Computational Geometry · Lecture

Range Searching

INSTITUT FÜR THEORETISCHE INFORMATIK · FAKULTÄT FÜR INFORMATIK

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18.11.2015

Geometry in Databases

In a personnel database, the employees of a company are anonymized and their monthly income and birth year are saved. We now want to perform a search: which employees have an income between 2,000 and 3,000 Euro and were born between 1960 and 1980?

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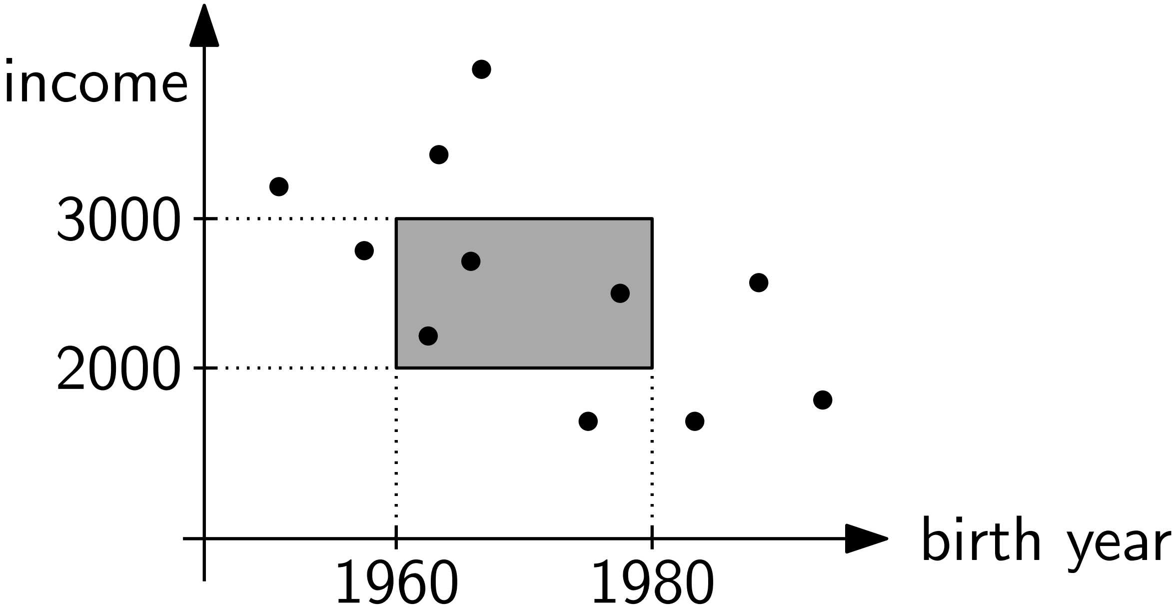

Geometric Interpretation:

Entries are points: (birth year, income level) and the query is an axis-parallel rectangle

