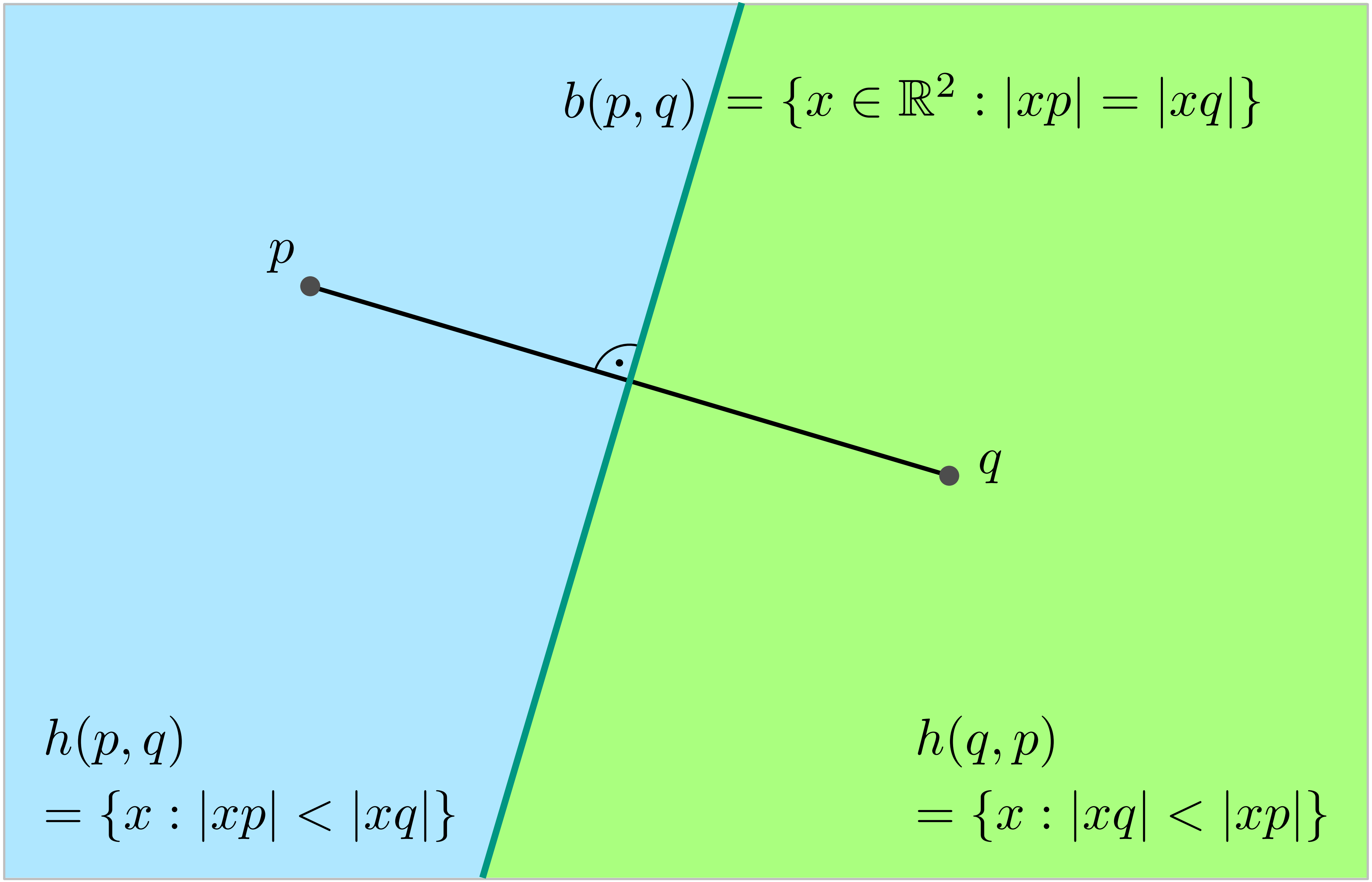

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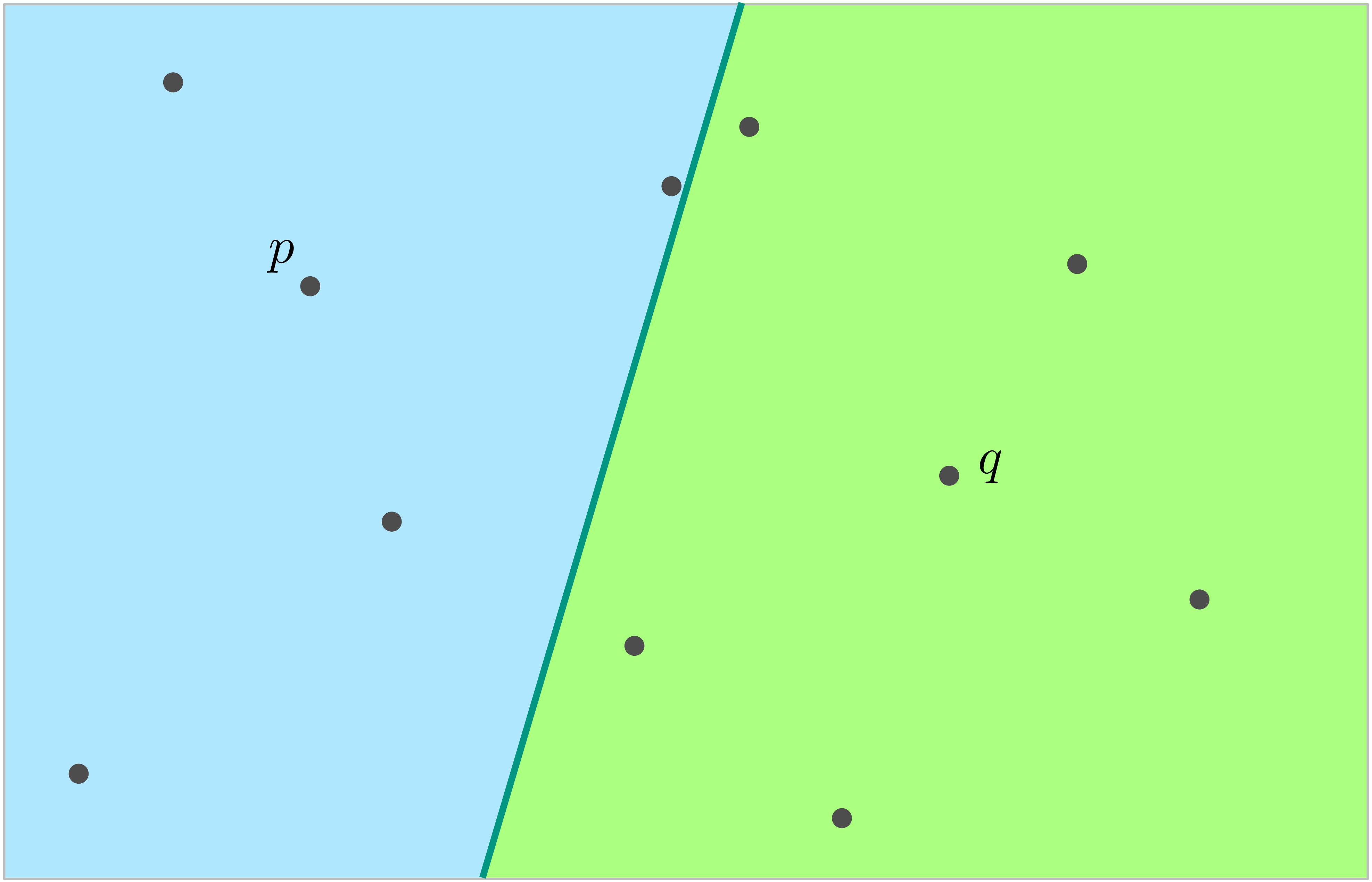

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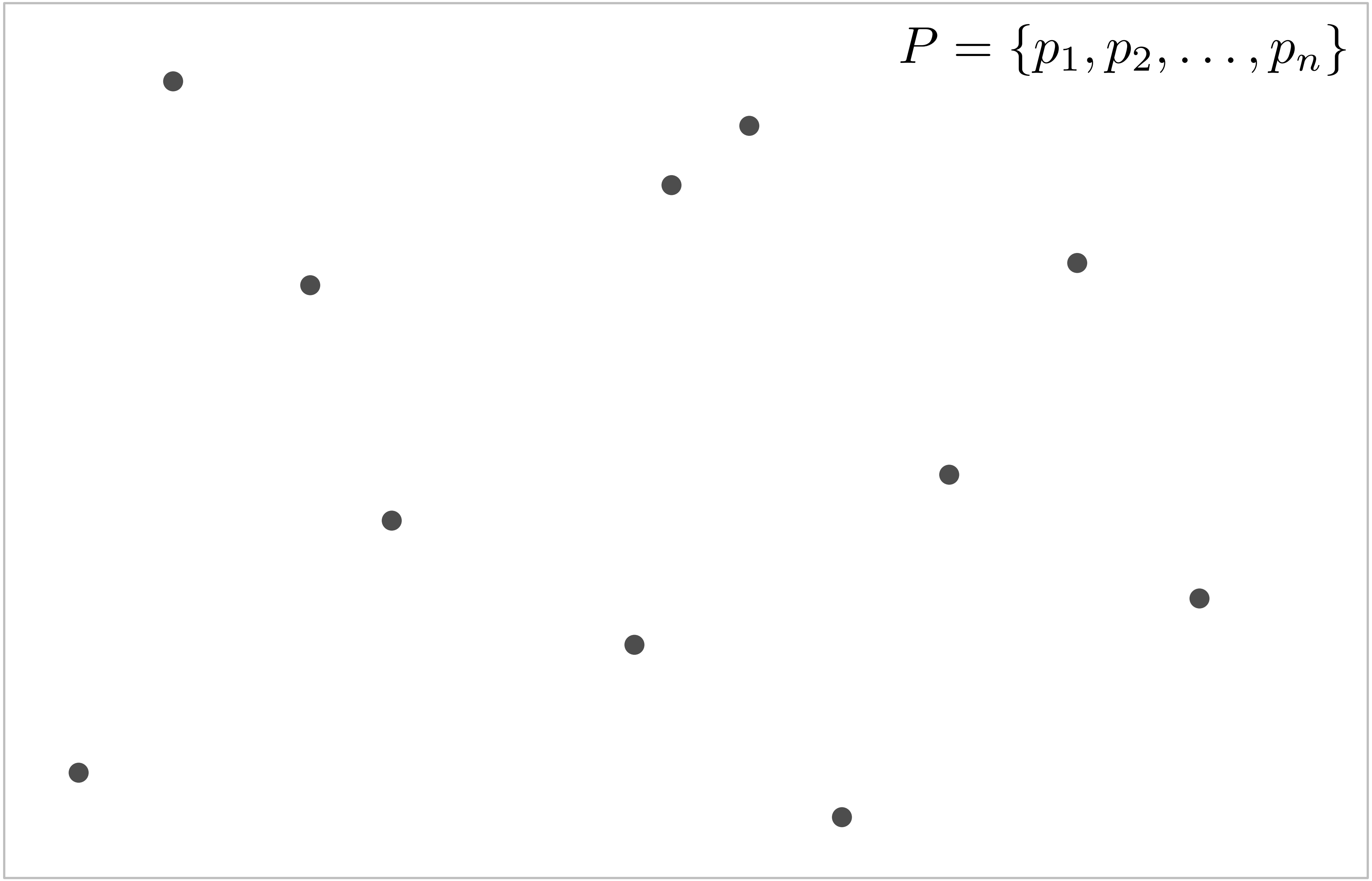

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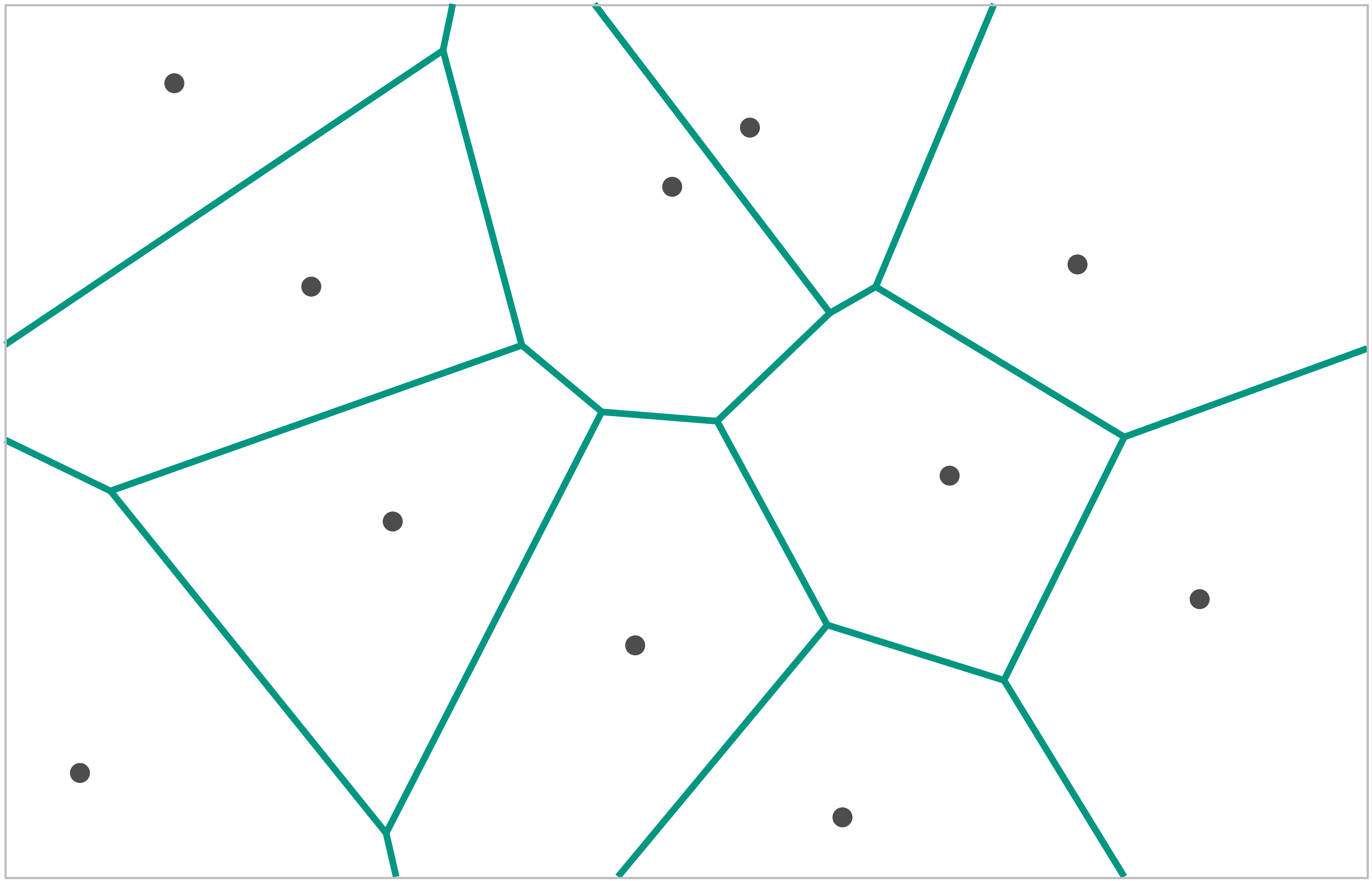

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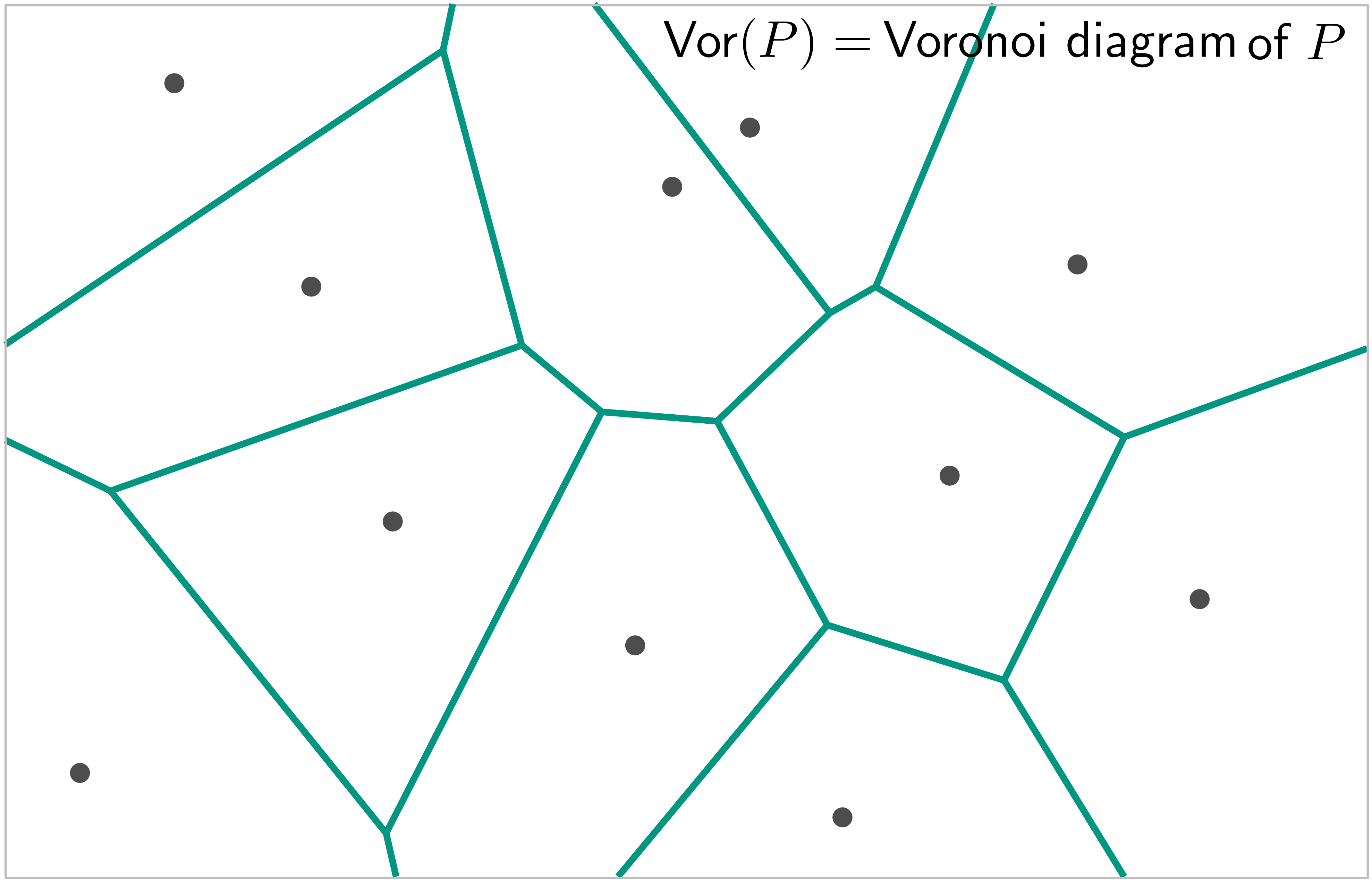

