Computational Geometry · Lecture
Introduction & Convex Hulls

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

Tamara Mchedlidze · Darren Strash

19.10.2015

AlgoGeom-Team

Lecturers

- Tamara Mchedlidze
- `mched@iti.uka.de`
- Room 307
- Office hours: by appointment

- Darren Strash
- `strash@kit.edu`
- Room 206
- Office hours: by appointment

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Exercise Leader

- Benjamin Niedermann
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Schedule

- Lecture: Mon. 15:45 – 17:15, SR 301
- Exercises: Wed. 15:45 – 17:15, SR 301 (starting Oct. 28)

Organization

Website

http://i11www.iti.kit.edu/teaching/winter2015/compgeom/

- Course Information
- Lecture Slides
- Exercises
- Additional Material

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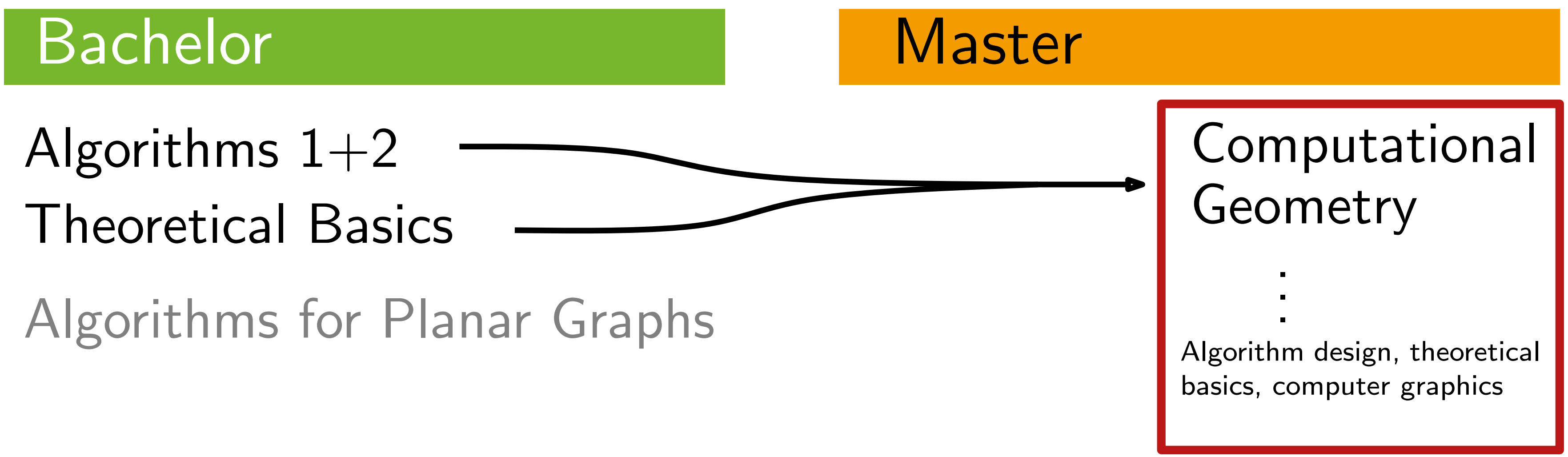
Computational Geometry in Computer Science Master's Studies



Organization

Exercises

- Every second Wednesday starting 28.10
- Exercise problems posted at least one week before an exercise session.
- Rinforce lecture material, help prepare for exam.

What will the exercises involve?

- Weekly work for about 45–60 minutes
- Active participation in exercises is expected
- Volunteers will work problems on the board
- Can hand in exercises for feedback
- Variations will be announced
  - Exercise on 16.12 instead of 23.12

Computational Geometry

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Objectives: At the end of the course you will be able to...

- explain concepts, structures, and problem definitions
- understand the discussed algorithms, and explain and analyze them
- select and adapt appropriate algorithms and data structures
- analyze new geometric problems and develop efficient solutions

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Course Time Breakdown: 5LP = 150h

Time in lectures and exercise sessions	ca. 35h
Preparation and review	ca. 25h
Working on exercises	ca. 20h
Project work	ca. 40h
Exam preparation	ca. 30h

Modules and Grading

Master's in Computer Science

- Computational Geometry (IN4INAG) [5 LP]
- Algorithm Engineering & Applications (IN4INAEA) [5 LP]
- Design and Analysis of Algorithms (IN4INDAA) [5 LP]
- Computer Graphics Algorithms (IN4INACG) [3 LP]

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Test Modalities

- Semester-long project in small teams (application-driven geometric algorithms)
  $\rightarrow$ 20% of grade
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More on this in 1–2 weeks

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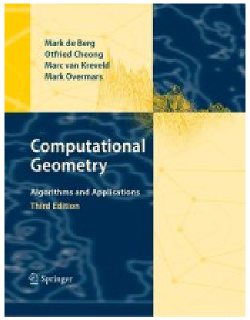
Class Materials



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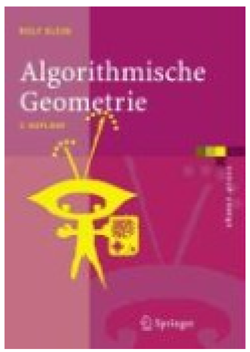
Computational Geometry: Algorithms and Applications
Springer, 3rd Edition, 2008



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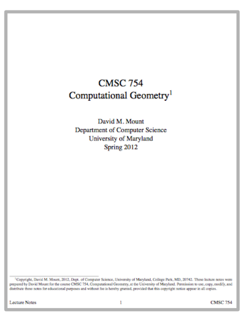
CMSC 754
Computational Geometry[1]

David M. Mount
Department of Computer Science
University of Maryland
Spring 2012

David Mount:
Computational Geometry
Lecture Notes CMSC 754, U. Maryland, 2012

`http://www.cs.umd.edu/class/spring2012/cmsc754/Lects/cmsc754-lects.pdf`

Both books are available in the library!

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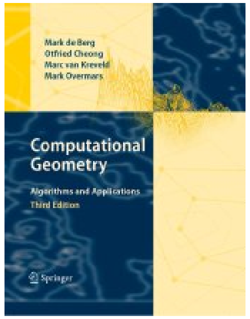

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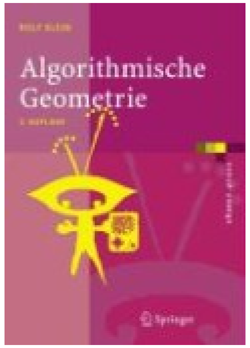

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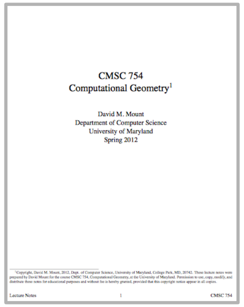

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What is Computational Geometry?

Algorithmische Geometrie

Als **Algorithmische Geometrie** (engl. *Computational Geometry*) bezeichnet man ein Teilgebiet der Informatik, das sich mit der algorithmischen Lösung geometrisch formulierter Probleme beschäftigt. Ein zentrales Problem ist dabei die Speicherung und Verarbeitung geometrischer Daten. Im Gegensatz zur Bildbearbeitung, deren Grundelemente Bildpunkte (Pixel) sind, arbeitet die algorithmische Geometrie mit geometrischen Strukturelementen wie Punkten, Linien, Kreisen, Polygonen und Körpern.

What is Computational Geometry?

Algorithmische Geometrie

Computational geometry is a branch of computer science that deals with algorithmic solutions to geometric problems. A central problem is the storage and processing of geometric data...such as points, lines, circles, polygons...

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Where is Computational Geometry Used?

- Computer Graphics and Image Processing
- Visualization
- Geographic Information Systems (GIS)
- Robotics
- ...

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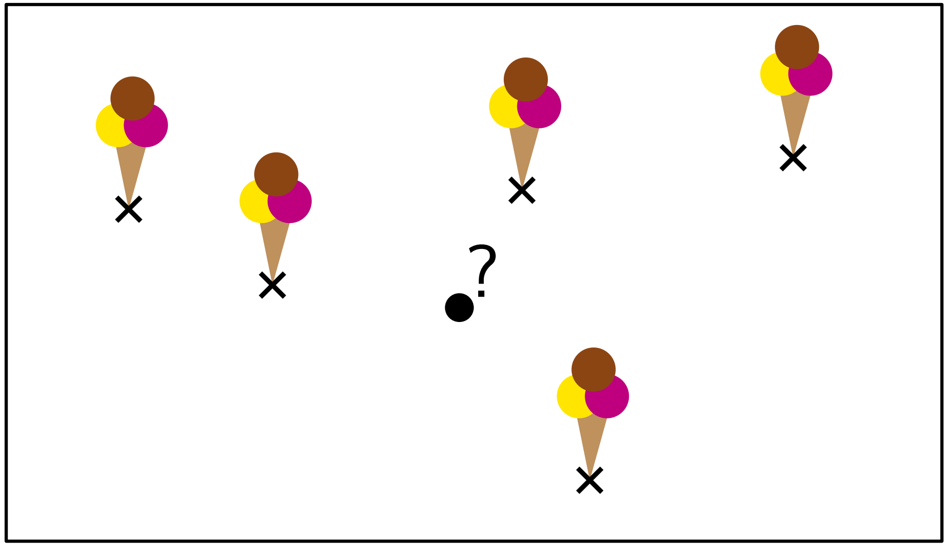
Central Themes

- Geometric algorithms and data structures
- Discrete and combinatorial geometric problems

Example 1

It's a hot 42°C summer day in Karlsruhe. Suppose you know the location of every ice cream shop in the city. How can you determine the closest ice cream shop for any location on a map?



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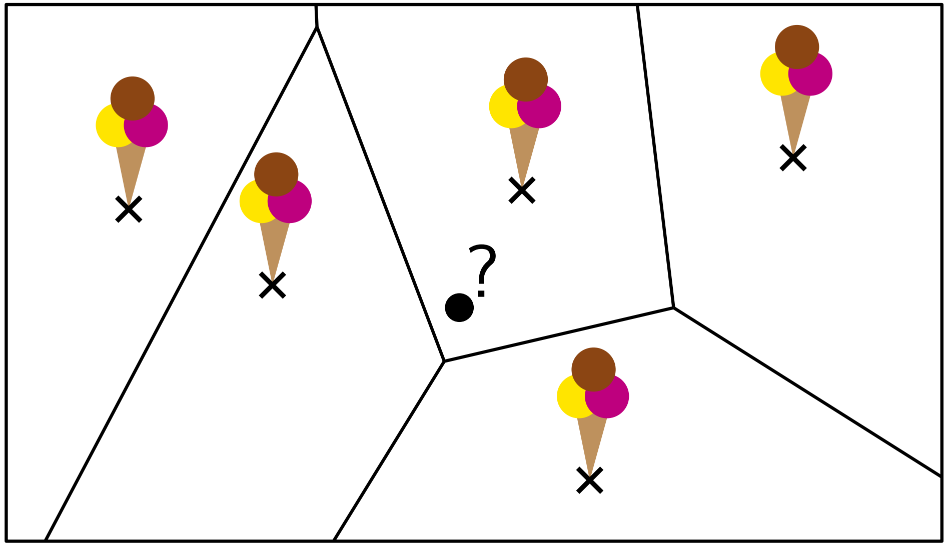

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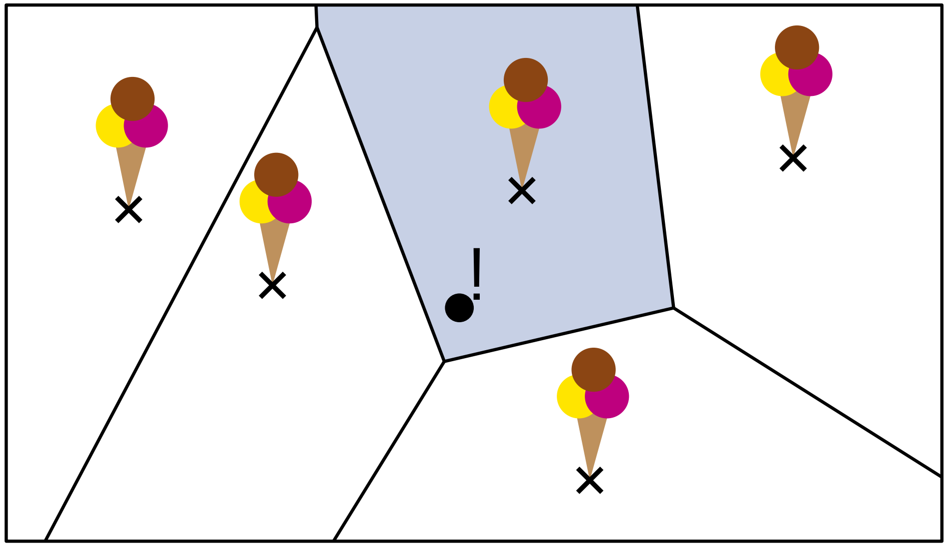

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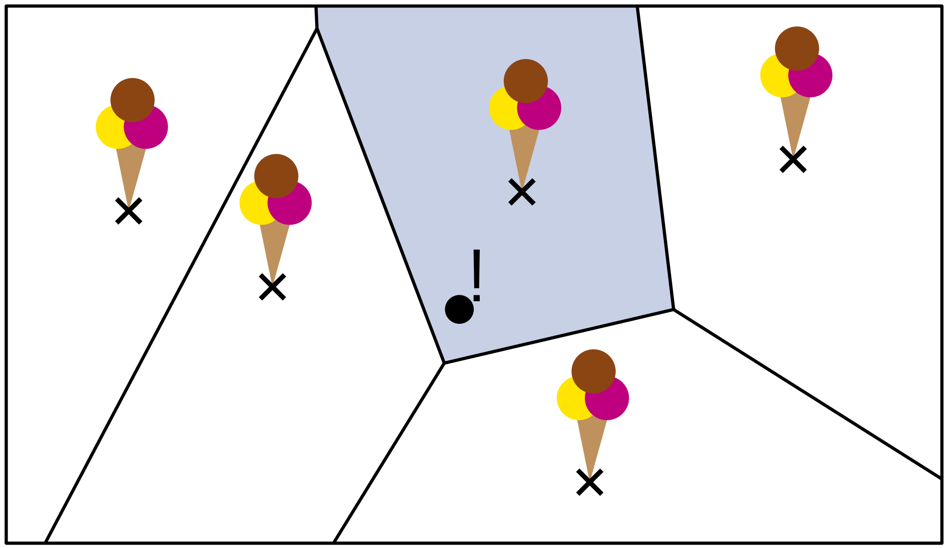

The solution is a *division* of $\mathbb{R}^2$, called a **Voronoi Diagram**.

Many applications, including: site planning, nearest-neighbor finding, robot motion planning, radio cells . . .

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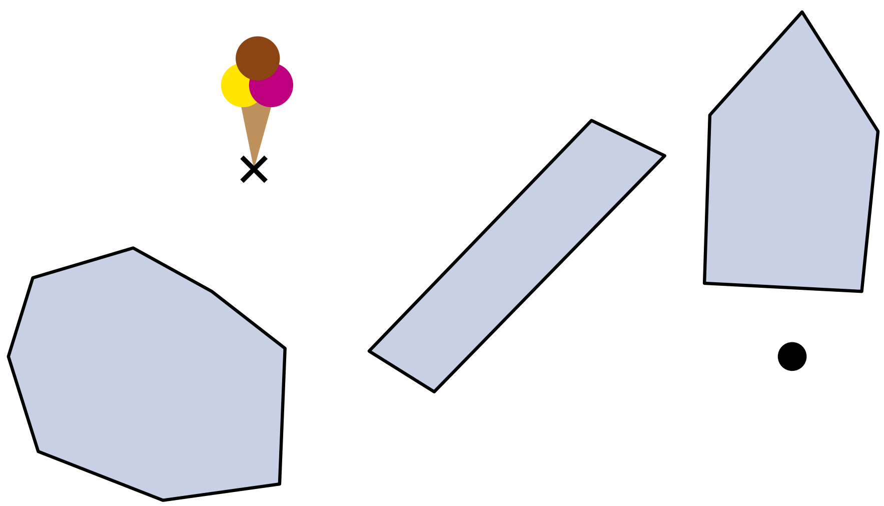
Now it is 50°C in Karlsruhe. We want to send a robot to buy an ice cream cone. How can the robot reach the destination without passing through houses, park benches, and trees?



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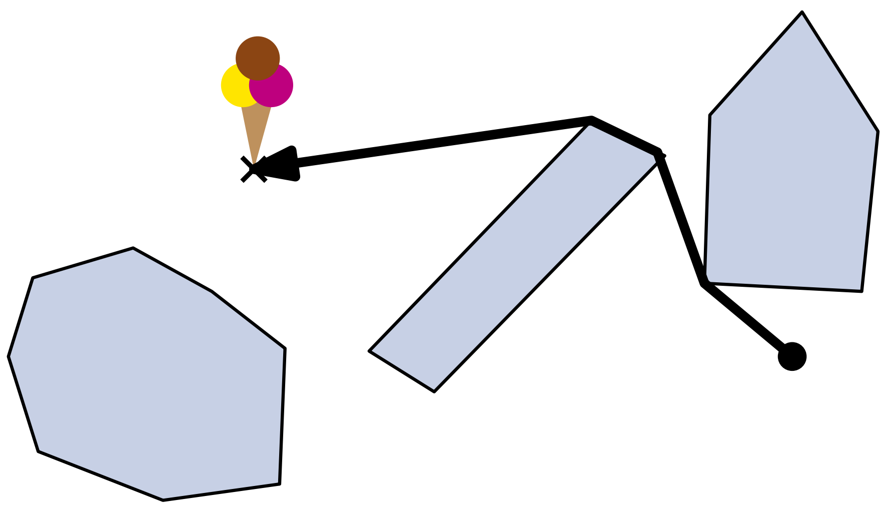

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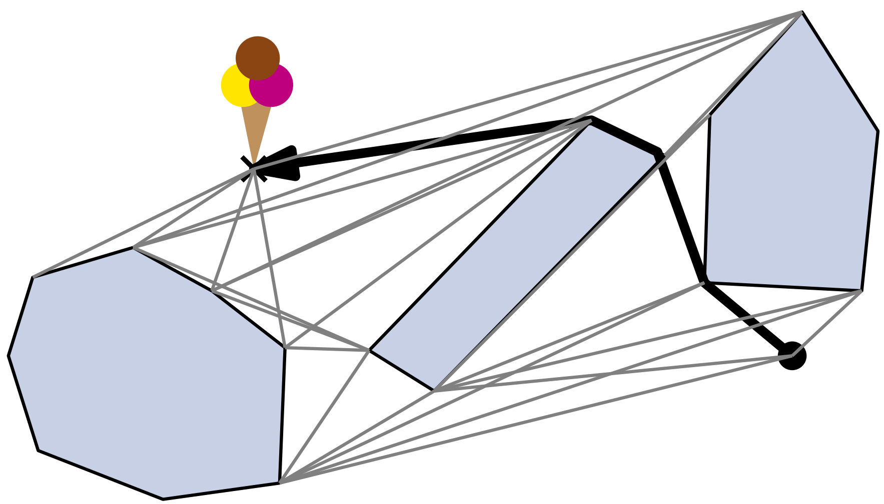
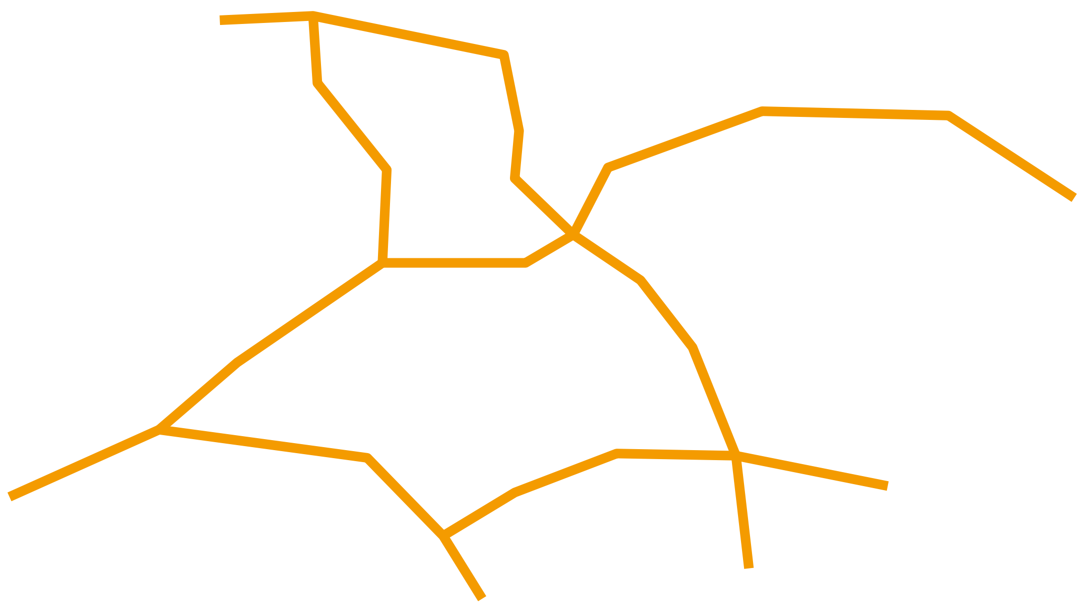


Motion planning problem in robotics:
Given a set of obstacles with a start and destination point, find a collision-free shortest route (e.g., using the **visibility graph).**

Example 3

Maps in geographic information systems consist of several levels (e.g., roads, water, borders, etc.). When superimposing several layers, what are the intersection points?

One example is to to view all roads and rivers as a set of links and ask for the bridges. For these, you have to find all intersections between the two layers.



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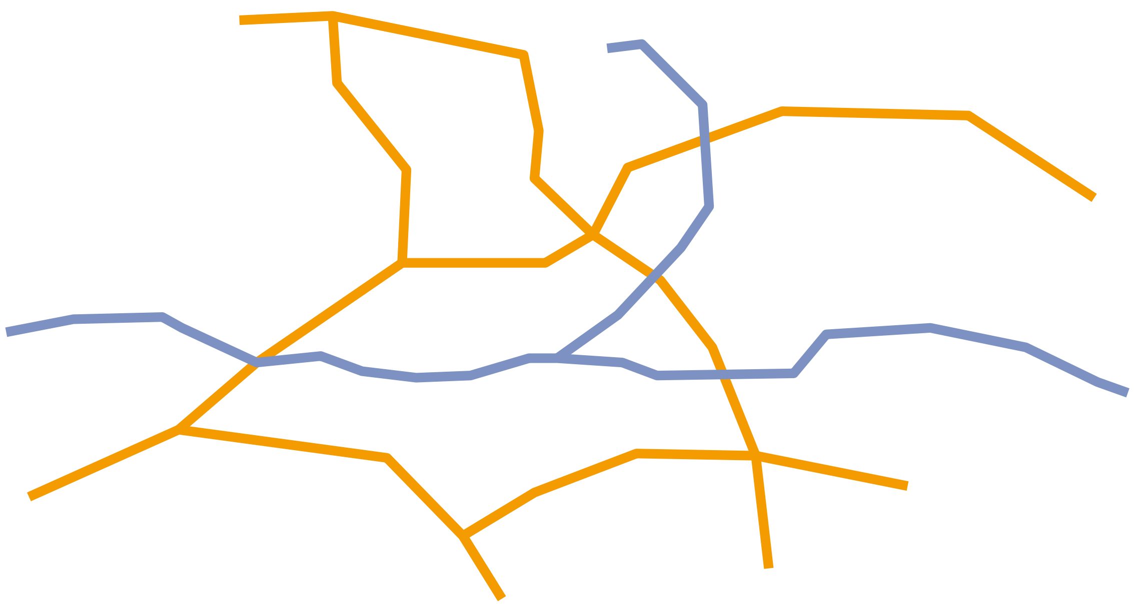

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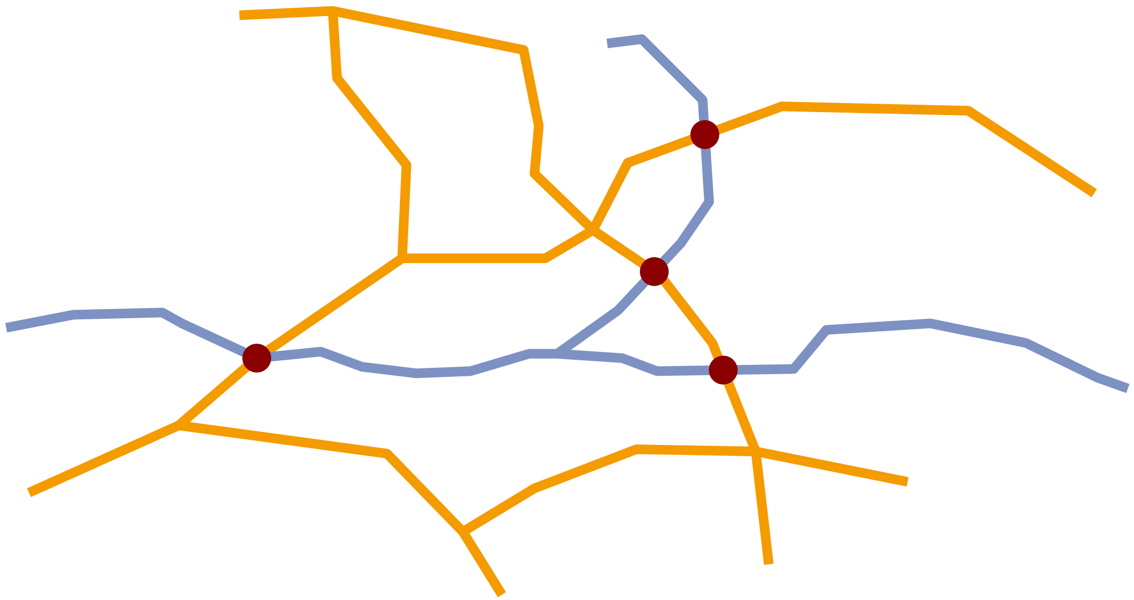