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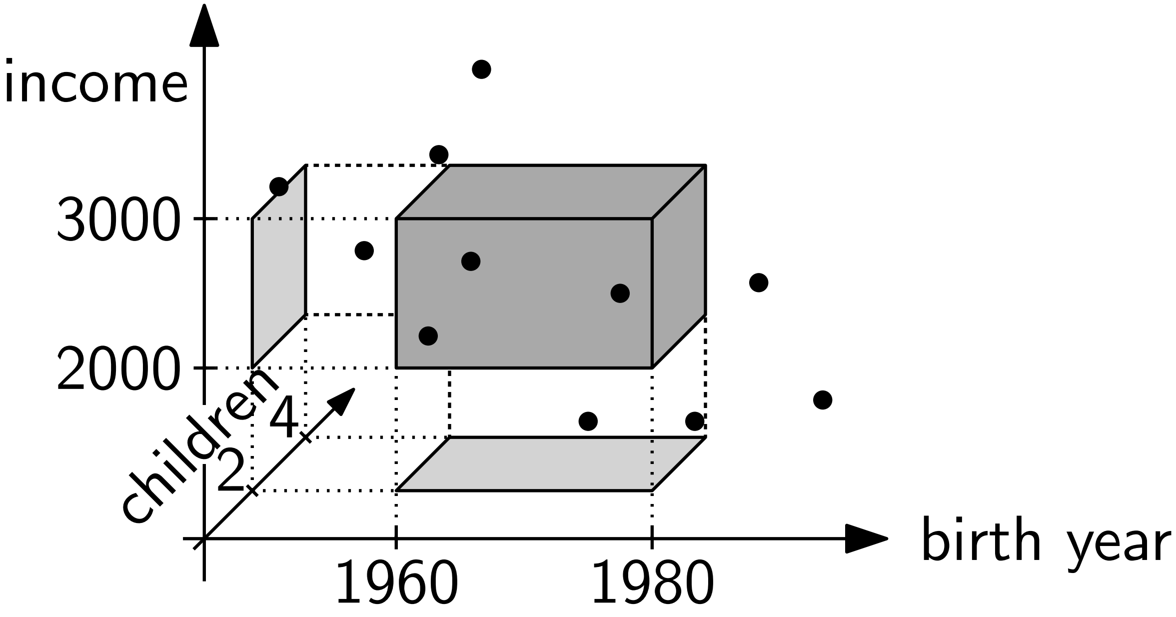

This problem can easily be generalized to d dimensions.

Orthogonal Range Queries

Given: n points in $\mathbb{R}^d$

Output: A data structure that efficiently answers queries of the form $[a_1, b_1] \times \cdots \times [a_d, b_d]$

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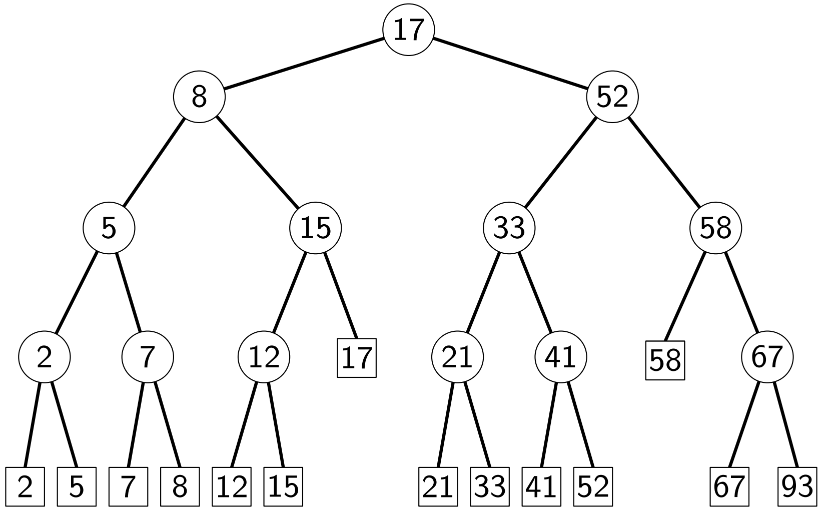

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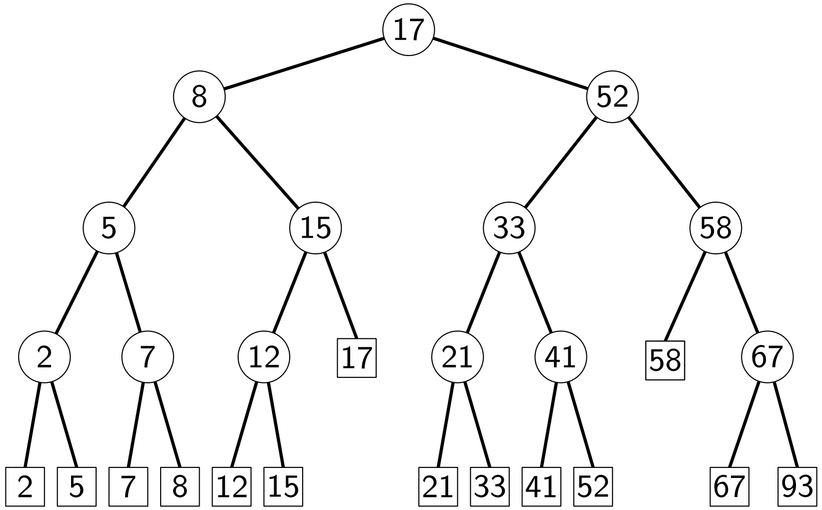