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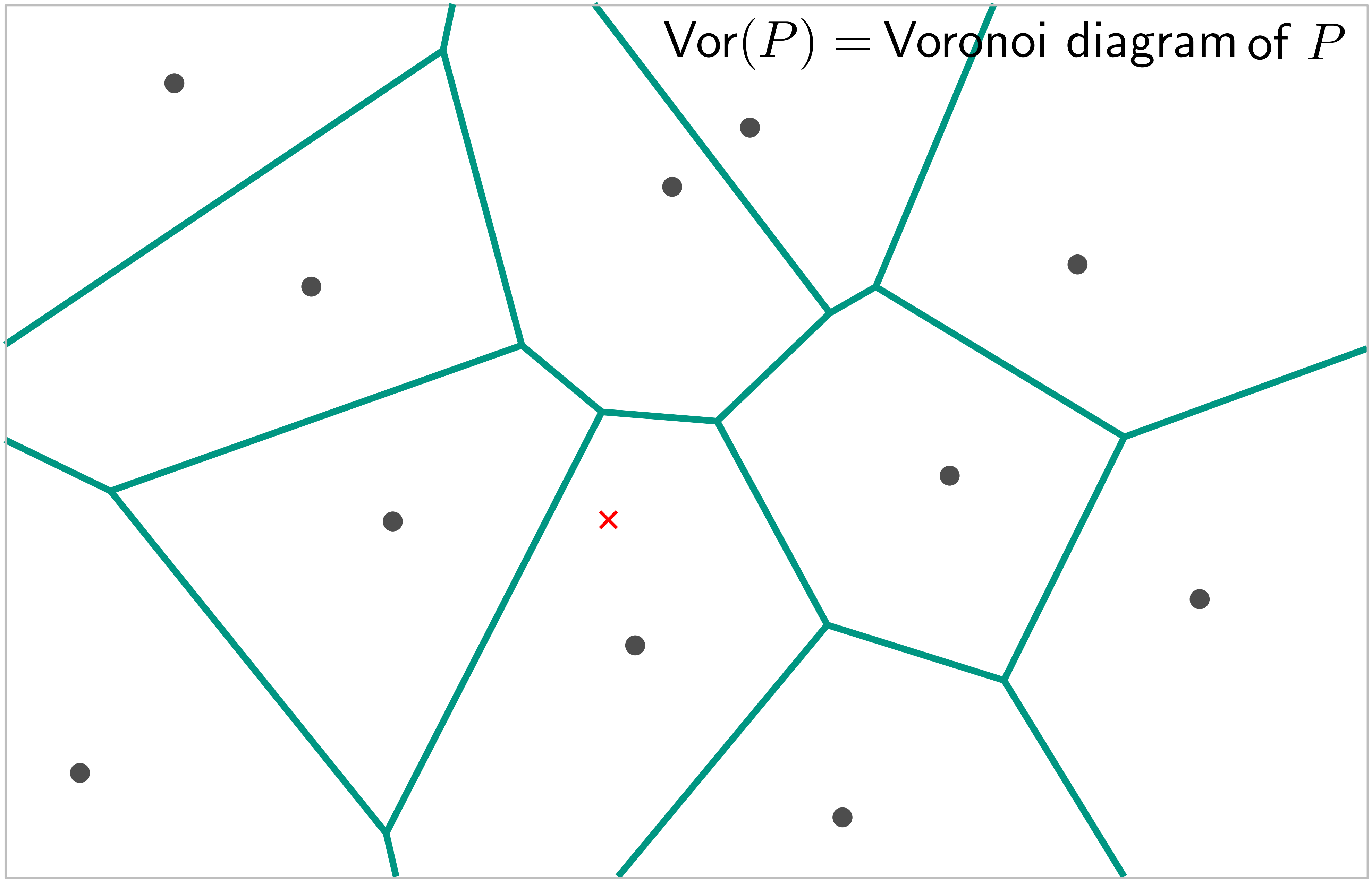

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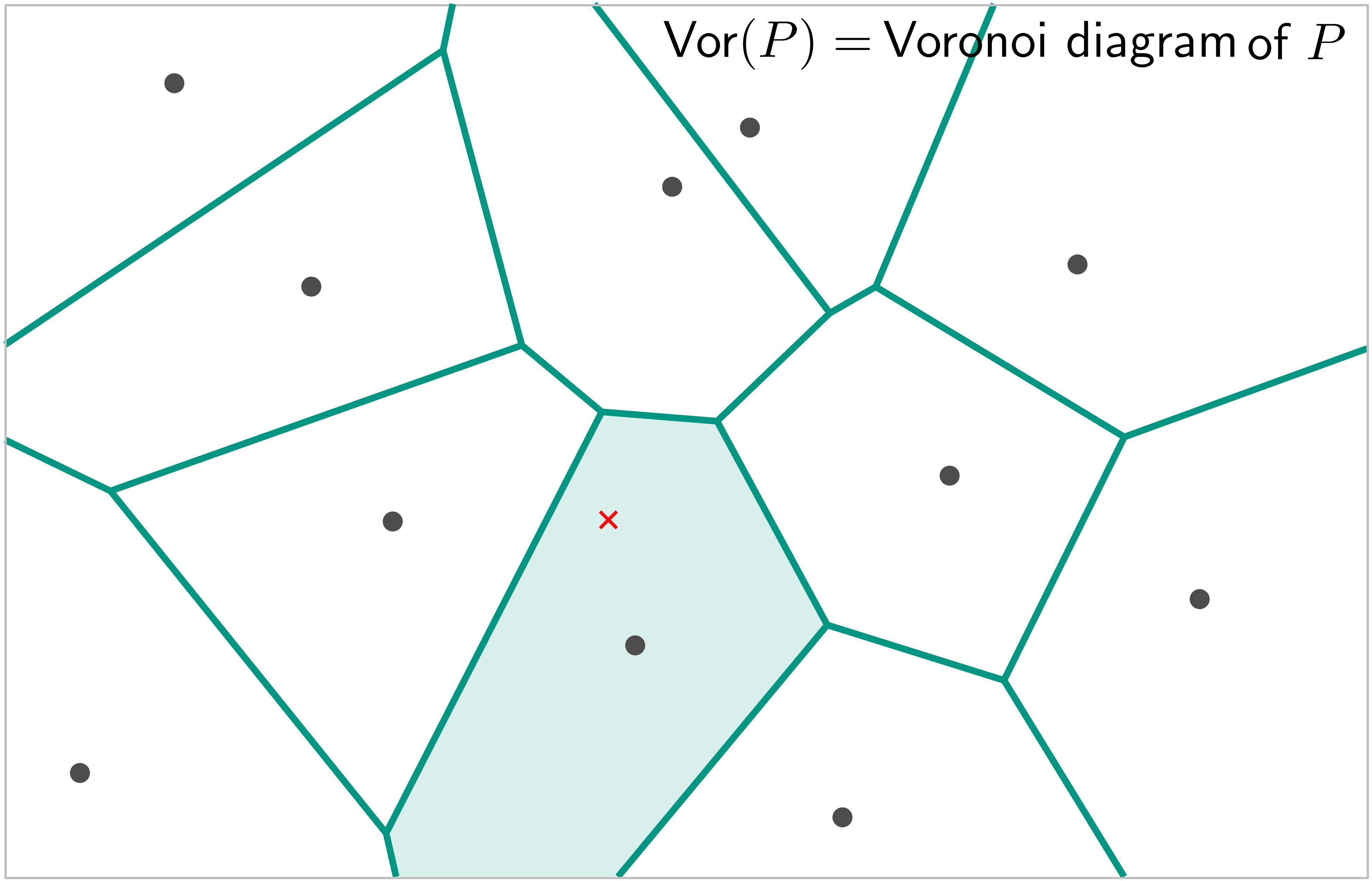

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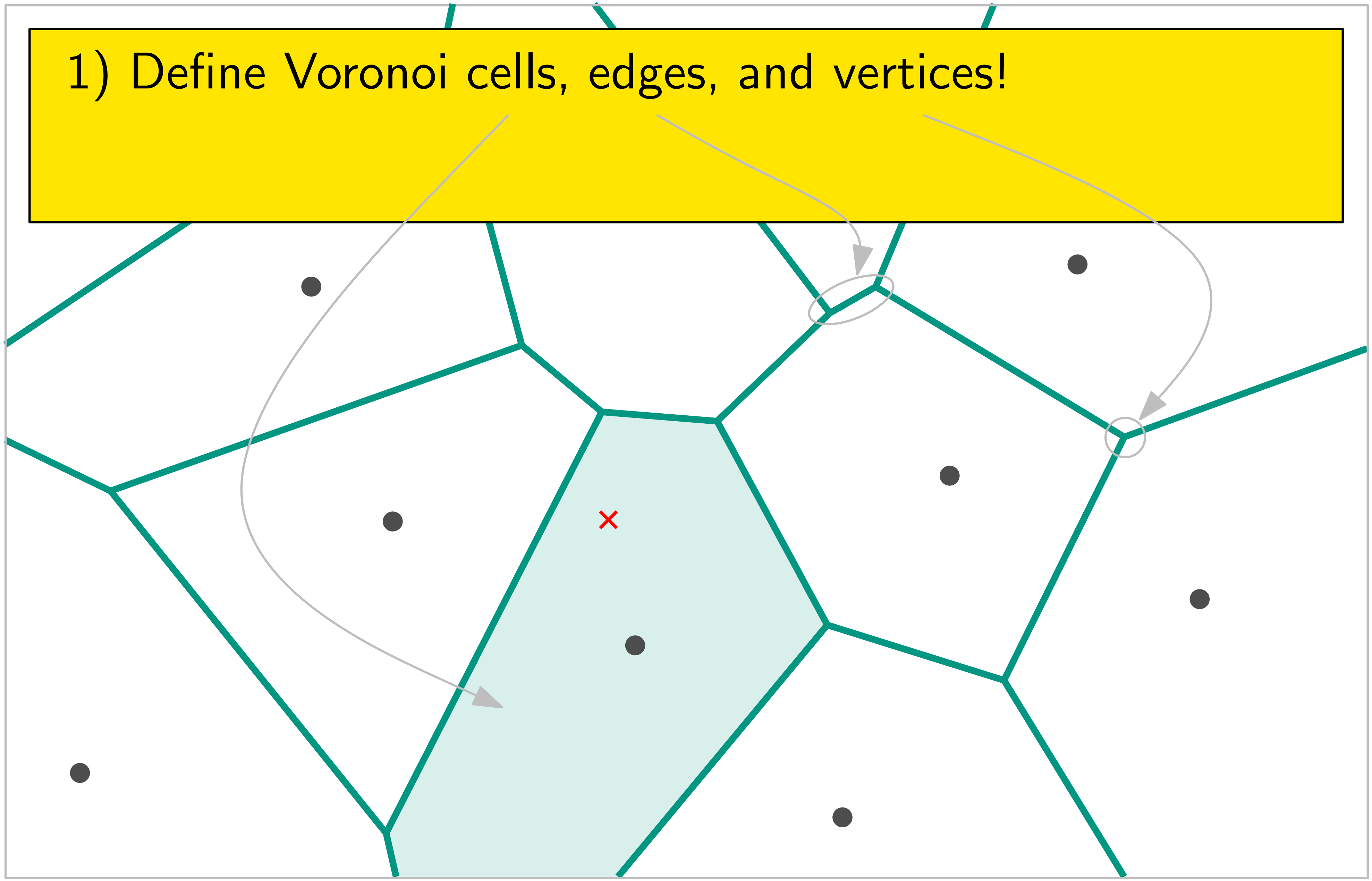

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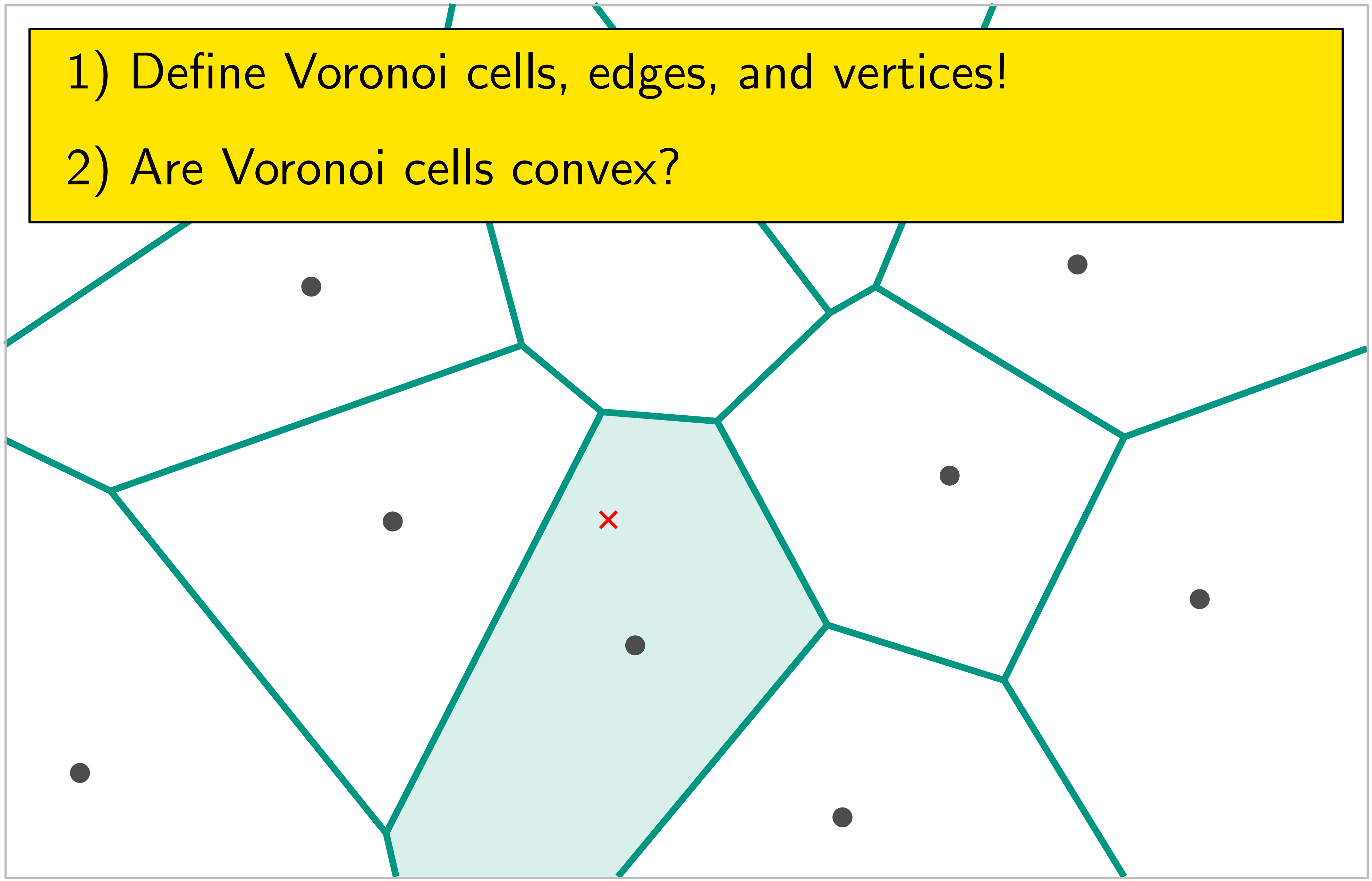

1) Define Voronoi cells, edges, and vertices!

2) Are Voronoi cells convex?

The Voronoi Diagram

Let P be a set of points in the plane and let $p, p', p'' \in P$.