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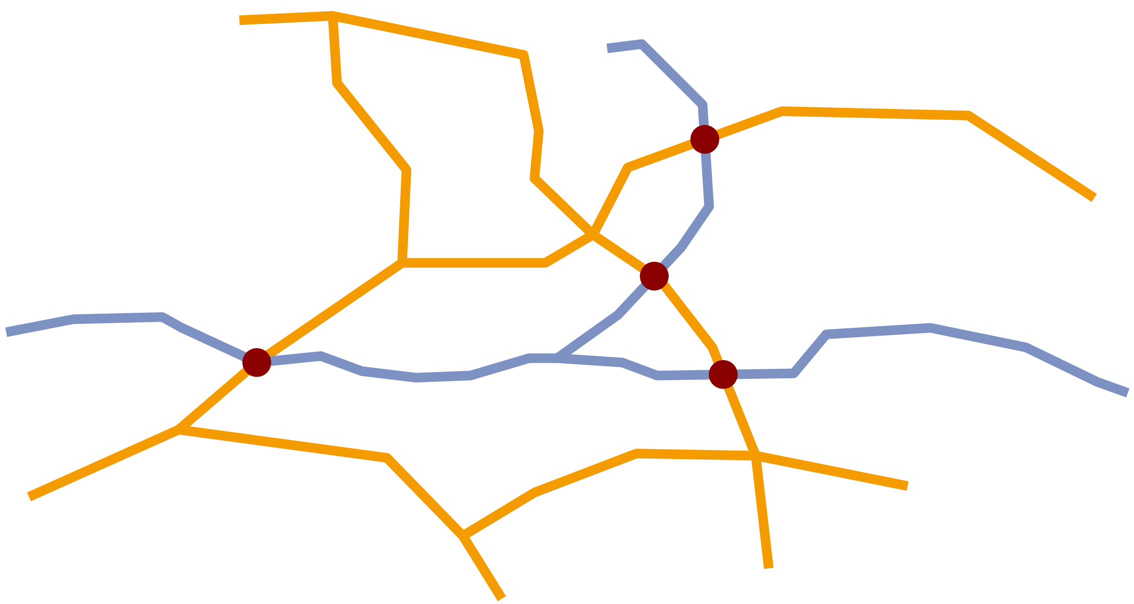
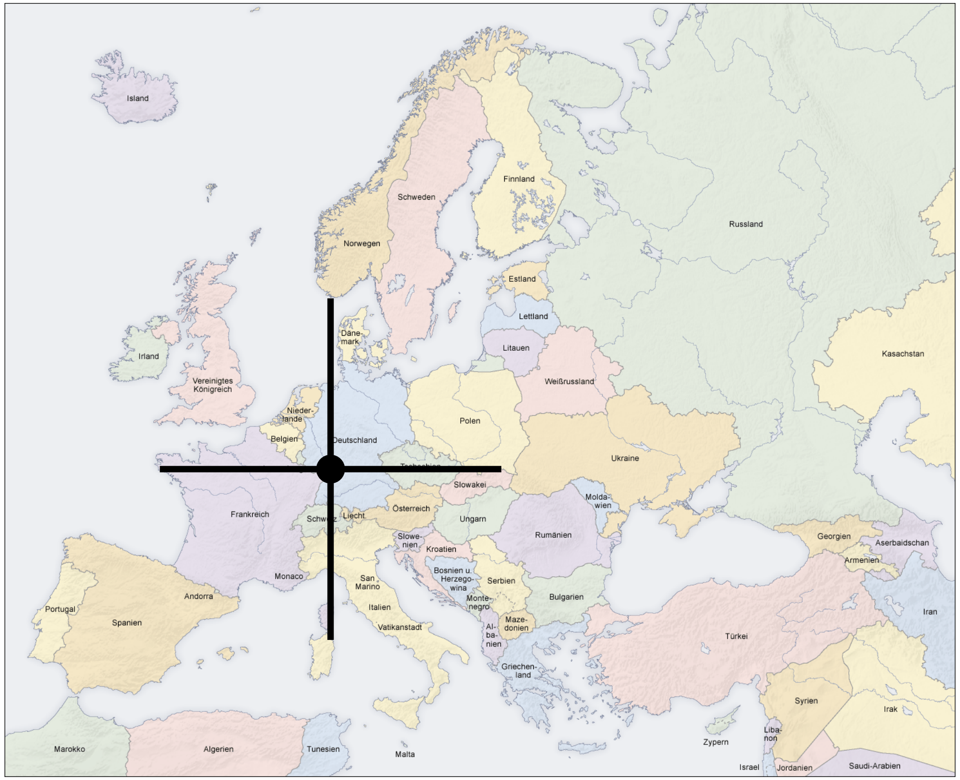


Testing all edge pairs is slow. How can you quickly find all intersections?

Example 4

Given a map and a query point q (e.g., a mouse click), determine the country containing q.



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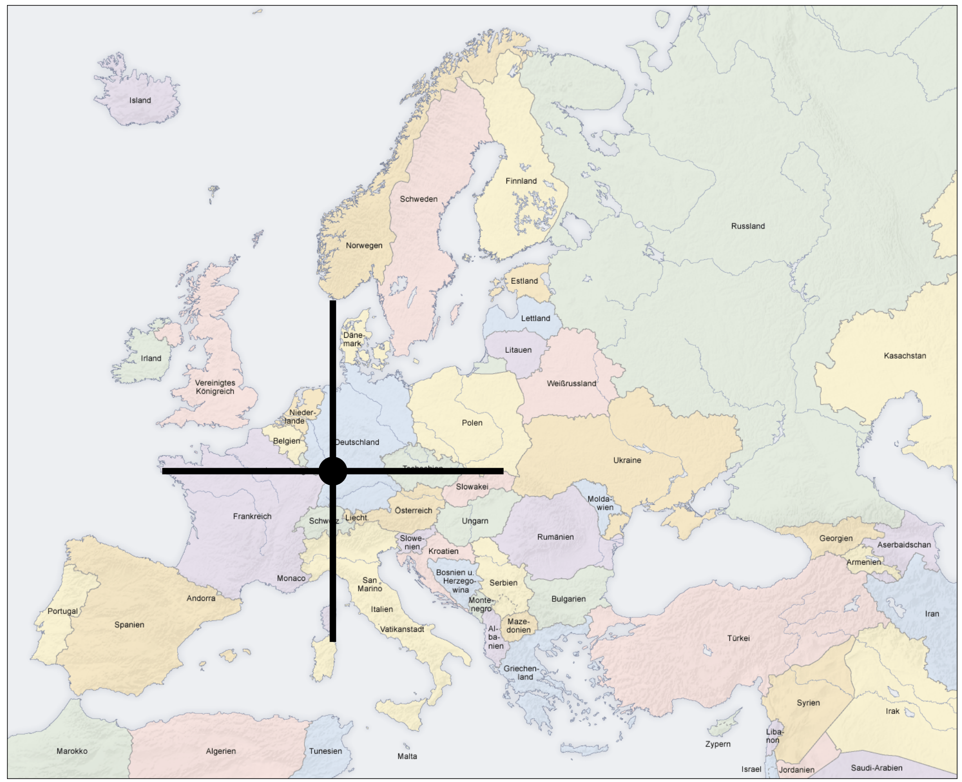

We want a fast data structure for answering point queries.

Example 5

A navigation system should display a current map. How can we effectively choose the data to display?

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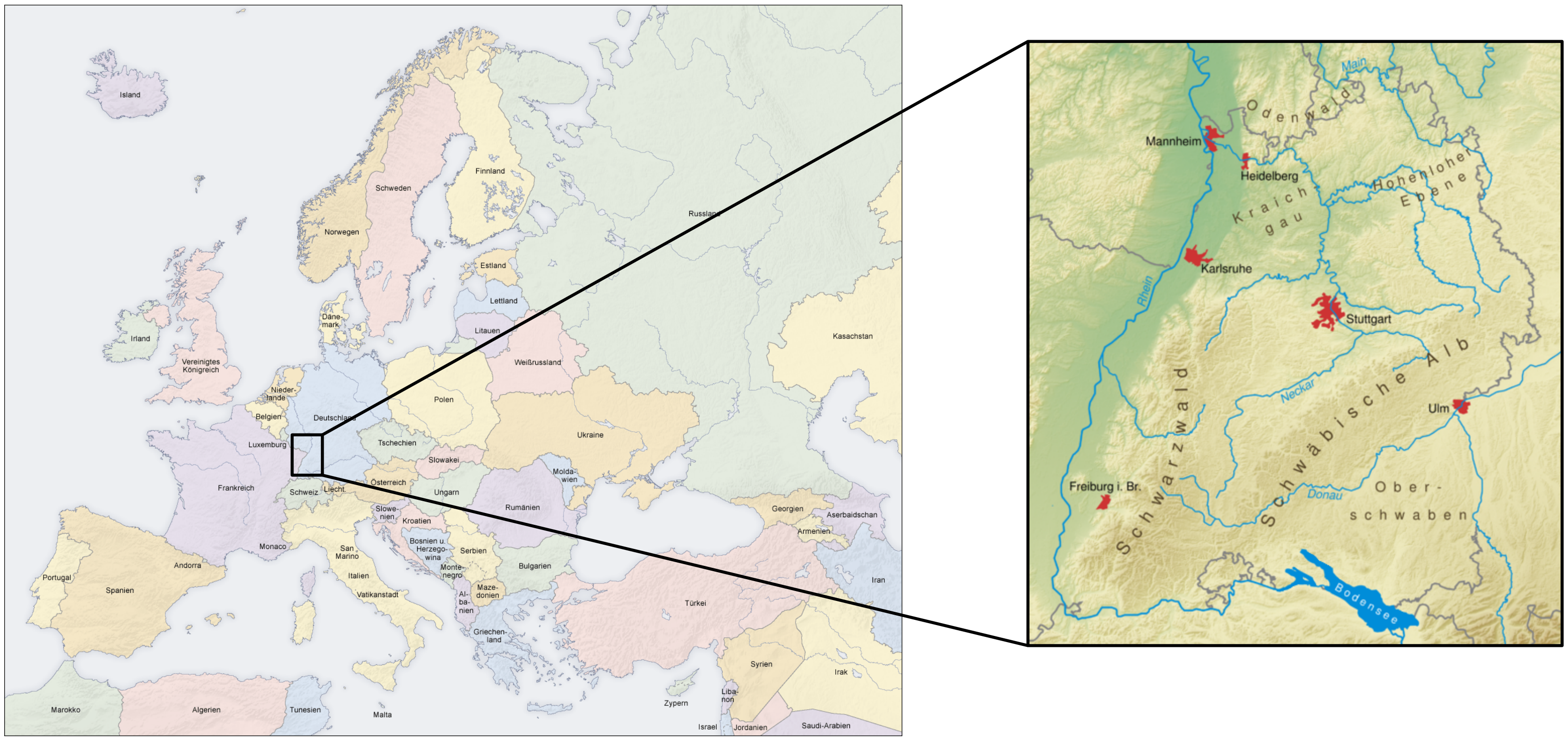

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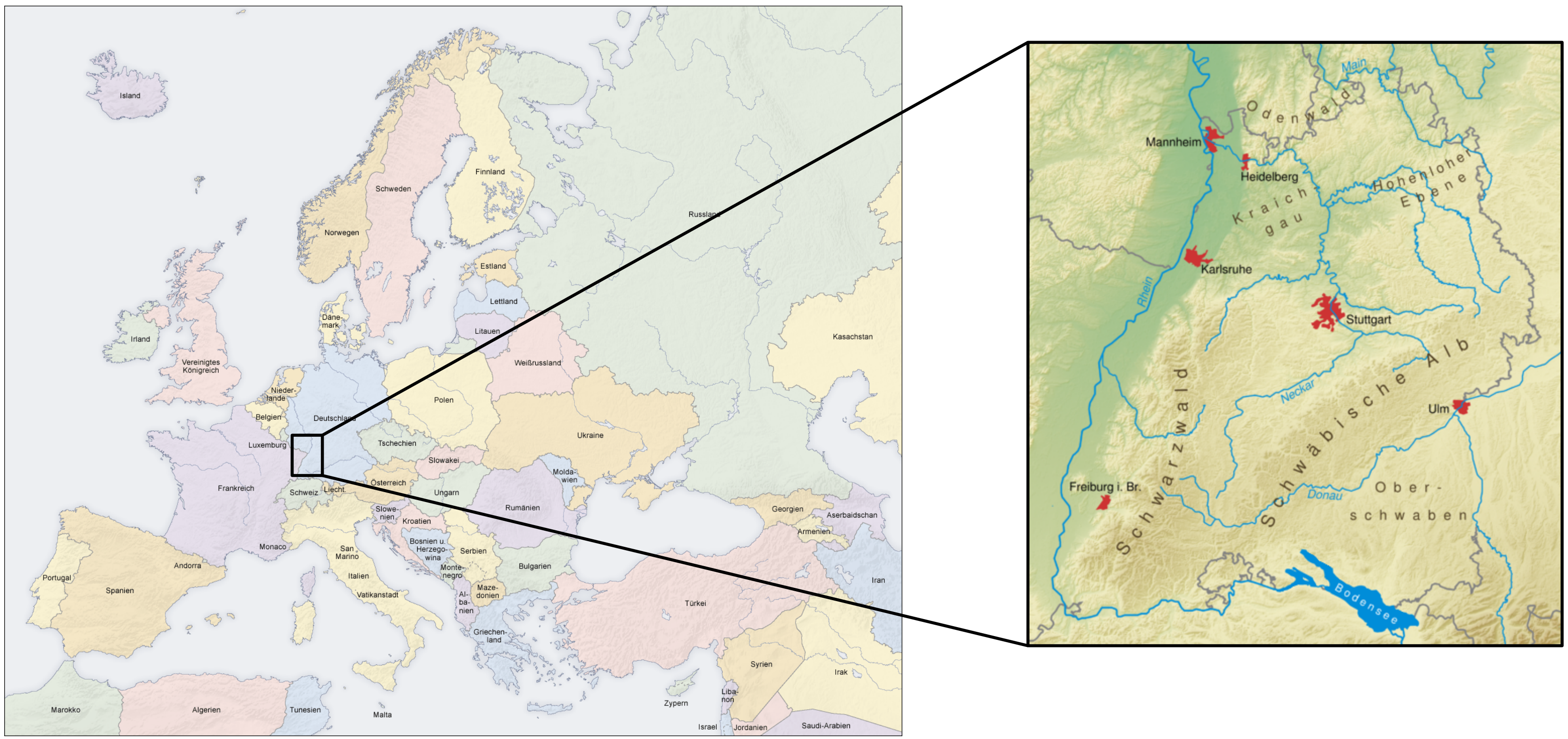


Evaluating each map feature is unrealistic.

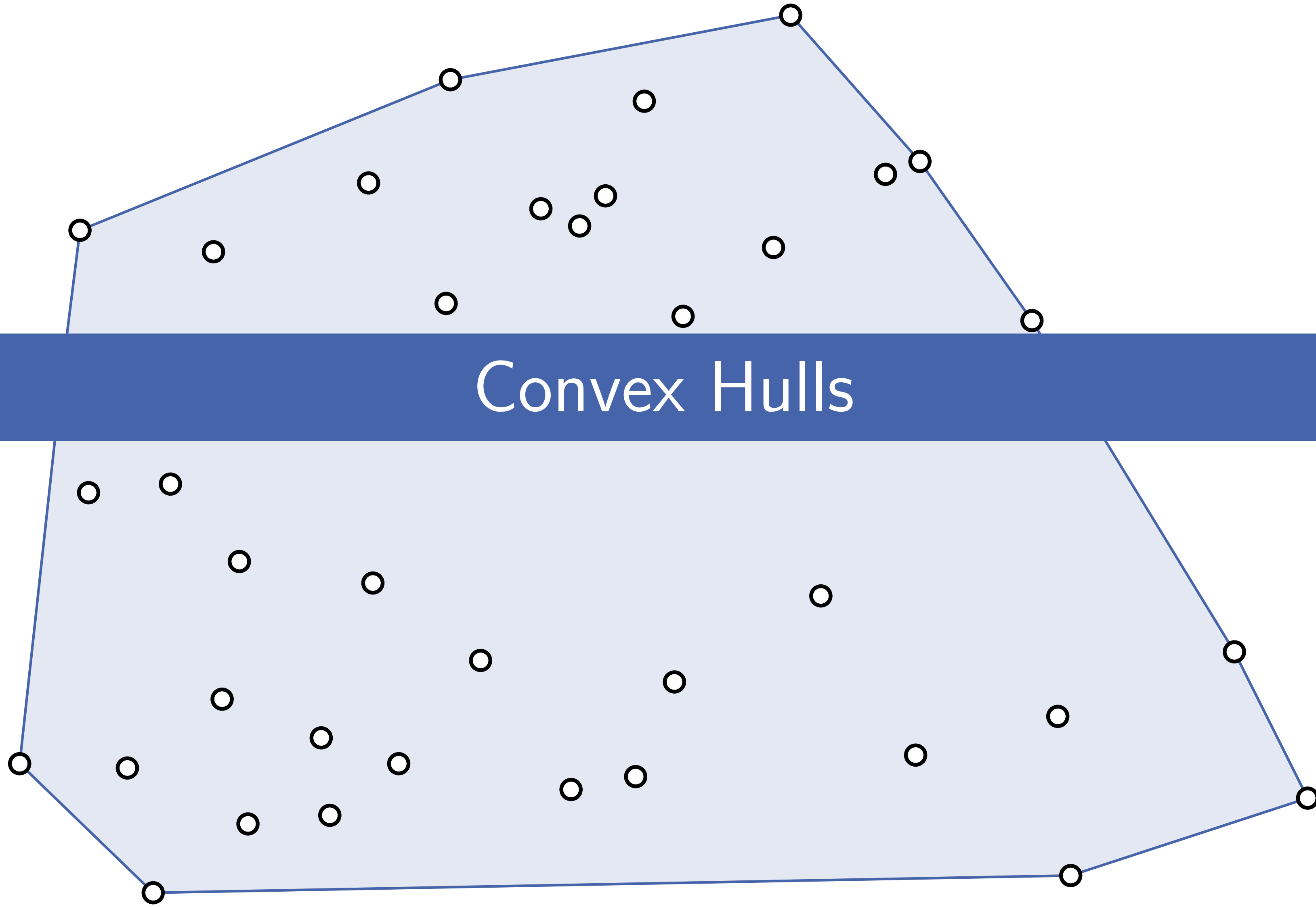
We want a fast data structure for answering range queries

Topics

We will cover the following topics:

- Convex Hulls
- Line Segment Intersection
- Polygon Triangulation
- Geometric Linear Programming
- Data Structures for Range Queries
- Data Structure for Point Location Queries
- Voronoi Diagrams and Delaunay Triangulation
- Duality of Points and Lines
- Quadtrees
- Well-Separated Pair Decompositions
- Visibility Graphs
- ...

Convex Hulls



Mixing Ratios

Given...

Mixture	fraction A	fraction B
s_1	$10\,\%$	$35\,\%$
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q_1: Yes! Ratio 1:1

q_2: No!

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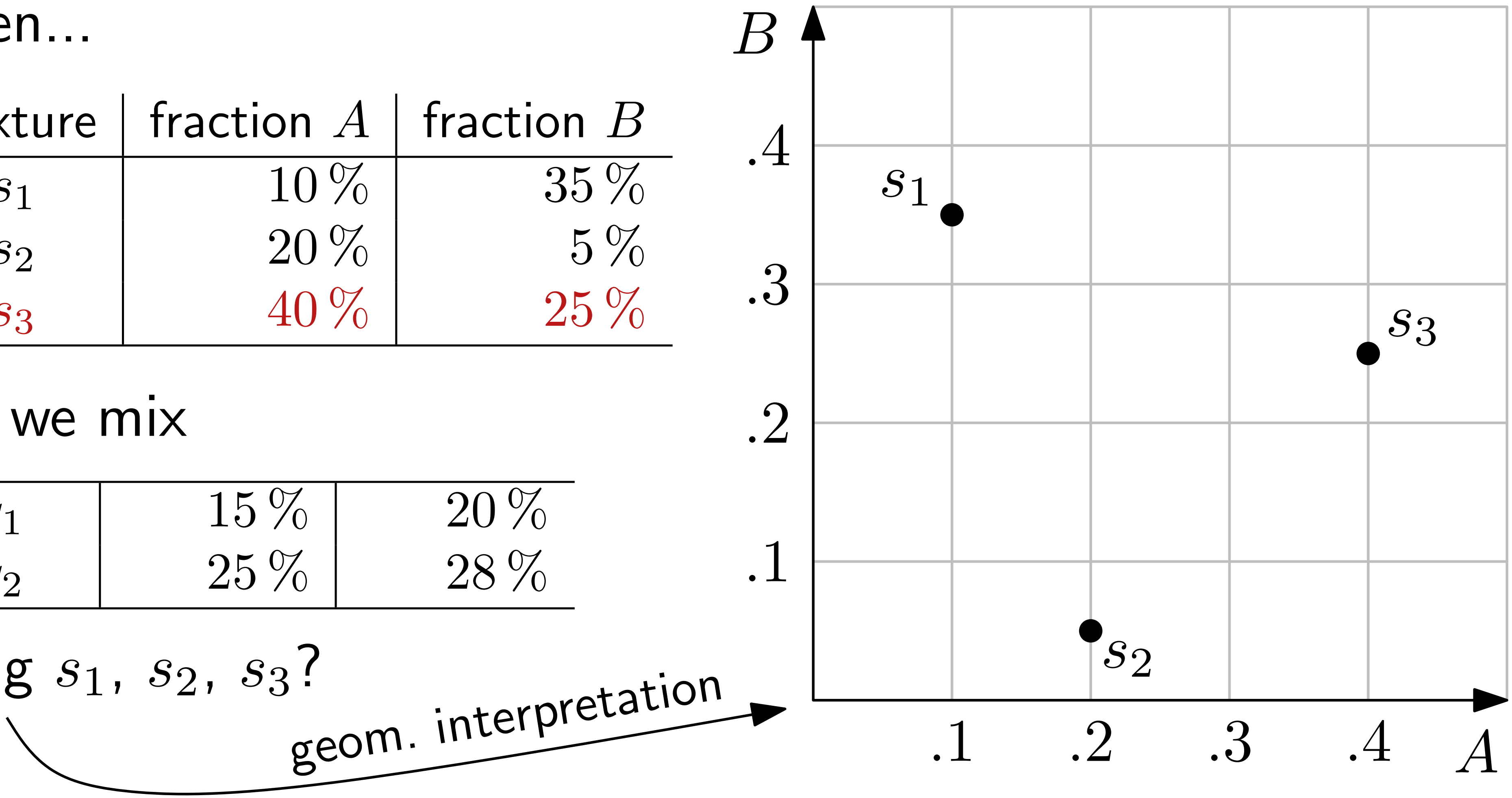

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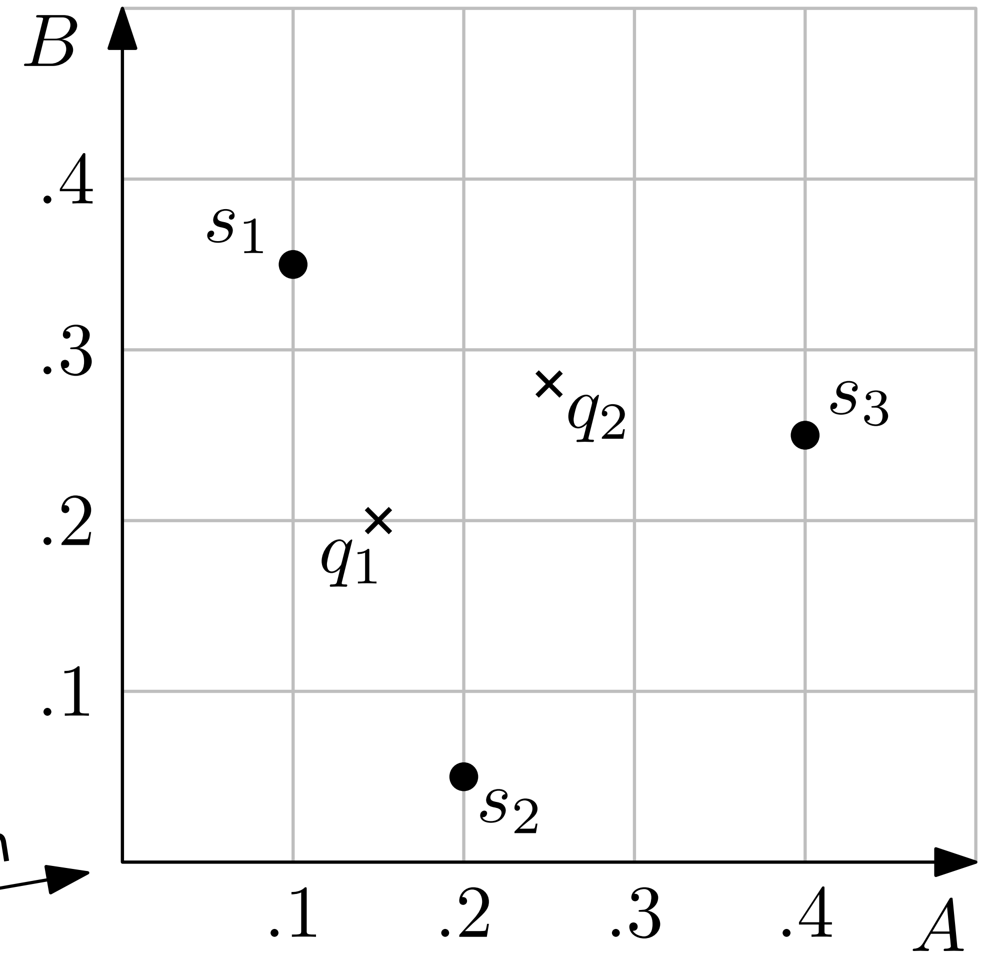