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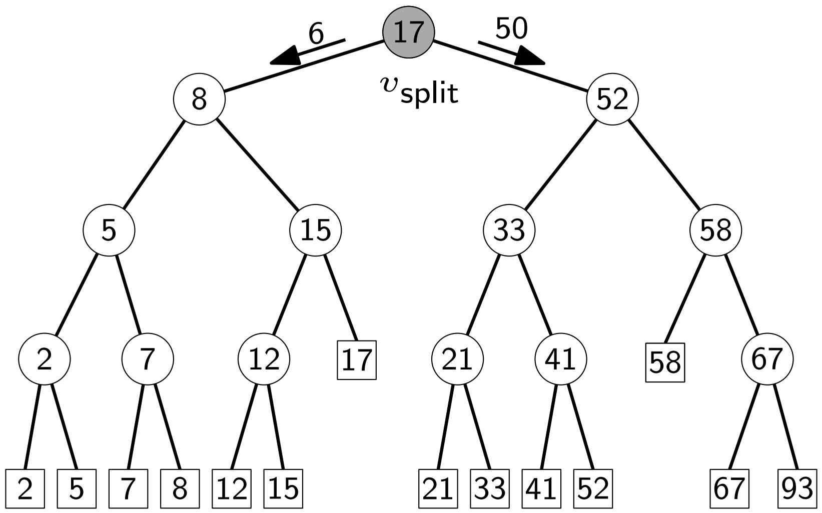

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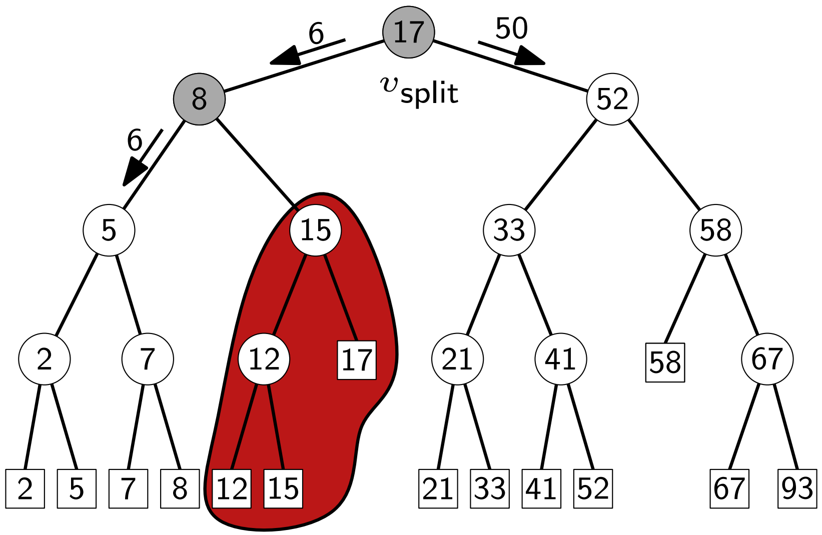