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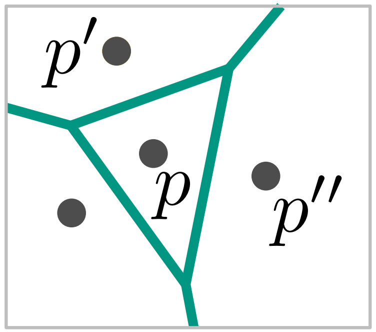

Vor(P)

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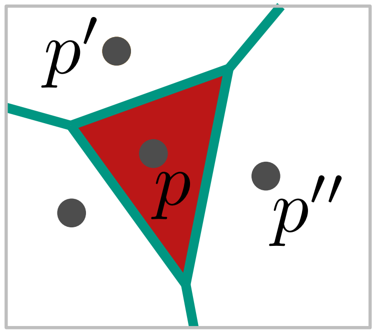

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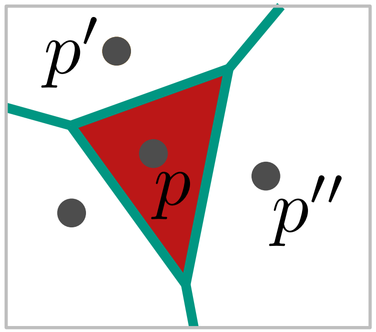

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- Voronoi cell

$\mathcal{V}(\{p\}) =$

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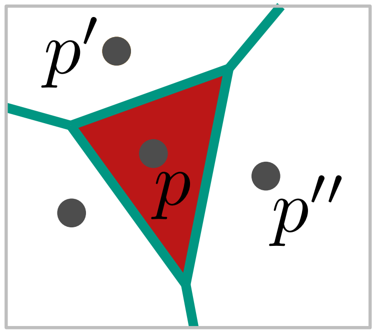

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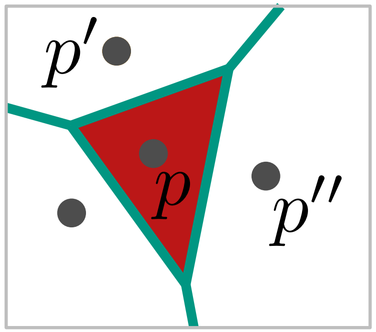

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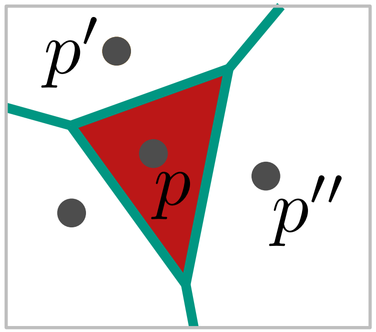

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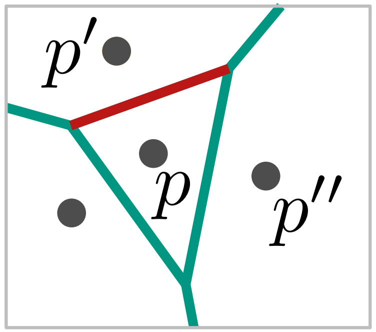

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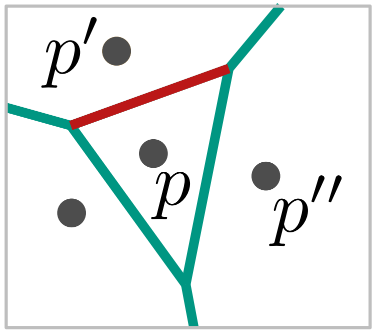
Voronoi Diagram  Vor(P)

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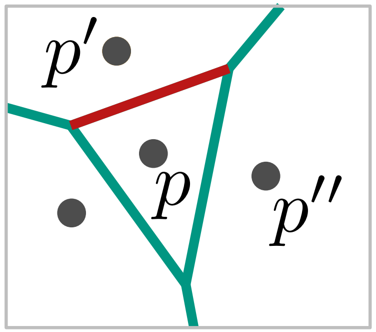

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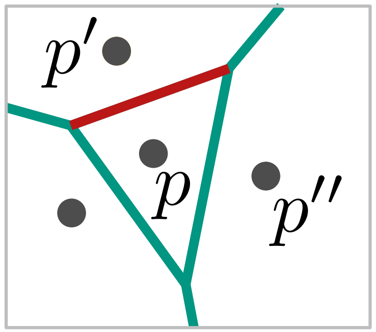

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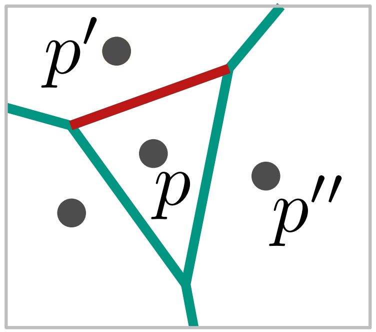

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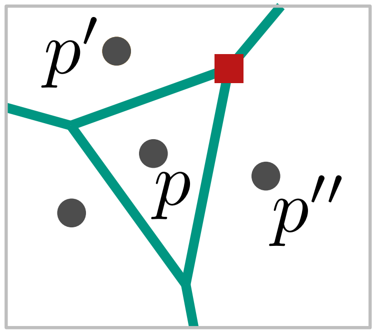

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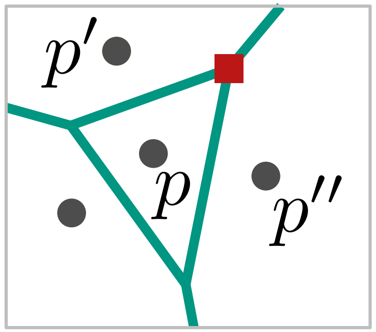

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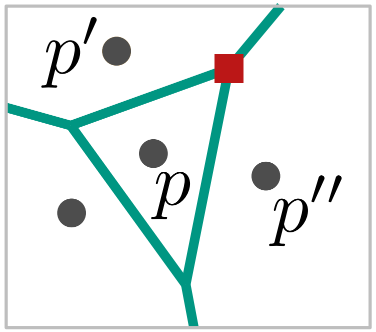

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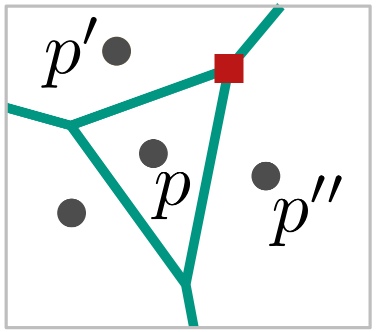

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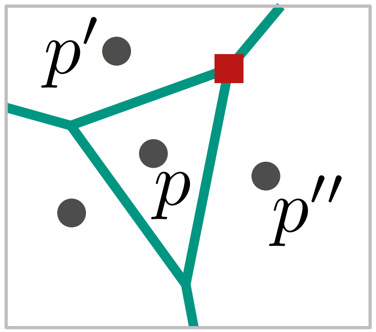

Vor(P) → subdivision

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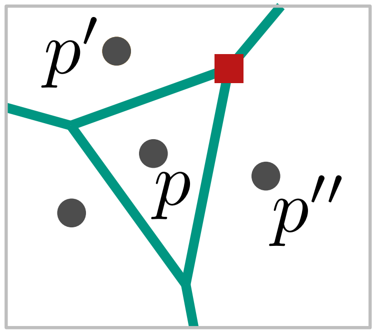

Vor(P) → subdivision
→ geometric graph

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Properties

Theorem 1: Let $P \subset \mathbb{R}^2$ be a set of n points. If all points are collinear, then $\mathrm{Vor}(P)$ consists of $n-1$ parallel lines. Otherwise $\mathrm{Vor}(P)$ is connected and its edges are either segments or half lines.

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Theorem 2: Let $P \subset \mathbb{R}^2$ be a set on n points. $\mathrm{Vor}(P)$ has at most $2n-5$ nodes and $3n-6$ edges.

Properties

Theorem 1: Let $P \subset \mathbb{R}^2$ be a set of n points. If all points are collinear, then $\mathrm{Vor}(P)$ consists of $n-1$ parallel lines. Otherwise $\mathrm{Vor}(P)$ is connected and its edges are either segments or half lines.

How many edges and vertices $\mathrm{Vor}(P)$ has?

Find a set P so that a cell in $\mathrm{Vor}(P)$ has linear complexity.

Can this happen with (almost) all cells?

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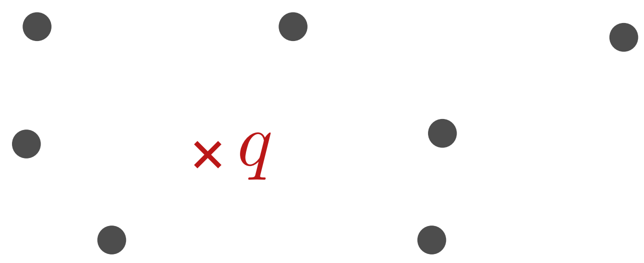
Exercise!

Characterization

Definition: Let q be a point. Define $C_P(q)$ to be the points in P that lie on the empty circle with center q.



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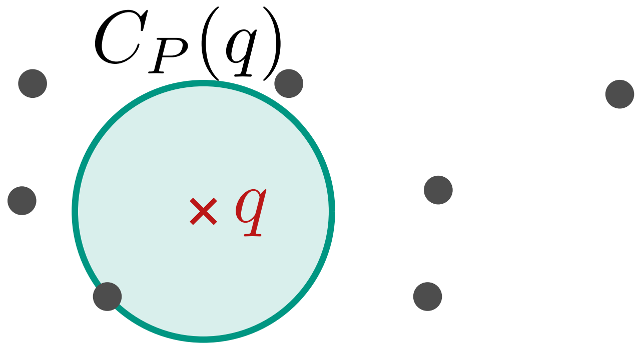

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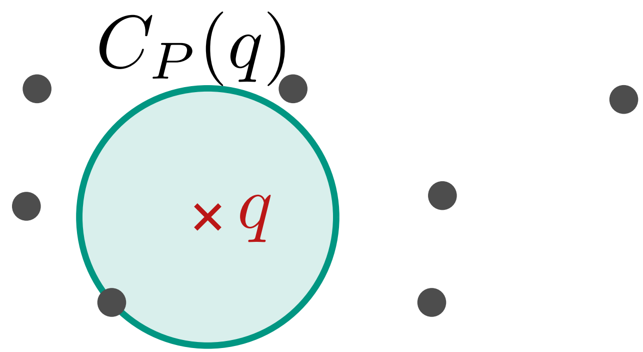
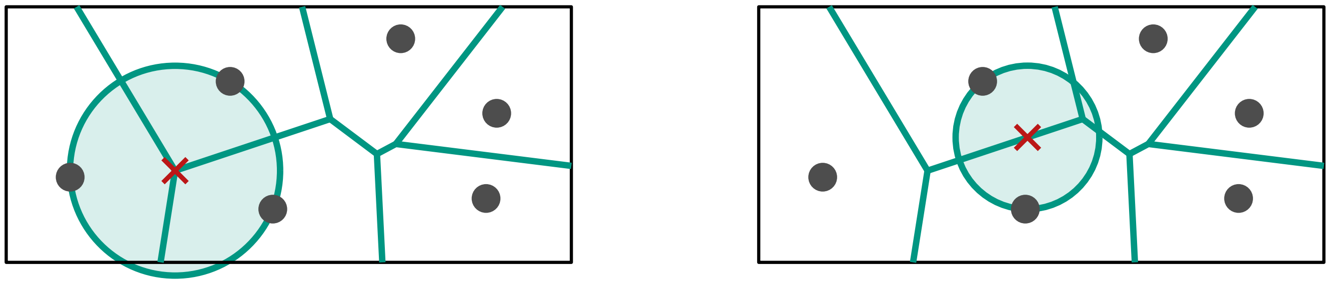


Theorem 3:
- A point q is a Voronoi vertex $\Leftrightarrow |C_P(q) \cap P| \geq 3$,
- the bisector $b(p_i, p_j)$ defines a Voronoi edge $\Leftrightarrow \exists q \in b(p_i, p_j)$ with $C_P(q) \cap P = \{p_i, p_j\}$.



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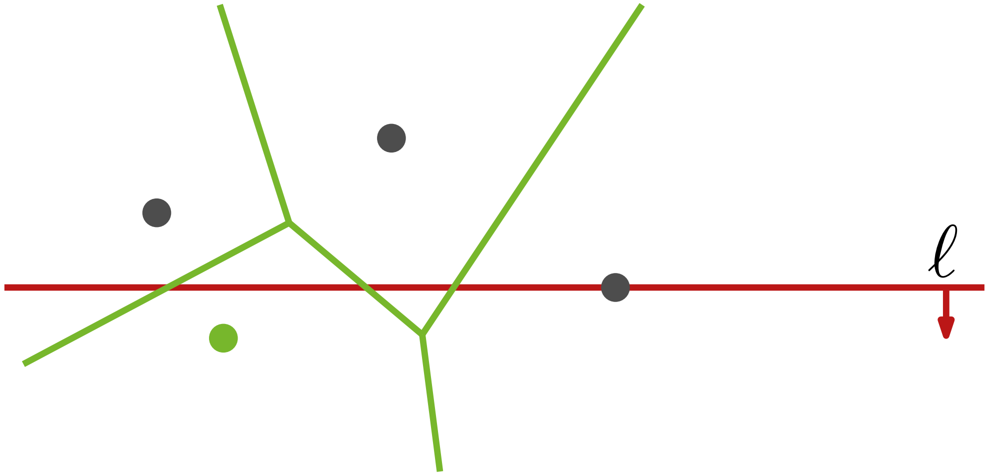

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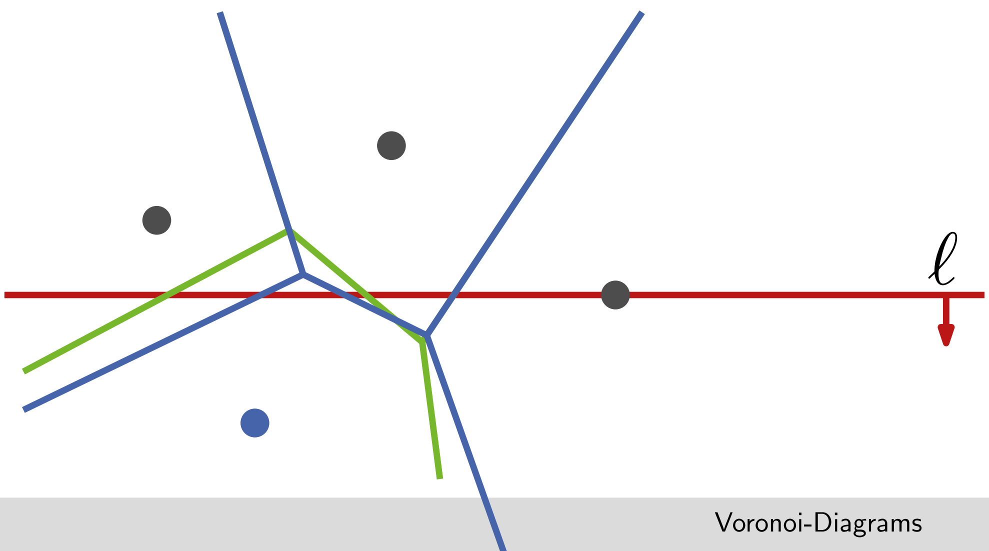

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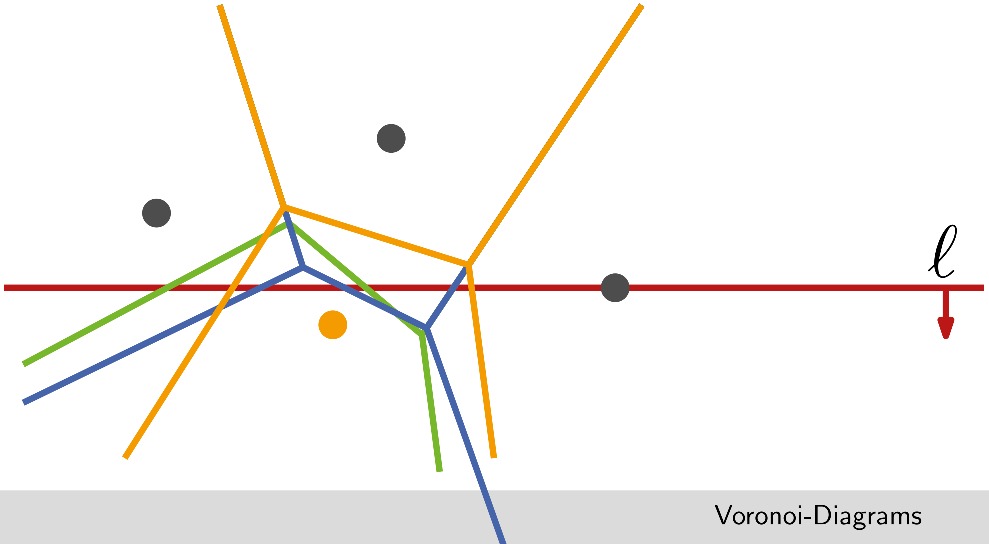

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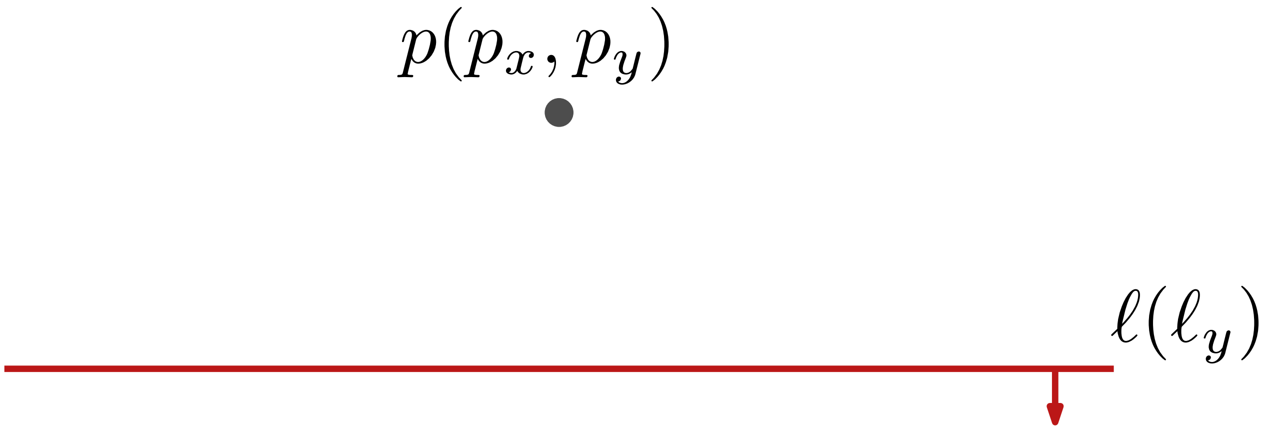

