Computational Geometry · Lecture

Range Searching II: Windowing Queries

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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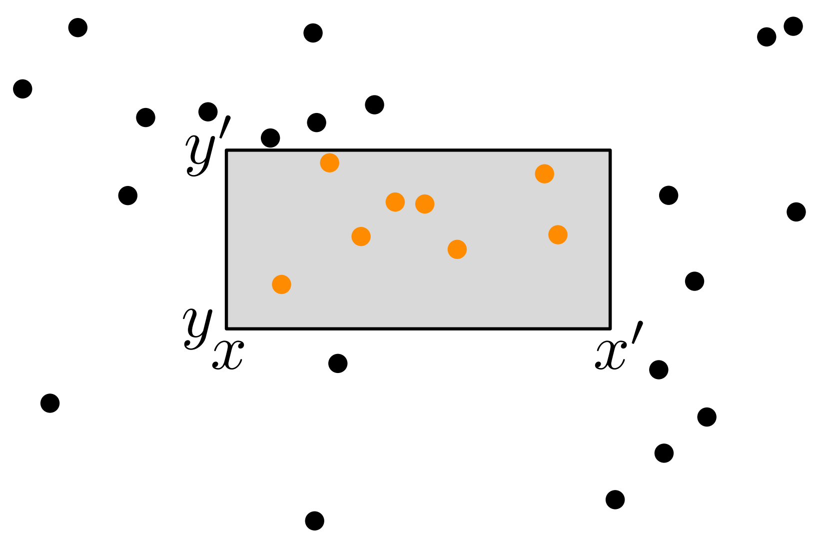
Object types in range queries



Setting so far:

- Input: set of points P (here $P \subset \mathbb{R}^2$)
- Output: all points in $P \cap [x, x'] \times [y, y']$
- Data structures: kd-trees or range trees

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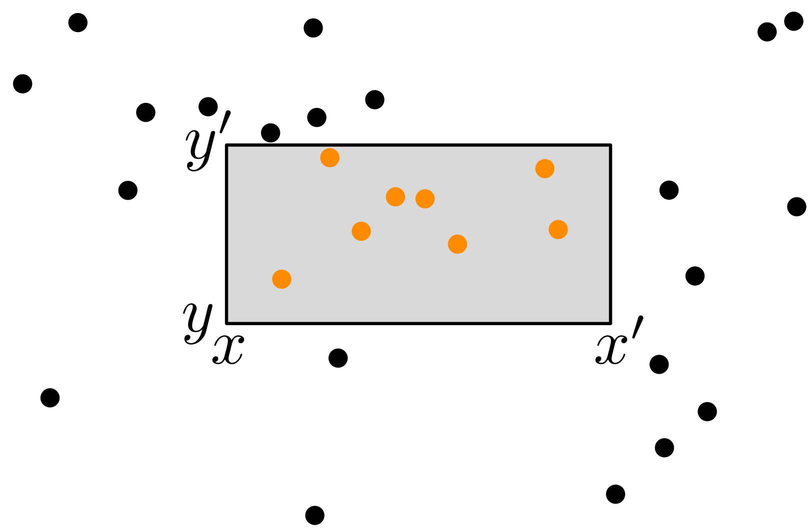

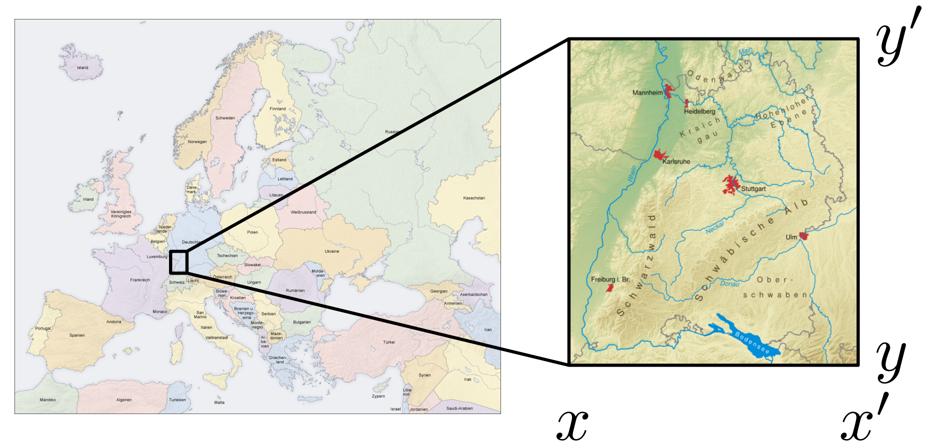

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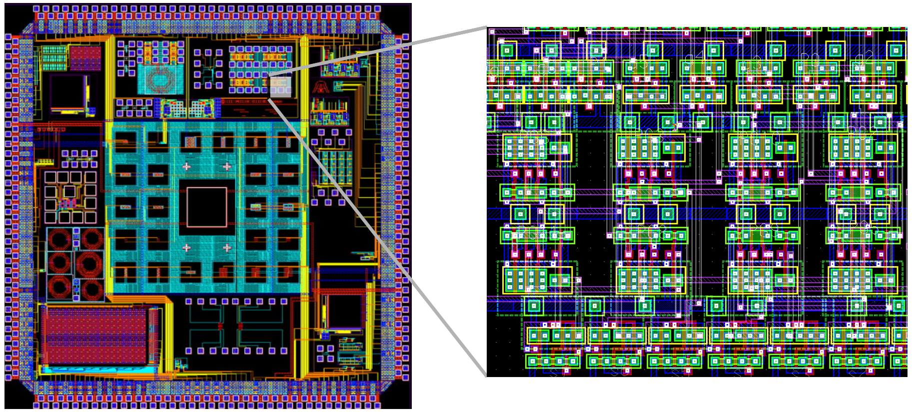
Further variant

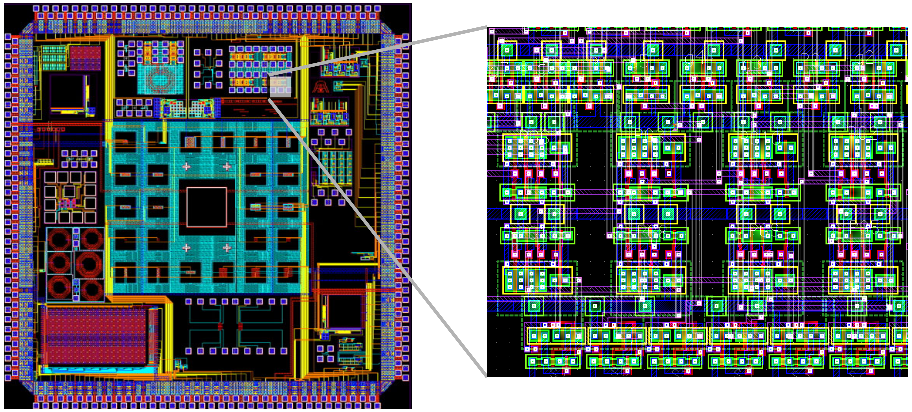
- Input: set of line segments S (here in $\mathbb{R}^2$)
- Output: all segments in $S \cap [x, x'] \times [y, y']$
- Data structures: ?