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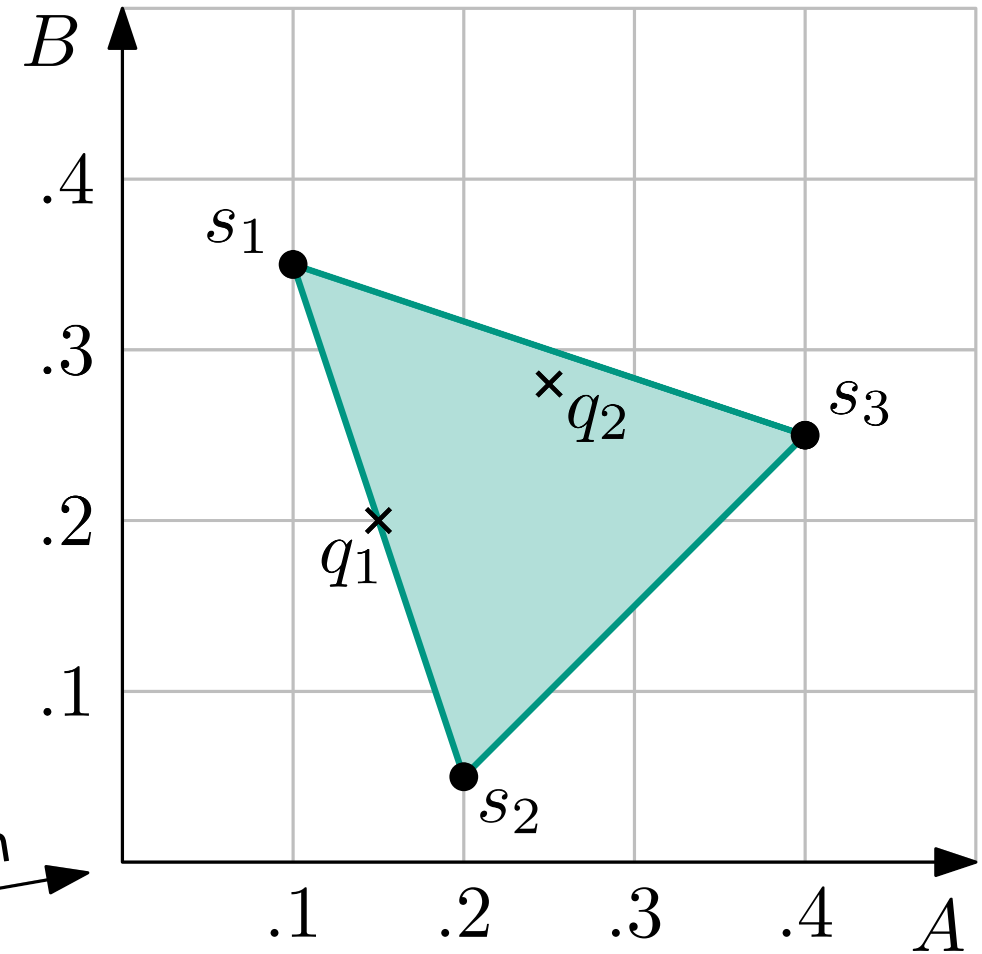

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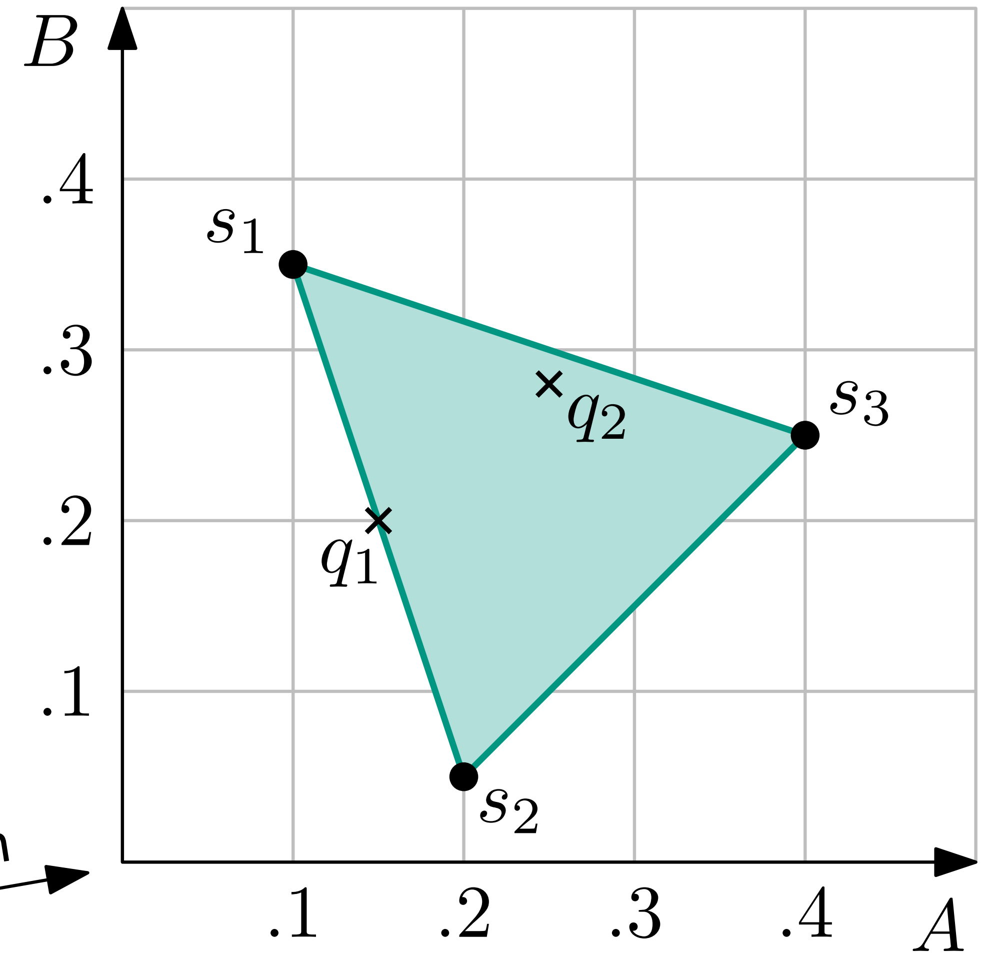

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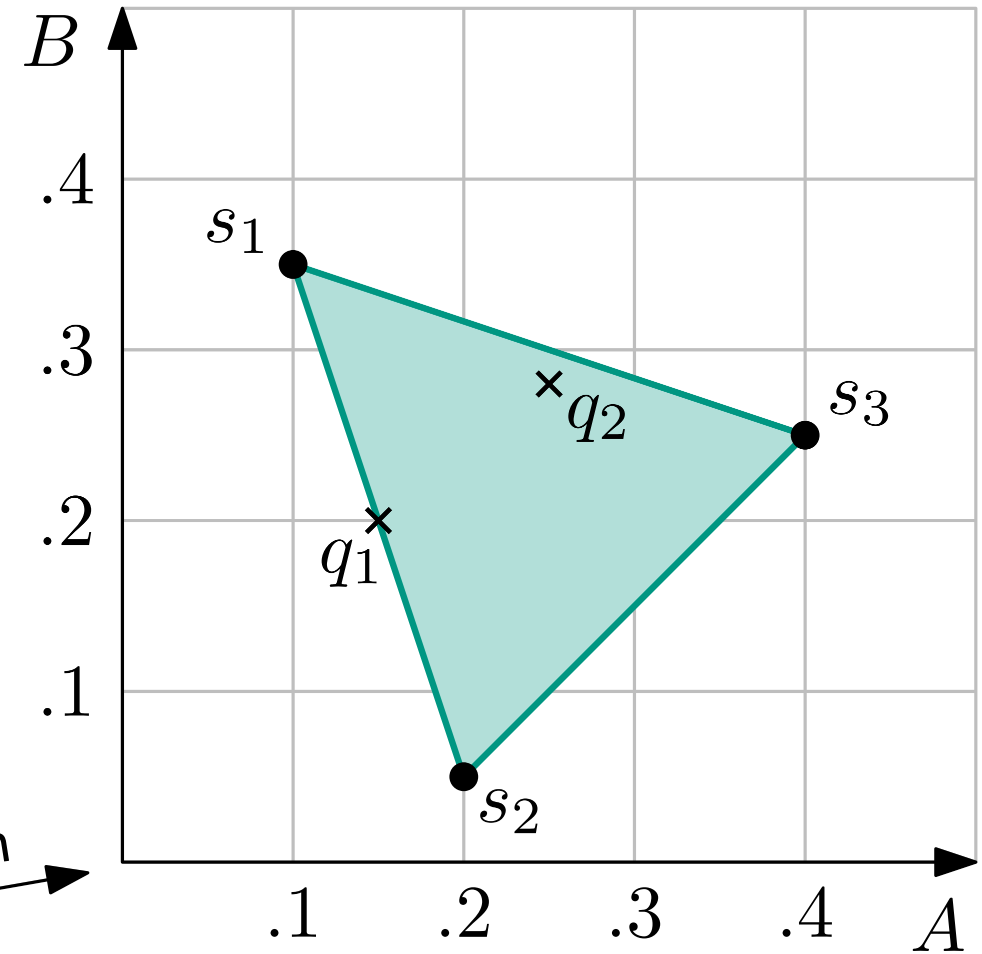

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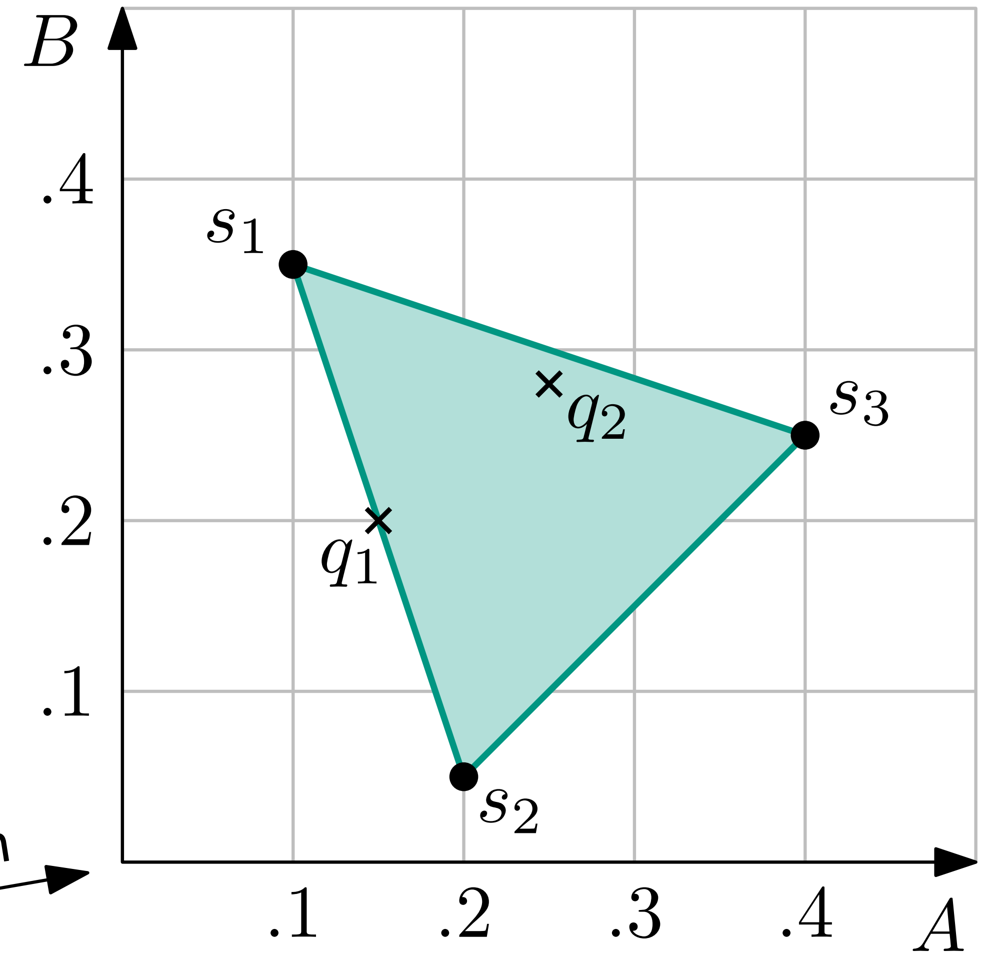

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Definition of Convex Hull

Def: A region $S \subseteq \mathbb{R}^2$ is called **convex**, when for two points $p, q \in S$ then line $\overline{pq} \in S$.
The **convex hull** $CH(S)$ of S is the smallest convex region containing S.

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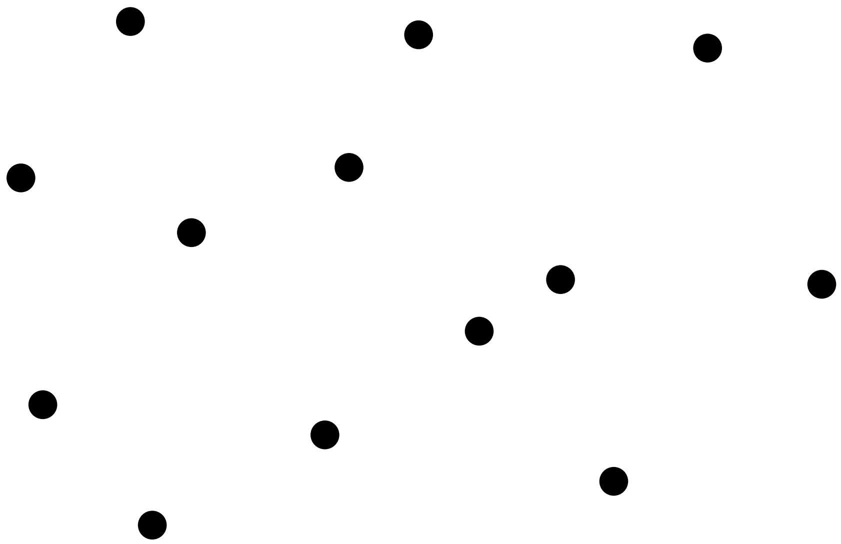

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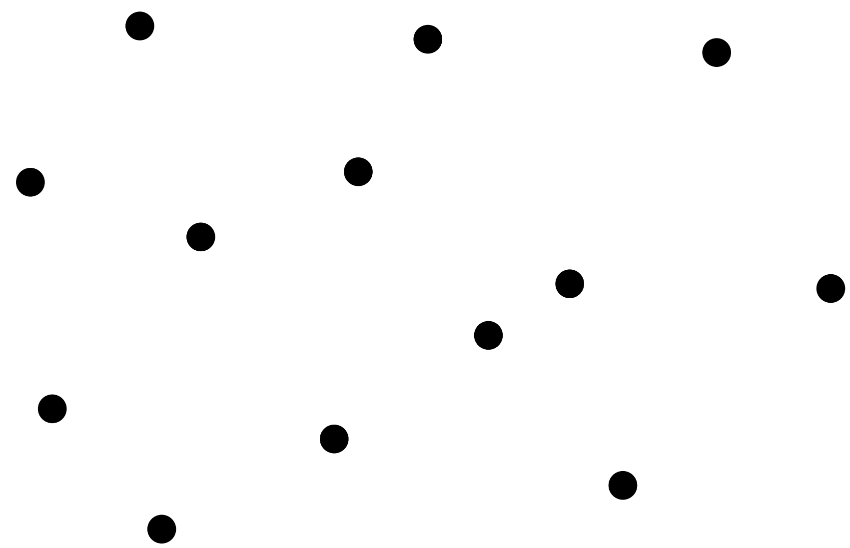

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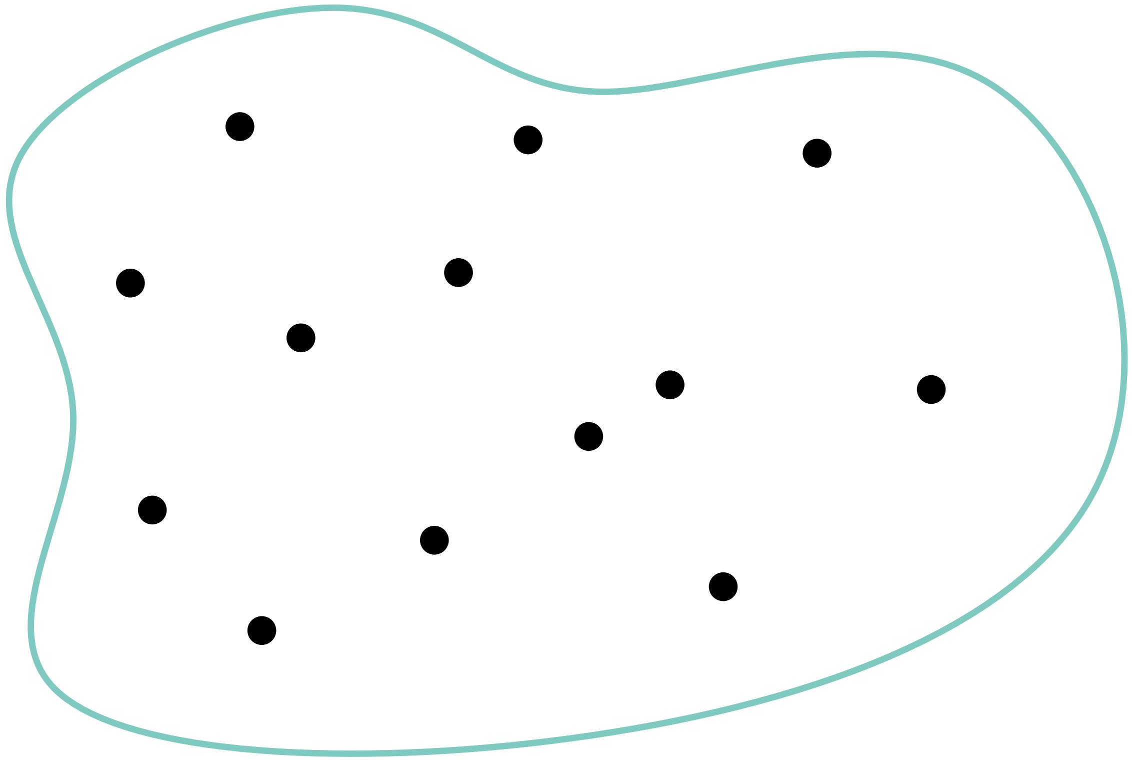

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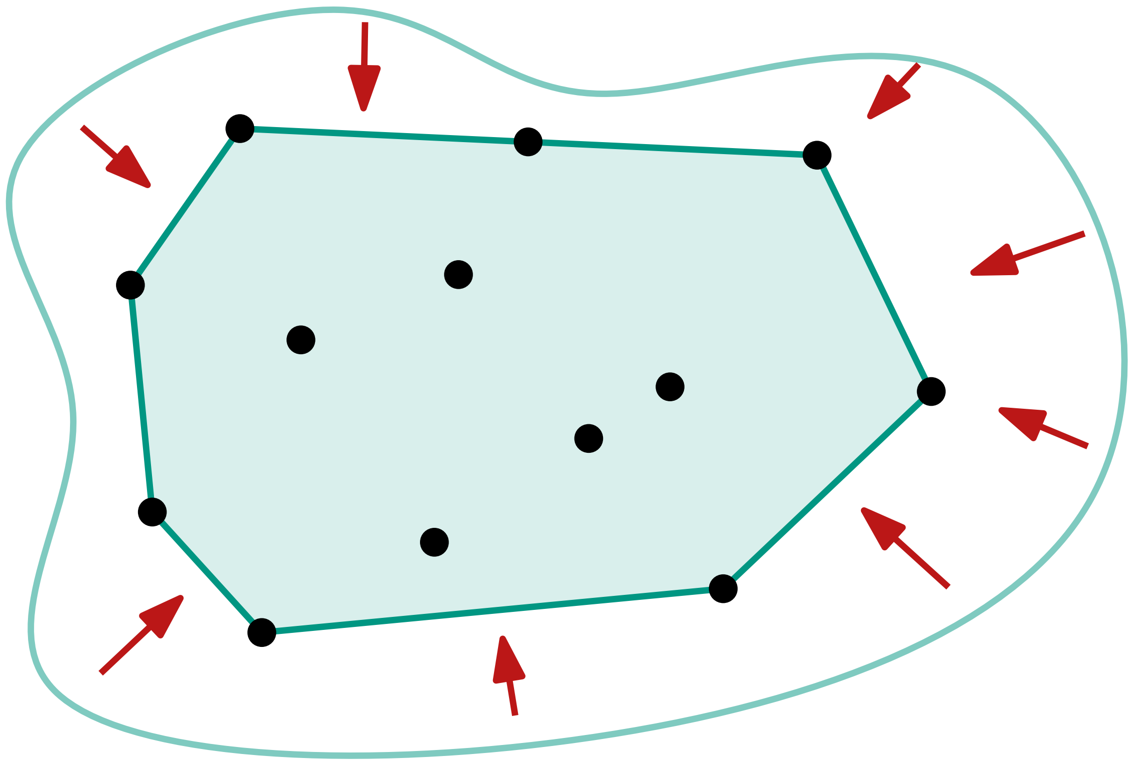
- and let it go!



Definition of Convex Hull

Def: A region $S \subseteq \mathbb{R}^2$ is called **convex**, when for two points $p, q \in S$ then line $\overline{pq} \in S$.
The **convex hull** $CH(S)$ of S is the smallest convex region containing S.

In physics:

- put a large rubber band around all points
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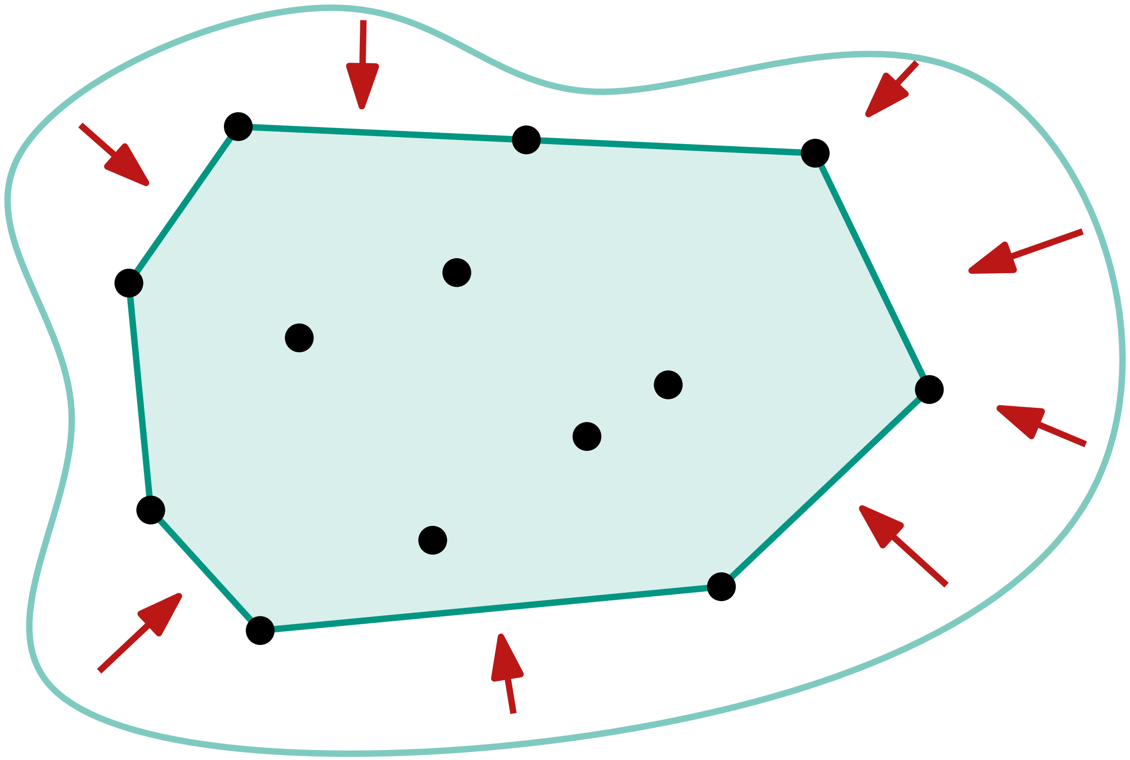

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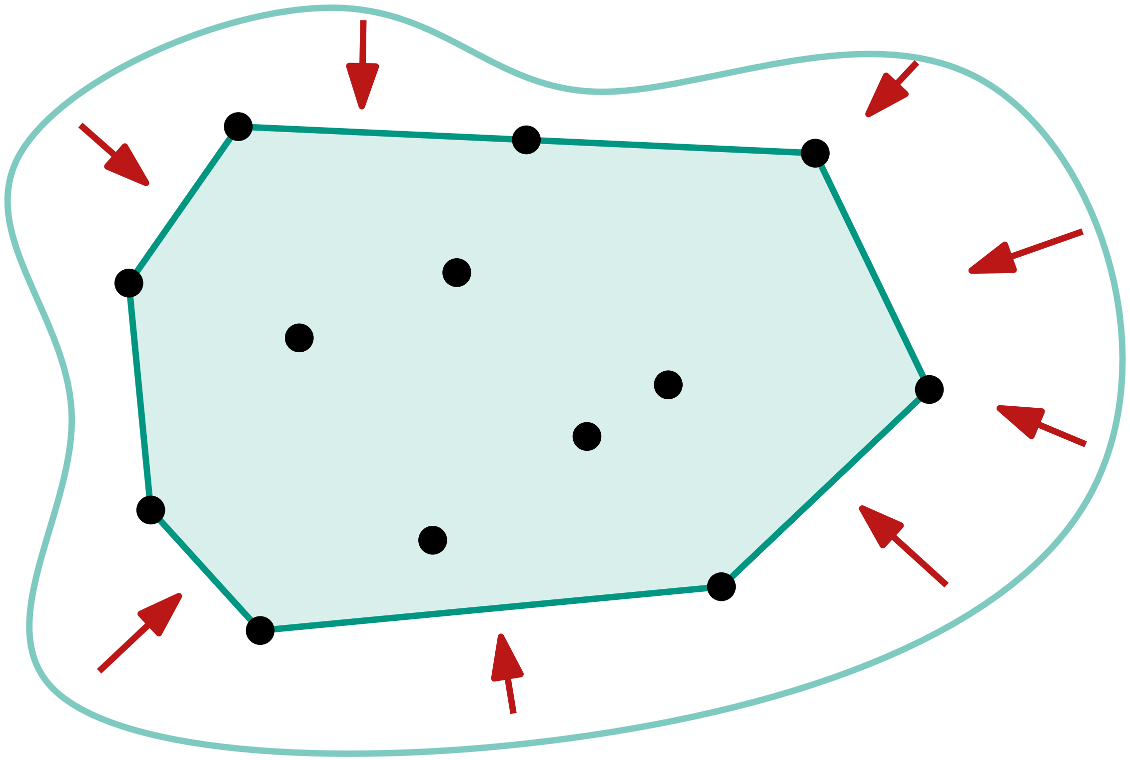


In physics:

- put a large rubber band around all points
- and let it go!
- unfortunately, does not help algorithmically

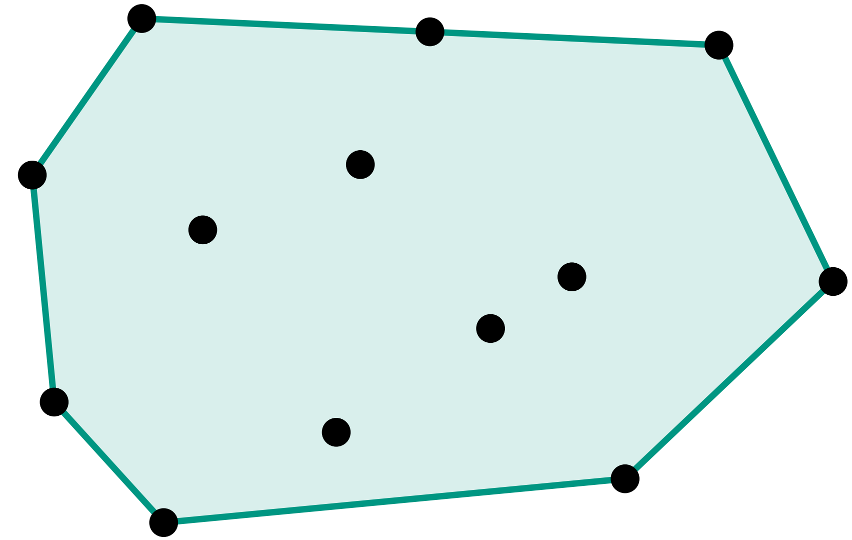
In mathematics:

- define $CH(S) = \bigcap_{C \supseteq S:\ C \text{ convex}} C$
- does not help :-(

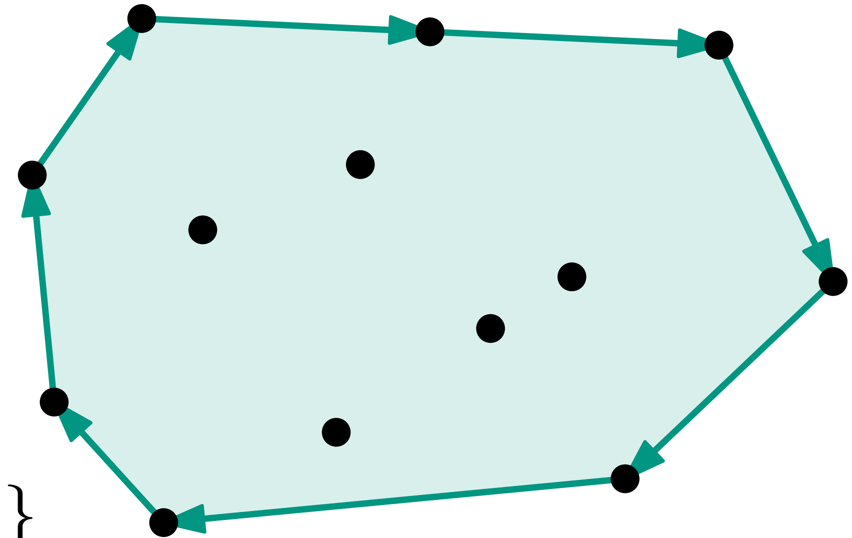
Algorithmic Approach

Lemma:

For a set of points $P \subseteq \mathbb{R}^2$, $CH(P)$ is a convex polygon that contains P and whose vertices are in P.