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$$p(p_x, p_y)$$

Points closer to p than ℓ are already processed.

$$\ell(\ell_y)$$

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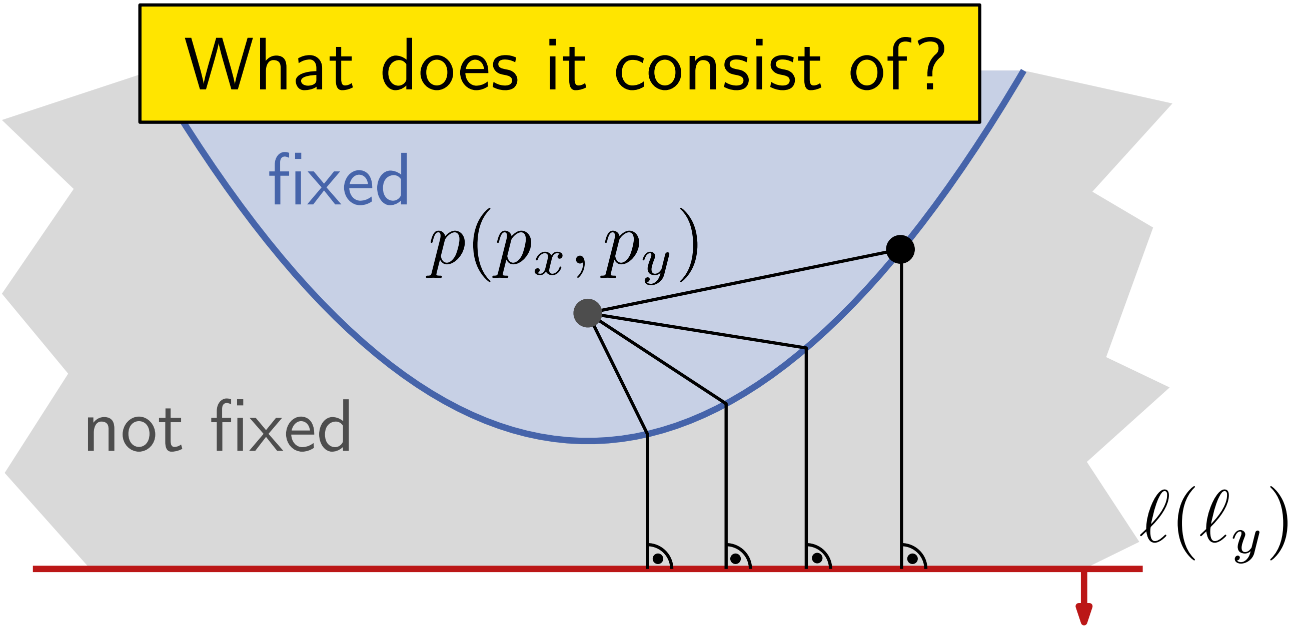

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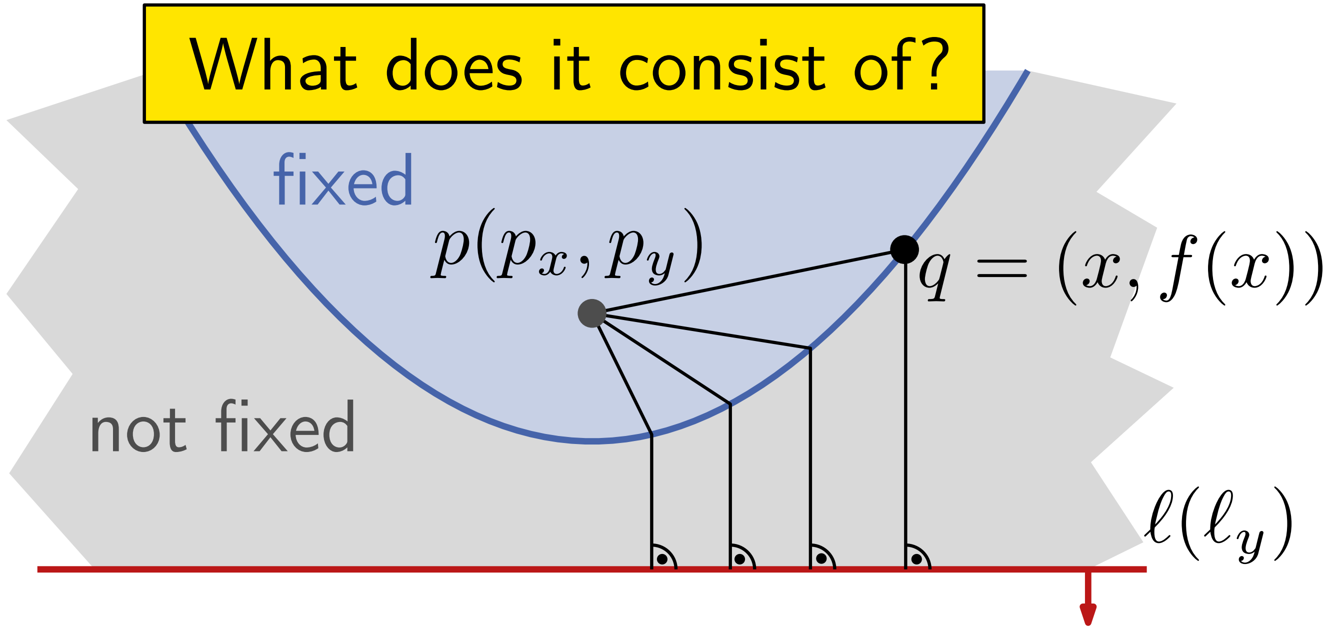

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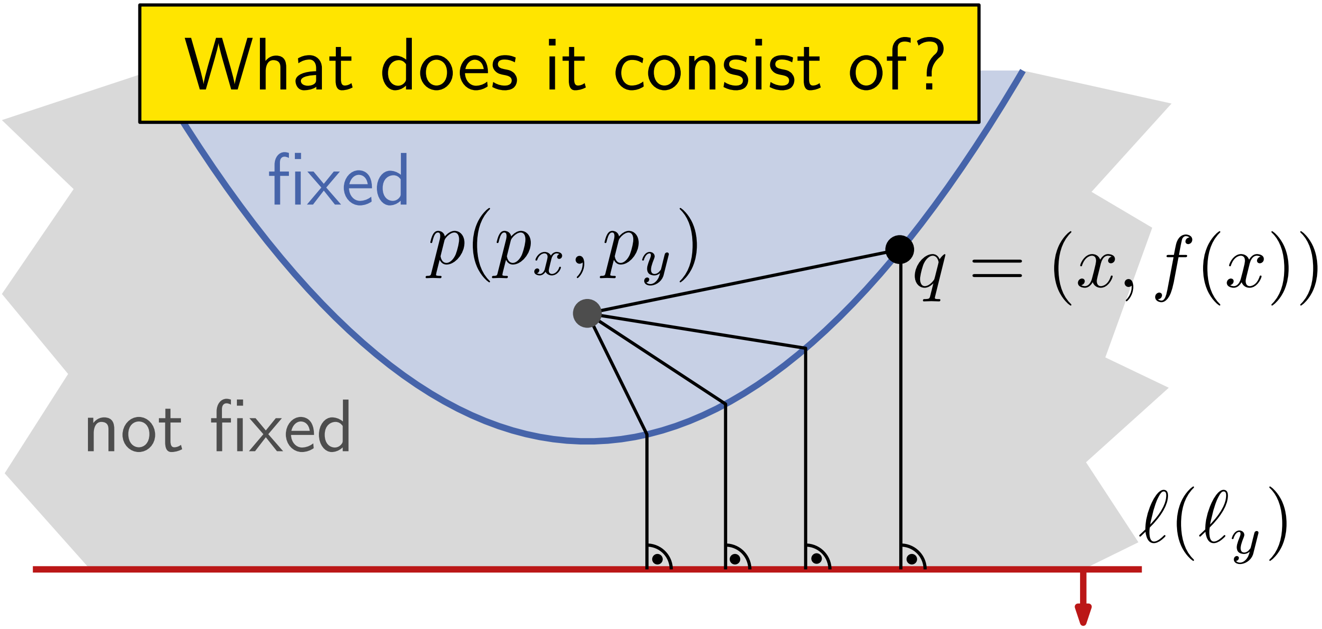

Enforcing the equality $|pq| = |q\ell|$ gives

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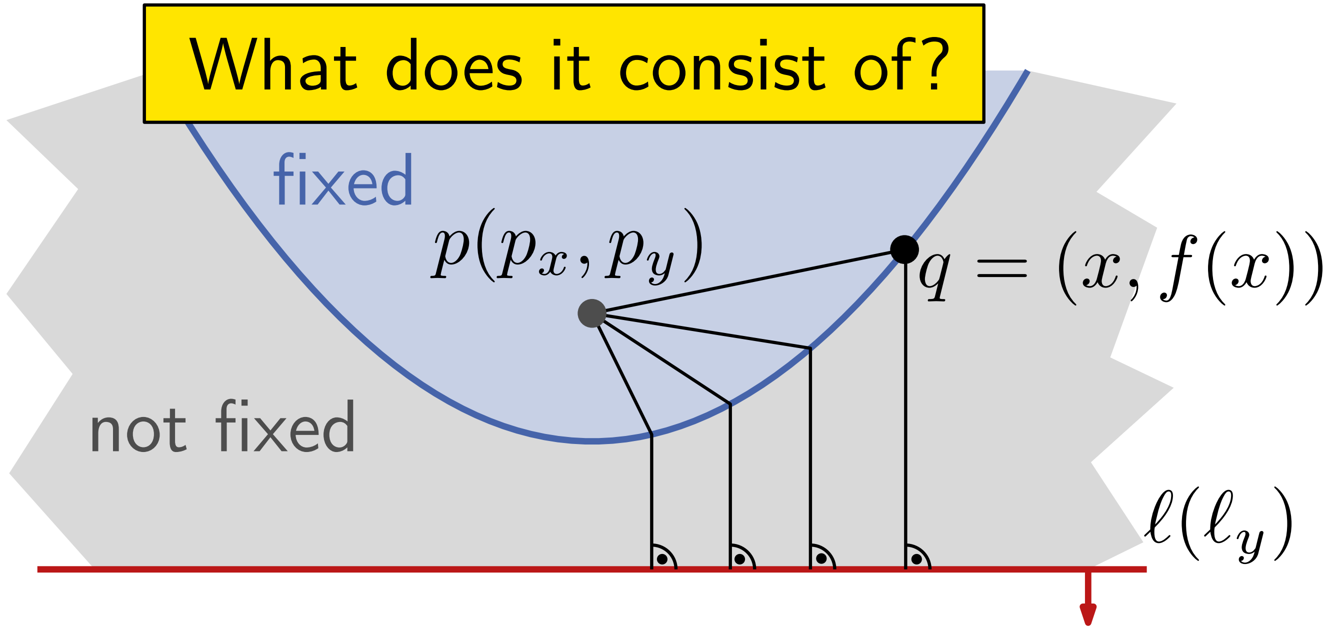

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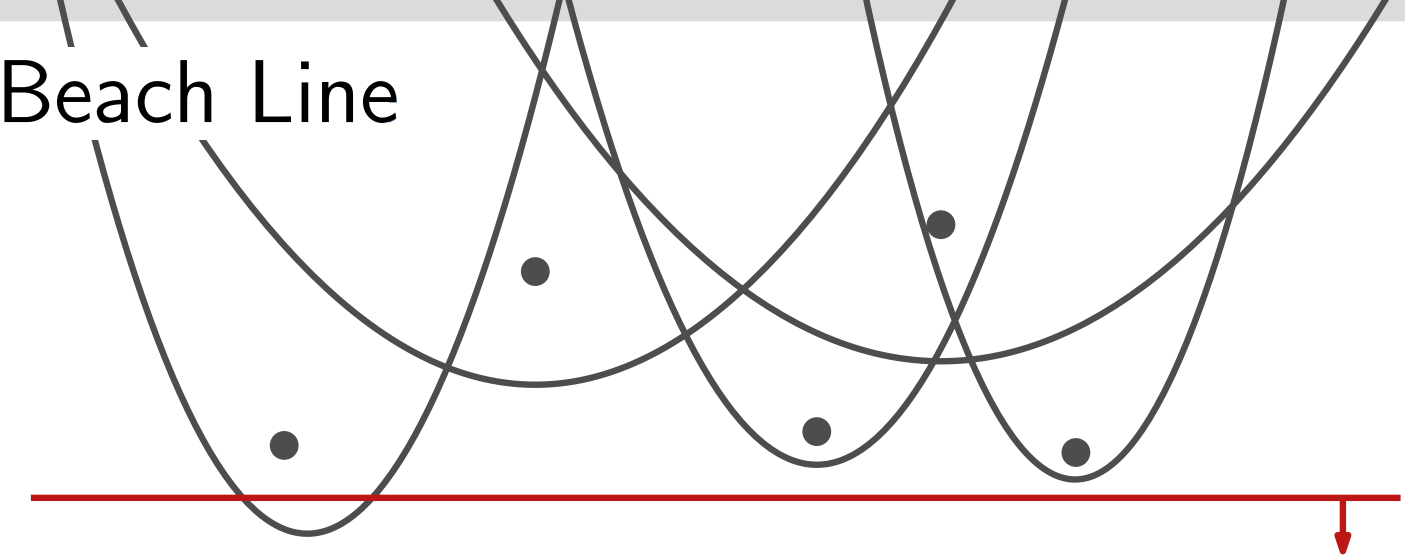
The Beach Line



Definition: The **beach line** β_ℓ is the lower envelope of parabolas f_p^ℓ for the points already found.

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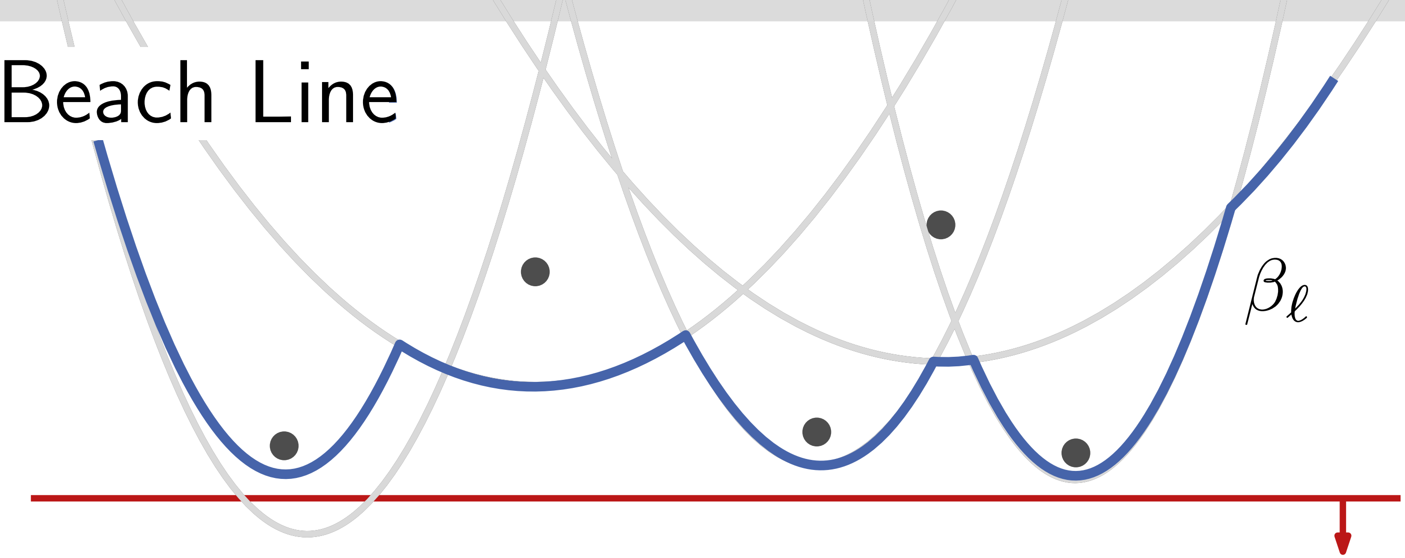

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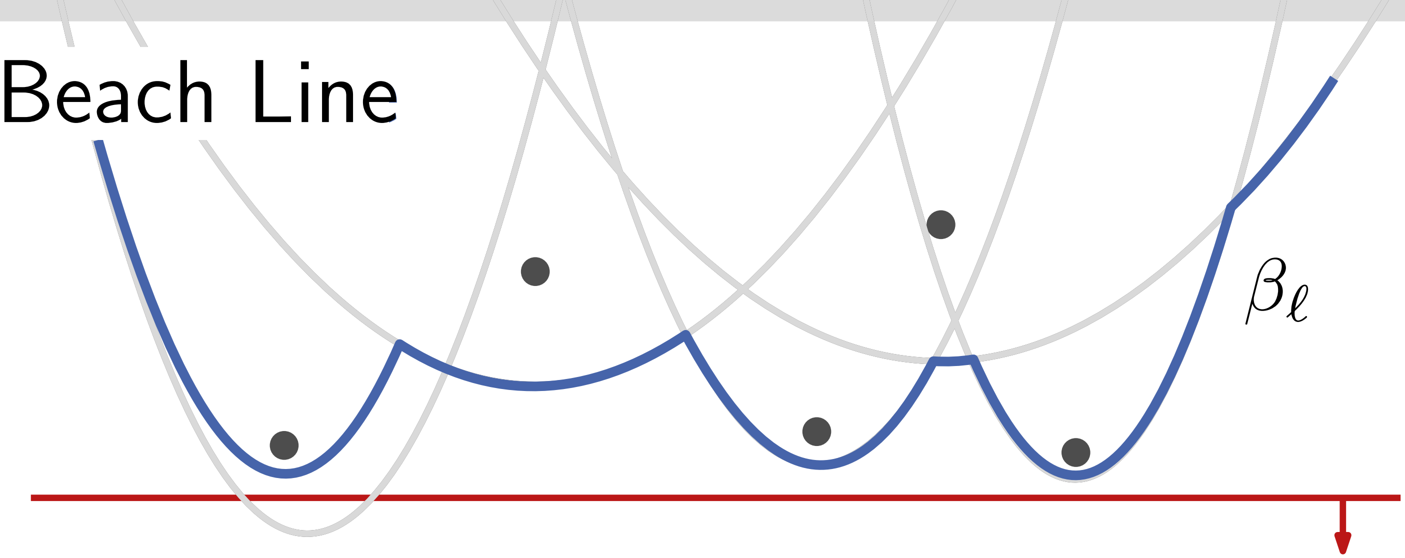