Axis-parallel line segments



special case (e.g., in VLSI design): all line segments are axis-parallel

Axis-parallel line segments



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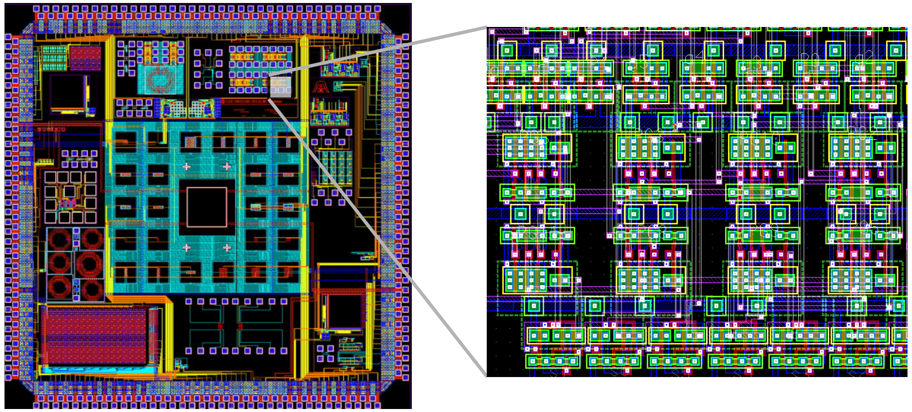
Axis-parallel line segments



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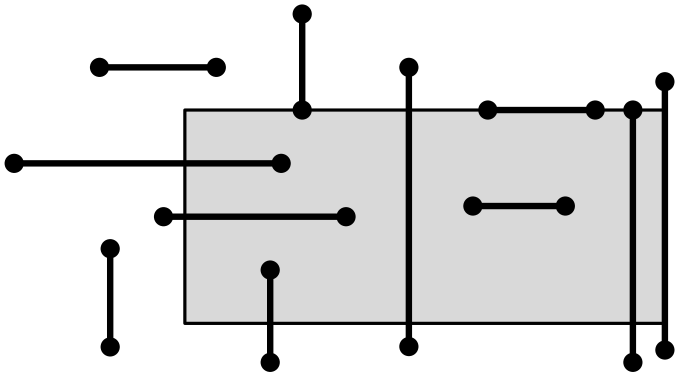
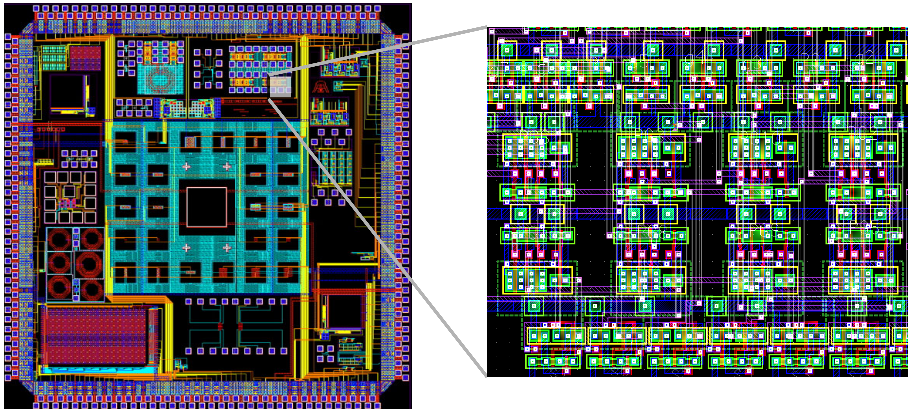


How to approach this case?

Axis-parallel line segments



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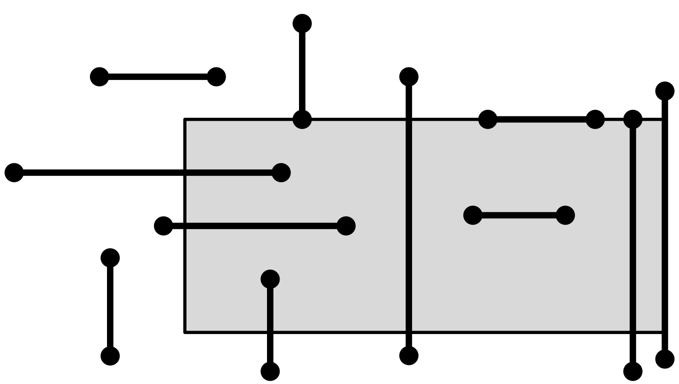


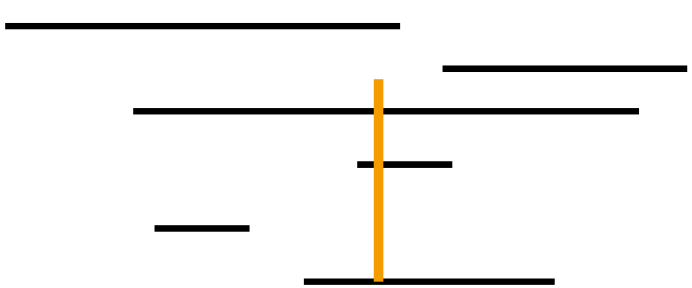
Case 1: ≥ 1 endpoint in R
$\rightarrow$ use range tree

Case 2: both endpoints $\notin R$
$\rightarrow$ intersect left or top edge of R

Case 2 in detail

Problem:

Given a set H of n horizontal line segments and a vertical query segment s, find all line segments in H that intersect s. (Vertical segments and a horizontal query are analogous.)



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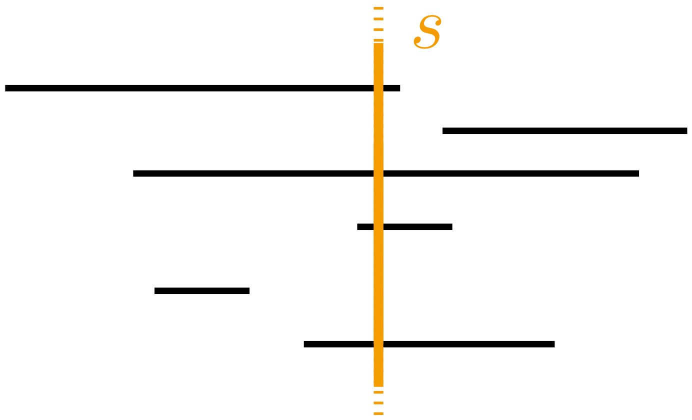
One level simpler: vertical line $s := (x = q_x)$

Given n intervals $I = \{[x_1, x'_1], [x_2, x'_2], \ldots, [x_n, x'_n]\}$ and a point q_x, find all intervals that contain q_x.