Algorithmic Approach



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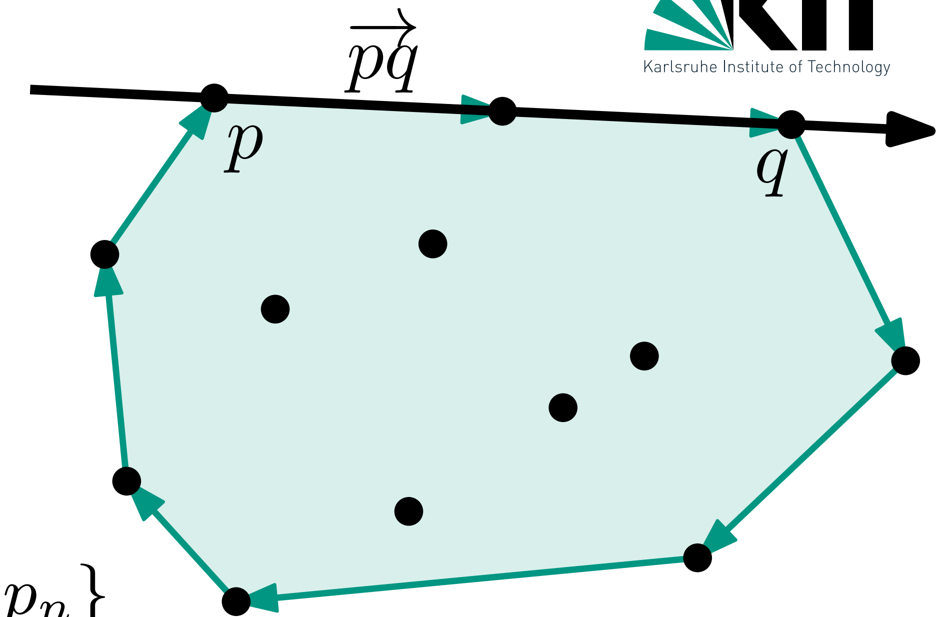
For a set of points $P \subseteq \mathbb{R}^2$, $CH(P)$ is a convex polygon that contains P and whose vertices are in P.

Input: A set of points $P = \{p_1, \ldots, p_n\}$

Output: List of vertices of $CH(P)$ in clockwise order

Algorithmic Approach



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Observation:

(p, q) is an edge of $CH(P) \Leftrightarrow$ each point $r \in P \setminus \{p, q\}$

- strictly right of the oriented line $\overrightarrow{pq}$ or
- on the line segment $\overline{pq}$

A First Algorithm

FirstConvexHull(P)

```
E ← ∅
foreach (p, q) ∈ P × P with p ≠ q do
    valid ← true
    foreach r ∈ P do
        if not (r strictly right of →pq or r ∈ ‾pq) then
            valid ← false
    if valid then
        E ← E ∪ {(p, q)}
construct sorted node list L of CH(P) from E
return L
```

A First Algorithm

KIT Karlsruhe Institute of Technology

FirstConvexHull(P)

$E \leftarrow \emptyset$
foreach $(p, q) \in P \times P$ with $p \neq q$ **do**
    valid $\leftarrow$ *true*
    foreach $r \in P$ **do**
        if not (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then**
            valid $\leftarrow$ *false*
    if *valid* **then**
        $E \leftarrow E \cup \{(p, q)\}$
construct sorted node list L of $CH(P)$ from E
return L

Check all possible edges (p, q)

A First Algorithm

FirstConvexHull(P)

$E \leftarrow \emptyset$
foreach $(p, q) \in P \times P$ with $p \neq q$ **do**
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Check all possible edges (p, q)

Test in $O(1)$ time with

$$\begin{vmatrix} x_r & y_r & 1 \\ x_p & y_p & 1 \\ x_q & y_q & 1 \end{vmatrix} < 0$$

$\rightarrow$ Exercise!

Running Time Analysis

FirstConvexHull(P)

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$\Theta(1)$

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construct sorted node list L of $CH(P)$ from E

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$\Theta(1)$ $\Theta(n)$

Running Time Analysis

FirstConvexHull(P)

$E \leftarrow \emptyset$

foreach $(p, q) \in P \times P$ with $p \neq q$ **do** $(n^2 - n)\cdot$
- *valid* $\leftarrow$ *true*
- **foreach** $r \in P$ **do** $\Theta(n)$
  - **if not** (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then** $\Theta(1)$
    - *valid* $\leftarrow$ *false*
- **if** *valid* **then**
  - $E \leftarrow E \cup \{(p, q)\}$

construct sorted node list L of $CH(P)$ from E

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Running Time Analysis

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Question: How do we implement this?

Running Time Analysis

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$\Theta(1)$ $\Theta(n)$ $\Theta(n^3)$ $O(n^2)$

Lemma: The convex hull of n points in the plane can be computed in $O(n^3)$ time.

Running Time Analysis

Karlsruhe Institute of Technology

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$\Theta(1)$ $\Theta(n)$ $\Theta(n^3)$

if *valid* **then**

$E \leftarrow E \cup \{(p, q)\}$

Can we do better?

construct sorted node list L of $CH(P)$ from E $O(n^2)$

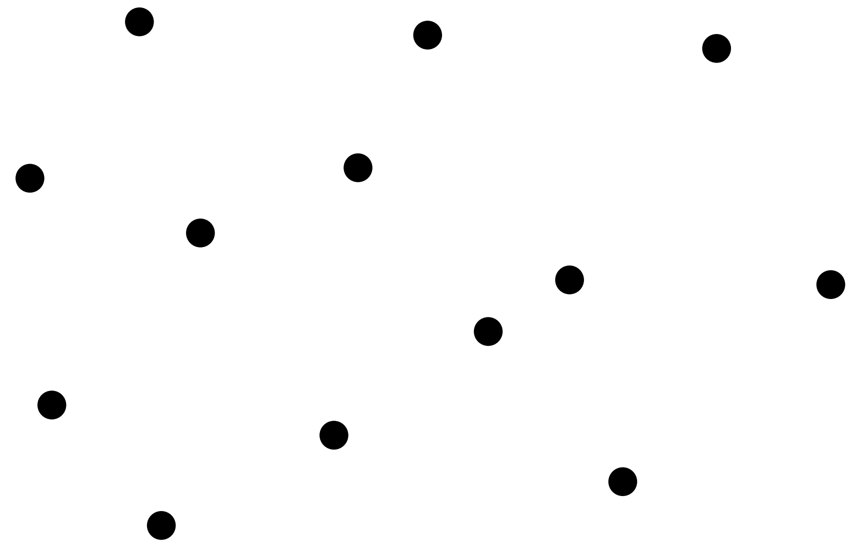
return L

Lemma: The convex hull of n points in the plane can be computed in $O(n^3)$ time.

Incremental Approach

Idea: For $i = 1, \ldots, n$ compute $CH(P_i)$ where $P_i = \{p_1, \ldots, p_i\}$

Question: Which ordering of the points is useful?



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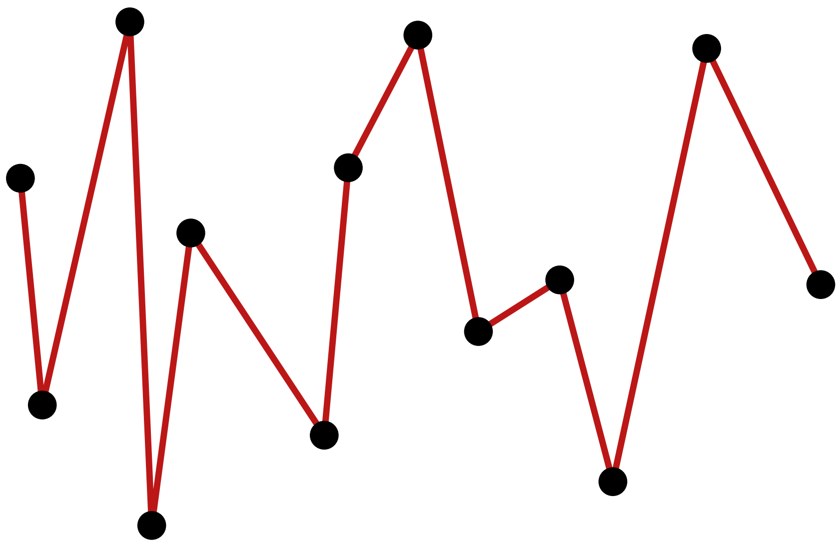
Question: Which ordering of the points is useful?

Answer: From left to right!



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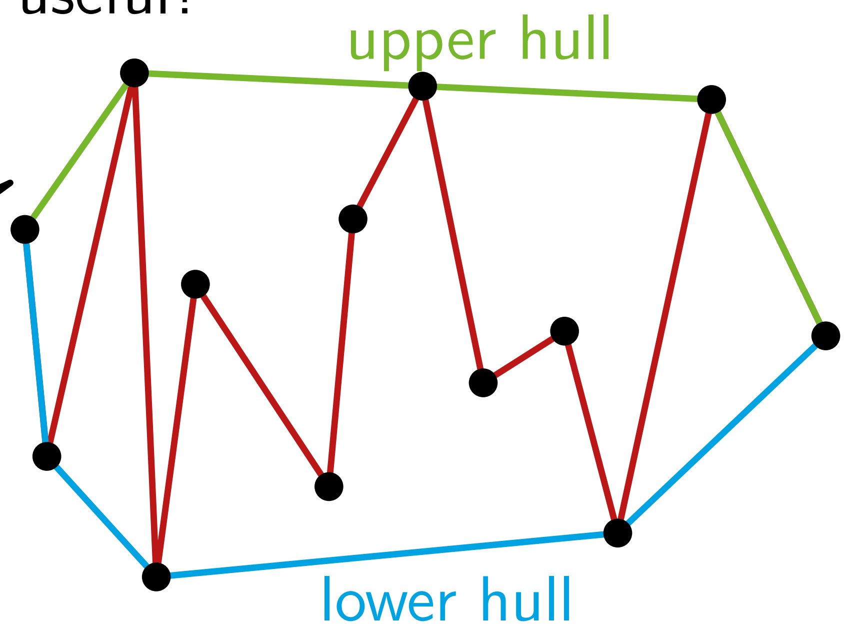
Consider the upper and lower hull separately



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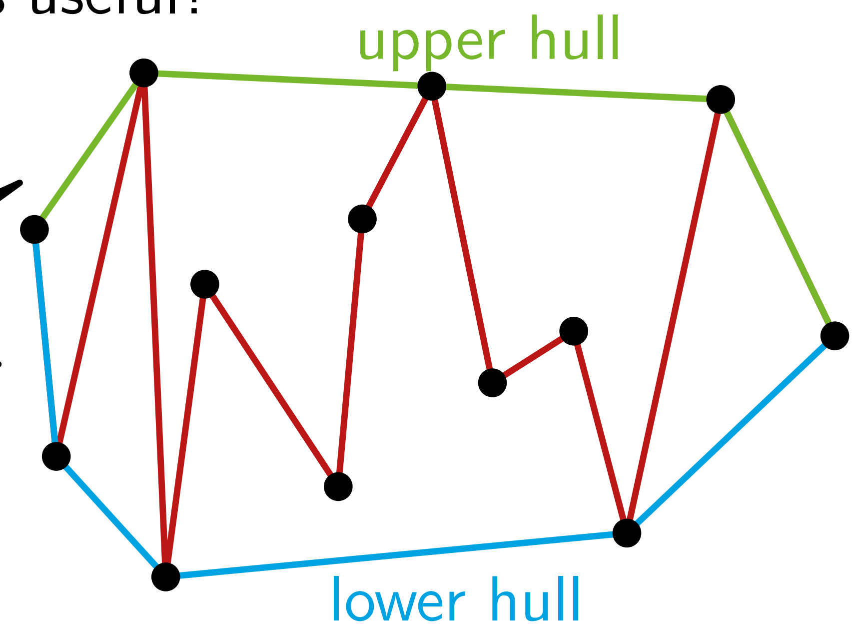

UpperConvexHull(P)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort P from left to right

$L \leftarrow \langle p_1, p_2 \rangle$

for $i \leftarrow 3$ **to** n **do**

L.append(p_i)

?

return L

Incremental Approach

Karlsruhe Institute of Technology

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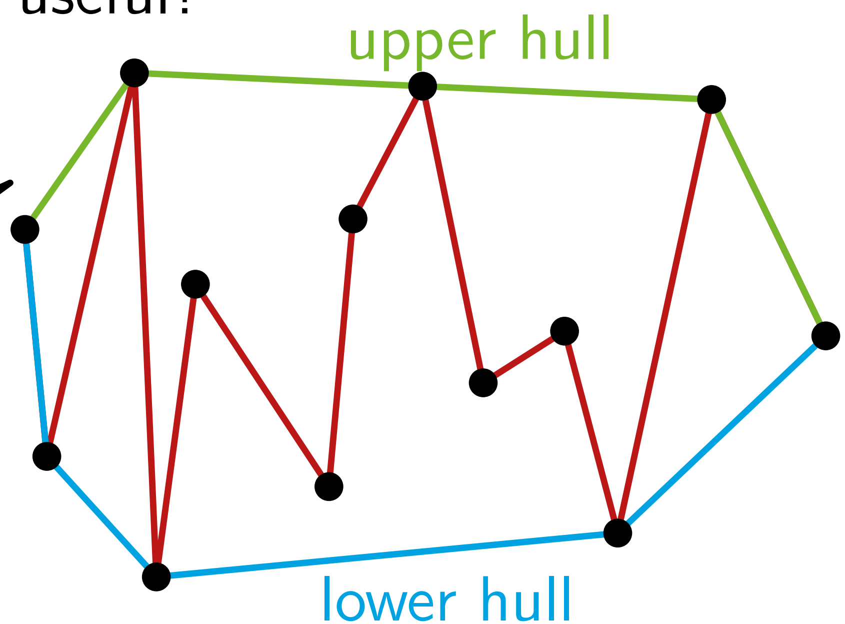

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$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort P from left to right
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for $i \leftarrow 3$ **to** n **do**
  L.append(p_i)
  while $|L| > 2$ **and** the last 3 points in L do not form right turn **do**
    remove the second-to-last point in L
return L

Incremental Approach

Idea: For $i = 1, \ldots, n$ compute $CH(P_i)$ where $P_i = \{p_1, \ldots, p_i\}$

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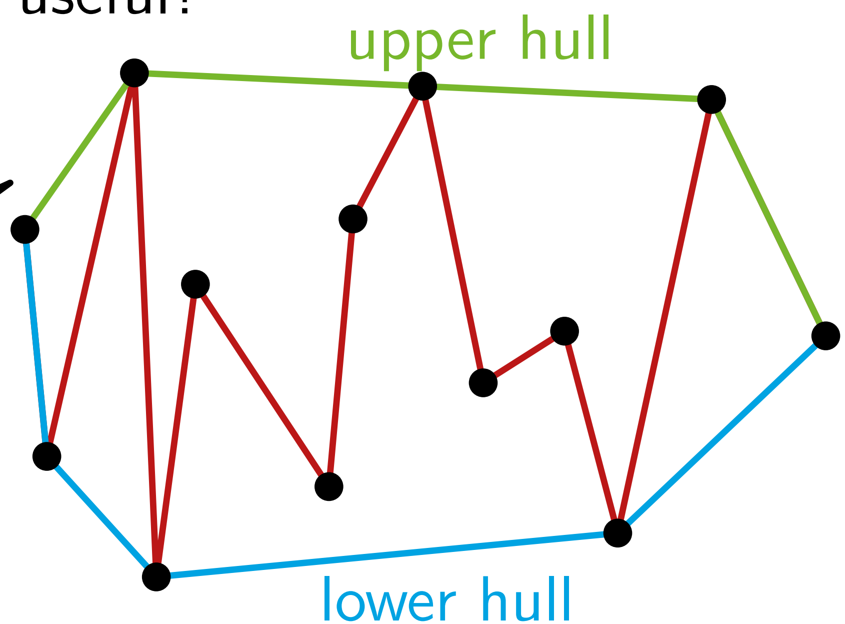

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return L

lower hull is handled similarly!

Running Time Analysis

UpperConvexHull(P)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort P from right to left

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Running Time Analysis

UpperConvexHull(P)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort P from right to left $\qquad O(n \log n)$

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Running Time Analysis

UpperConvexHull(P)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort P from right to left — $O(n \log n)$

$L \leftarrow \langle p_1, p_2 \rangle$

for $i \leftarrow 3$ **to** n **do** — $(n-2)\cdot$

  L.append(p_i)

  while $|L| > 2$ **and** last 3 points in L do not form right turn **do**

    remove the second-to-last point from L — ?

return L

Running Time Analysis

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return L

Amortized Analysis

- Each point is inserted into L exactly once
- A point in L is removed at most once from L
- $\Rightarrow$ Running time of the **for** loop including the **while** loop is $O(n)$

Running Time Analysis

UpperConvexHull(P)

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$\quad\quad$ remove the second-to-last point from L $\quad$ ~~?~~

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Running Time Analysis

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Amortized Analysis

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Theorem 1: The convex hull of n points in the plane can be computed in $O(n \log n)$ time. $\rightarrow$ *Graham's Scan*.

Alternative Approach: Gift Wrapping

Karlsruhe Institute of Technology

Idea: Begin with a point p_1 of $CH(P)$, then find the next edge of $CH(P)$ in clockwise order.

Alternative Approach: Gift Wrapping

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GiftWrapping(P)

$p_1 = (x_1, y_1) \leftarrow$ rightmost point in P; $p_0 \leftarrow (x_1, \infty)$; $j \leftarrow 1$
while true **do**
  $p_{j+1} \leftarrow \arg\max\{\angle p_{j-1}, p_j, q \mid q \in P \setminus \{p_{j-1}, p_j\}\}$
  if $p_{j+1} = p_1$ **then** break **else** $j \leftarrow j + 1$
return $(p_1, \ldots, p_{j+1})$

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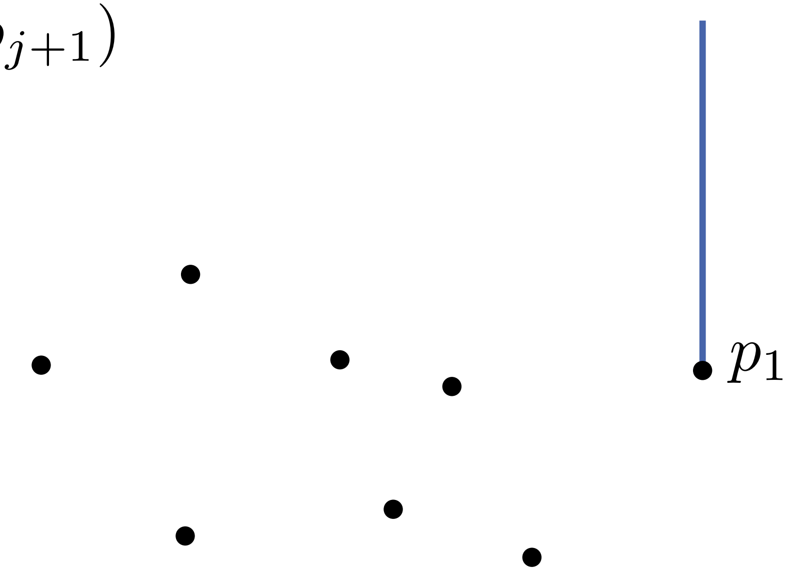

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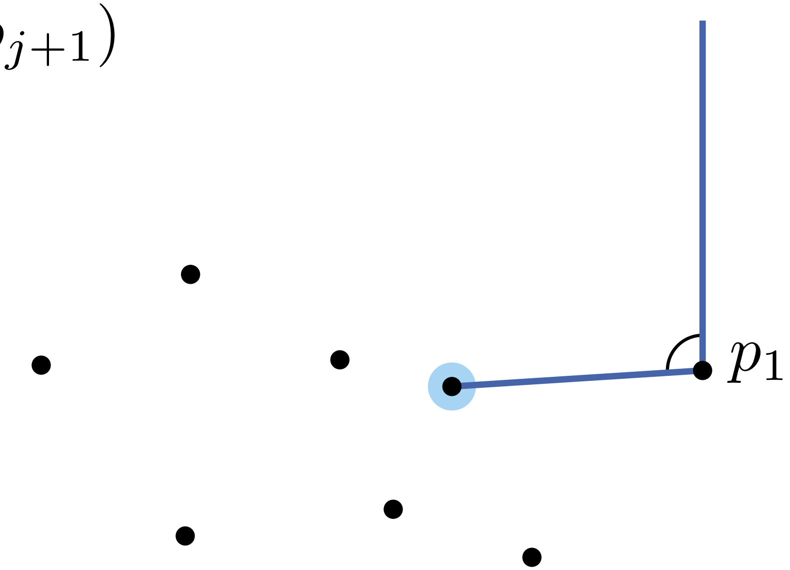

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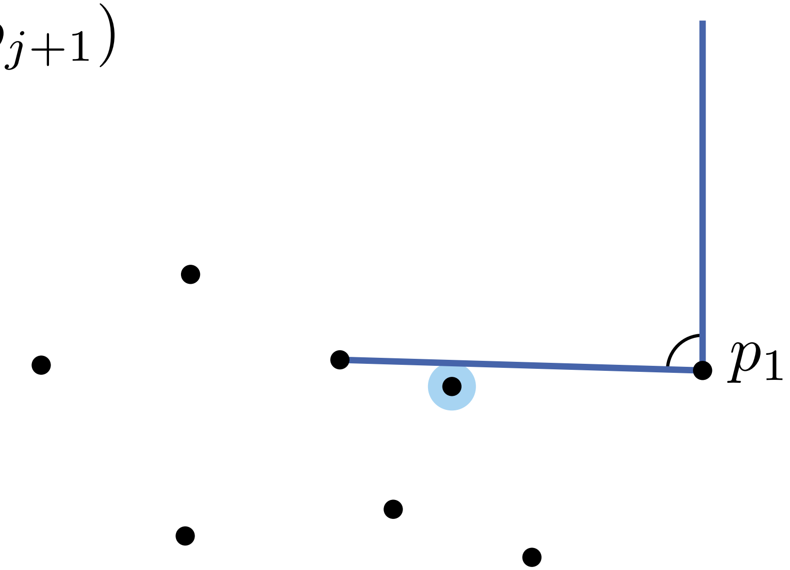

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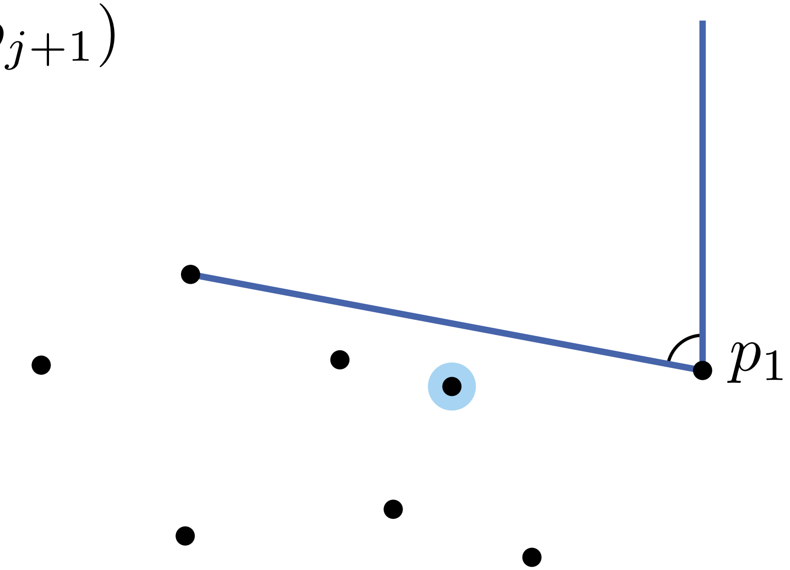