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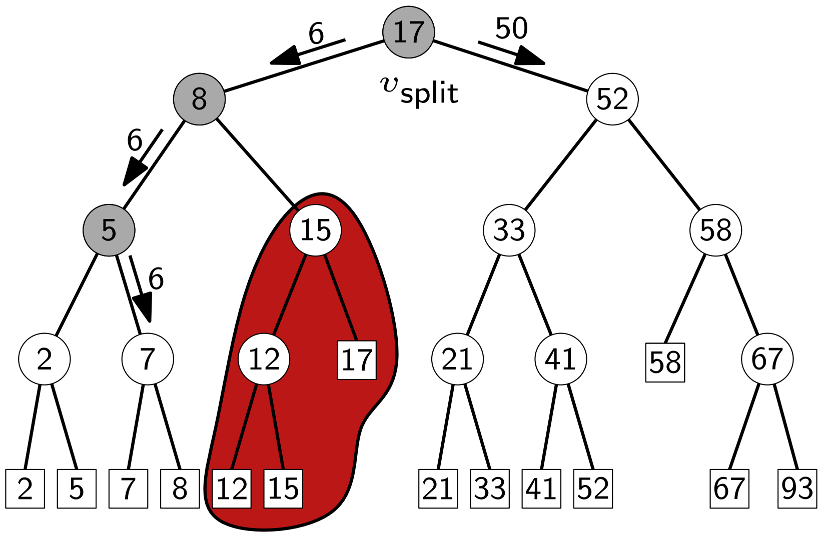

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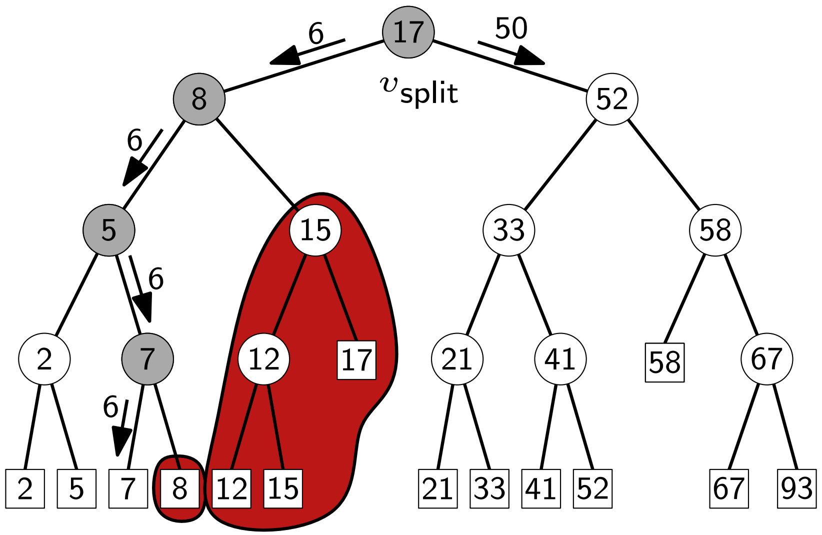

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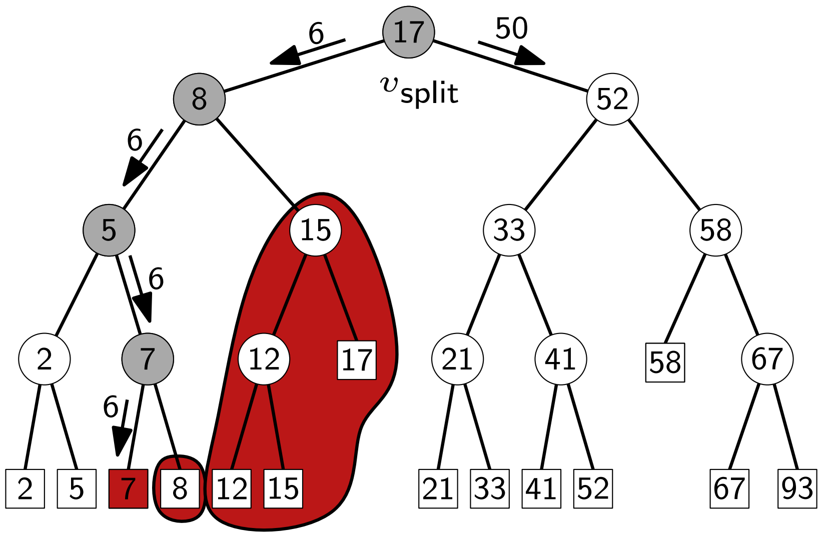