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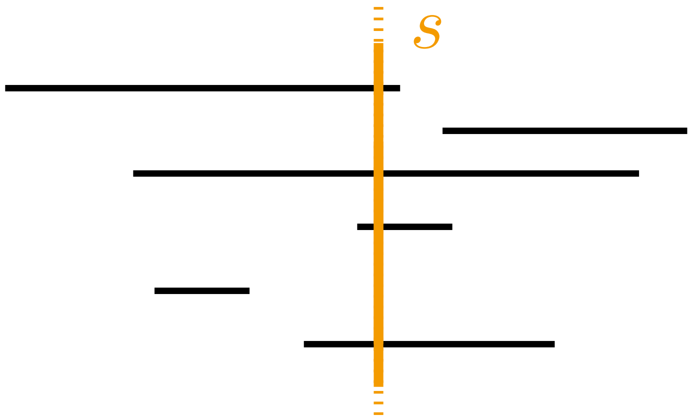


What do we need for an appropriate data structure?

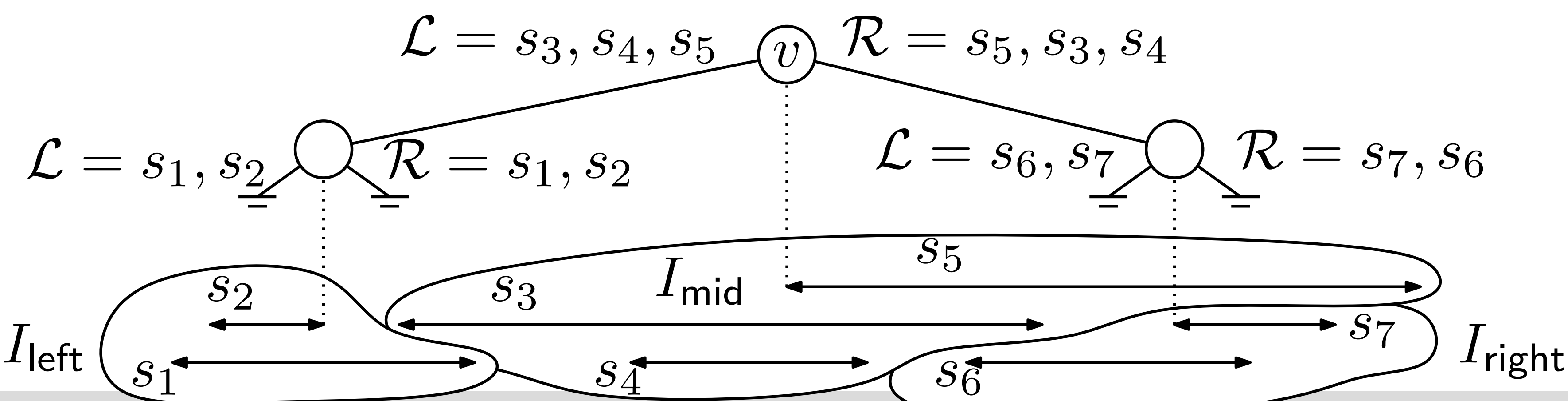
Interval Trees

Construction of an interval tree $\mathcal{T}$

- if $I = \emptyset$ then $\mathcal{T}$ is a leaf
- else let x_{mid} be the median of the endpoints of I and define

$$\begin{aligned} I_{\text{left}} &= \{[x_j, x'_j] \mid x'_j < x_{\text{mid}}\} \\ I_{\text{mid}} &= \{[x_j, x'_j] \mid x_j \le x_{\text{mid}} \le x'_j\} \\ I_{\text{right}} &= \{[x_j, x'_j] \mid x_{\text{mid}} < x_j\} \end{aligned}$$

  $\mathcal{T}$ consists of a node v for x_{mid} and
  - lists $\mathcal{L}(v)$ and $\mathcal{R}(v)$ for I_{mid} sorted by left and right interval endpoints, respectively
  - left child of v is an interval tree for I_{left}
  - right child of v is an interval tree for I_{right}



Properties of interval trees

Lemma 1: An interval tree for n intervals needs $O(n)$ space and has depth $O(\log n)$. It can be constructed in time $O(n \log n)$.

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How does the query work?

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```
QueryIntervalTree(v, q_x)
  if v no leaf then
    if q_x < x_mid(v) then
      search in L from left to right for intervals containing q_x
      QueryIntervalTree(lc(v), q_x)
    else
      search in R from right to left for intervals containing q_x
      QueryIntervalTree(rc(v), q_x)
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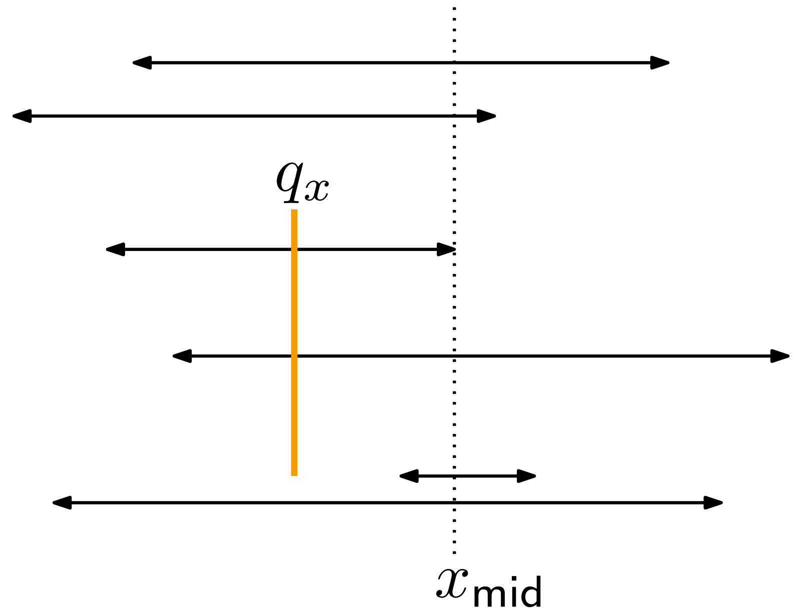
  - **else**
    - search in $\mathcal{R}$ from right to left for intervals containing q_x
    - QueryIntervalTree$(rc(v), q_x)$

Lemma 2: Using an interval tree we can find all k intervals containing a query point q_x in $O(\log n + k)$ time.

From lines to line segments

How can we adapt the idea of an interval tree for query segments $q_x \times [q_y, q'_y]$ instead of a query line $x = q_x$?