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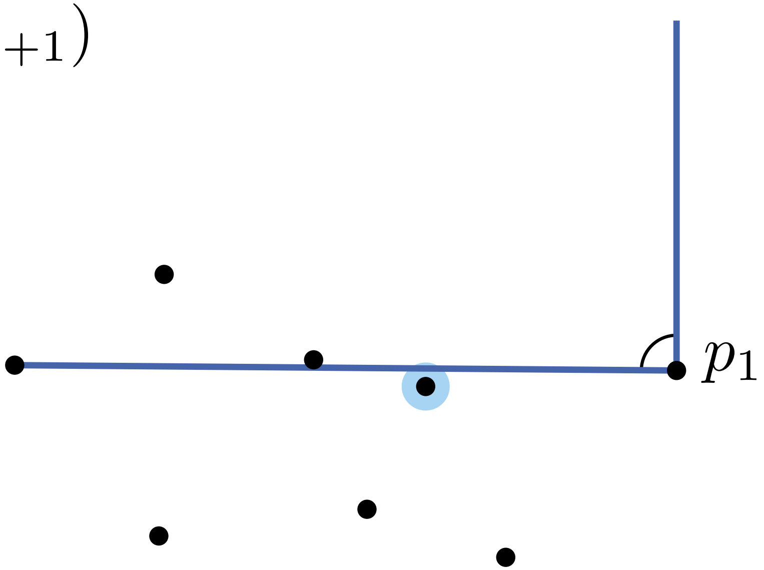

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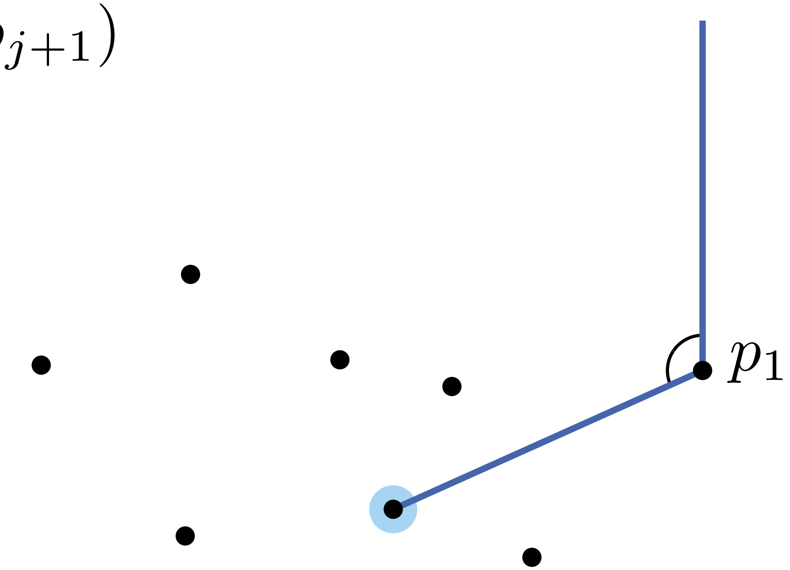

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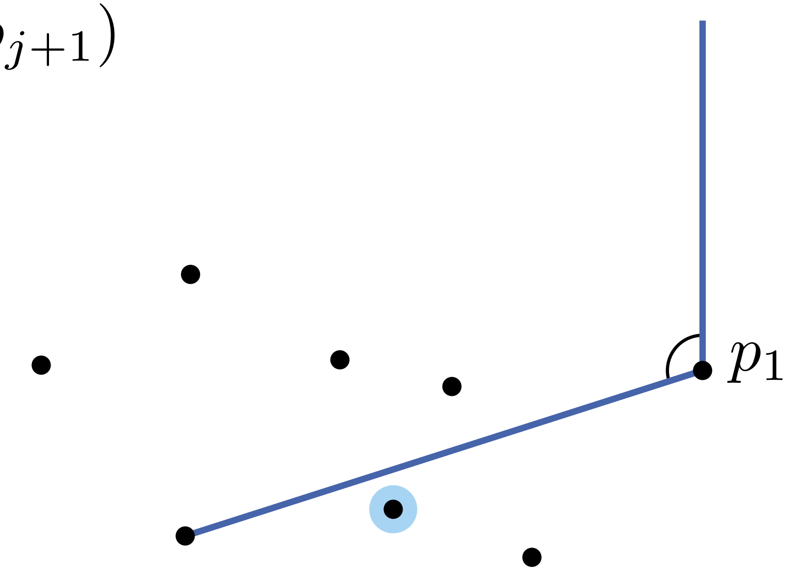

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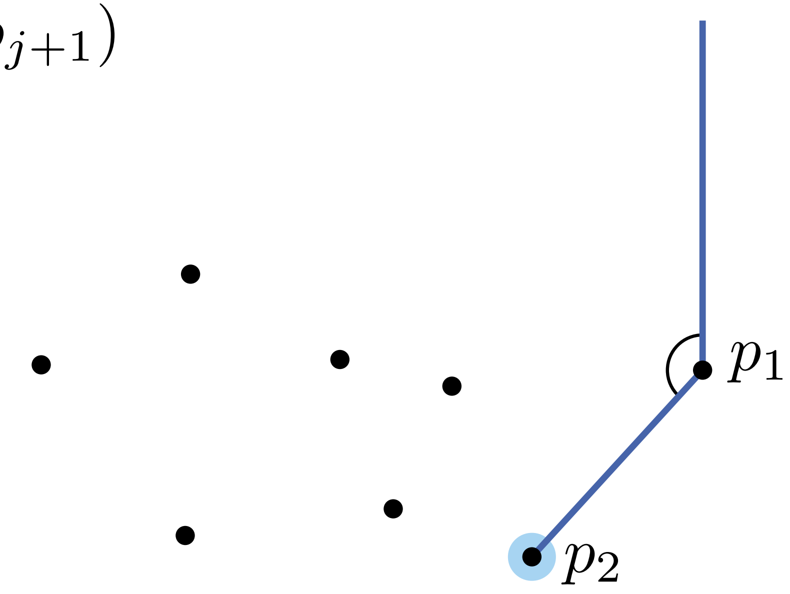

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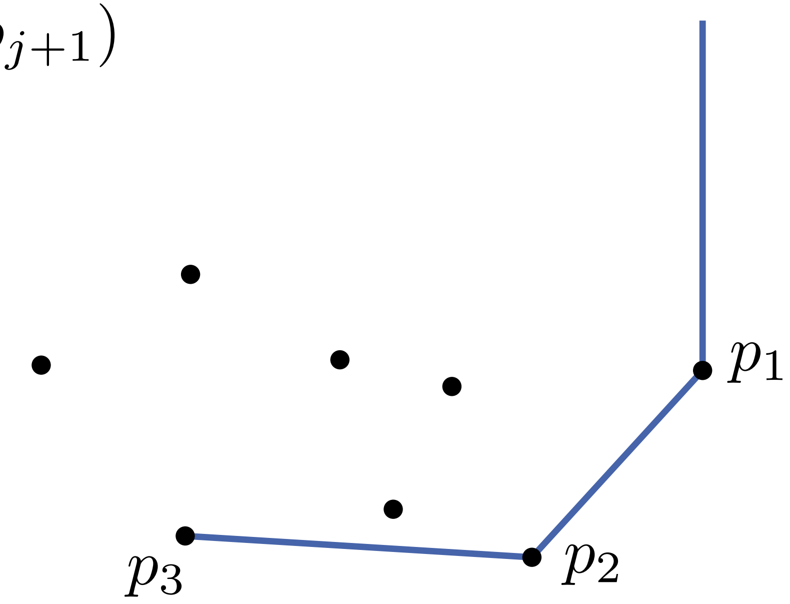

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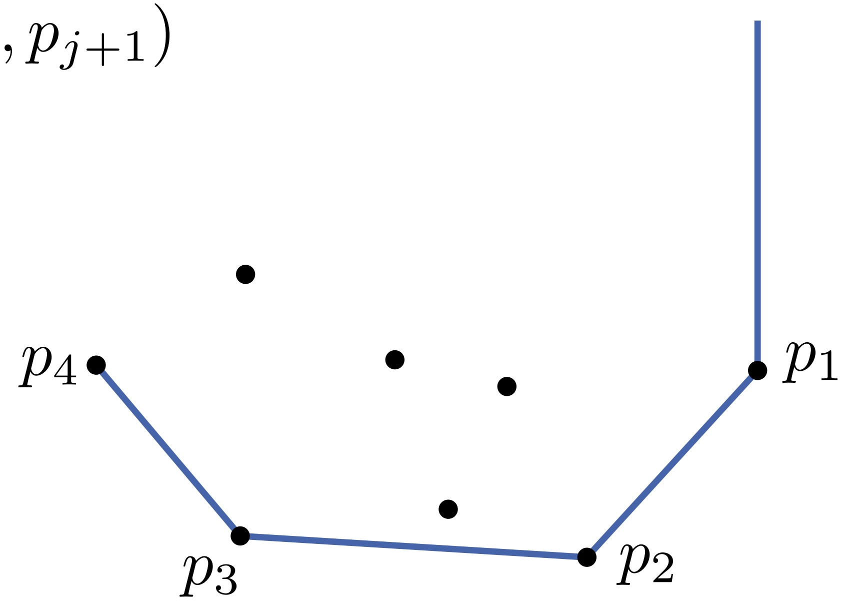

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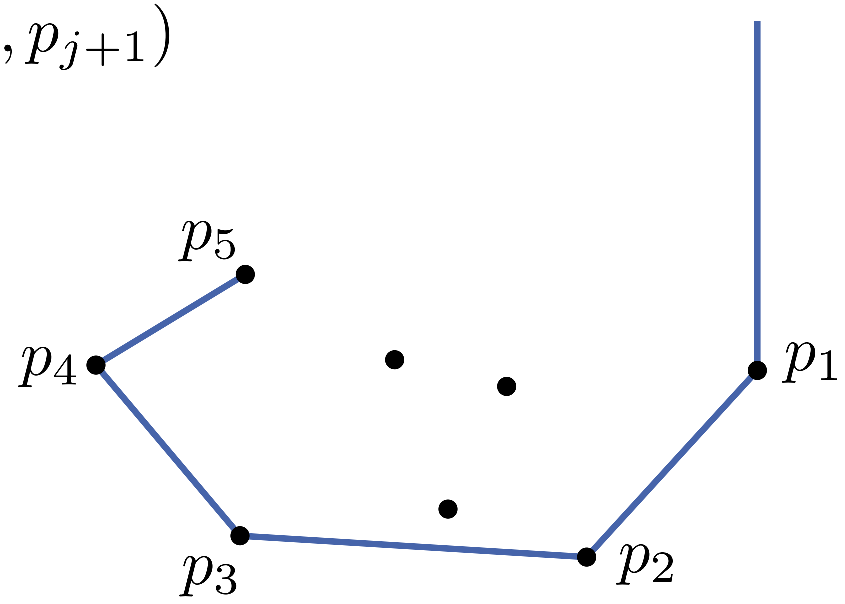

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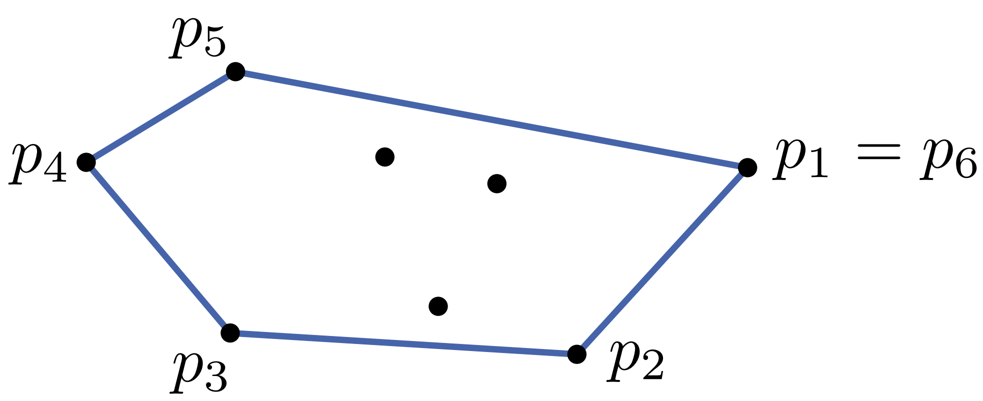

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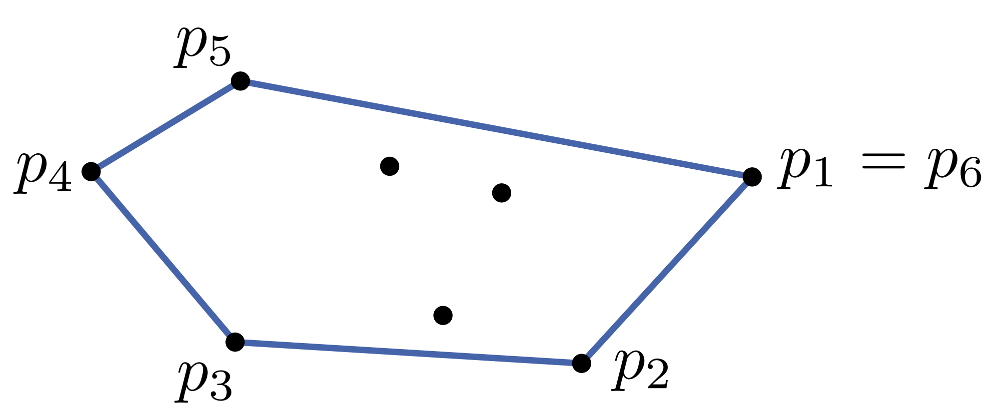

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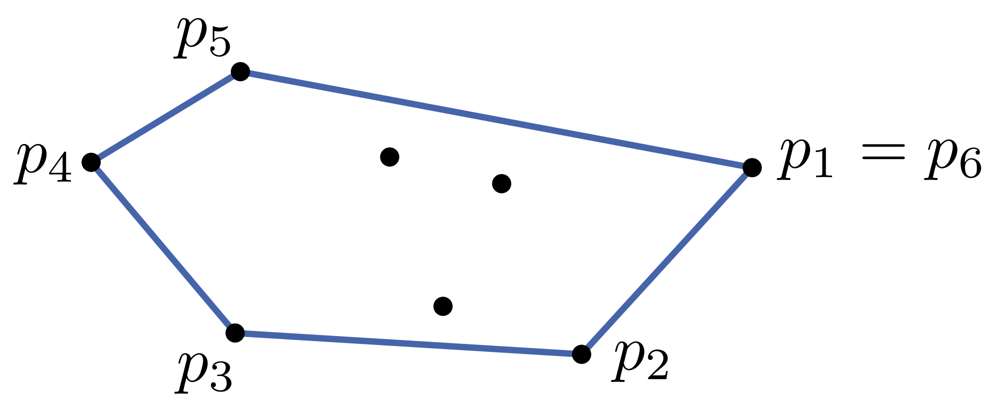

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while true **do** $O(h)$

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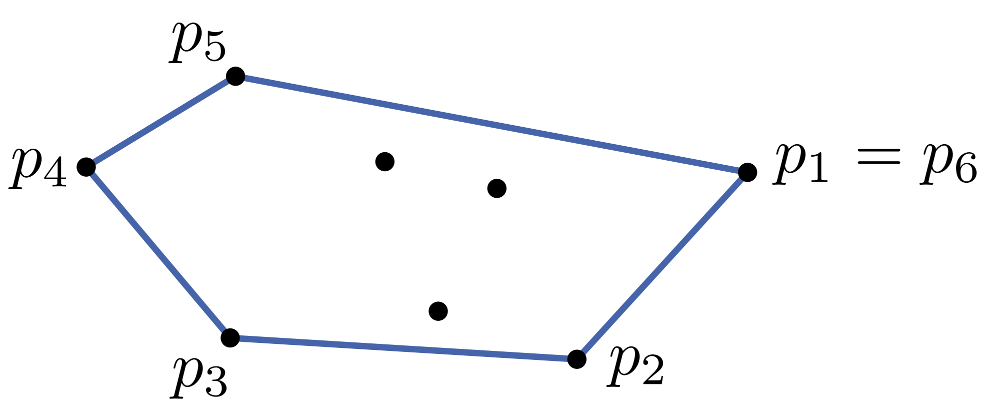

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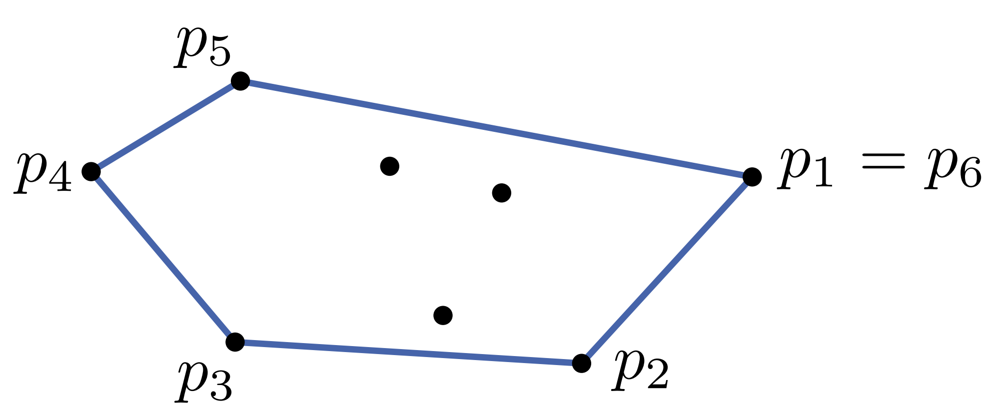
$O(n \cdot h)$



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while true **do**	$O(h)$	$O(n \cdot h)$
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Theorem 2: The convex hull $CH(P)$ of n points P in $\mathbb{R}^2$ can be computed in $O(n \cdot h)$ time using *Gift Wrapping* (also called *Jarvis' March*), where $h = |CH(P)|$.

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Total: $O(n \cdot h)$

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$\rightarrow$ more on that in the exercises!

Comparison

Which algorithm is better?

- Graham's Scan: $O(n \log n)$ time
- Jarvis' March: $O(n \cdot h)$ time

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Idea: Combine the two approaches into an optimal algorithm!

Chan's Algorithm

Suppose we know h:

ChanHull(P,h)

Divide P into sets P_i with $\leq h$ nodes
for i from 1 to $\lceil n/h \rceil$ **do**
  Compute with GrahamScan $CH(P_i)$

$p_1 = (x_1, y_1) \leftarrow$ rightmost point in P
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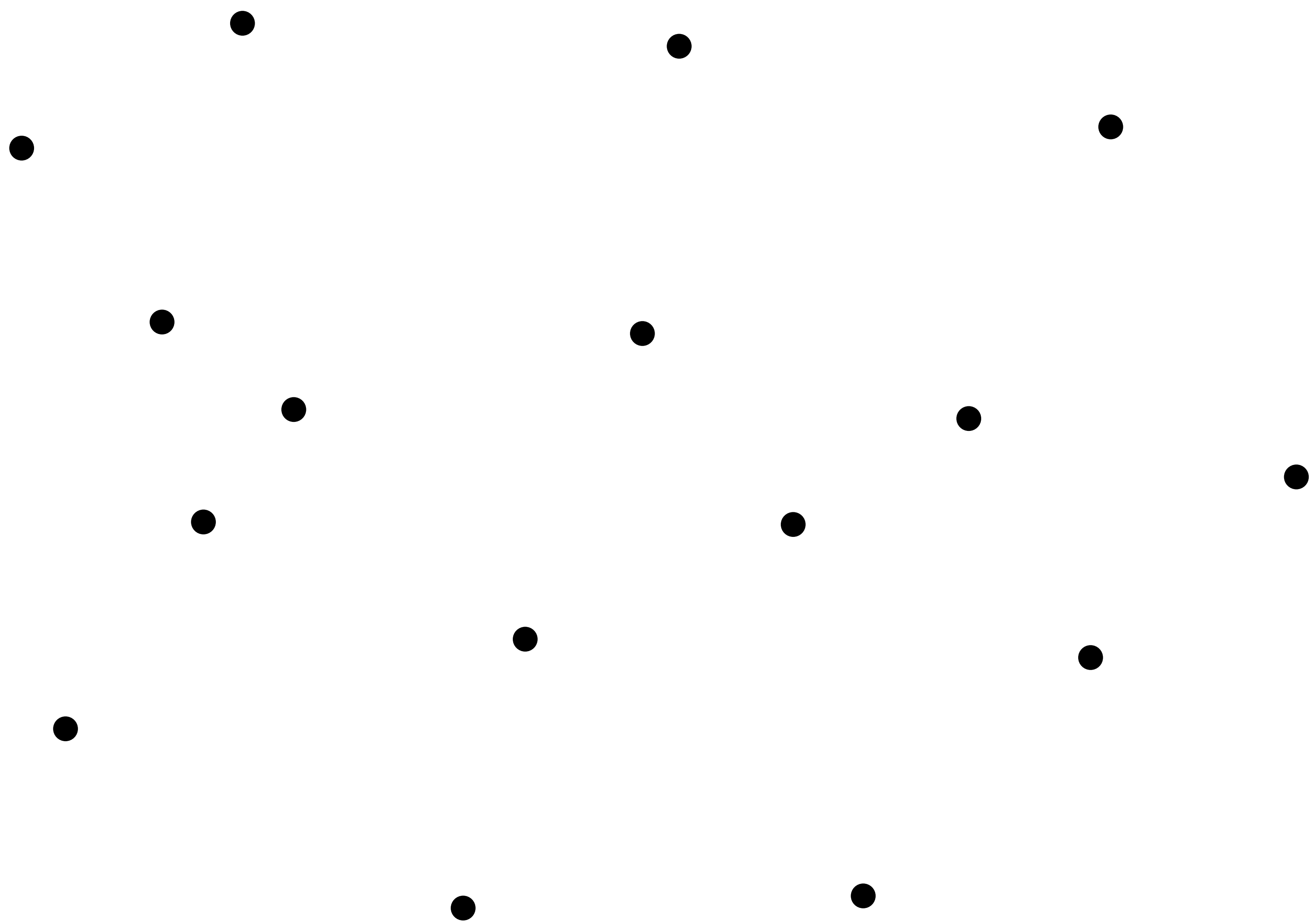
Gift Wrapping

Example

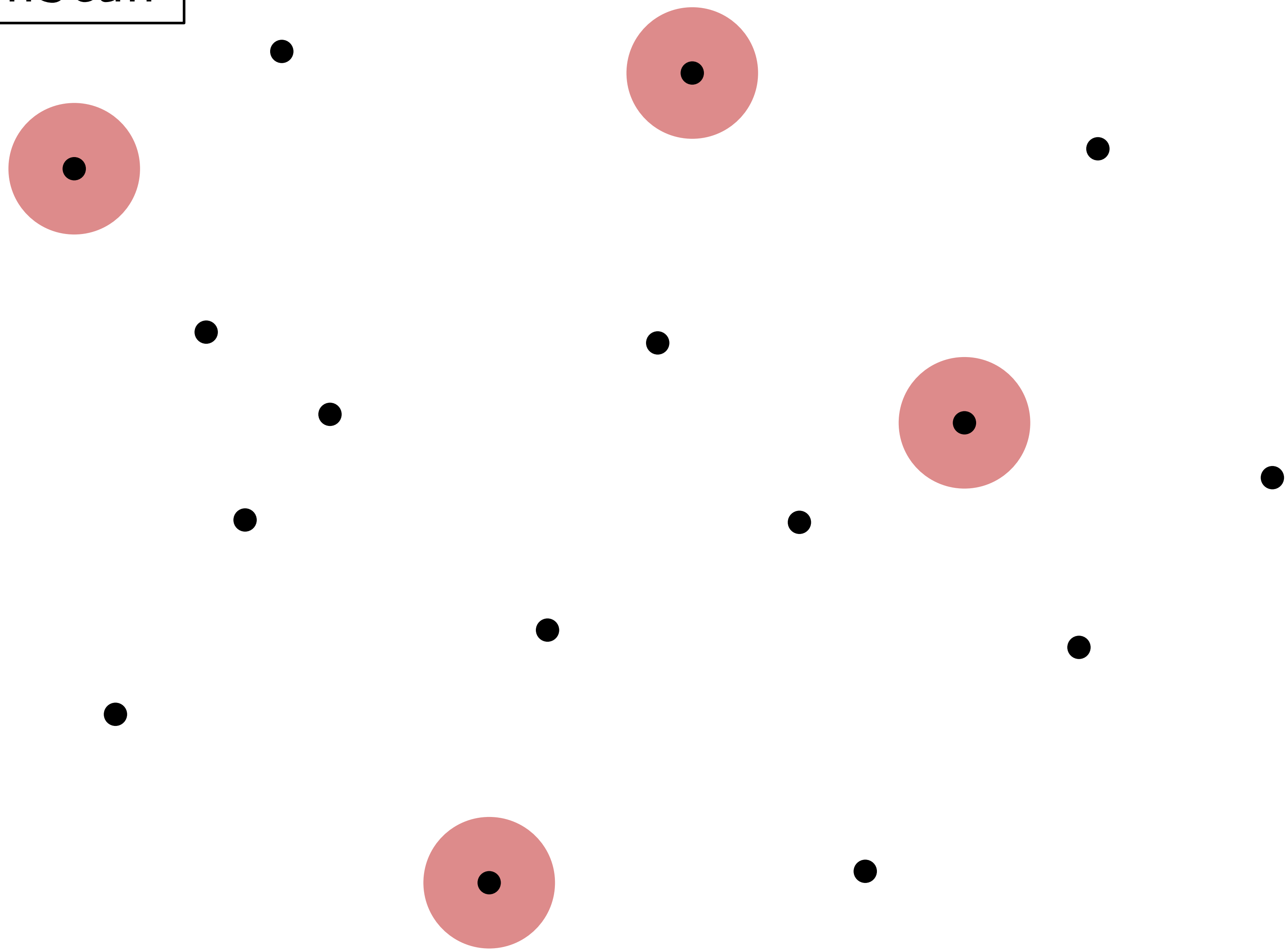
GrahamScan



n = 16

Example

GrahamScan



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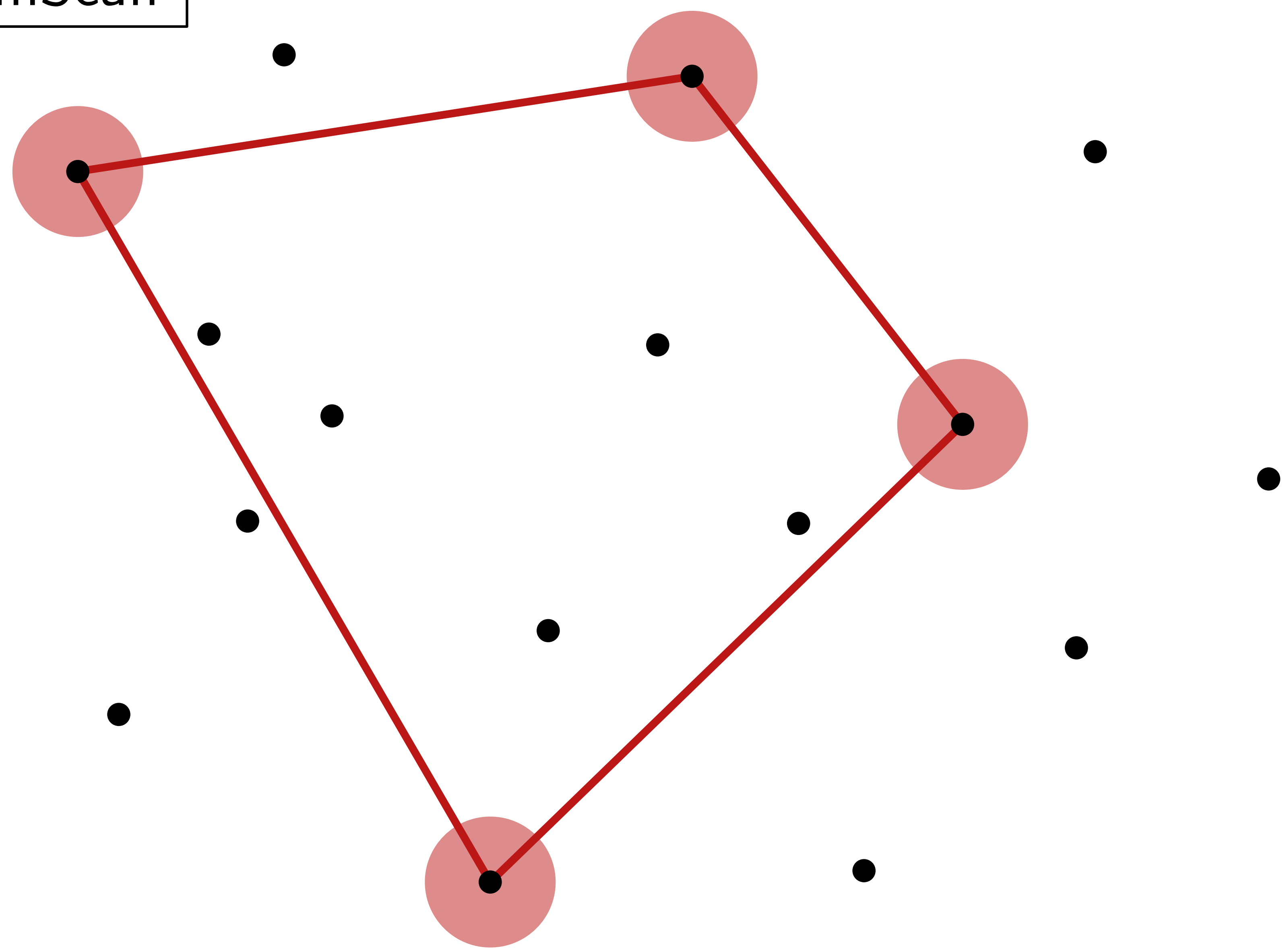