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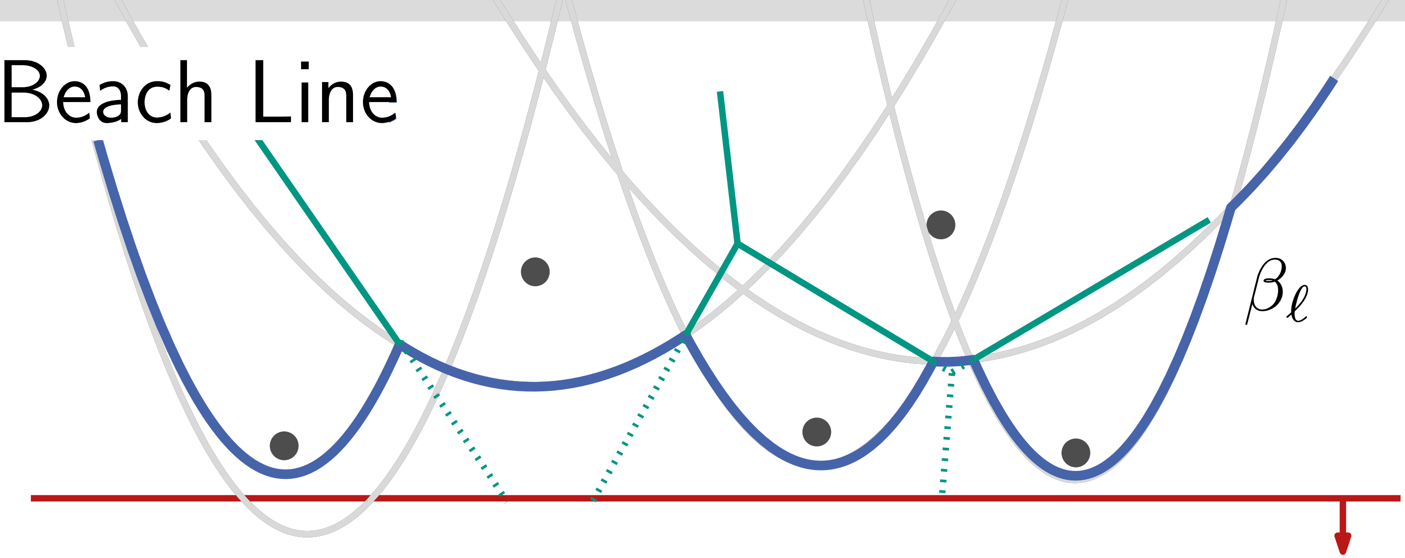

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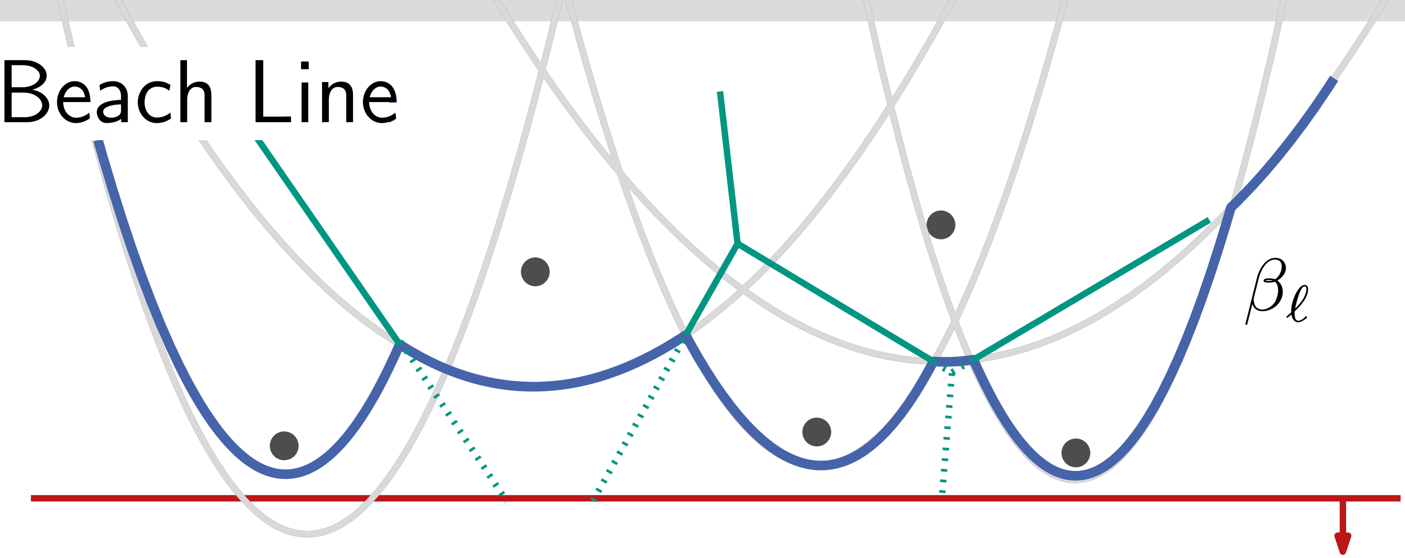

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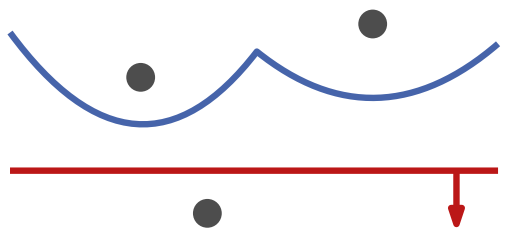
Goal: Store (implicit) contour β_ℓ instead of $\mathrm{Vor}(P) \cap \ell$

Before we proceed...

Demo

http://www.diku.dk/hjemmesider/studerende/duff/Fortune/

Point Events



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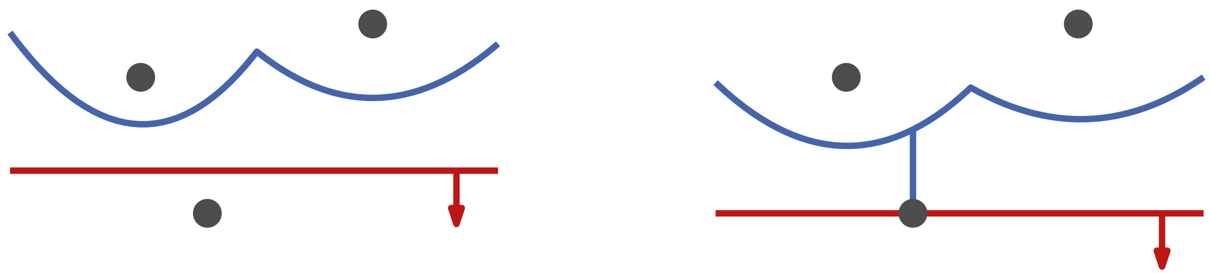

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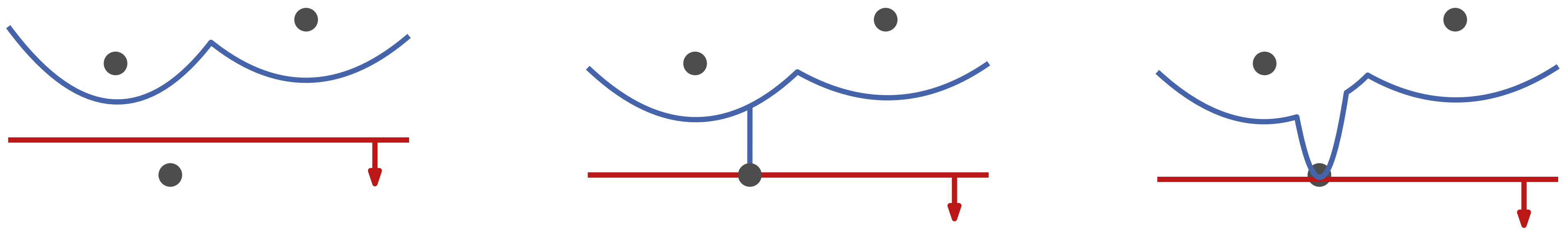

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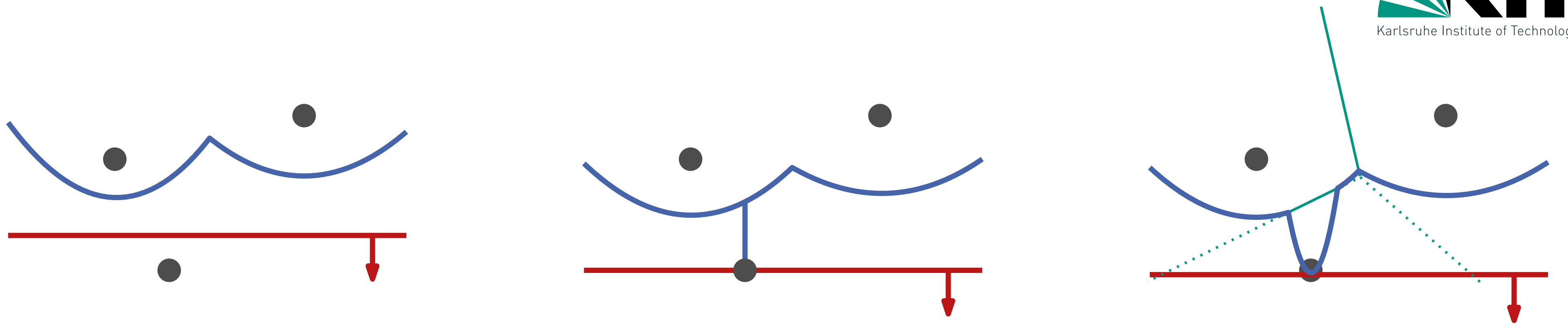

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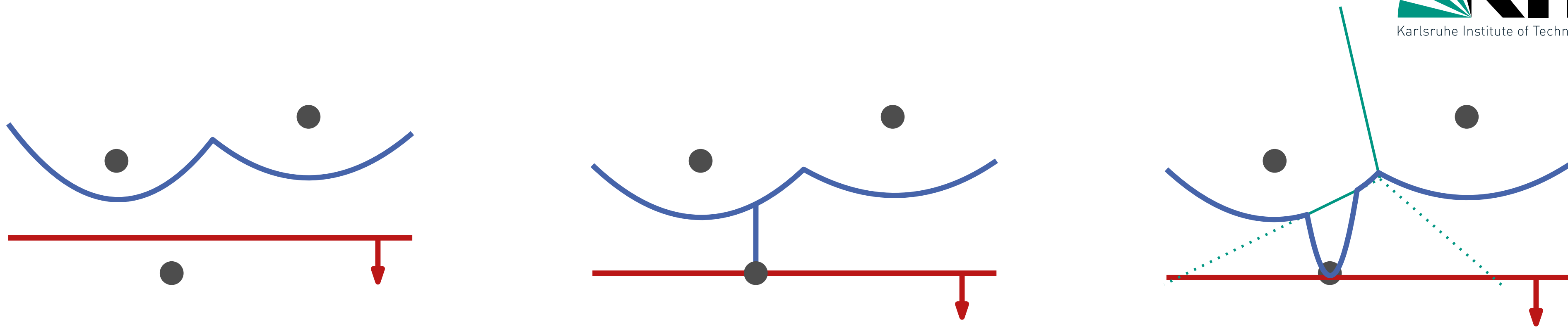

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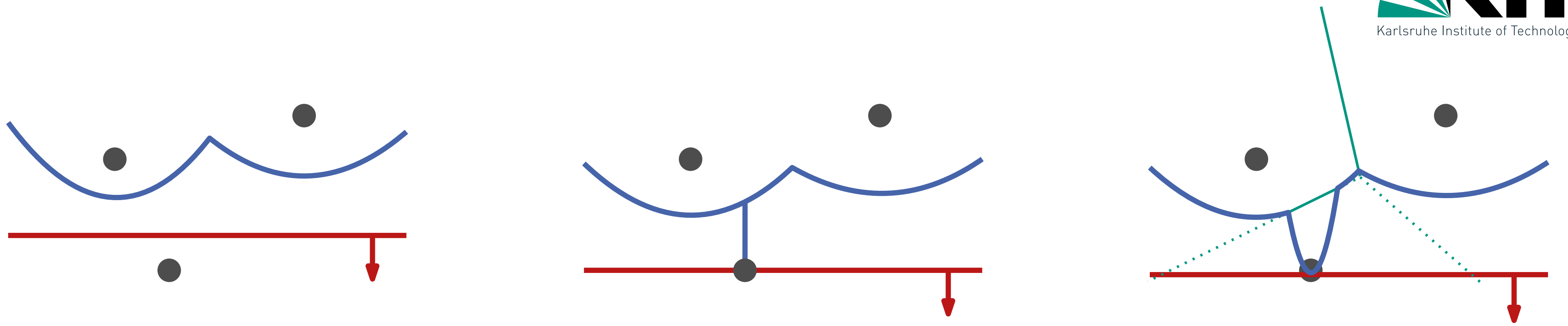

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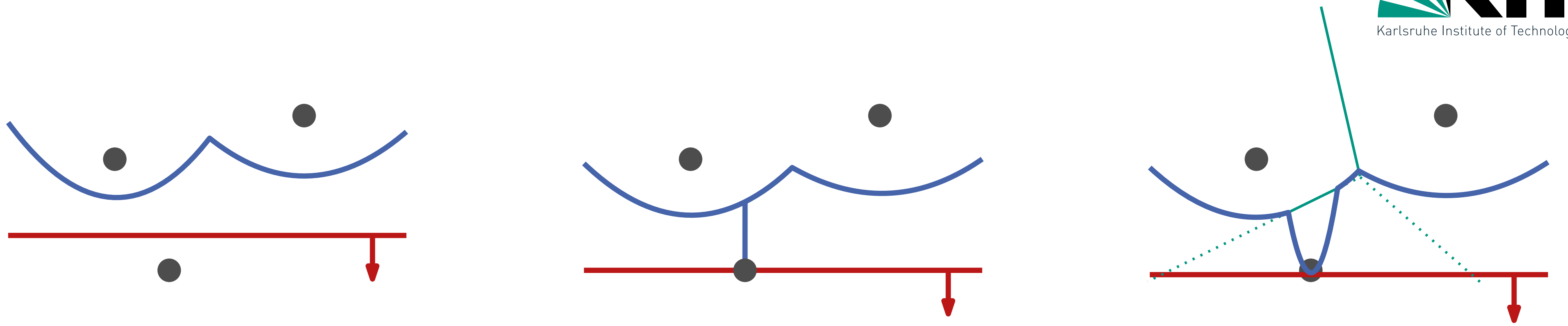

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More about this in exercises...