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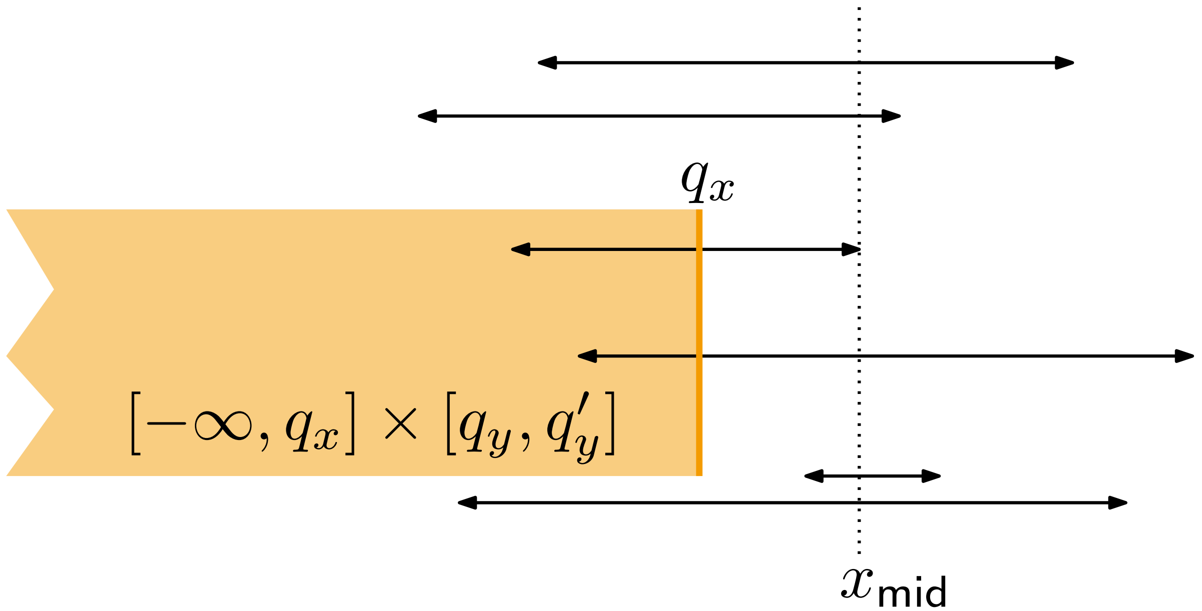

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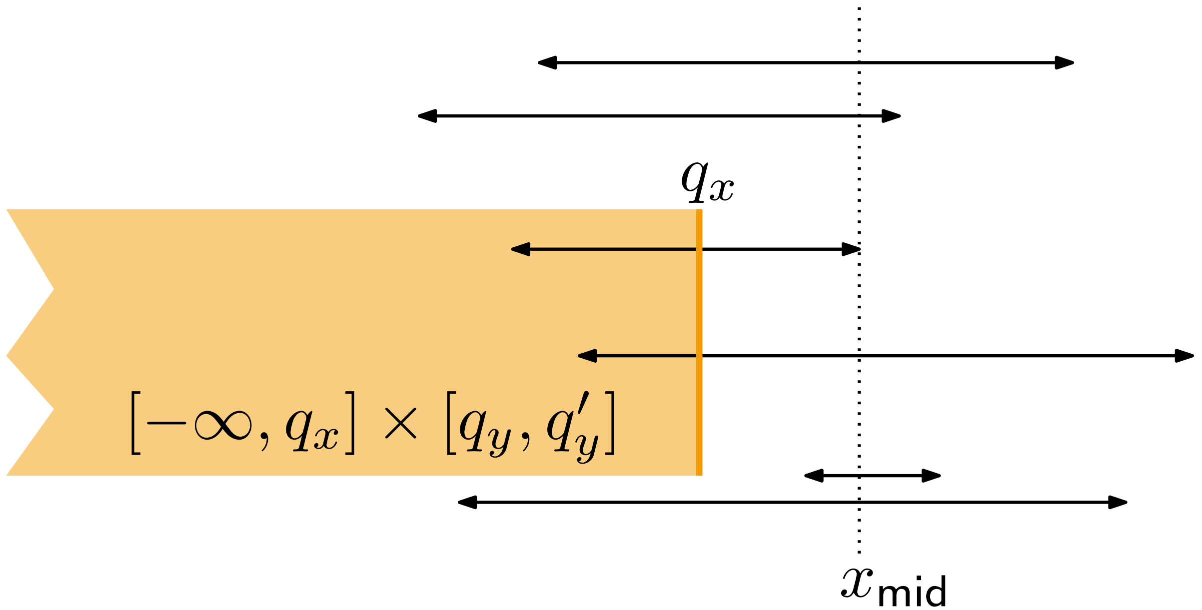

The correct line segments in I_{mid} can easily be found using a range tree instead of simple lists.

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Karlsruhe Institute of Technology

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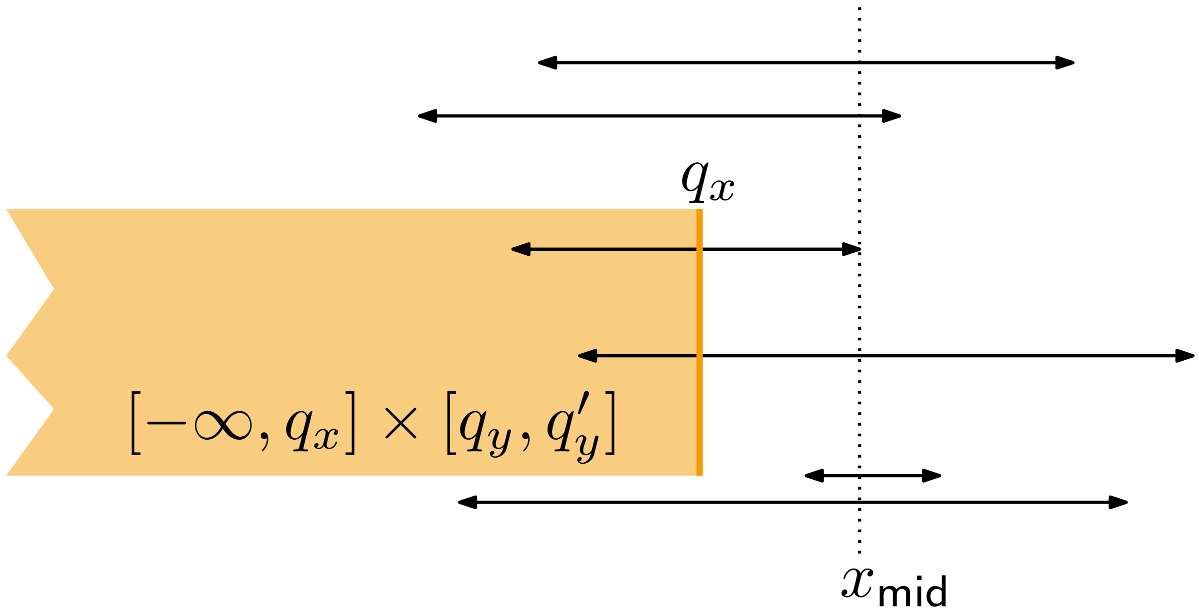
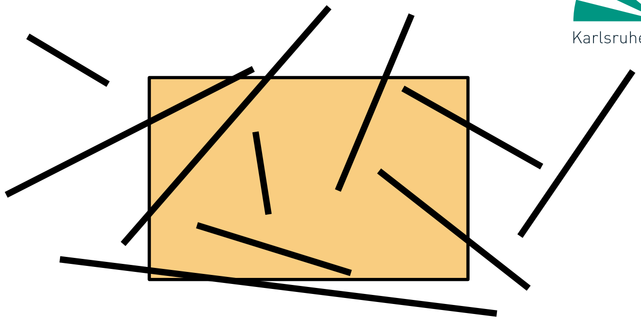


The correct line segments in I_{mid} can easily be found using a range tree instead of simple lists.