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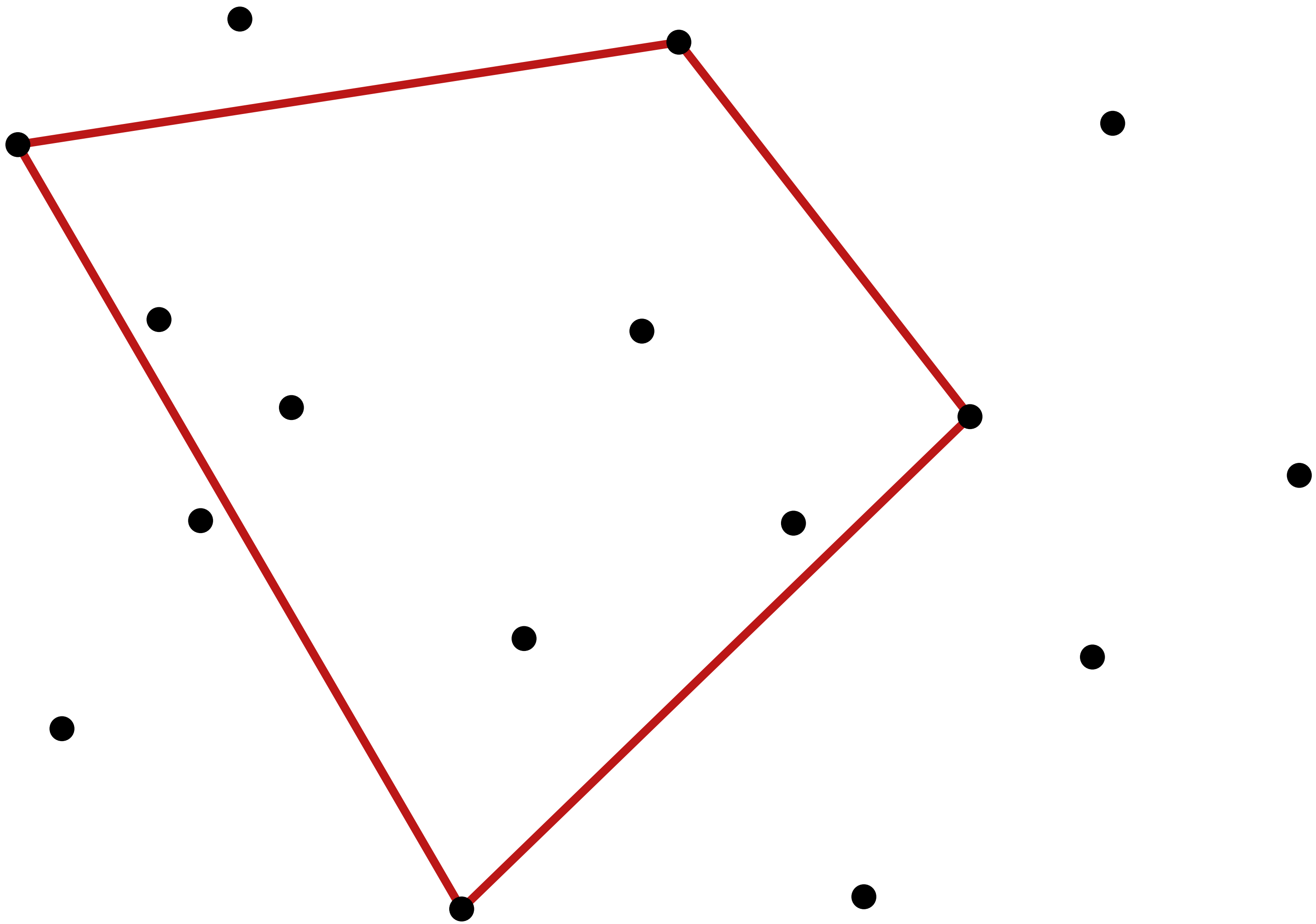

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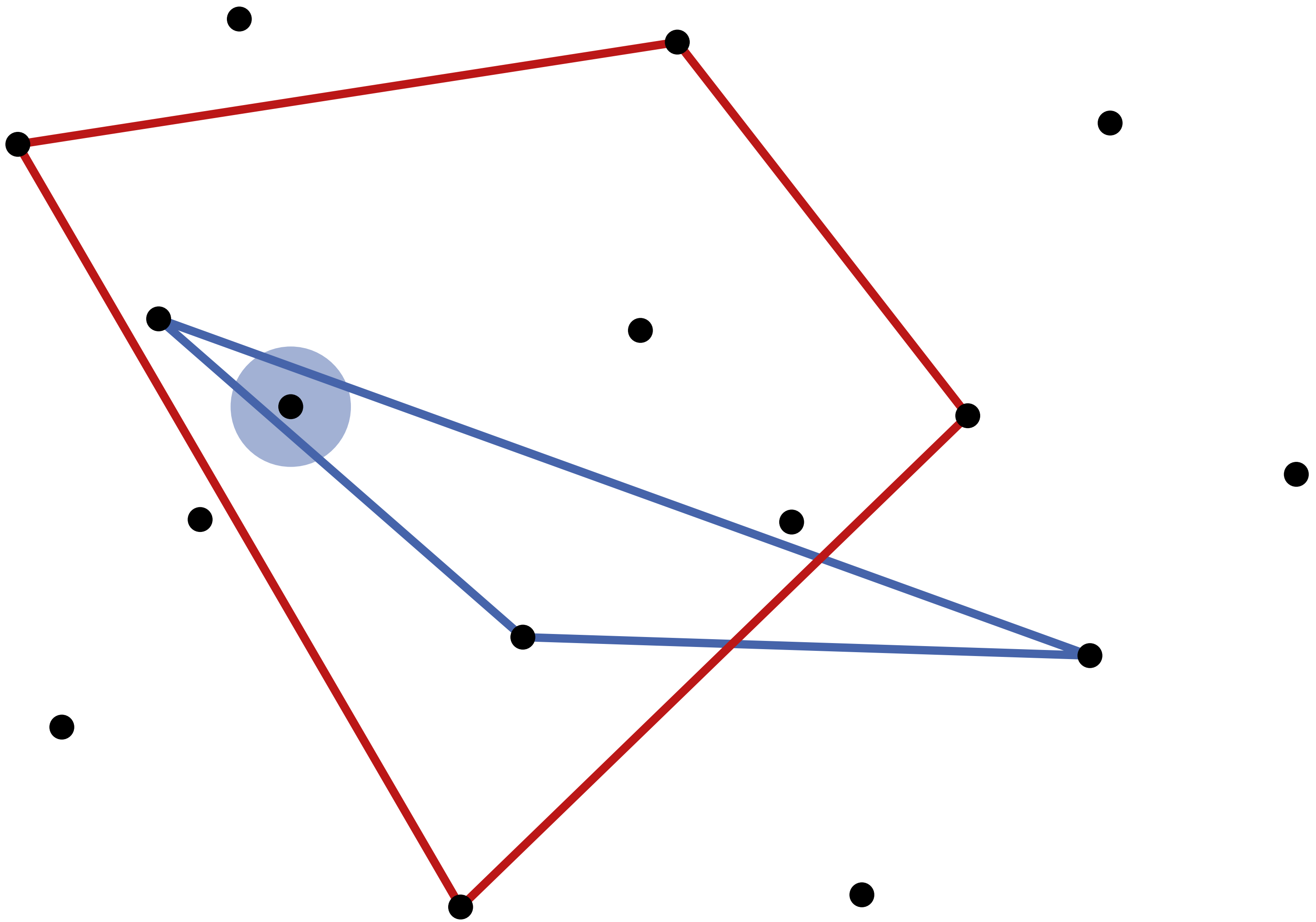

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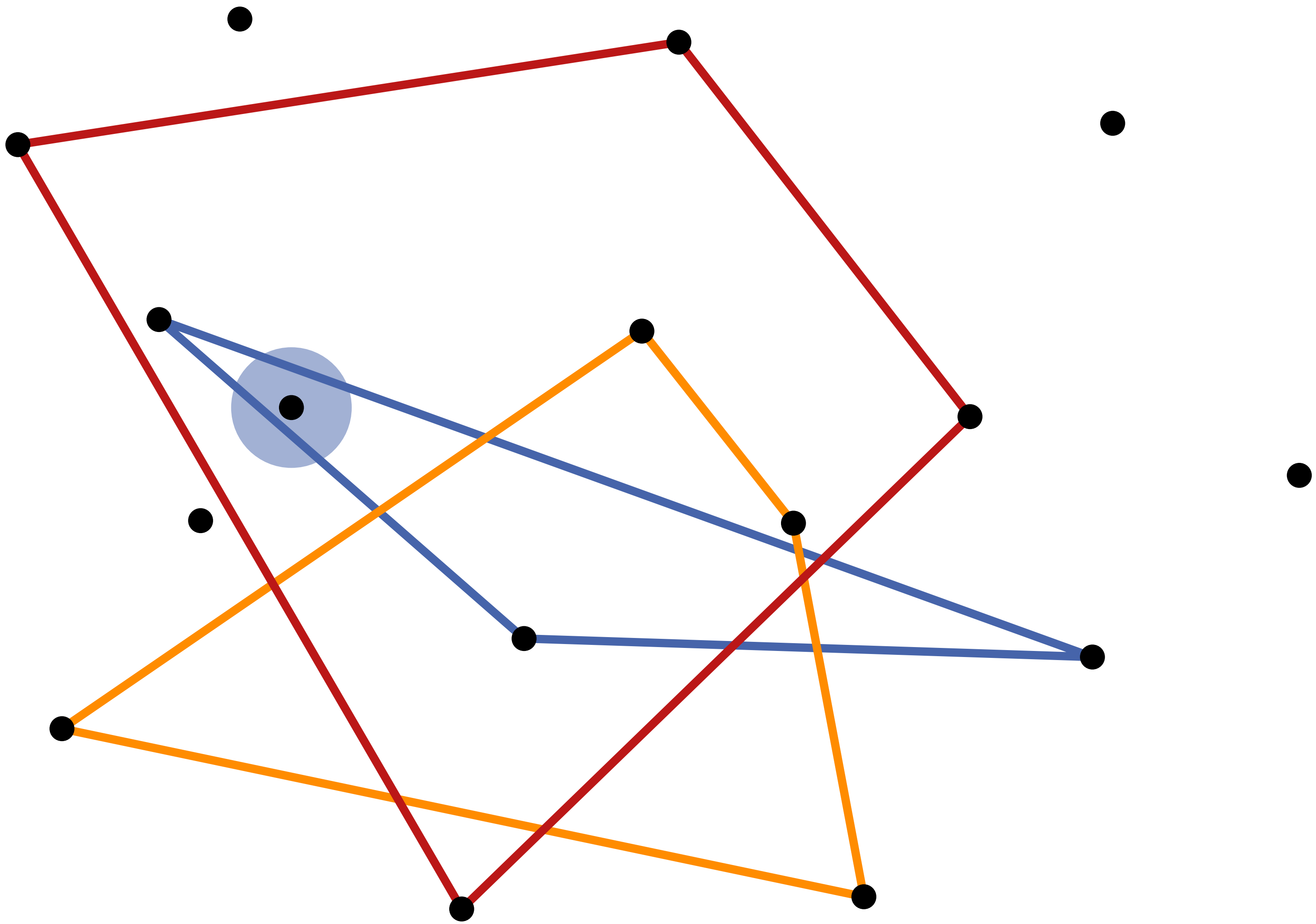

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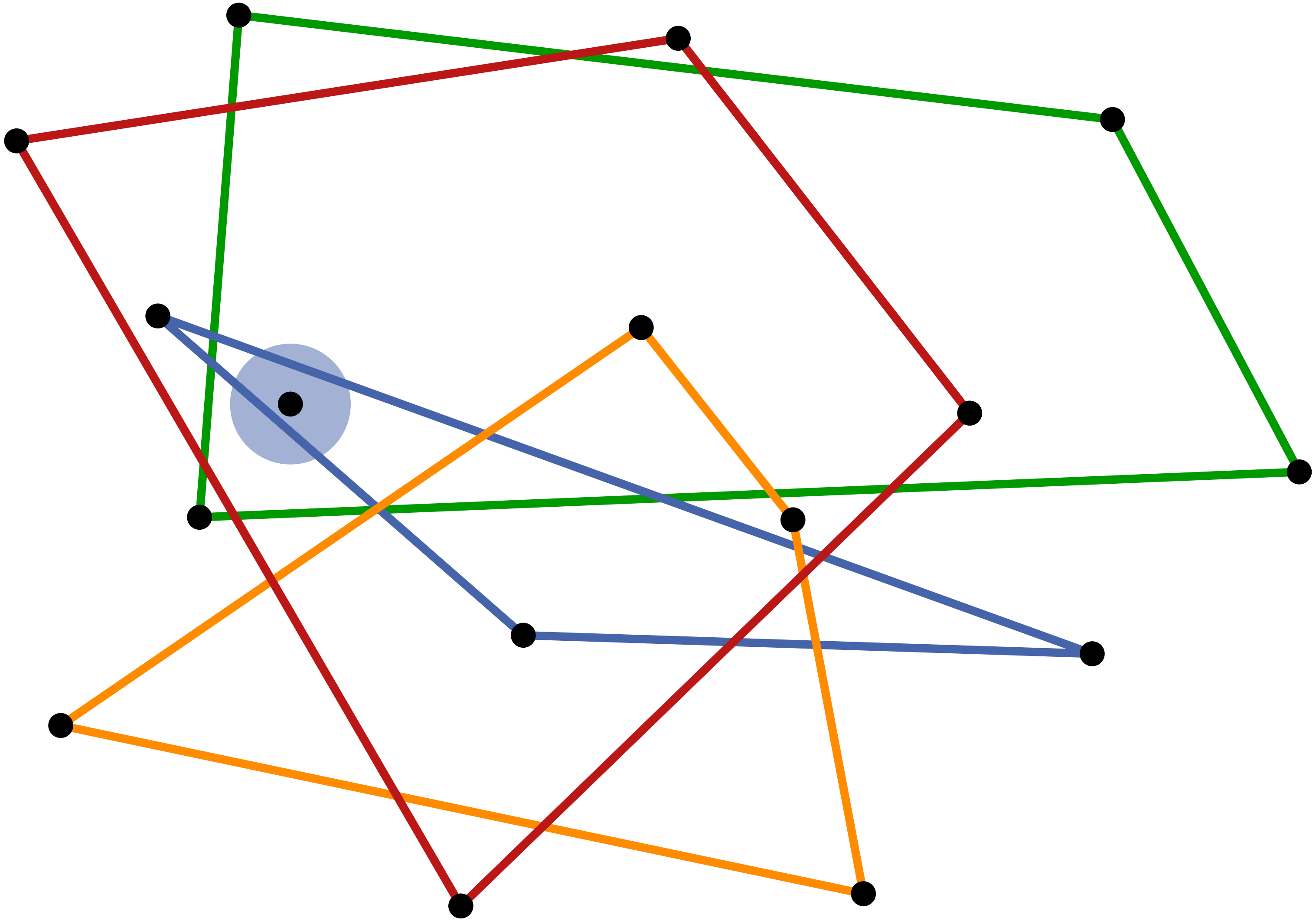

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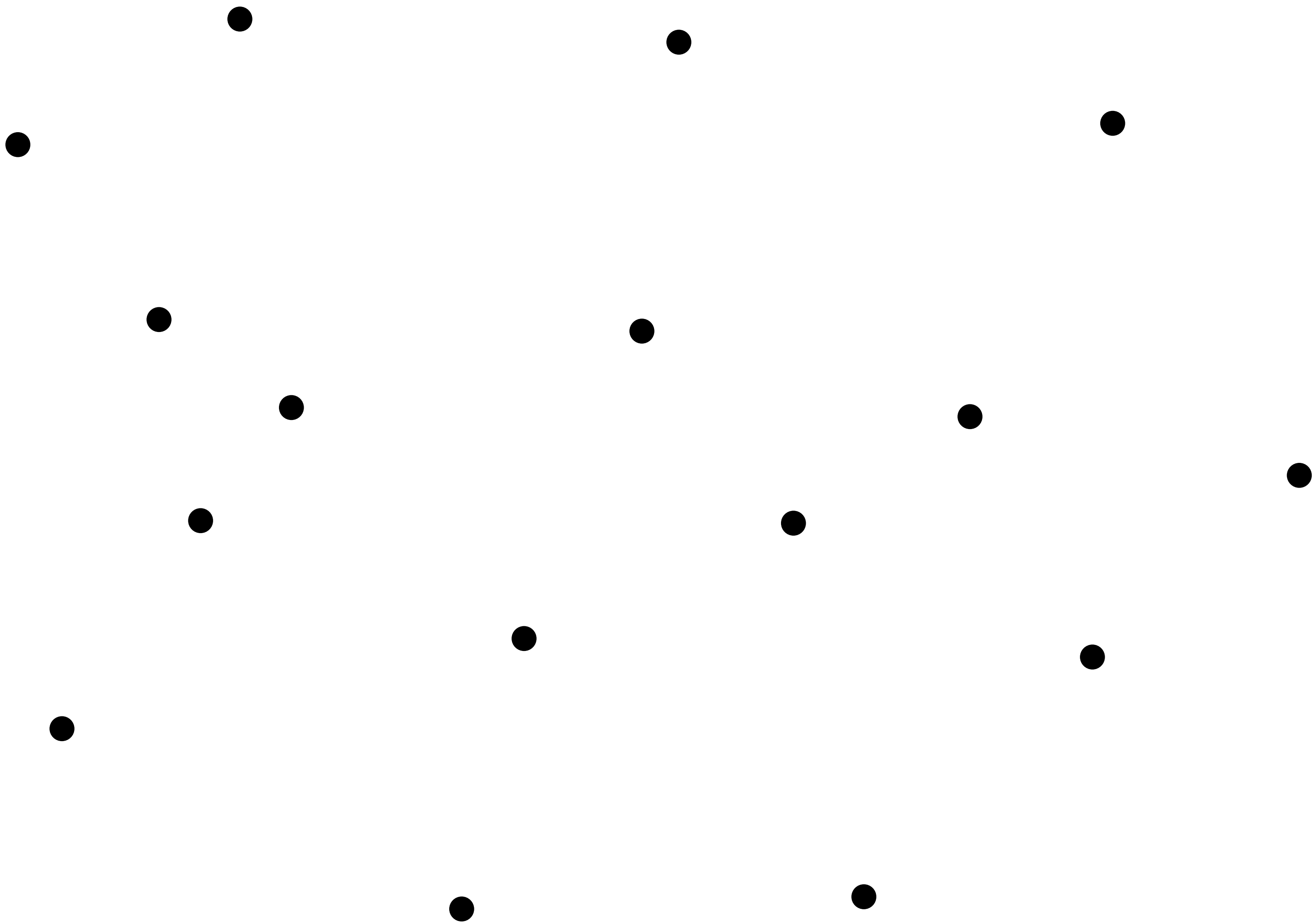

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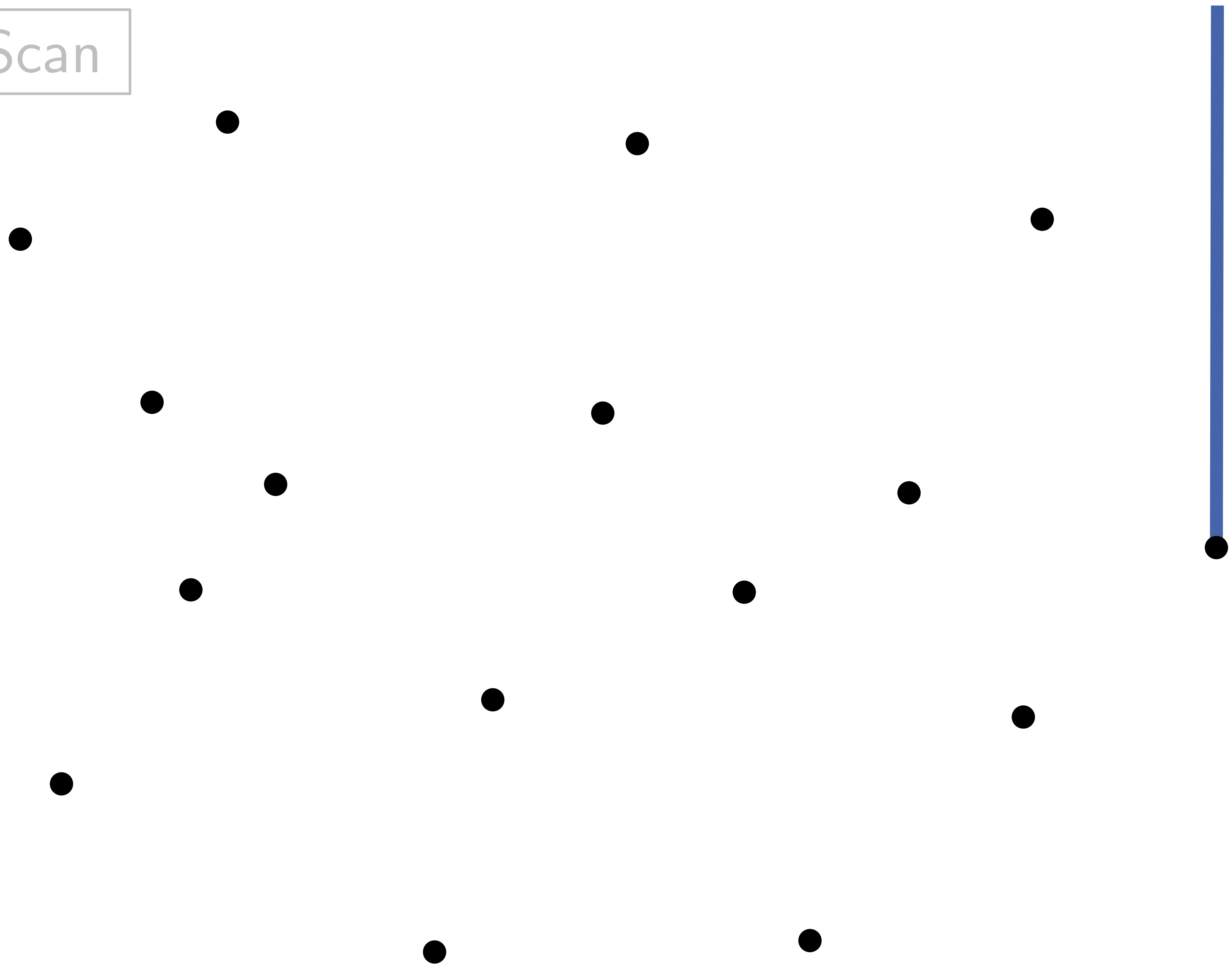