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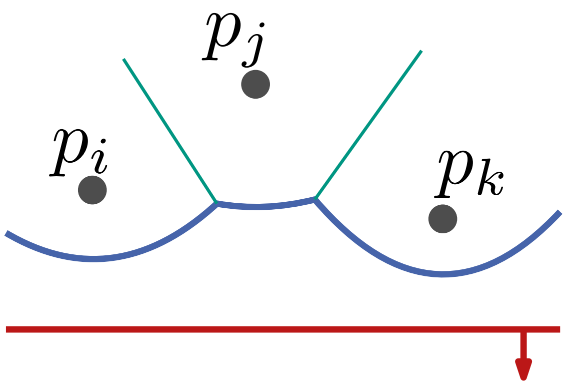

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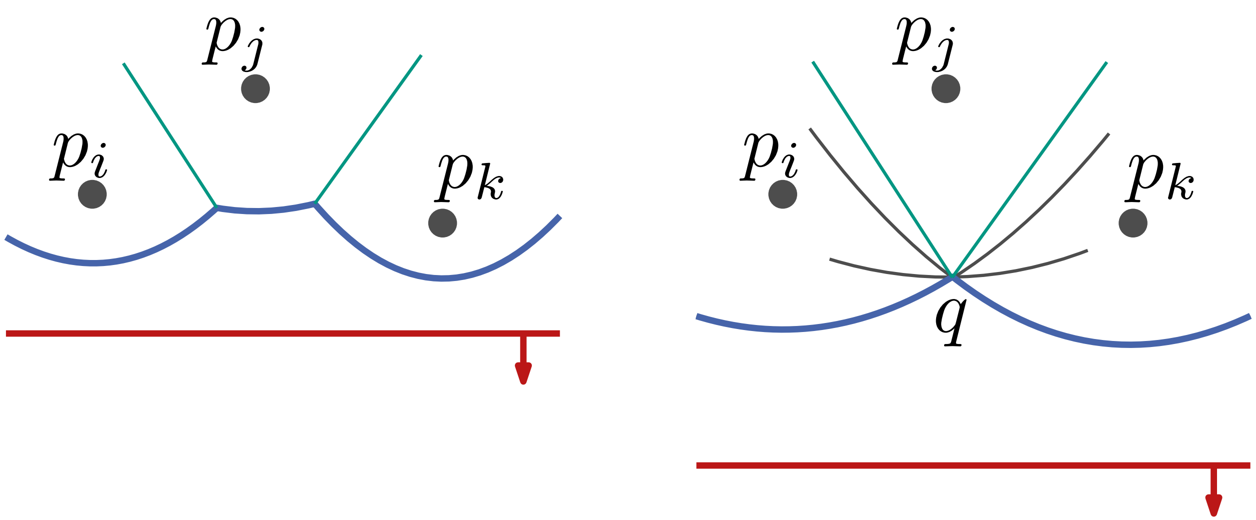

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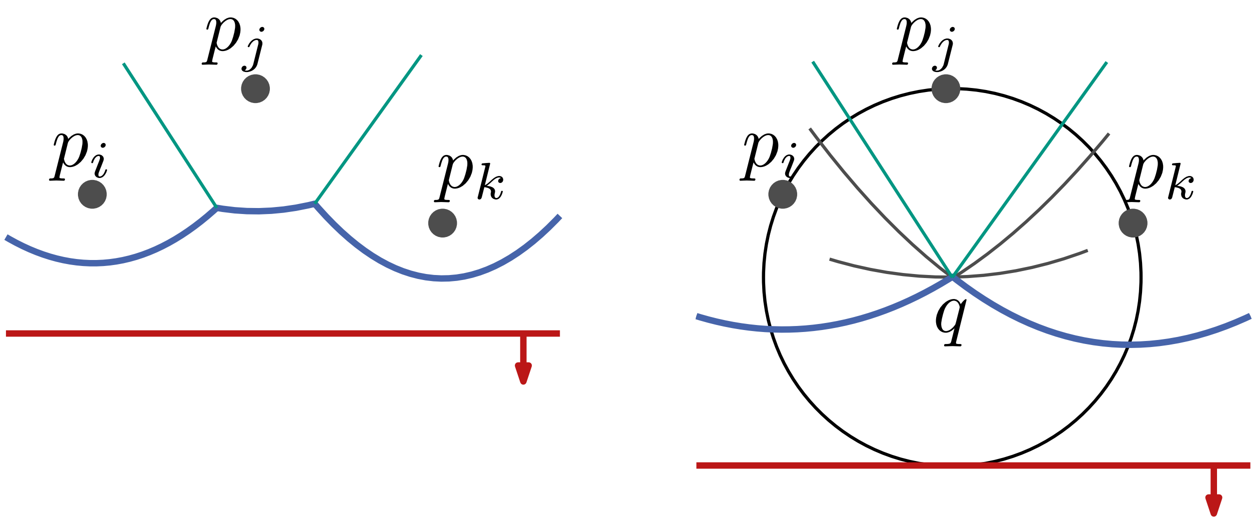

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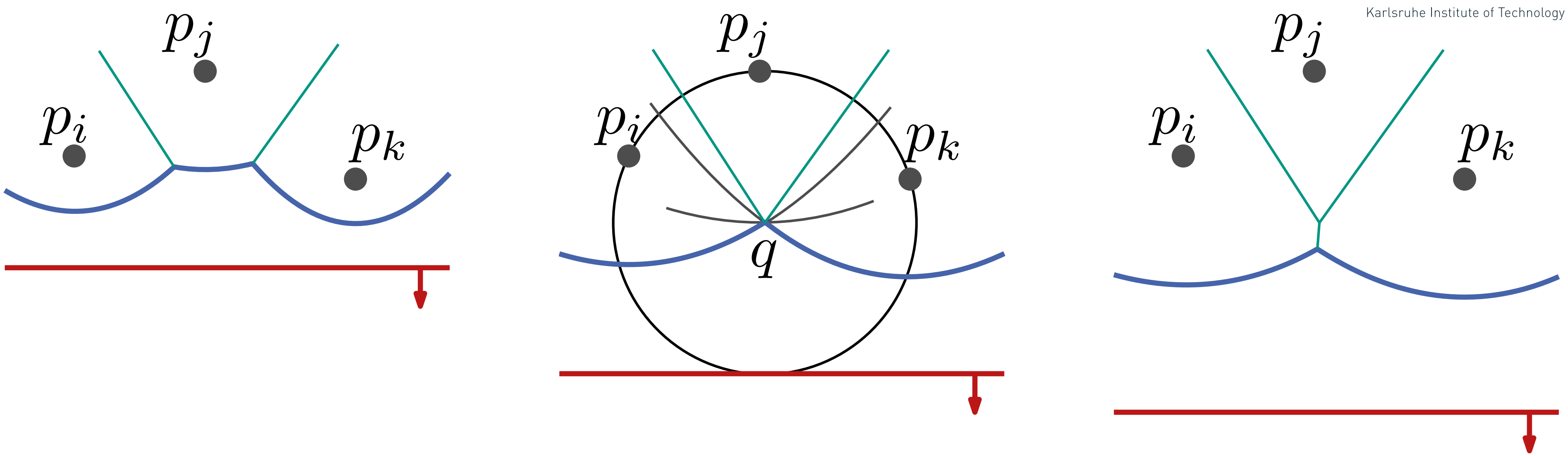

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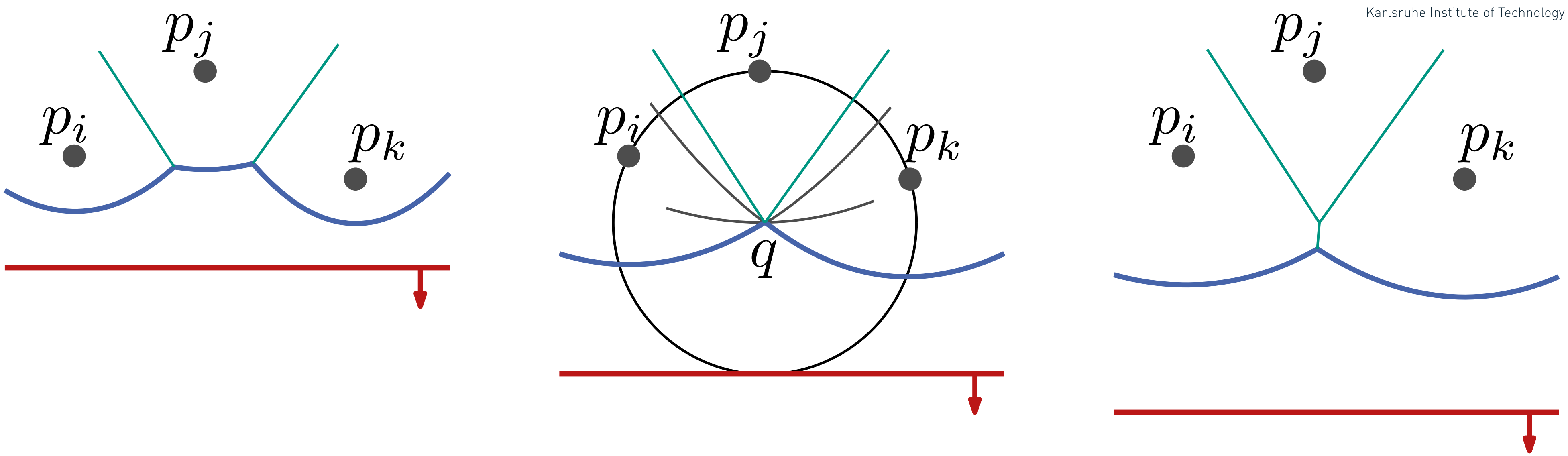

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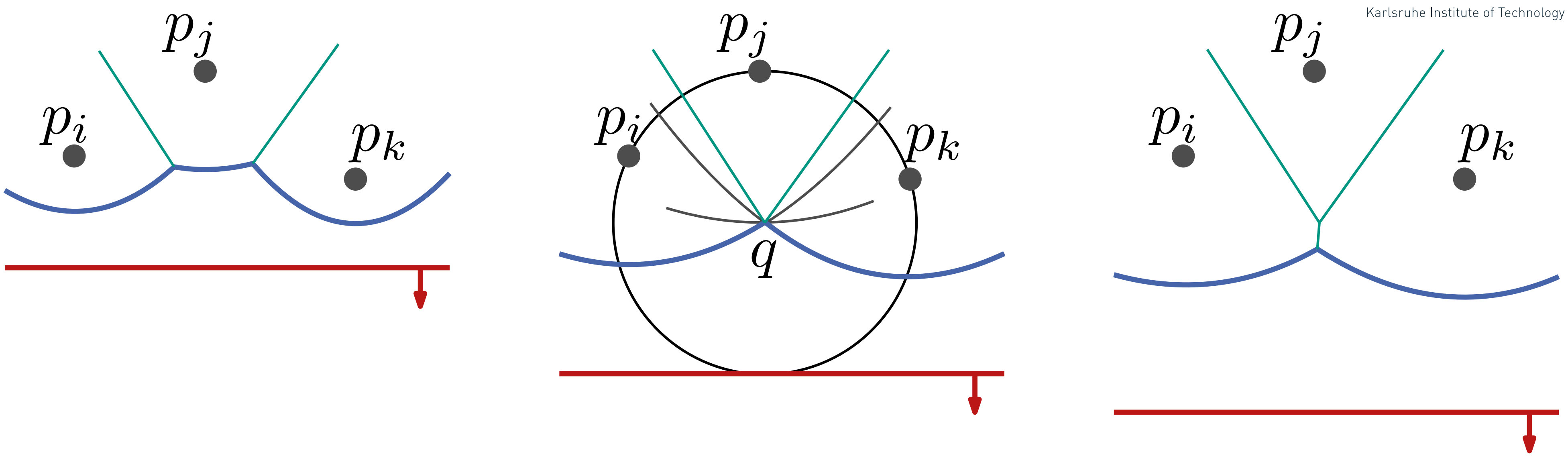

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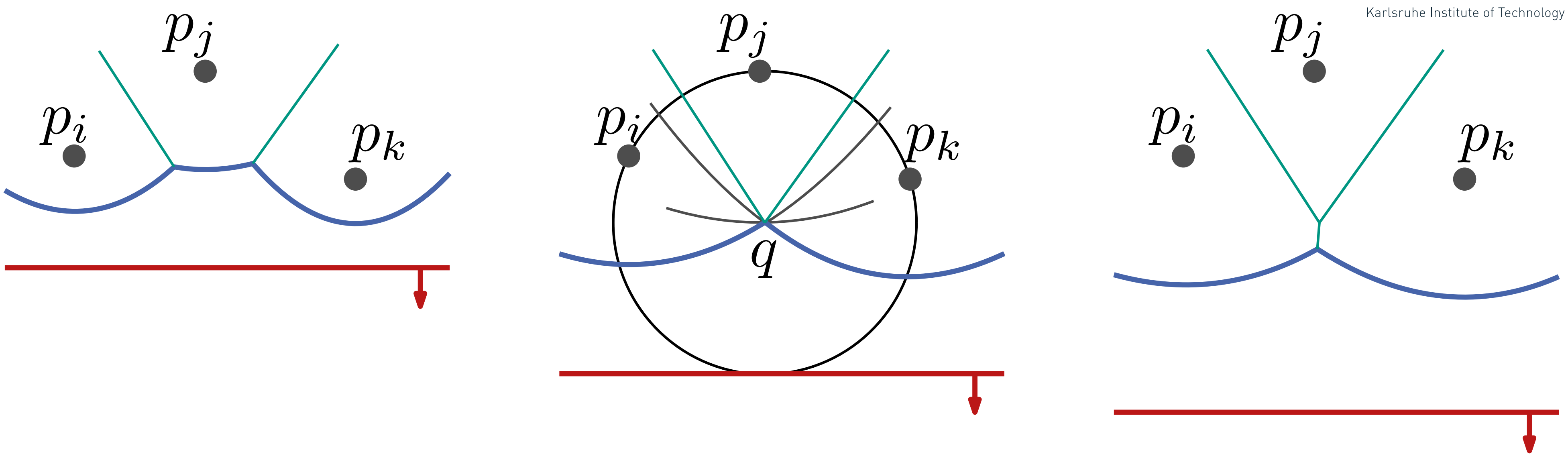

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Lemma 3: For each Voronoi vertex there is a circle event.

Data Structures

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  Warning: Include a bounding box to avoid half-lines

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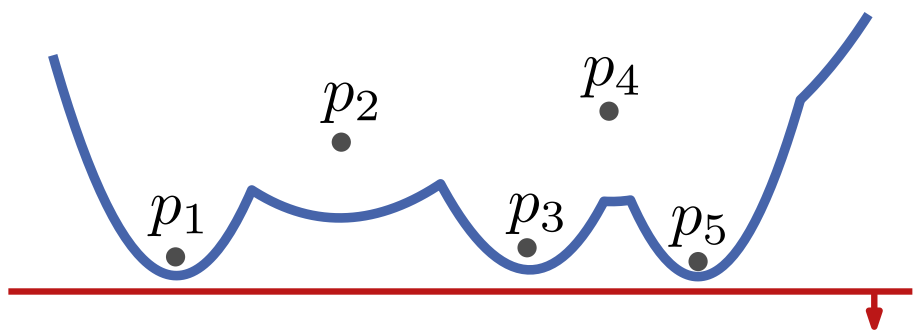

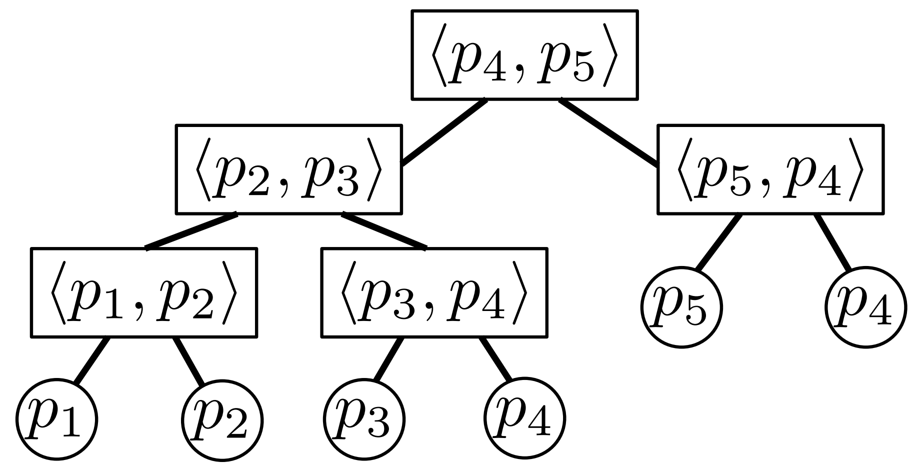

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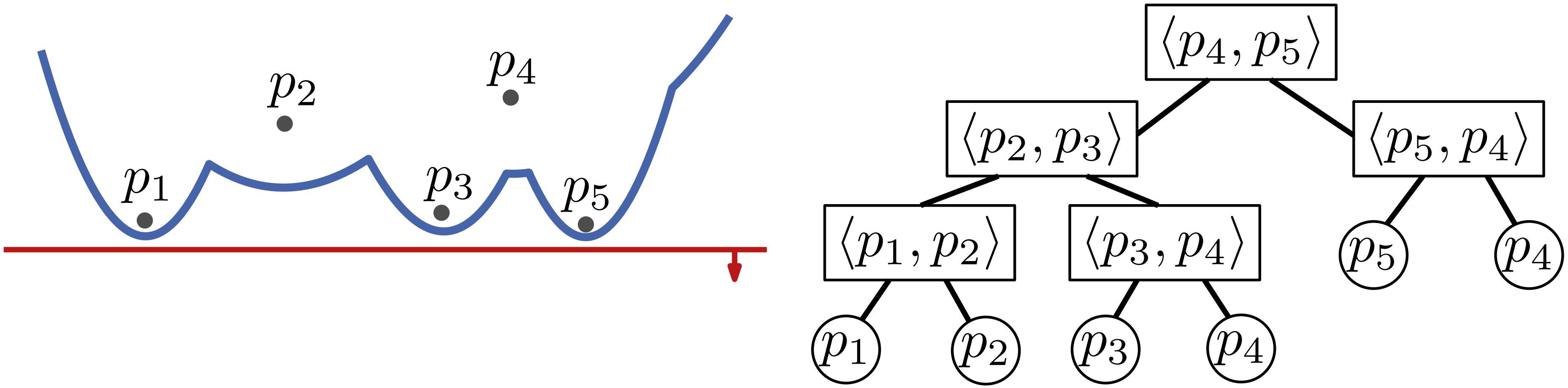

- Priority queue $\mathcal{Q}$ for the point and circle events
  - Pointer from circle event to corresponding leaf in $\mathcal{T}$ and vice versa