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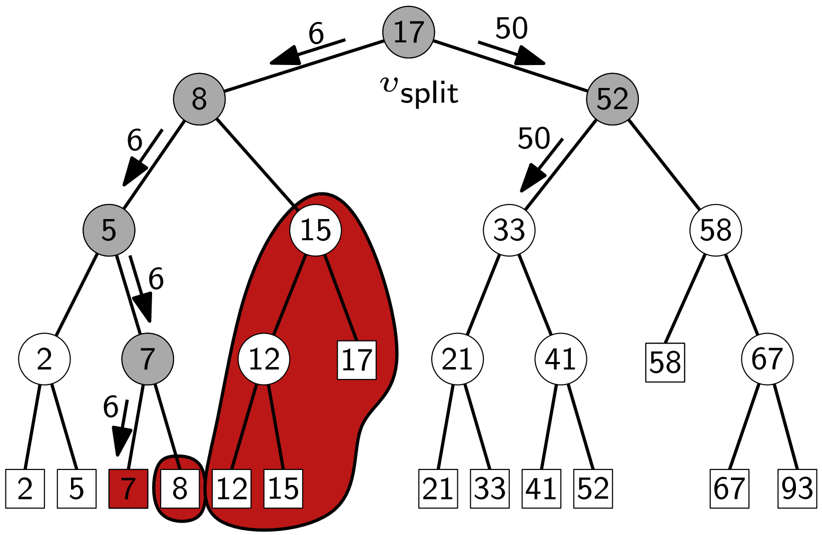

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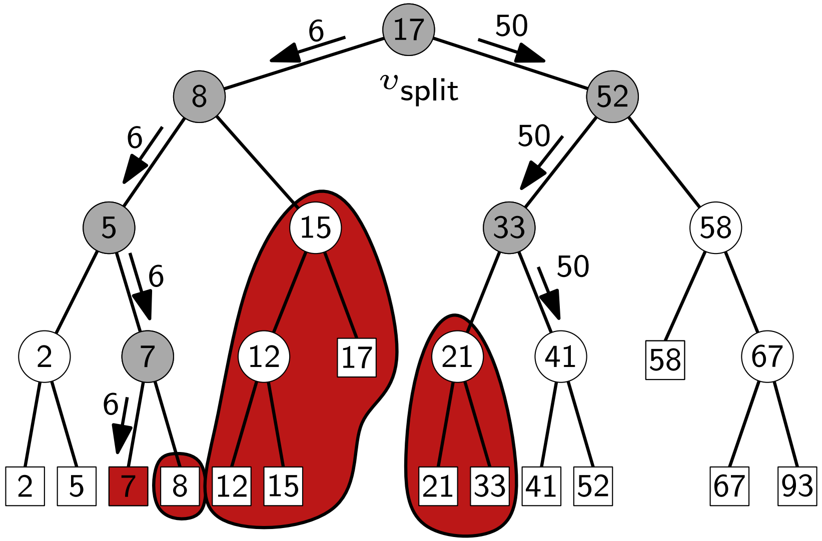

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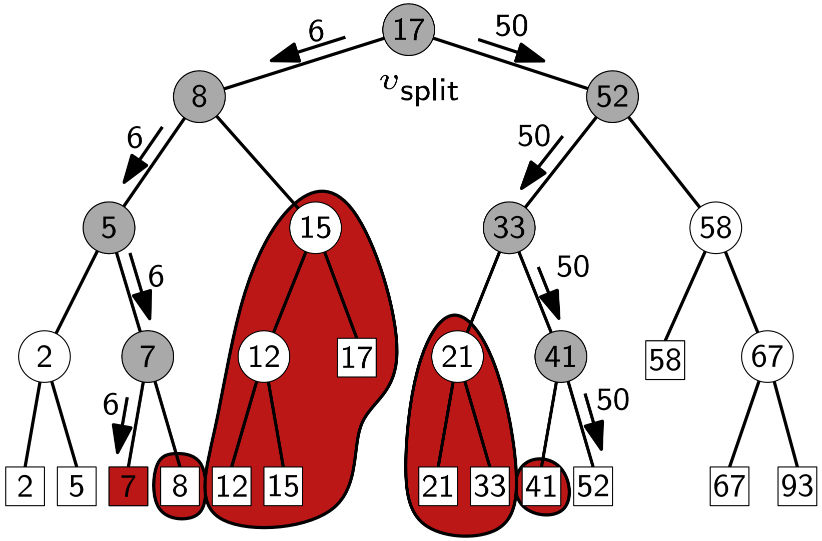