Theorem 1: Let S be a set of horizontal (axis-parallel) line segments in the plane. All k line segments that intersect a vertical query segment (an axis-parallel rectangle R) can be found in $O(\log^2(n) + k)$ time. The data structure requires $O(n \log n)$ space and construction time.

Arbitrary line segments

Map data often contain arbitrarily oriented line segments.



Problem:

Given n disjoint line segments and an axis-parallel rectangle $R = [x, x'] \times [y, y']$, find all line segments that intersect R.

How to proceed?

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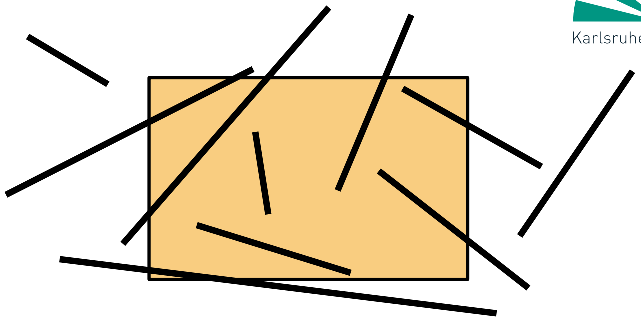

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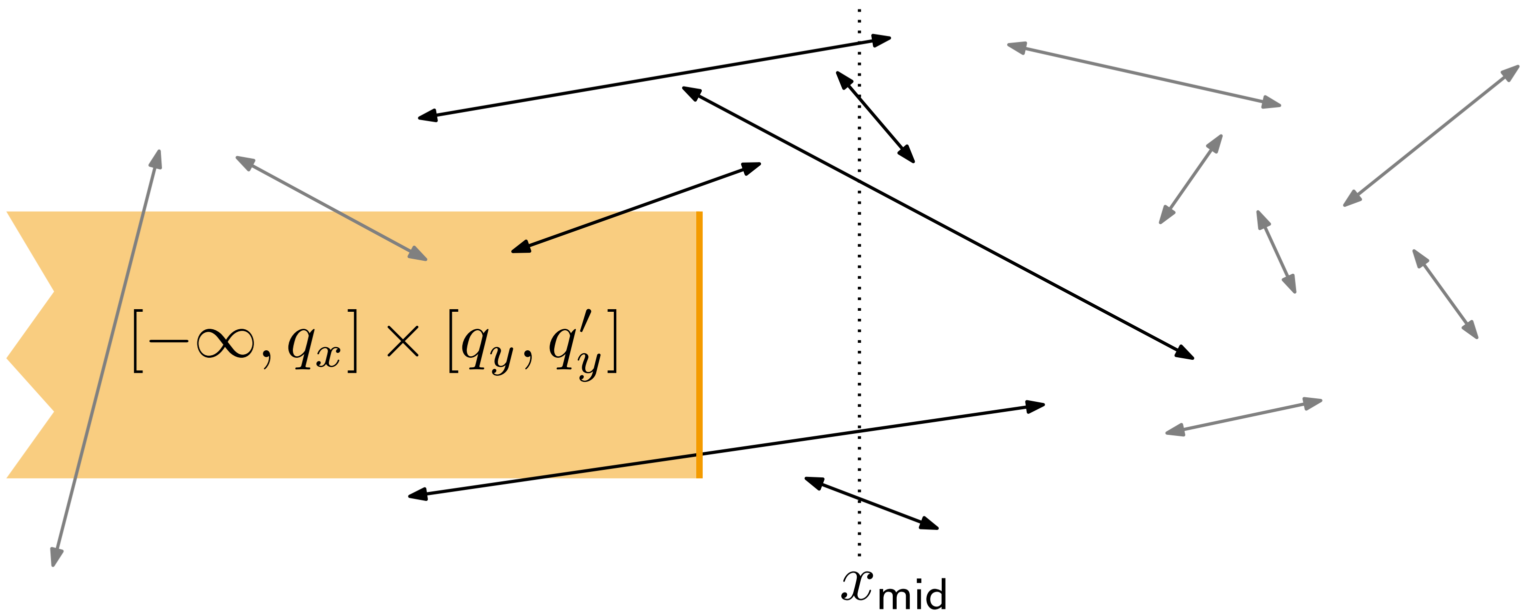
Decomposition into elementary intervals

Interval trees don't really help here



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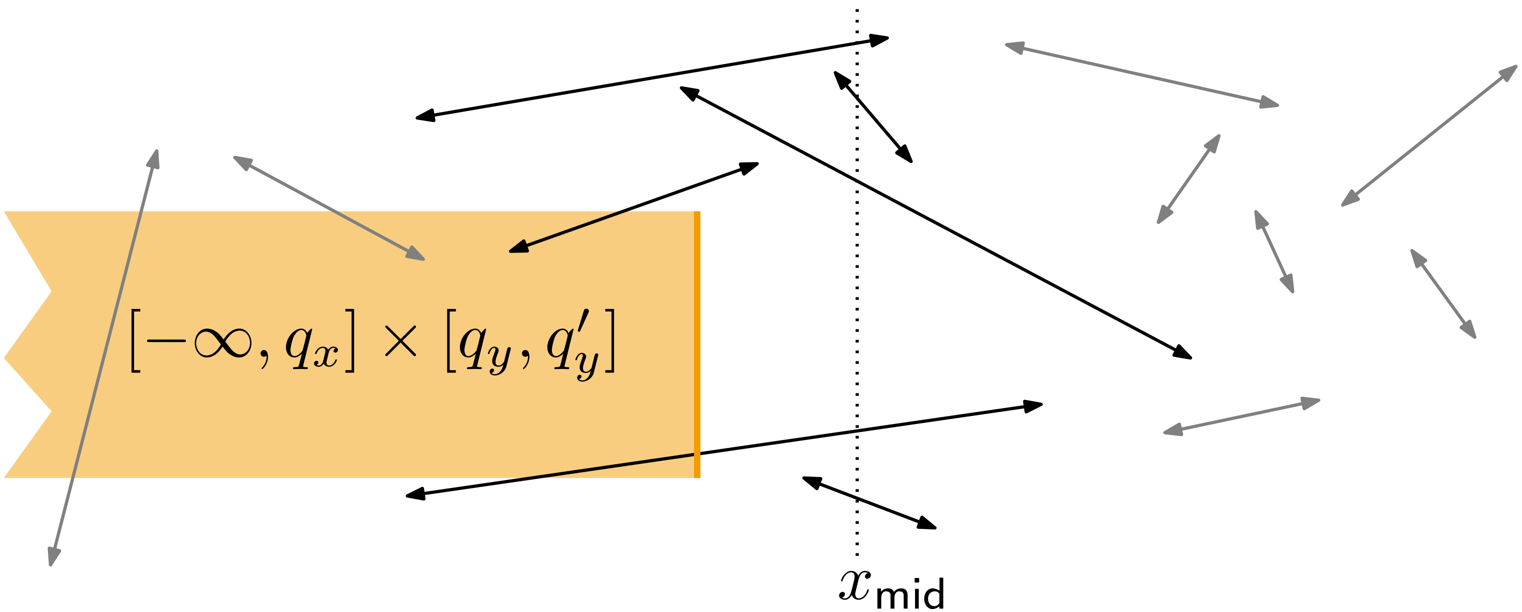