Fortune's Sweep Algorithm

Karlsruhe Institute of Technology

VoronoiDiagram($P \subset \mathbb{R}^2$)

```
Q ← new PriorityQueue(P)    // Point events sorted by y
T ← new BalancedBinarySearchTree()    // sweep status (β)
D ← new DCEL()    // DS for Vor(P)
while not Q.empty() do
    p ← Q.ExtractMax()
    if p point event then
        HandlePointEvent(p)
    else
        α ← arcs of β to be removed
        HandleCircleEvent(α)
Handle interior remaining nodes of T (half-lines of Vor(P))
return D
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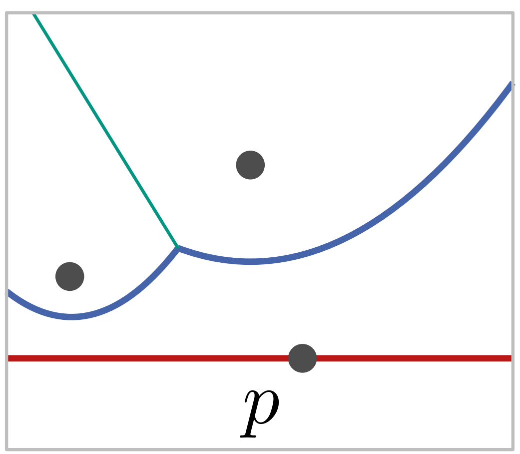
Handing Point Events

HandlePointEvent(point p)



Handing Point Events

HandlePointEvent(point p)

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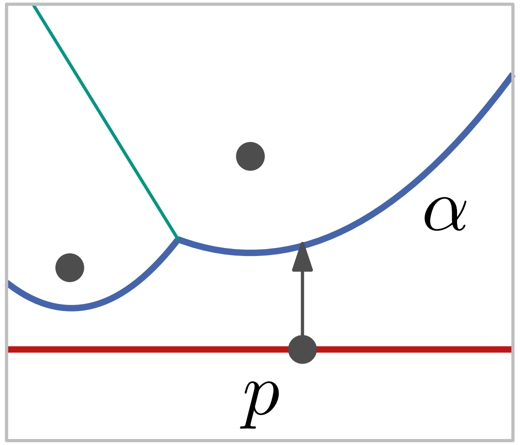

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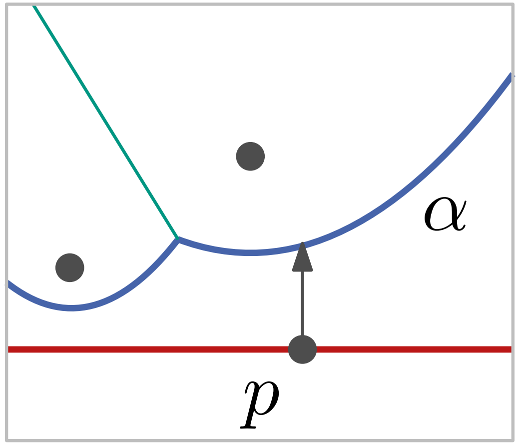

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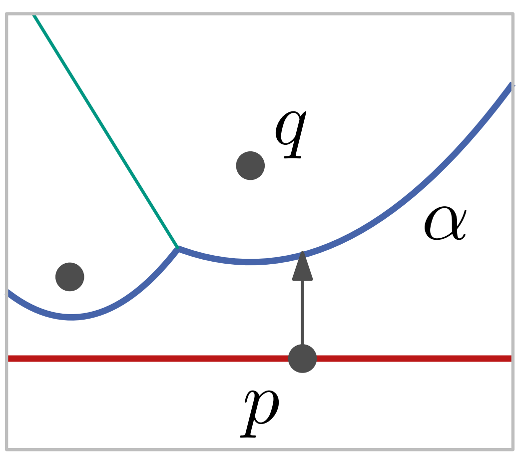

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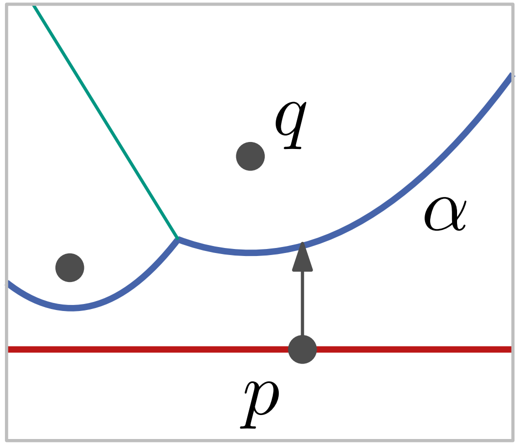

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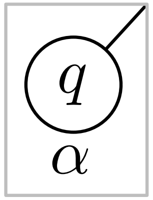

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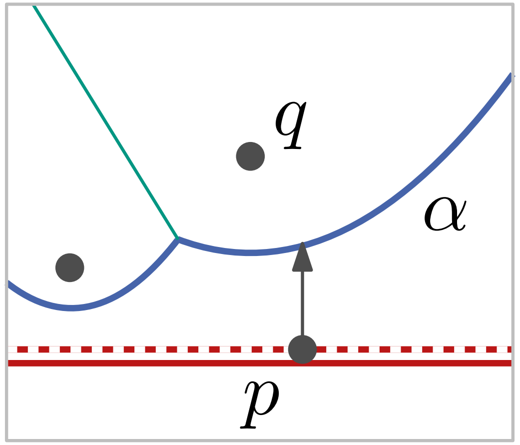

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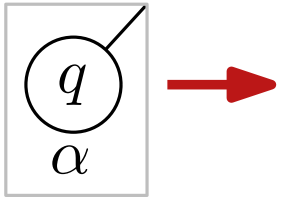

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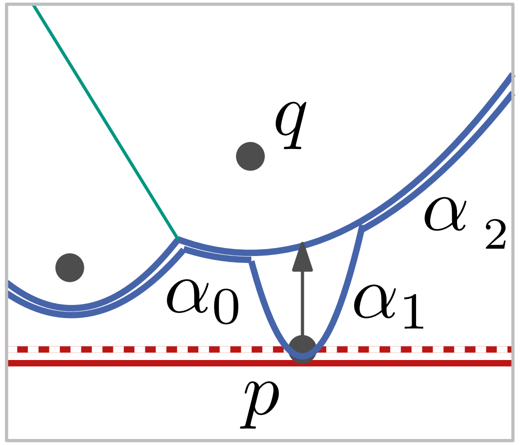

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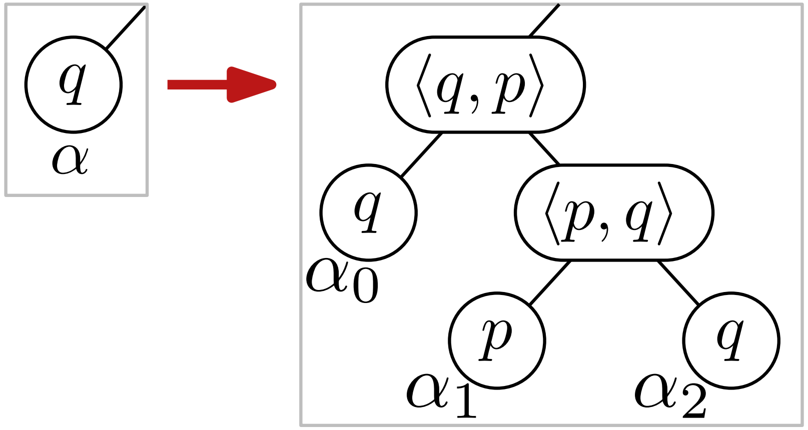

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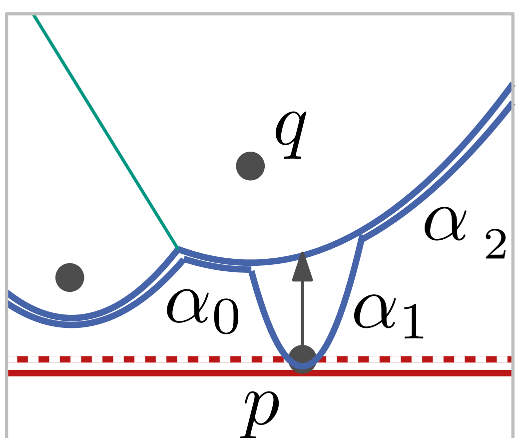

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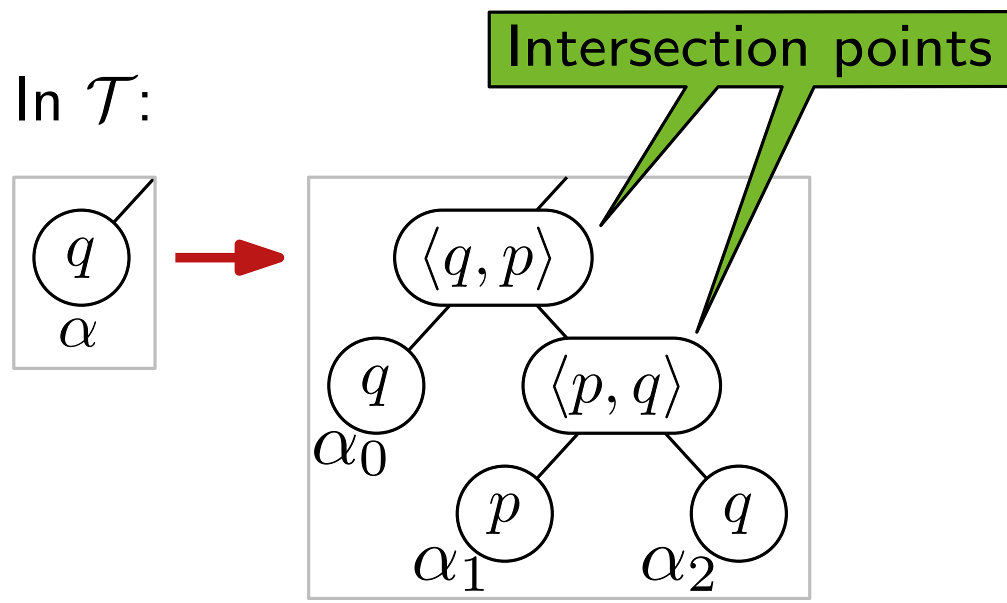

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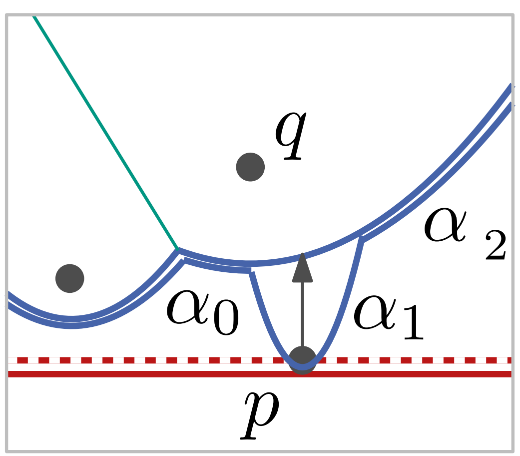

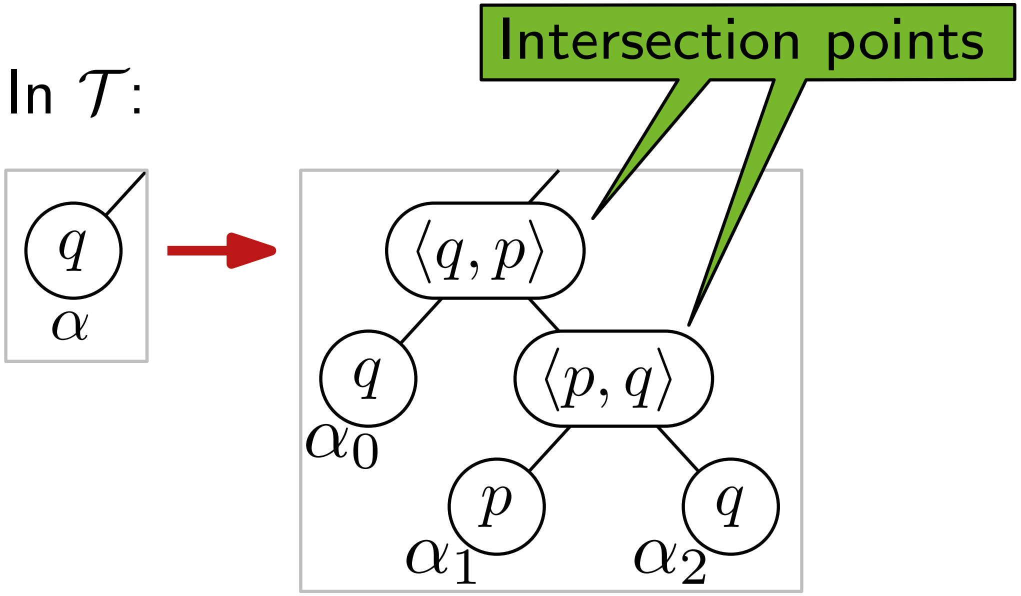

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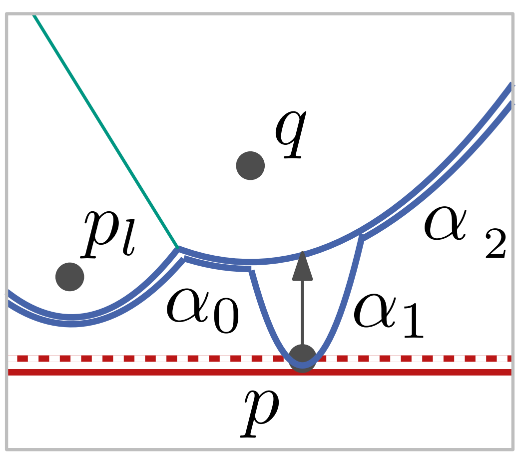
Neighbors of β



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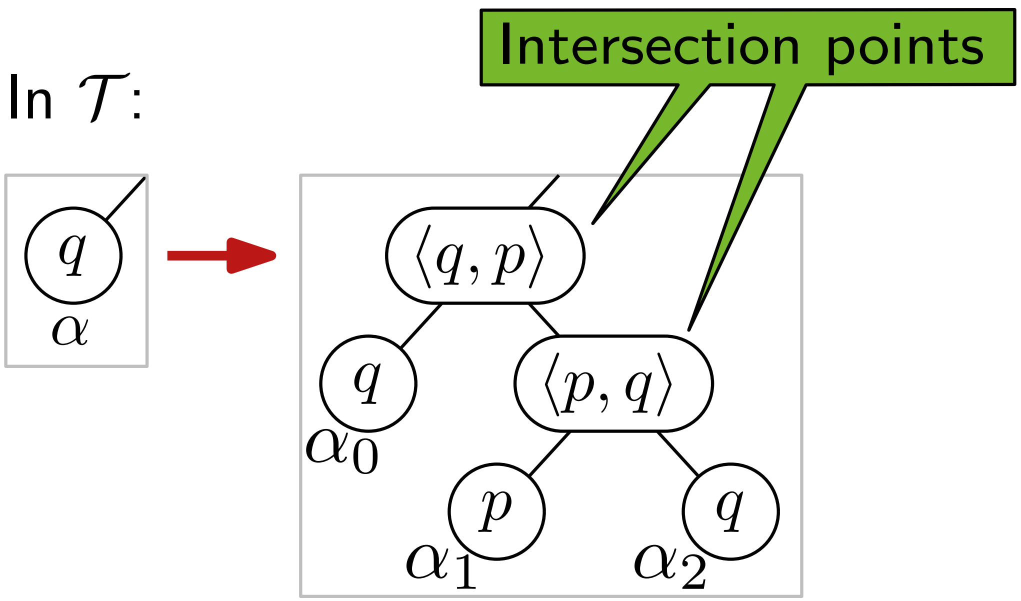

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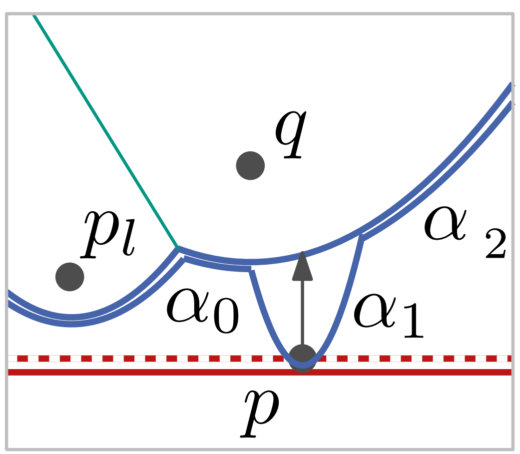

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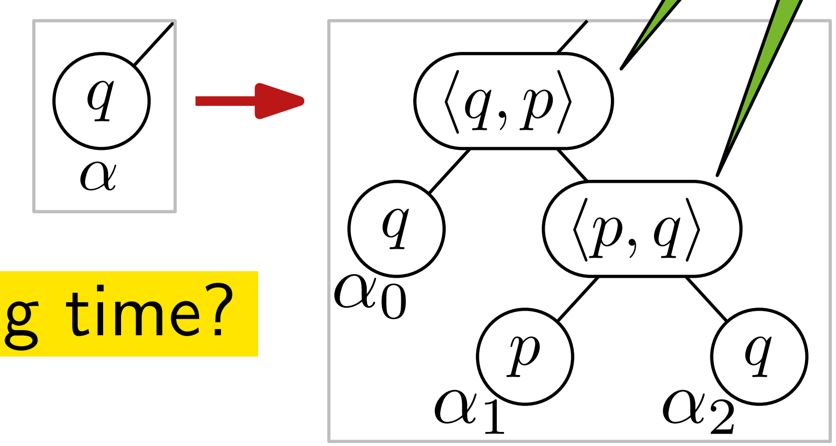
Intersection points



Running time?

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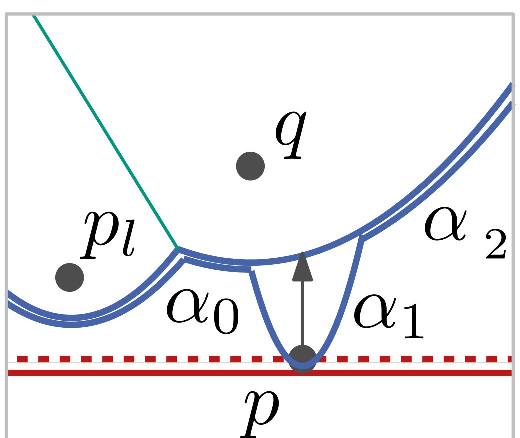