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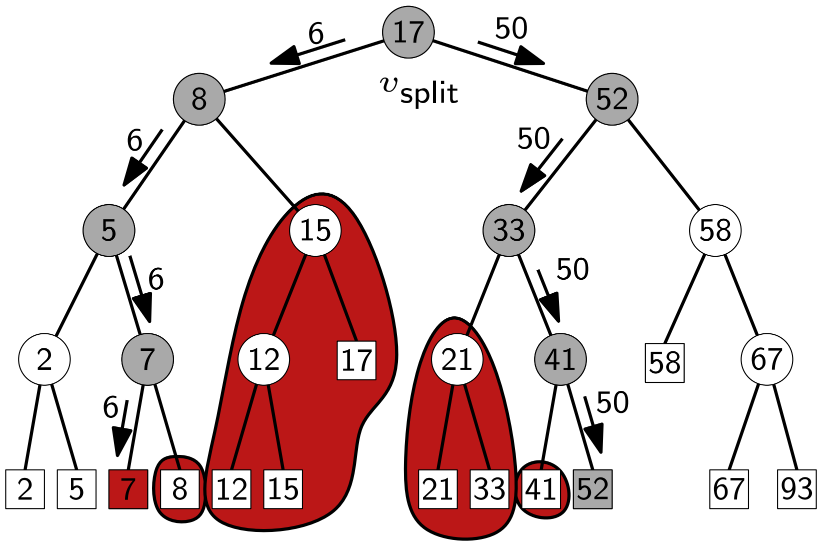

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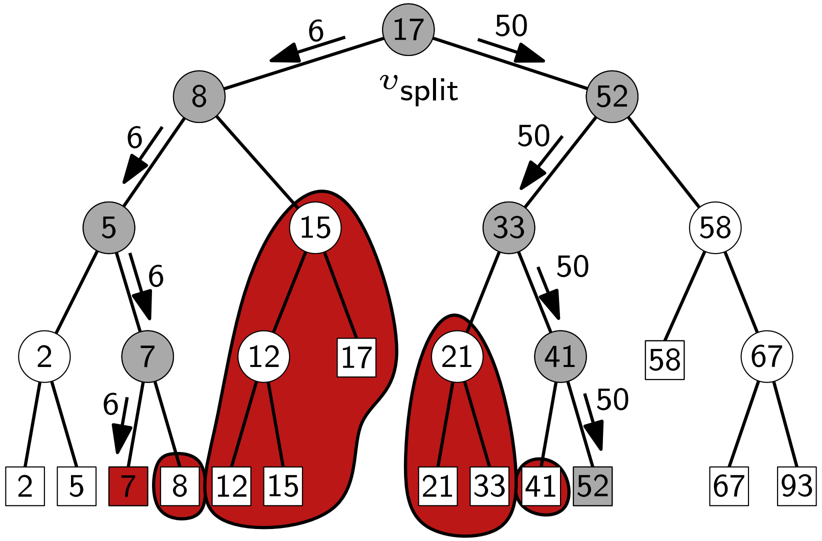

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Answer:

Points in the leaves between the search paths, (i.e., {7,8,12,15,17,21,33,41})