Identical 1d base problem:

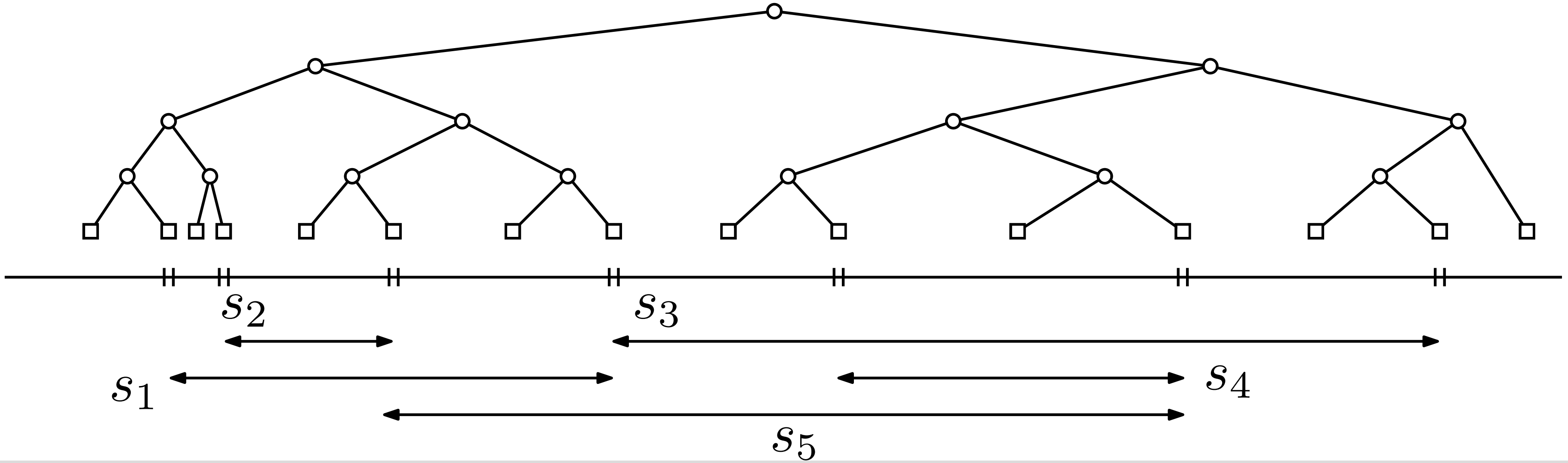
Given n intervals $I = \{[x_1, x'_1], [x_2, x'_2], \ldots, [x_n, x'_n]\}$ and a point q_x, find all intervals that contain q_x.

- sort all x_i and x'_i in list $p_1, \ldots, p_{2n}$
- create sorted elementary intervals $(-\infty, p_1), [p_1, p_1], (p_1, p_2), [p_2, p_2], \ldots, [p_{2n}, p_{2n}], (p_{2n}, \infty)$

Segment trees

Idea for data structure:

- create binary search tree with elementary intervals in leaves
- for all points q_x in the same elementary interval the answer is the same
- leaf μ for elementary interval $e(\mu)$ stores interval set $I(\mu)$
- query requires $O(\log n + k)$ time



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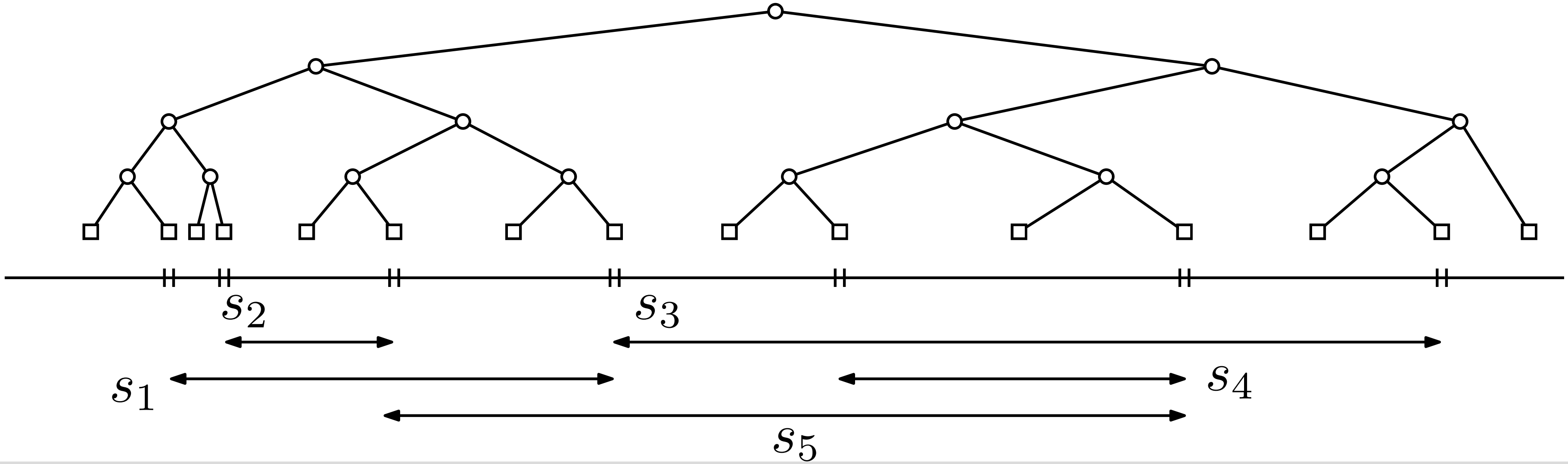
Any problem?