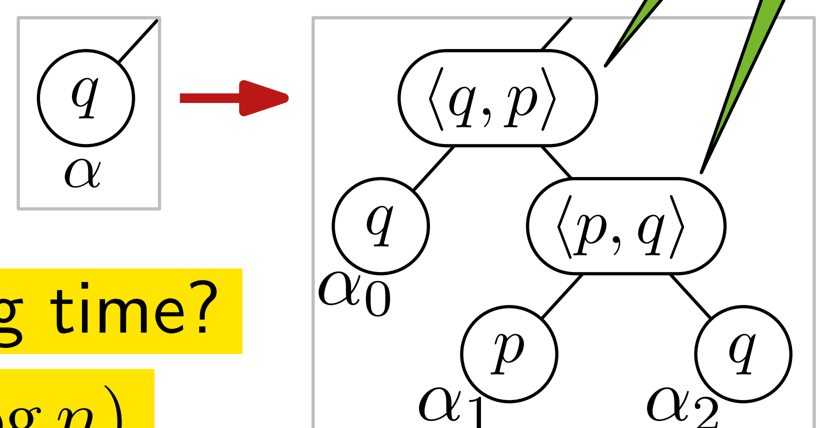


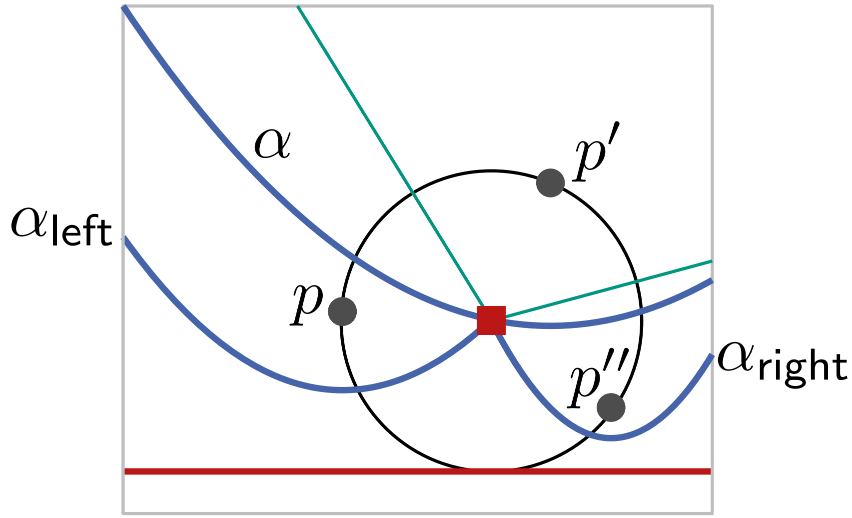


Running time?

$O(\log n)$

Handling Circle Events

HandleCircleEvent(arc α)



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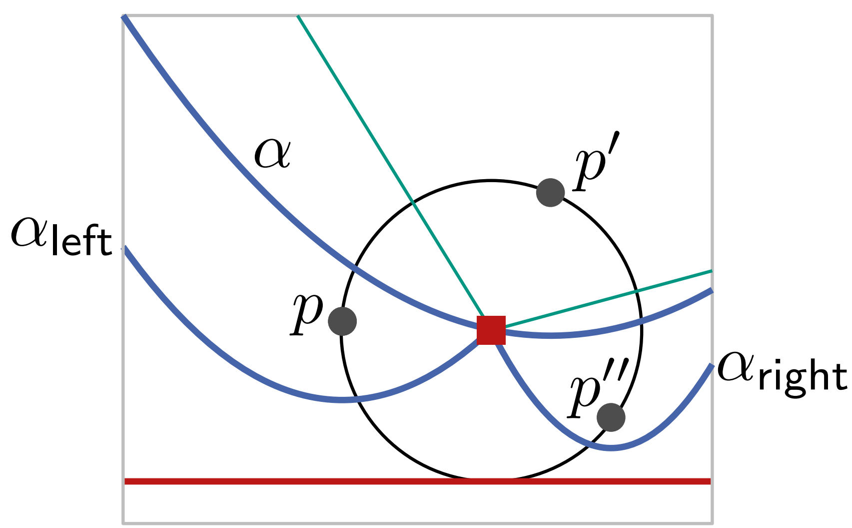

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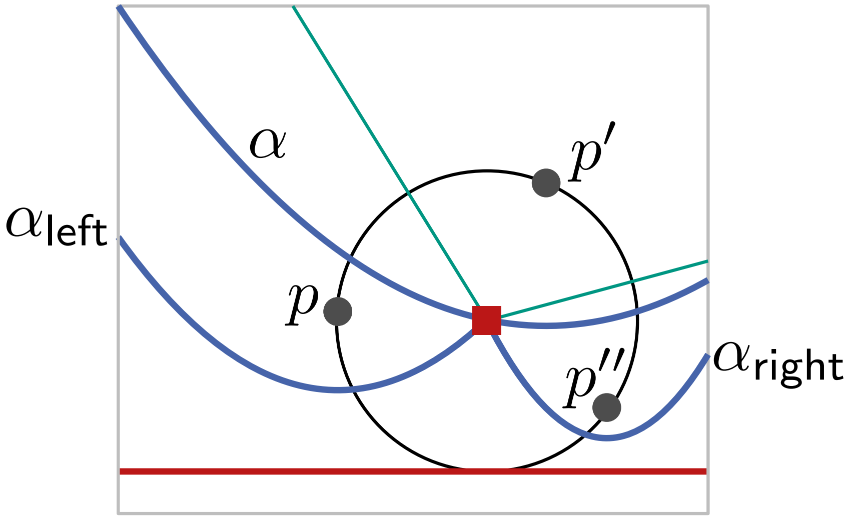

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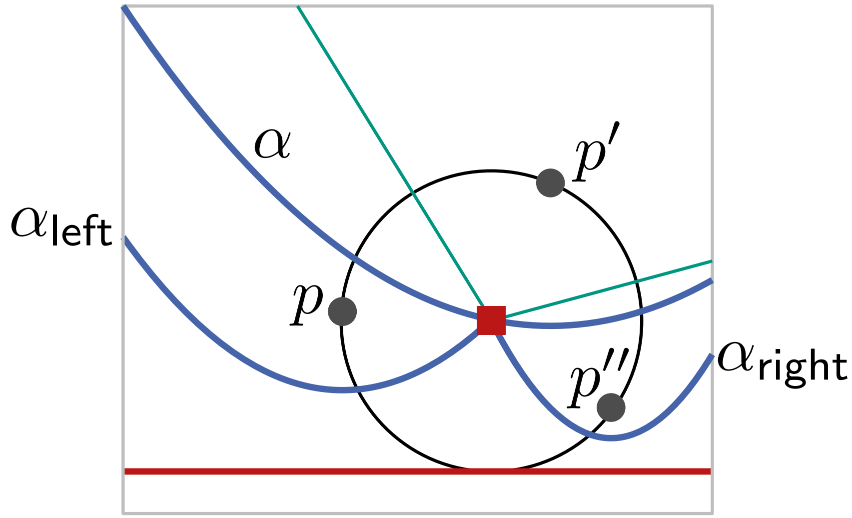

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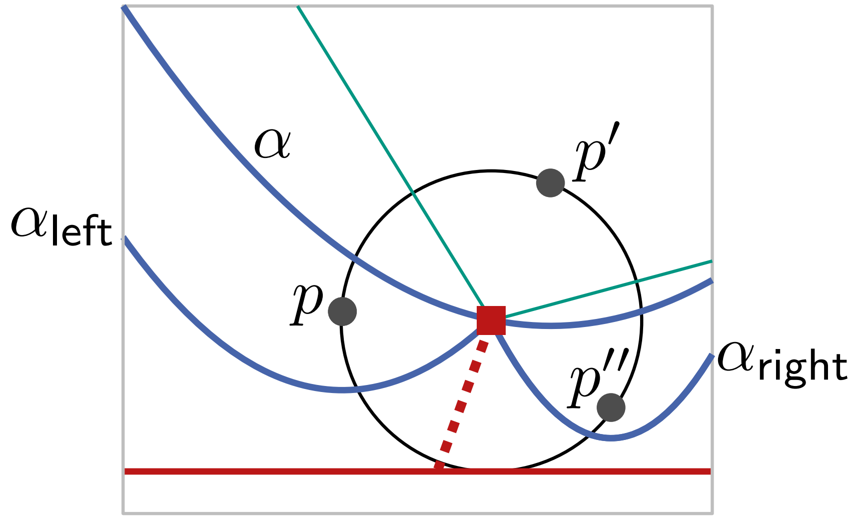

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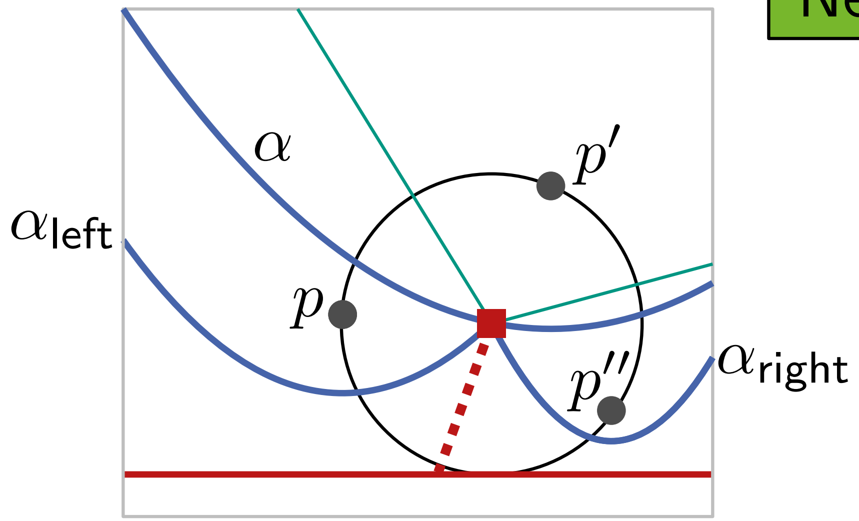

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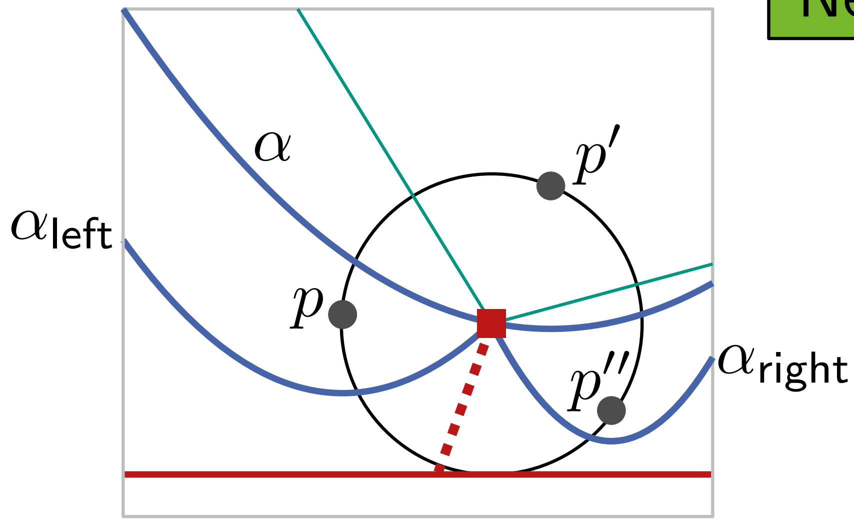

Running time?

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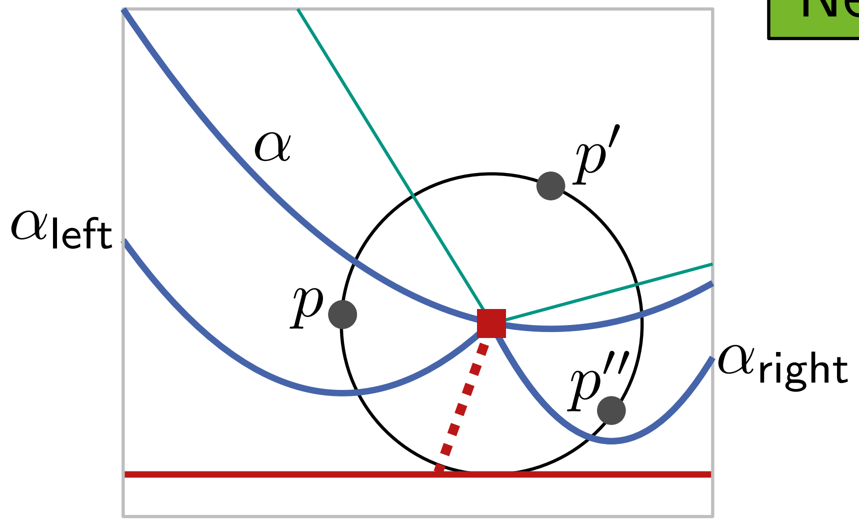

Running time?

$O(\log n)$

Fortune's Sweep Algorithm

Theorem 4: For a set P of n points, Fortune's sweep algorithm computes the Voronoi Diagram $\mathrm{Vor}(P)$ in $O(n \log n)$ time and $O(n)$ space.

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Karlsruhe Institute of Technology

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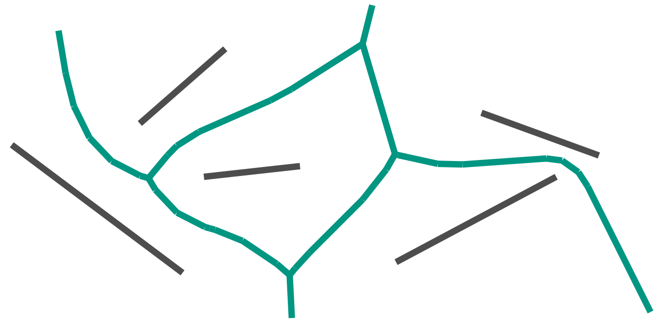
Yes! For example, we can design an algorithm to compute the Voronoi diagram for line segments with the same running time and space.

Other metrics like L_p or additive/multiplicative weighted Voronoi diagrams are possible.



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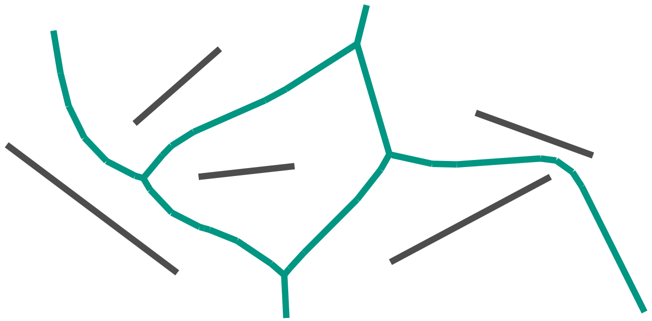

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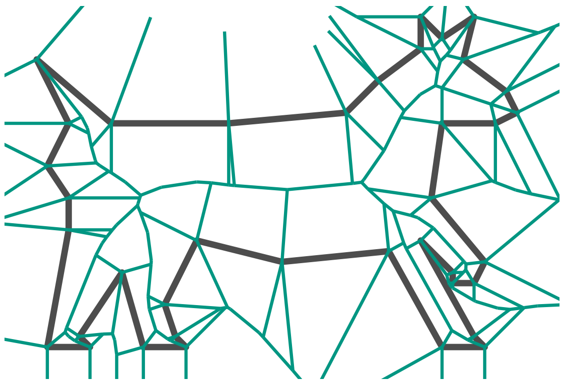

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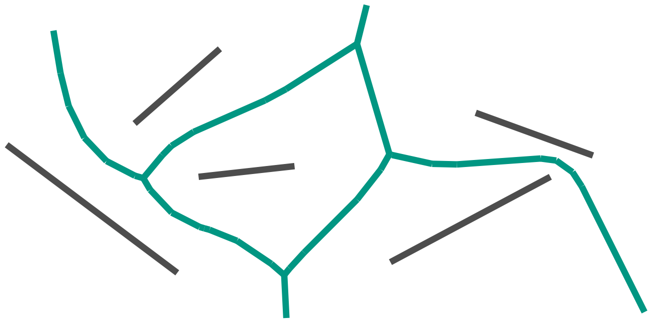

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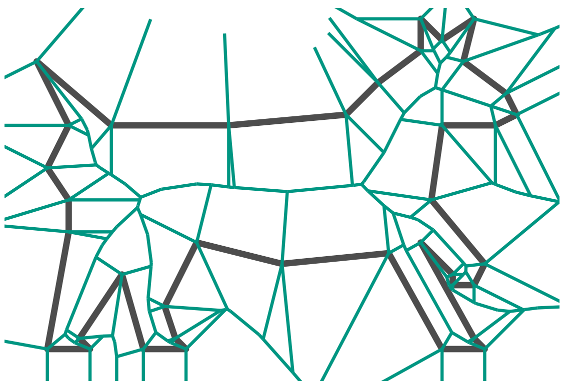

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KIT Karlsruhe Institute of Technology

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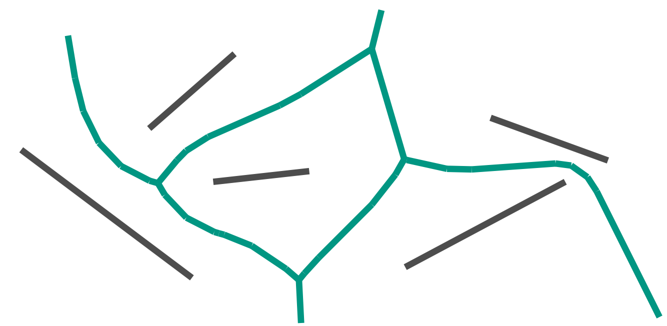

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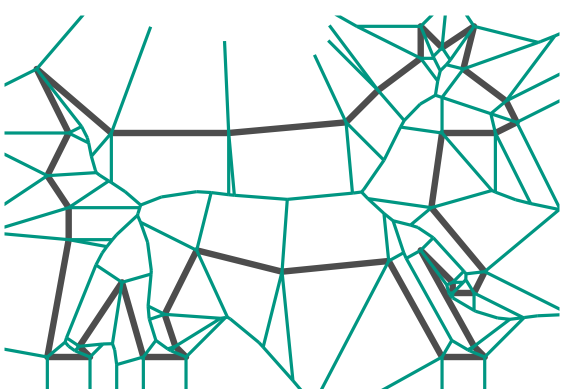

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The complexity of $\mathrm{Vor}(P)$ increases to $\Theta(n^{\lceil d/2 \rceil})$ and the running time to $O(n \log n + n^{\lceil d/2 \rceil})$.

From Voronoi to Art...

Geogebra