1dRangeQuery

FindSplitNode(T, x, x')

```
    $v \leftarrow$ root$(T)$
    while $v$ not a leaf and $(x' \le x_v$ or $x > x_v)$ do
        if $x' \le x_v$ then $v \leftarrow$ lc$(v)$ else $v \leftarrow$ rc$(v)$
    return $v$
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    if $v_{\text{split}}$ is leaf then report $v_{\text{split}}$
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        while $v$ not a leaf do
            if $x \le x_v$ then
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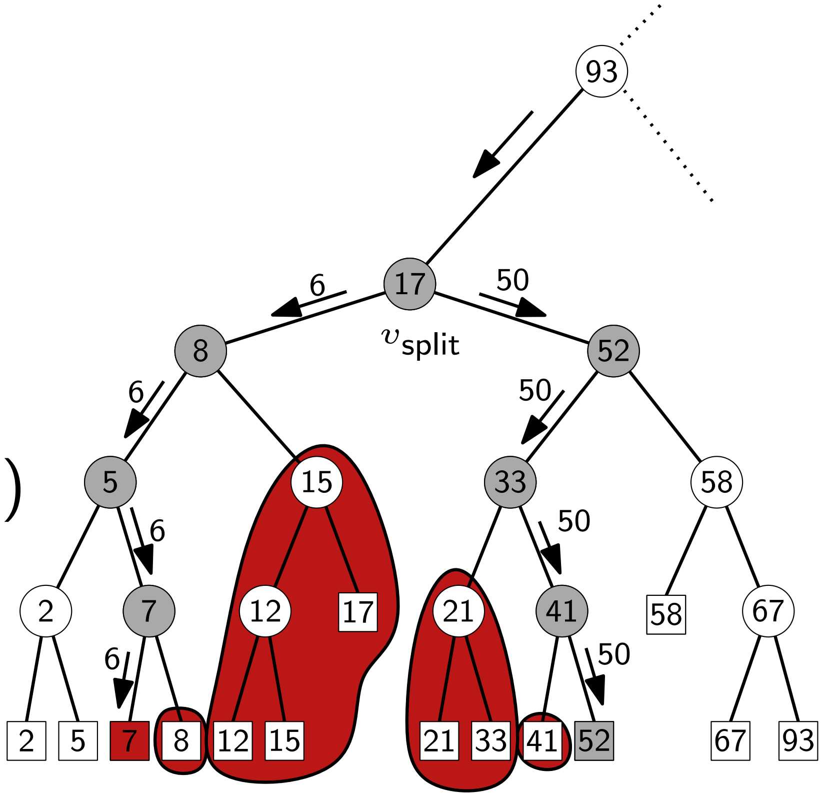

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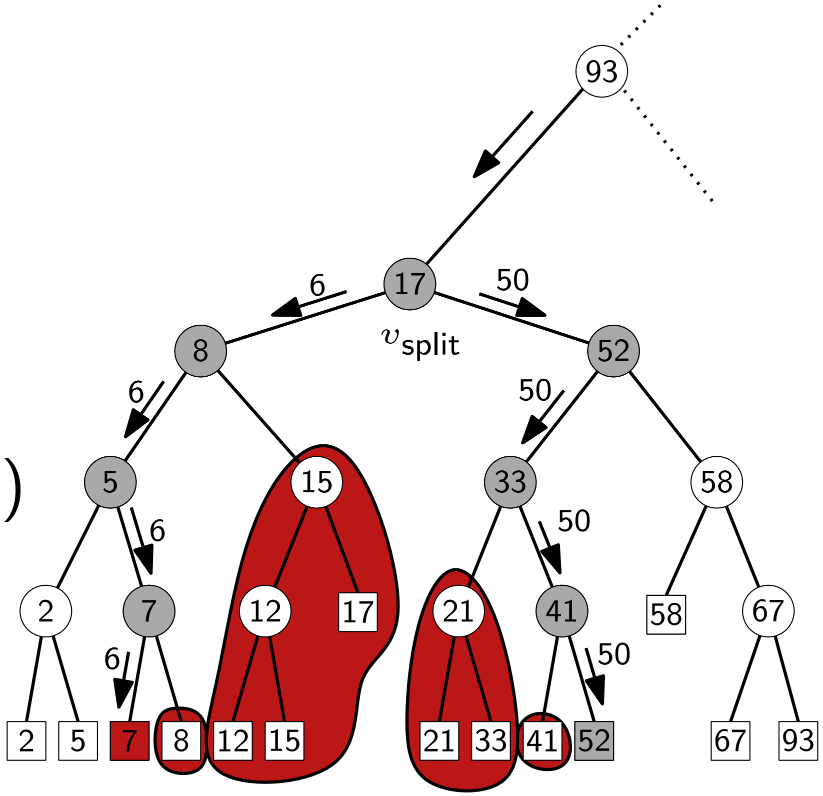
Can find *canonical subset* in linear time



Analysis of 1dRangeQuery

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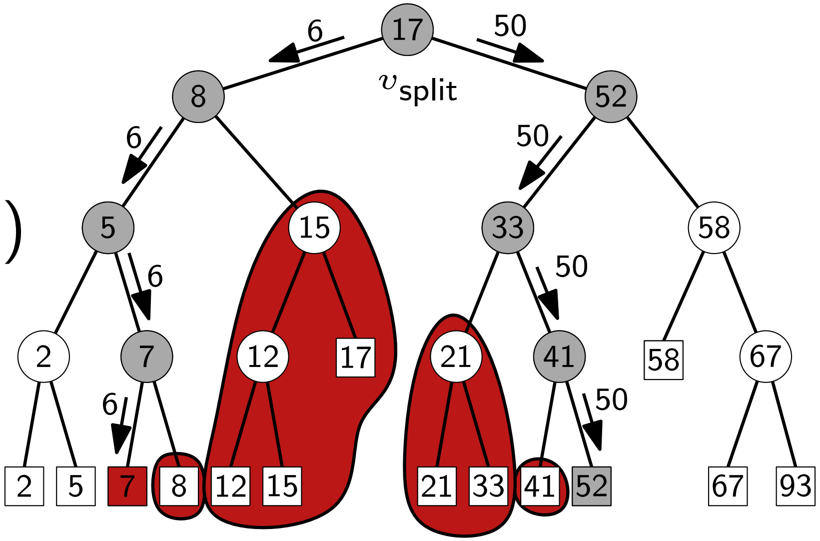