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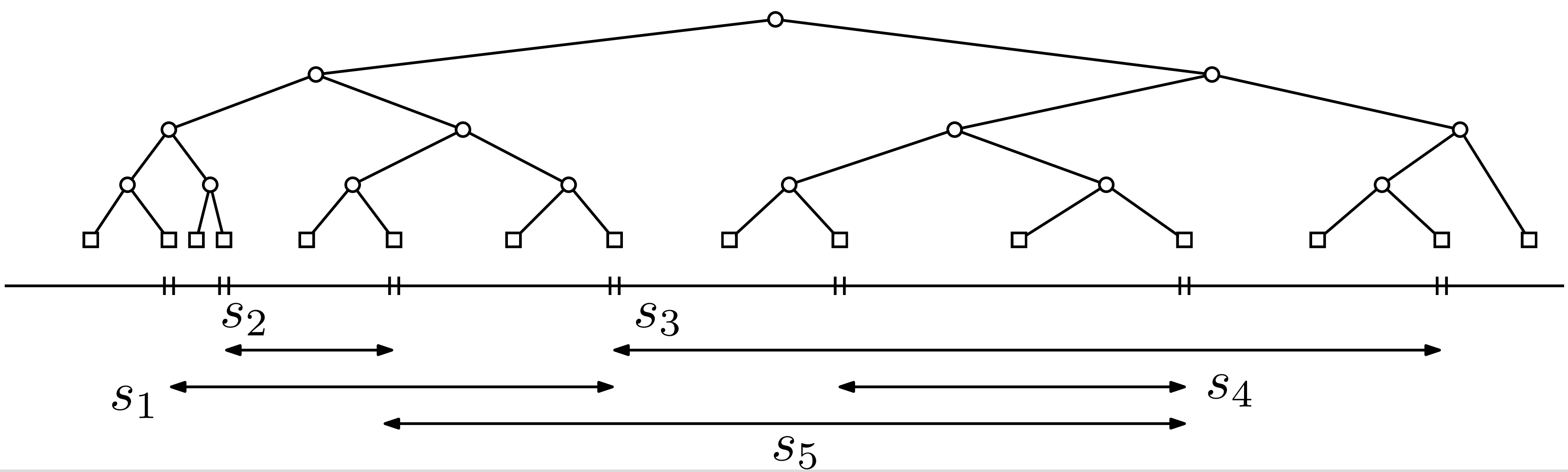
Problem: Storage space is worst-case quadratic

$\rightarrow$ store intervals as high up in the tree as possible



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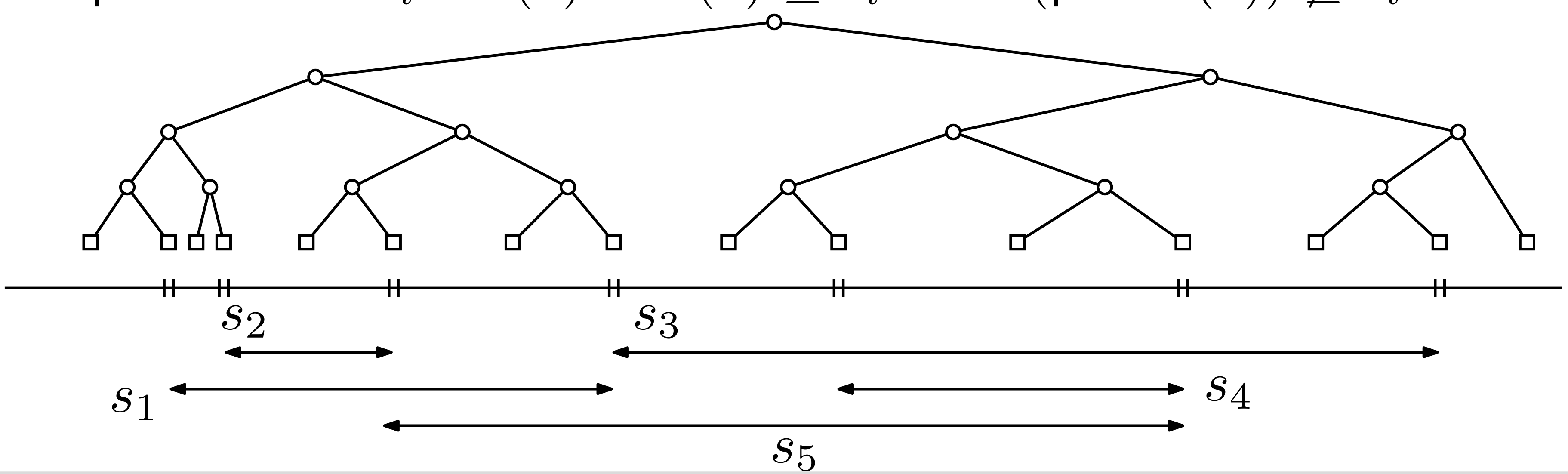
- node v represents interval $e(v) = e(lc(v)) \cup e(rc(v))$
- input interval $s_i \in I(v) \Leftrightarrow e(v) \subseteq s_i$ and $e(\text{parent}(v)) \not\subseteq s_i$



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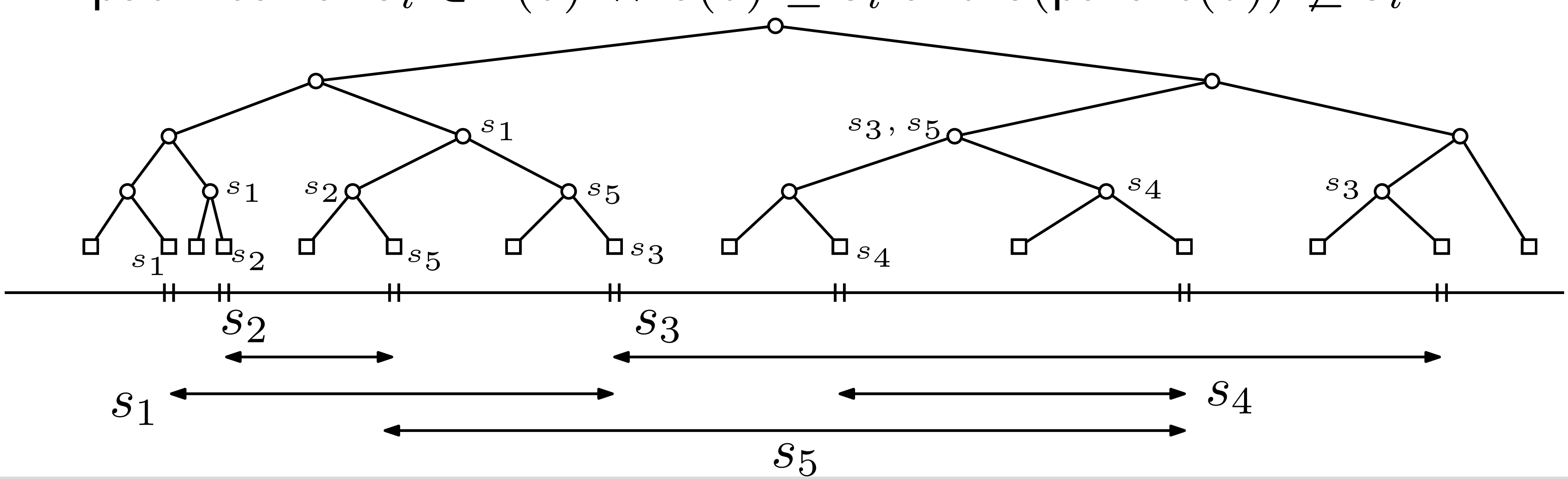