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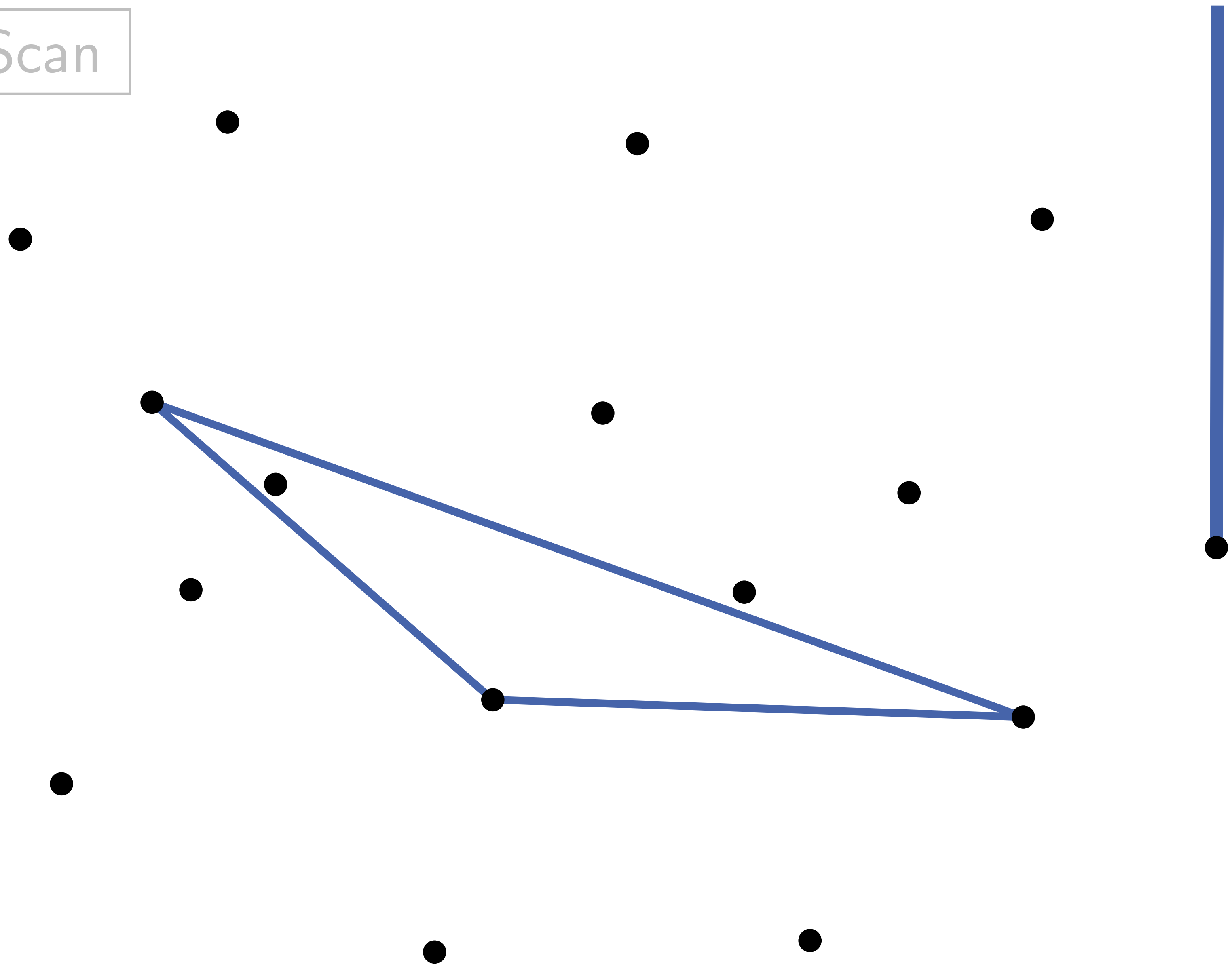

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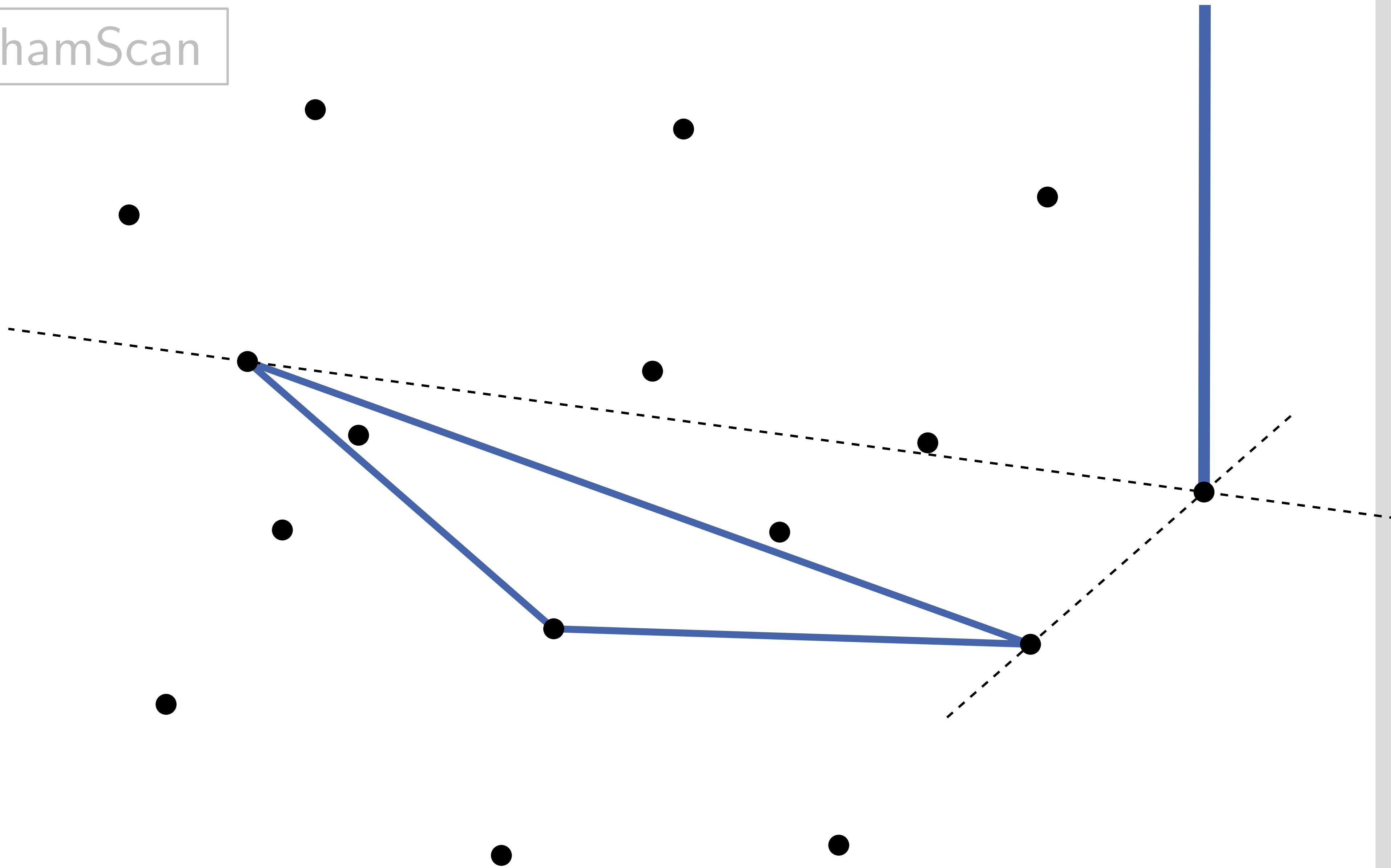

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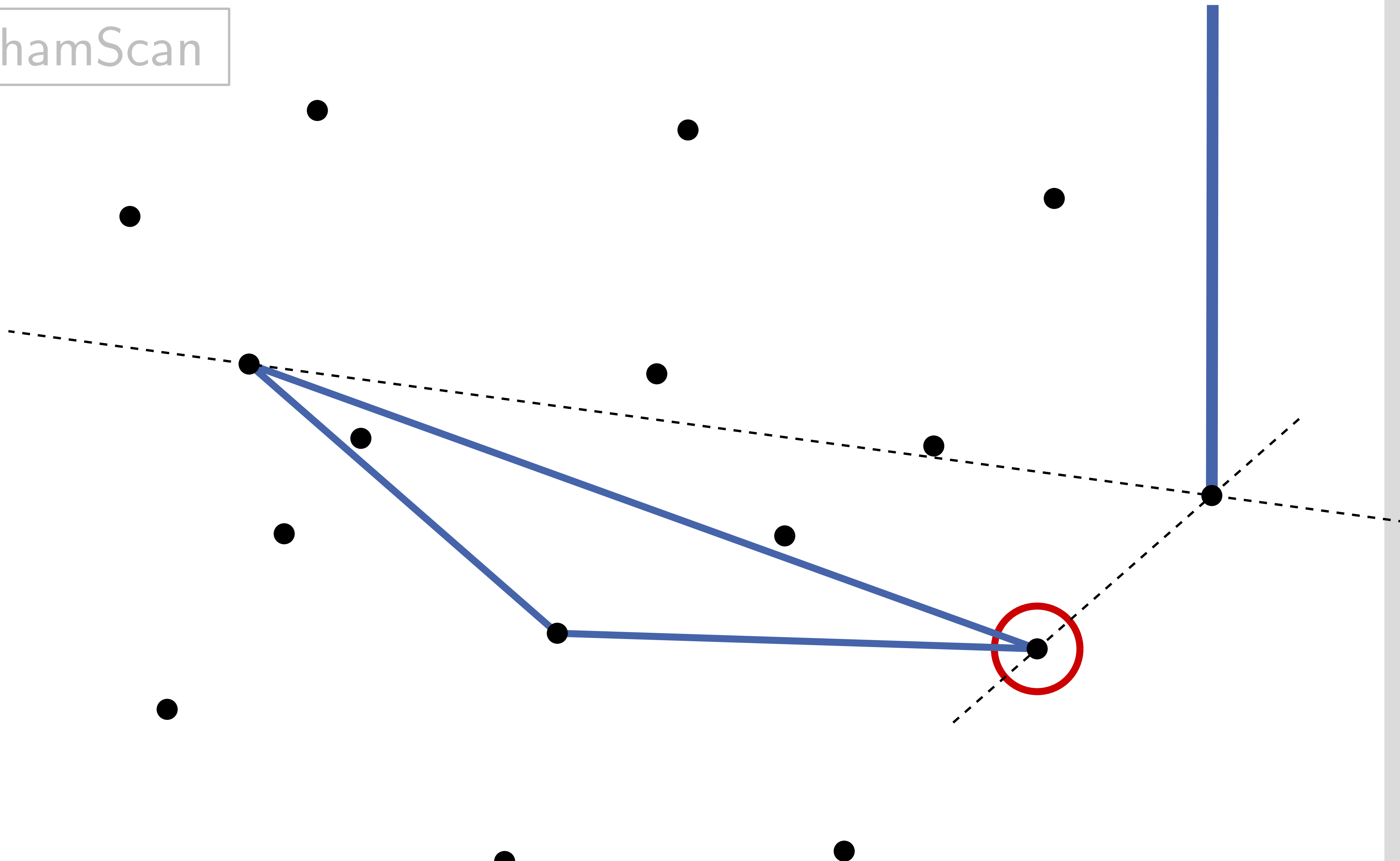

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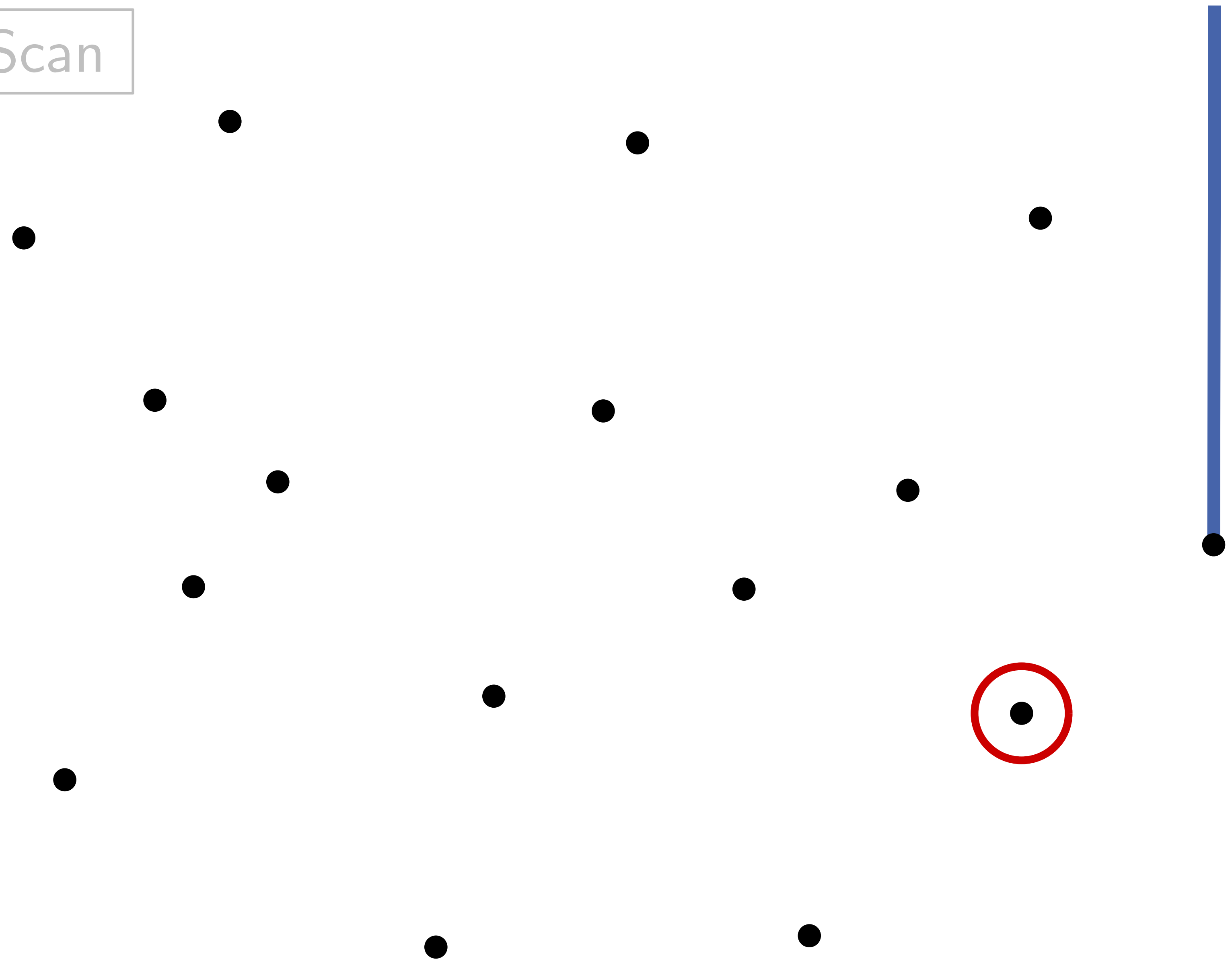

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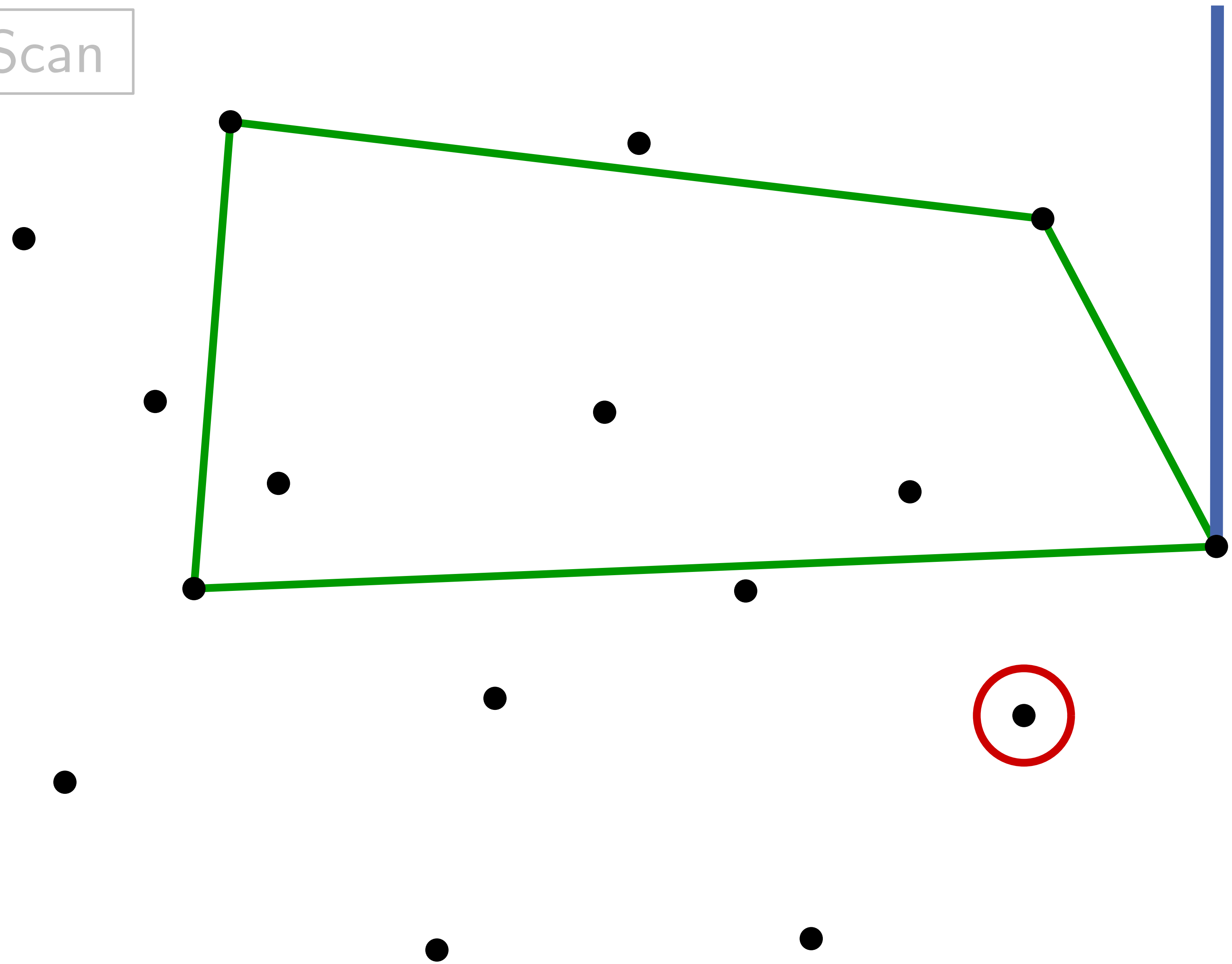

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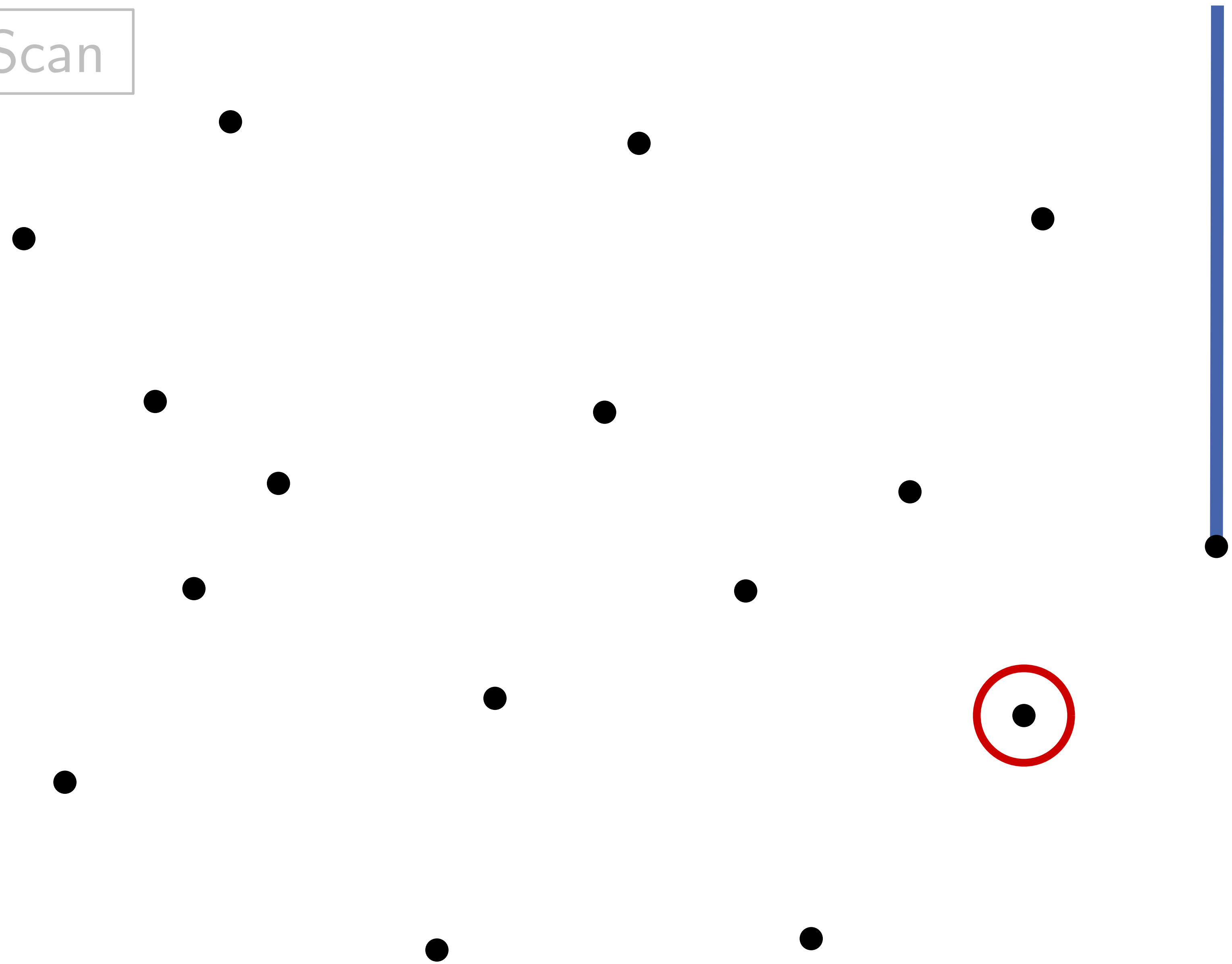

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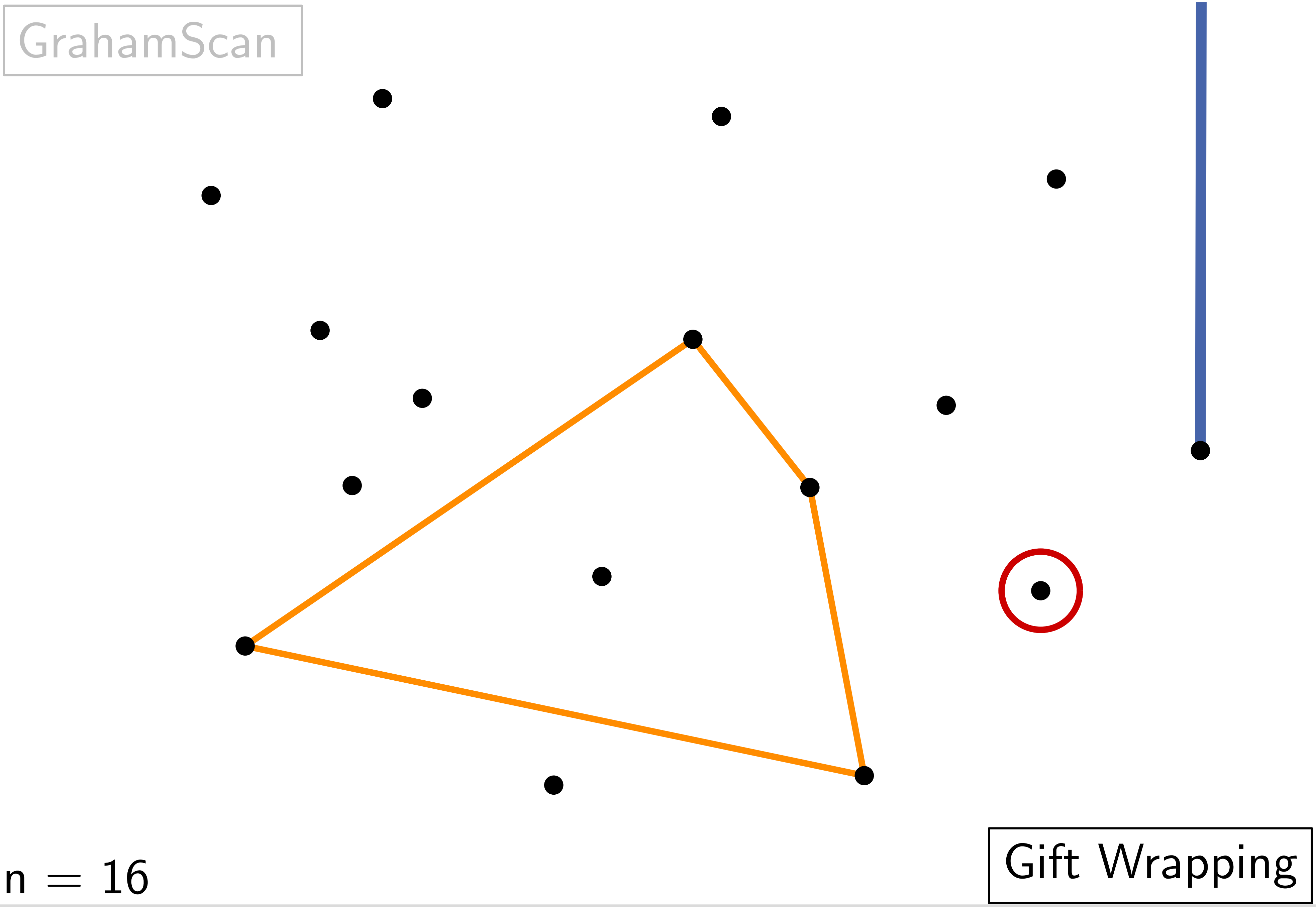