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Sketch of proof:

InsertSegmentTree($v, [x, x']$)

if $e(v) \subseteq [x, x']$ **then**
  store $[x, x']$ in $I(v)$
else
  if $e(lc(v)) \cap [x, x'] \neq \emptyset$ **then**
    InsertSegmentTree($lc(v)), [x, x']$)
  if $e(rc(v)) \cap [x, x'] \neq \emptyset$ **then**
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Queries in segment trees

QuerySegmentTree(v, q_x)

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return all intervals in I(v)
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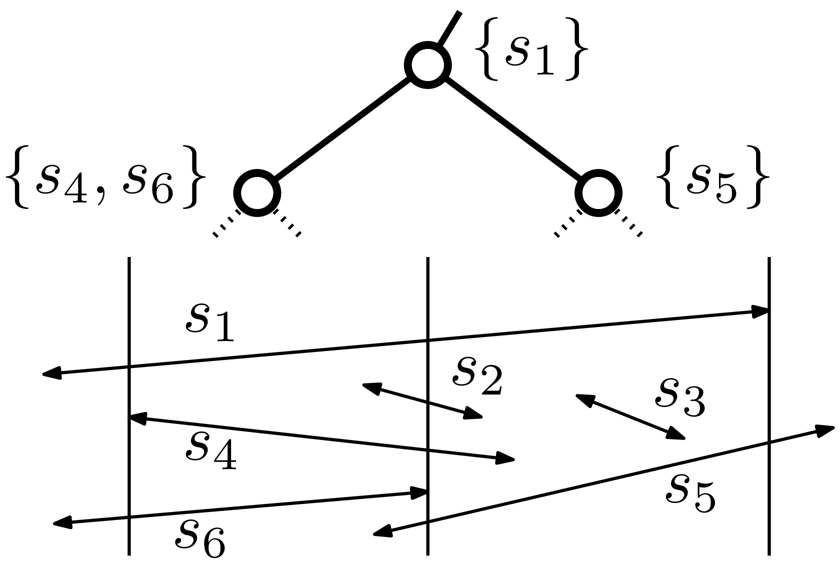
$\rightarrow$ *all* intervals stored in a positive node v contain q_x – in an interval tree one would have to continue searching

Back to arbitrary line segments

- create segment tree for the x intervals of the line segments
- each node v corresponds to a vertical strip $e(v) \times \mathbb{R}$
- line segment s is in $I(v)$ iff s crosses the strip of v but not the strip of parent(v)
- at each node v on the search path for the vertical segment $s' = q_x \times [q_y, q'_y]$ all segments in $I(v)$ cover the x-coordinate q_x



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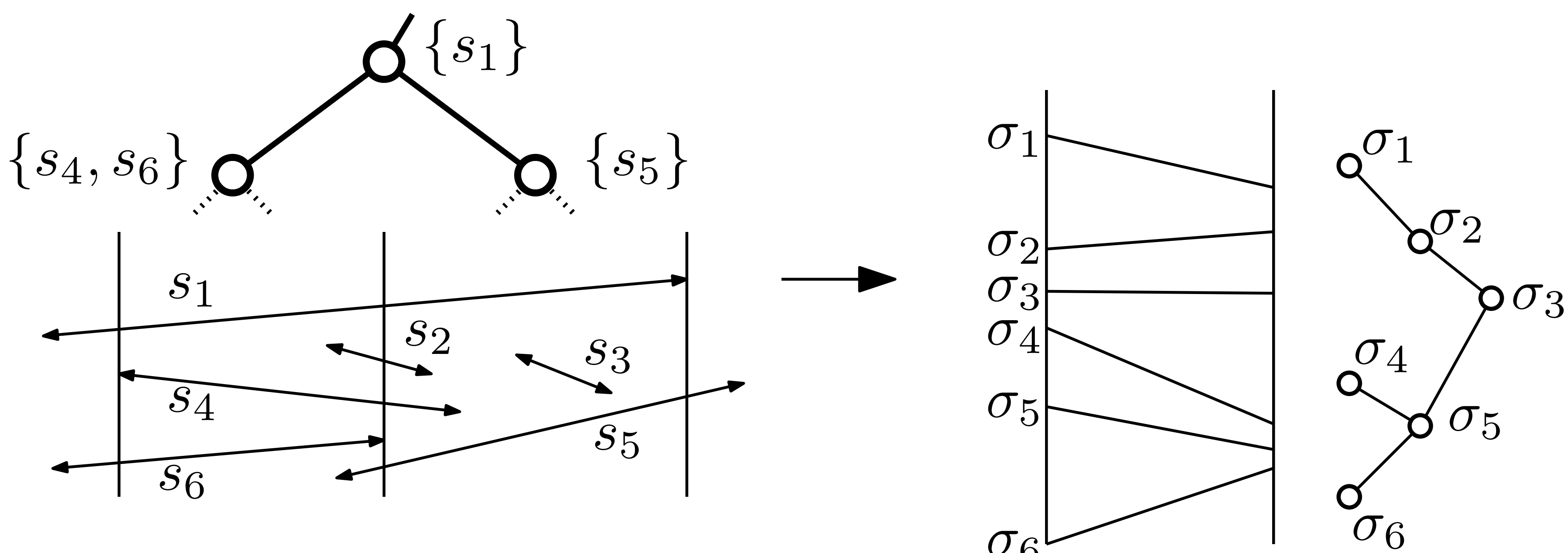
- at each node v on the search path for the vertical segment $s' = q_x \times [q_y, q'_y]$ all segments in $I(v)$ cover the x-coordinate q_x
- find segments in the strip that cross s' using a vertically sorted auxiliary binary search tree



Summary

Theorem 2: Let S be a set of interior-disjoint line segments in the plane. All k segments that intersect a vertical query segment (an axis-parallel query rectangle R) can be found in time $O(k + \log^2 n)$. The corresponding data structure uses $O(n \log n)$ space and $O(n \log^2 n)$ construction time.

Summary

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Remark:
The construction time for the data structure can be improved to $O(n \log n)$.

Discussion

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We have used range trees with $O(n \log n)$ space as auxiliary data structure in the interval trees. Using modified heaps this can be reduced to $O(n)$, see Chapter 10.2 in [BCKO'08].

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What to do for non-rectangular query regions?

By triangulating the query polygon, the problem can be reduced to triangular queries. Suitable data structures can be found, e.g., in chapter 16 of [BCKO'08].