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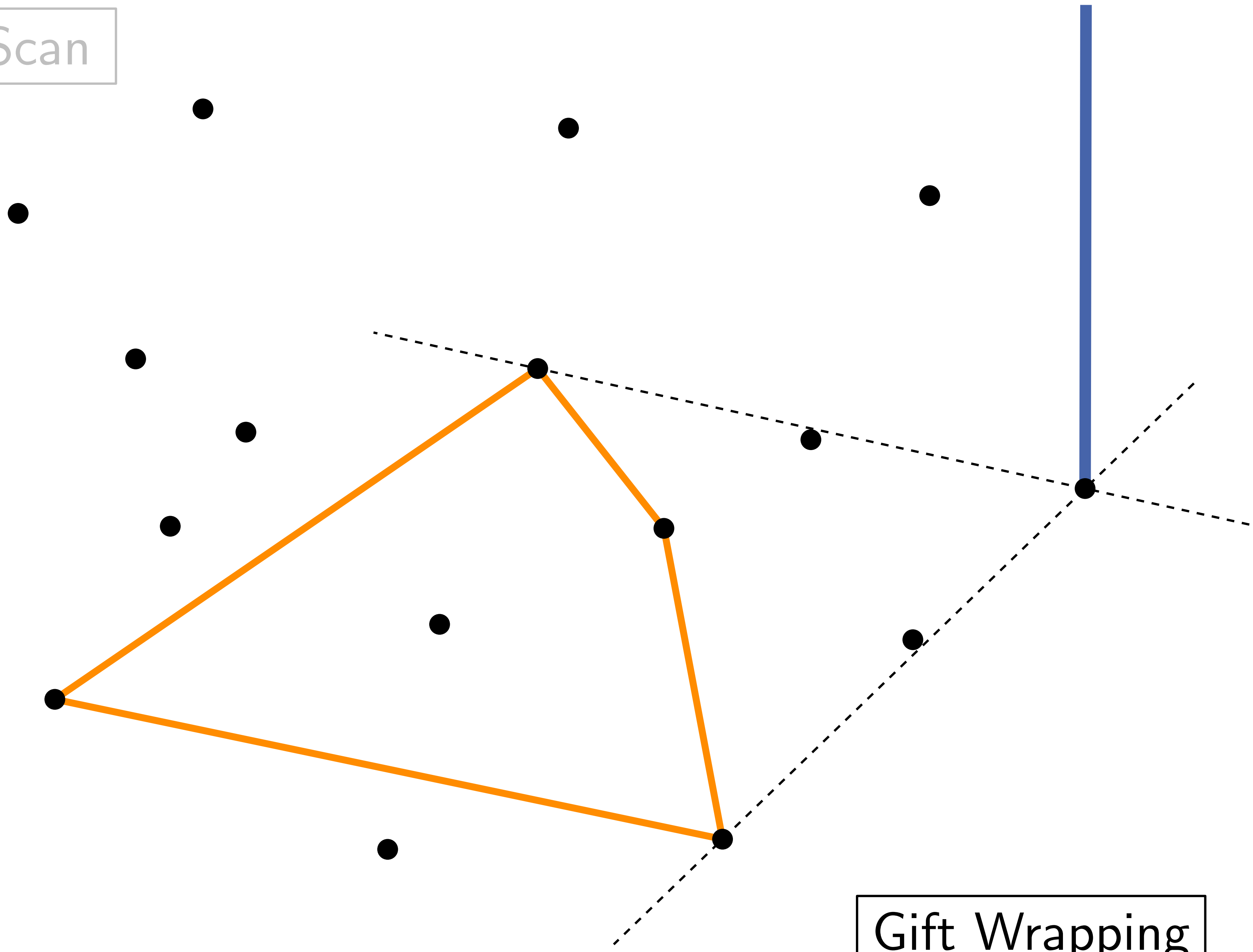

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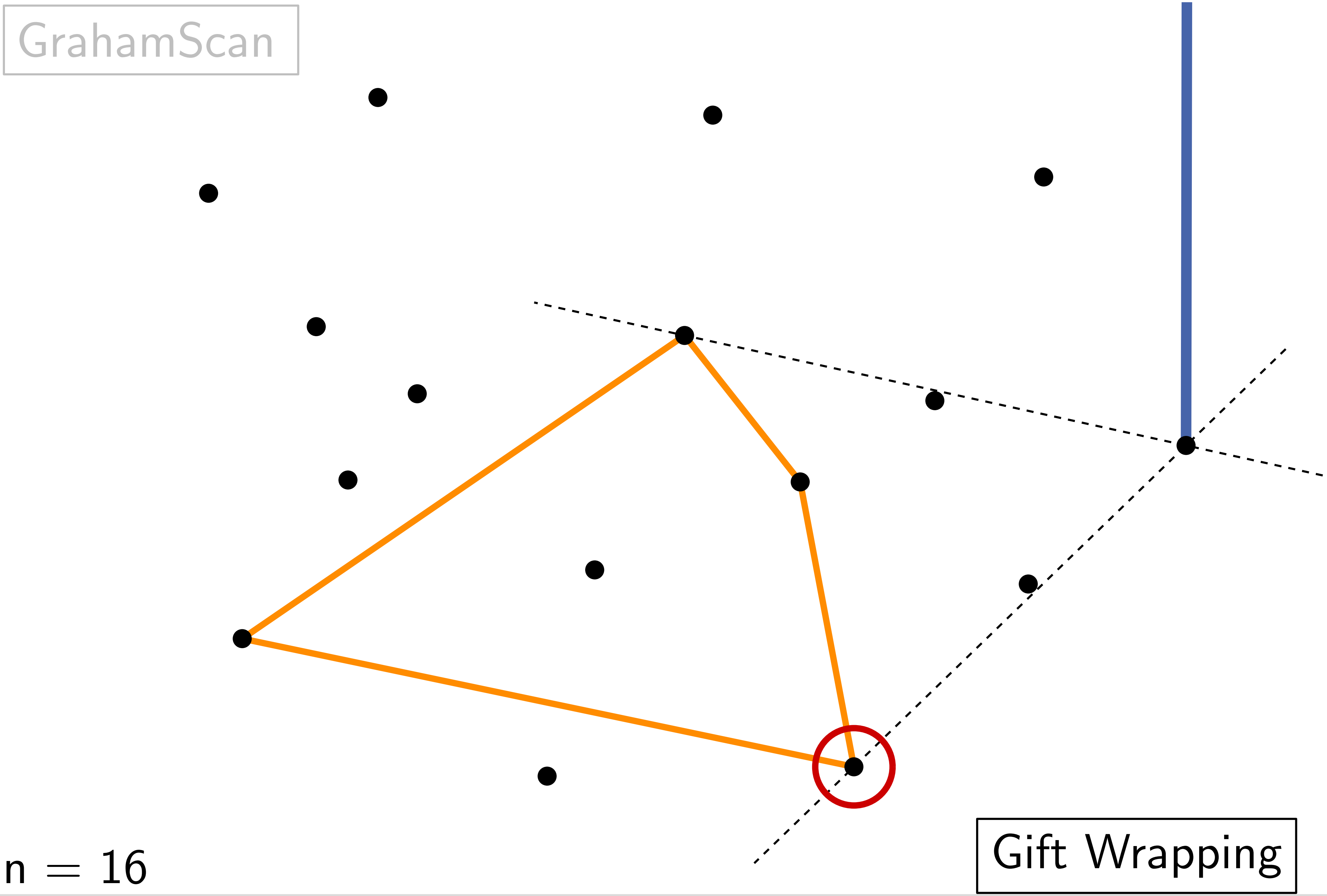

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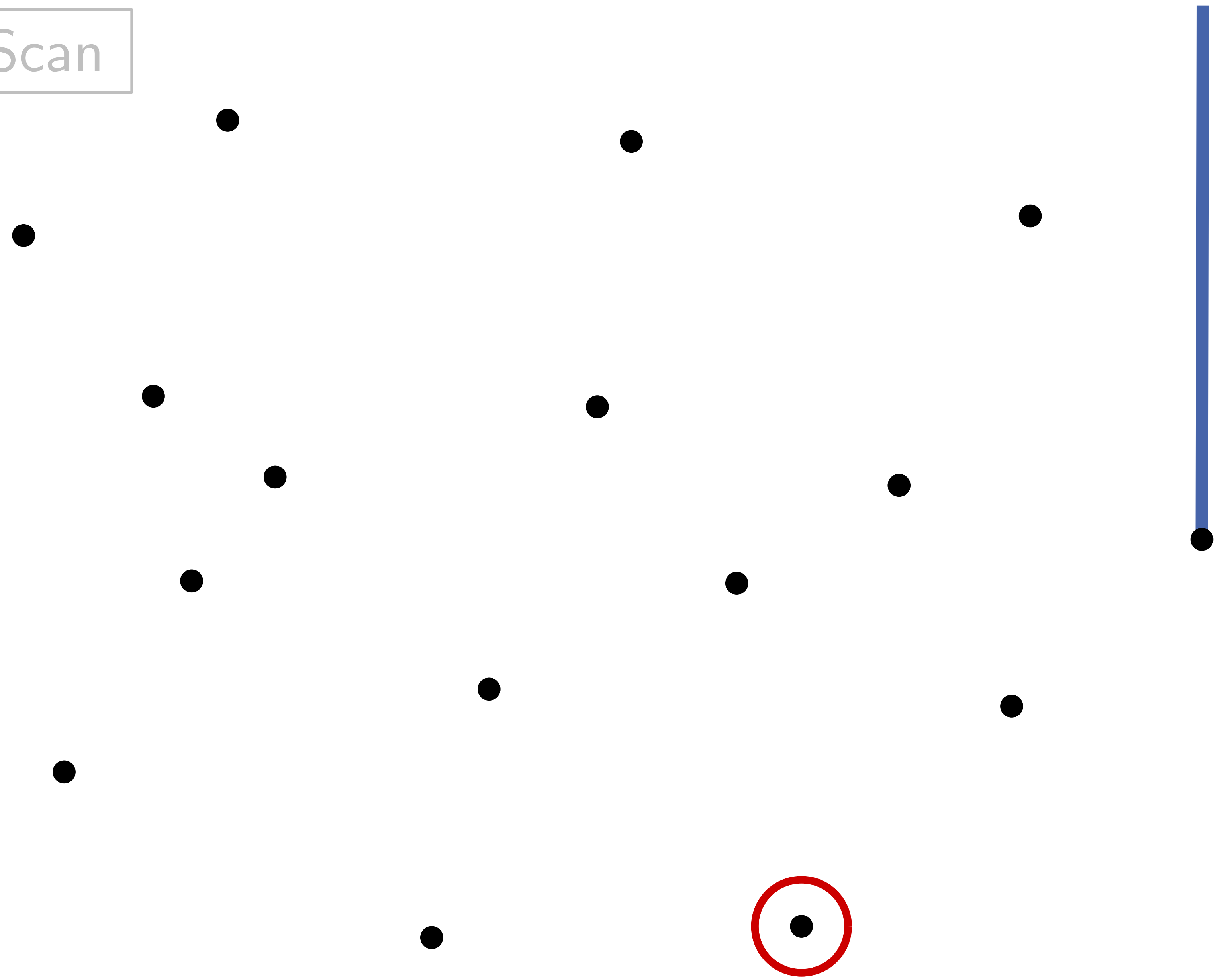

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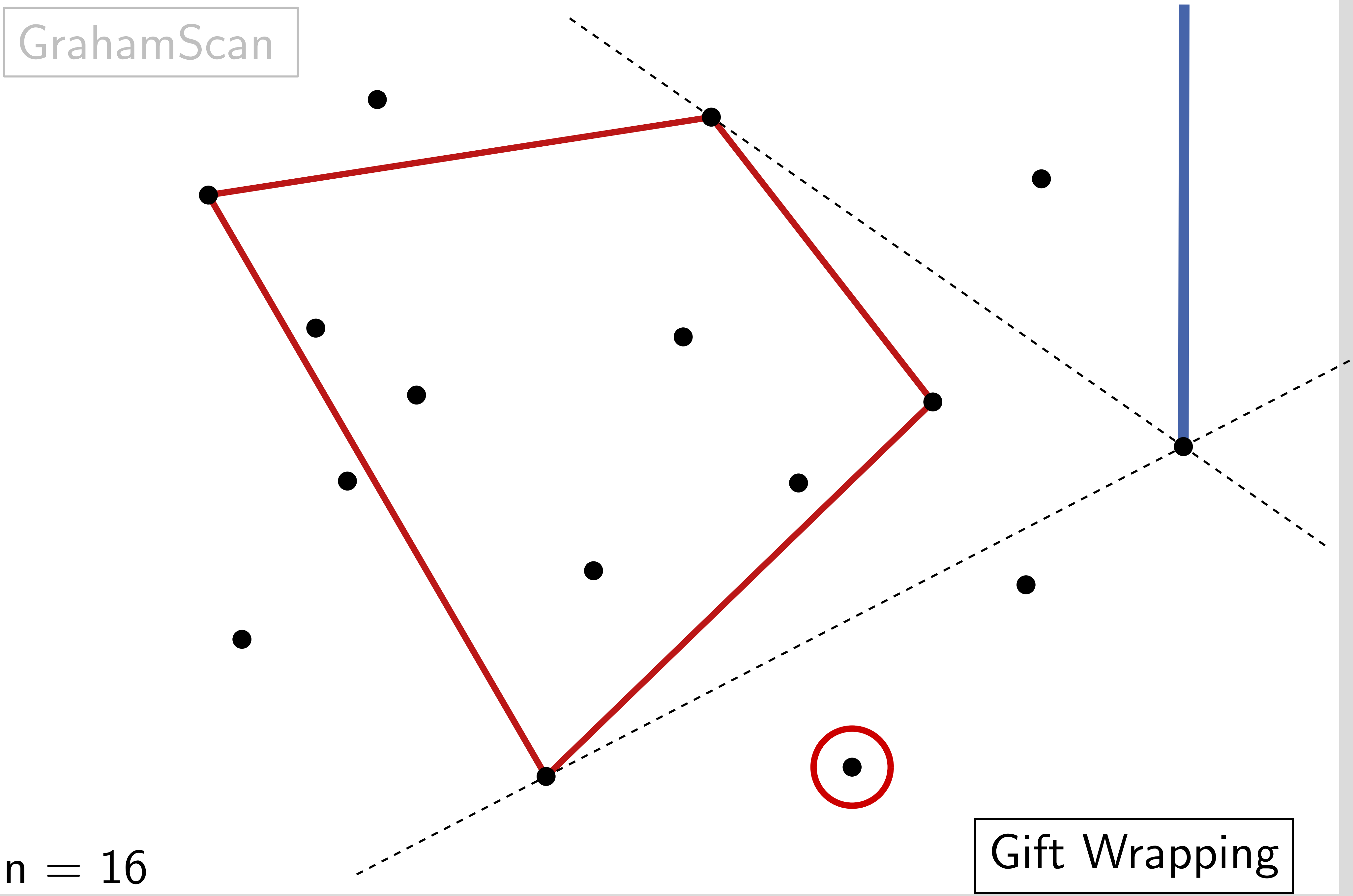

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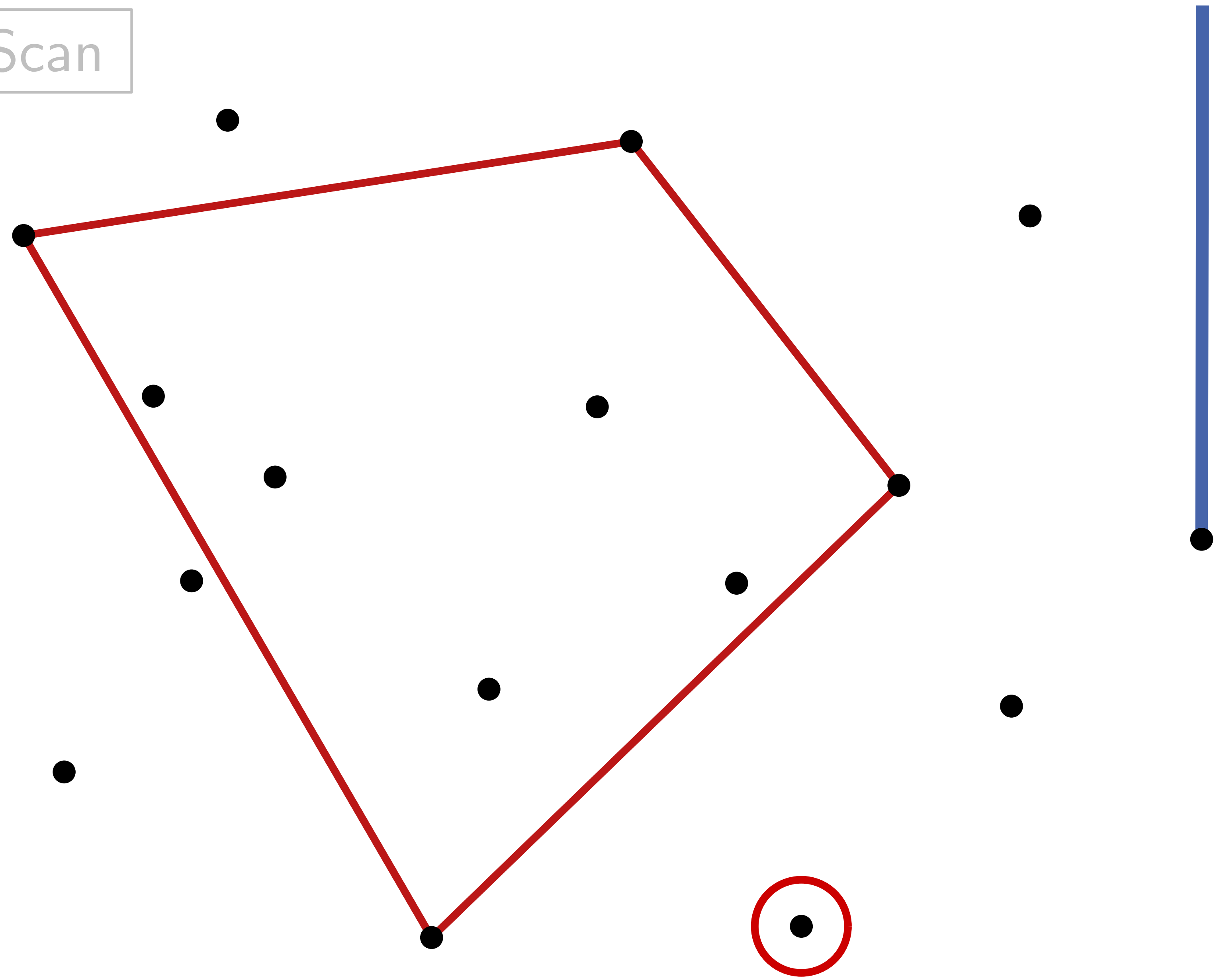

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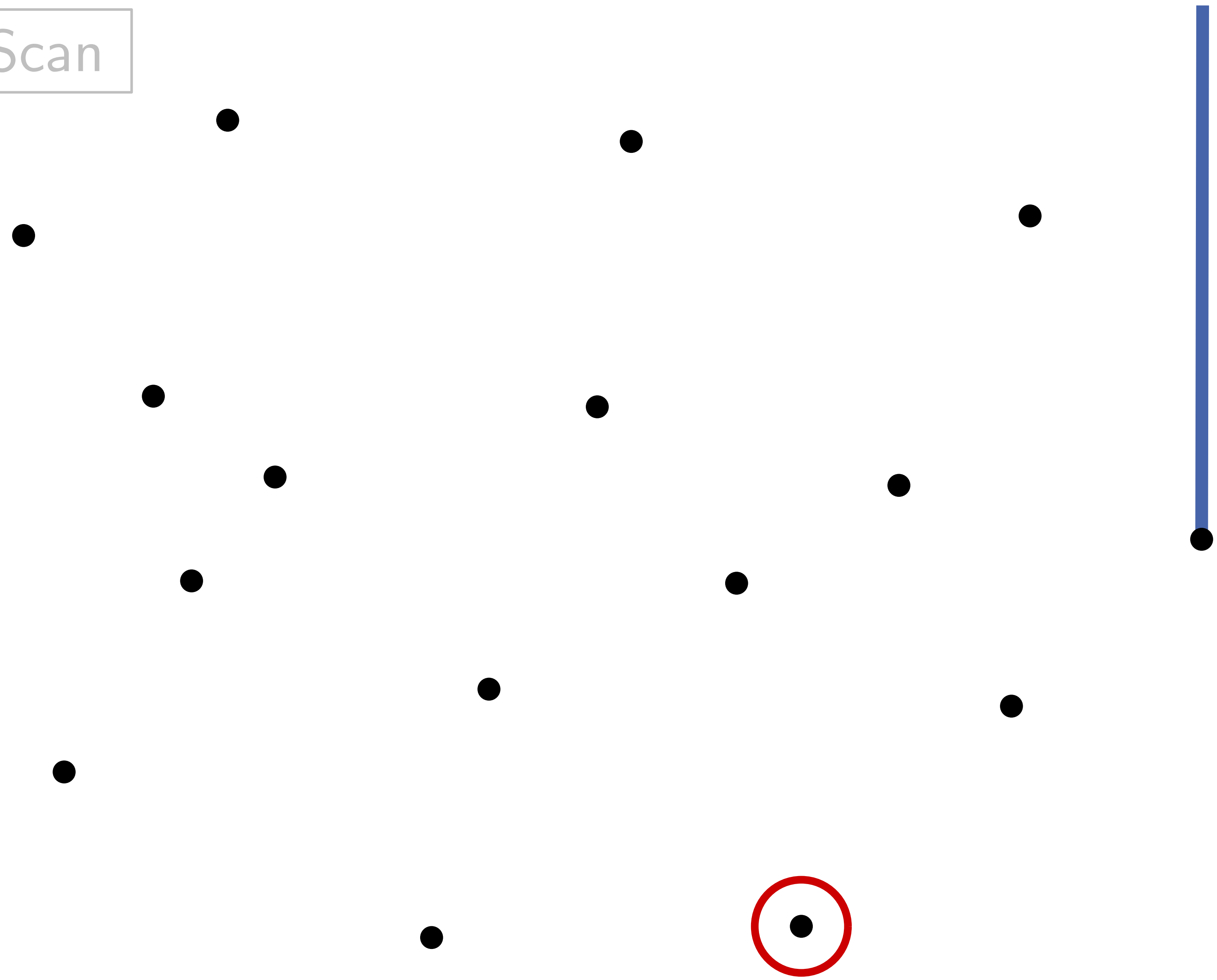

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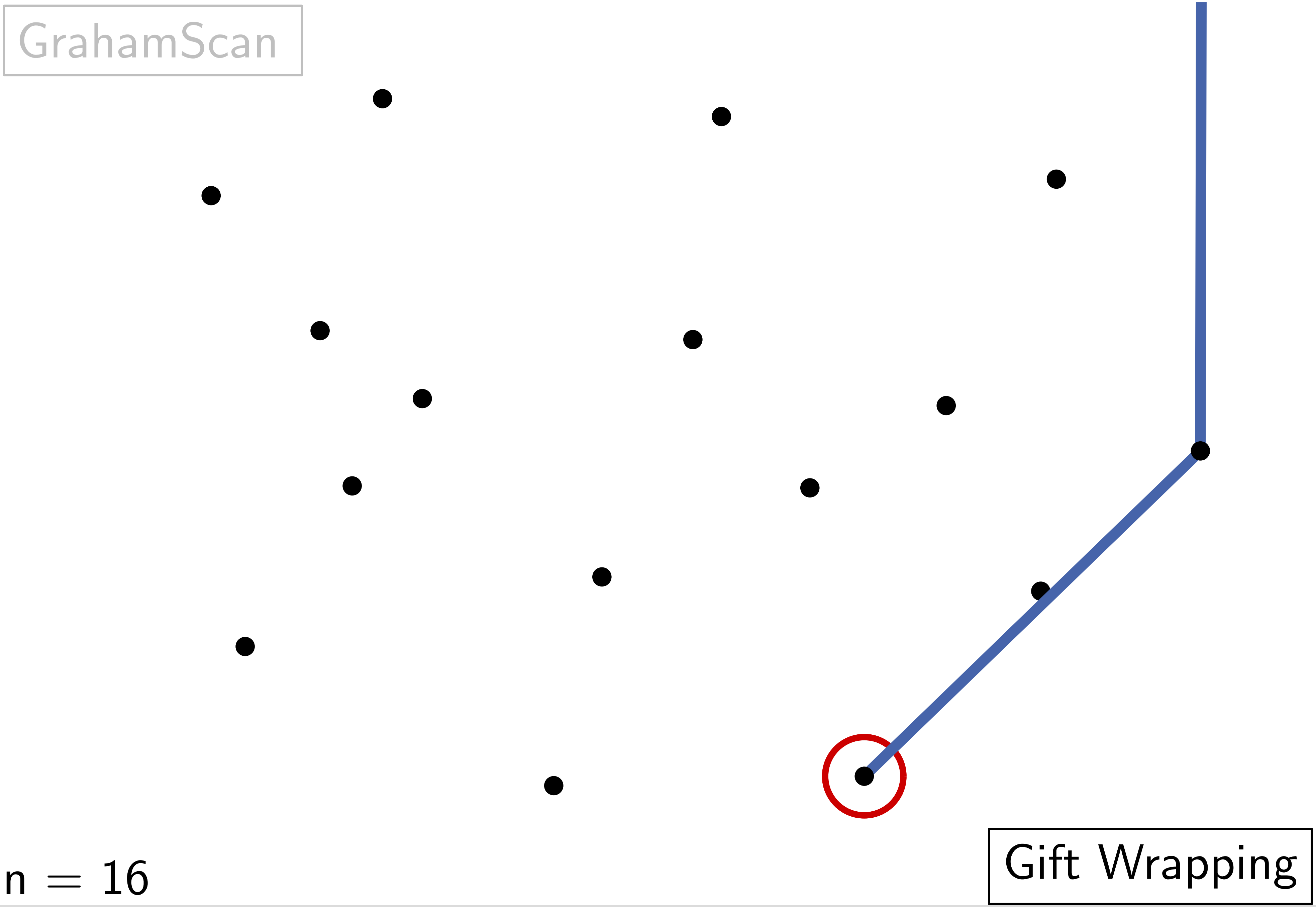

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Suppose we know h:

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  if $p_{j+1} = p_1$ **then return** $(p_1, \ldots, p_{j+1})$

return failure

$\mathcal{O}(n \log m)$

Total: $\mathcal{O}(n \log m)$

What to do with m?

Suggestions?

What to do with m?

FullChanHull(P)

for $t = 0, 1, 2, \ldots$ **do**

$m = \leftarrow \min\{n, 2^{2^t}\}$

result $\leftarrow$ ChanHull(P, m)

if result $\neq$ failure **then** break

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Running time:

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Theorem 3: The convex hull $CH(P)$ of n points P in $\mathbb{R}^2$ can be computed in $O(n \log h)$ time with Chan's Algorithm, where $h = |CH(P)|$.

Discussion

Karlsruhe Institute of Technology

Is it possible to compute faster than $O(n \log n)$ or $O(n \log h)$ time?

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What about the robustness of the algorithms?

- Regarding robustness: imprecision of floating-point arithmetic
- FirstConvexHull possibly produces a valid polygon
- Graham and Jarvis always provide a polygon, but it may have minor defects

Designing Geometric Algorithms–Guidelines

1.) Eliminate degenerate cases ($\rightarrow$ *general position*)

- unique x-coordinates
- no three collinear points
- ...

2.) Adjust degenerate inputs

- integrate into existing solutions (e.g., compute lexicographic order if x-coordinates are not unique)
- may require special treament

3.) Implementation

- primitive operations (available in libraries?)
- robustness