Theorem 1: A set of n points in $\mathbb{R}$ can preprocessed in $O(n \log n)$ time and stored in $O(n)$ space so that we can answer range queries in $O(k + \log n)$ time, where k is the number of reported points.

Orthogonal Range Queries for $d = 2$

Given: Set P of n points in $\mathbb{R}^2$

Goal: A data structure to efficiently answer range queries of the form $R = [x, x'] \times [y, y']$

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Solutions:

- *one* search tree, alternate search for x and y coordinates
  → kd-**Tree**
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Temporary assumption: general position, that is no two points have the same x- or y-coordinates

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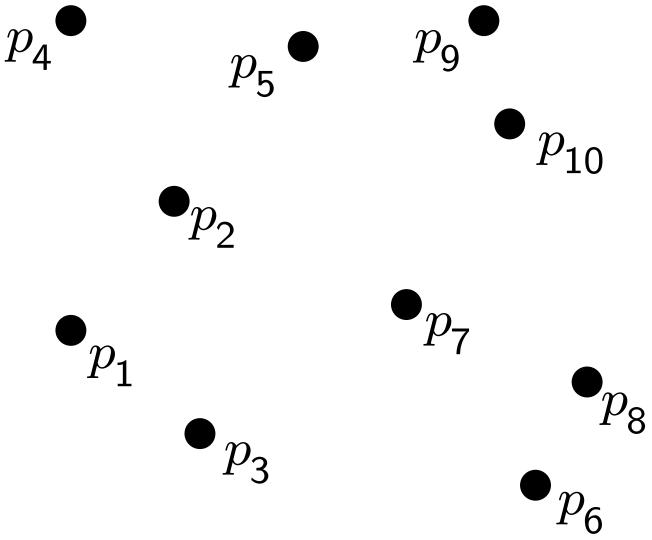

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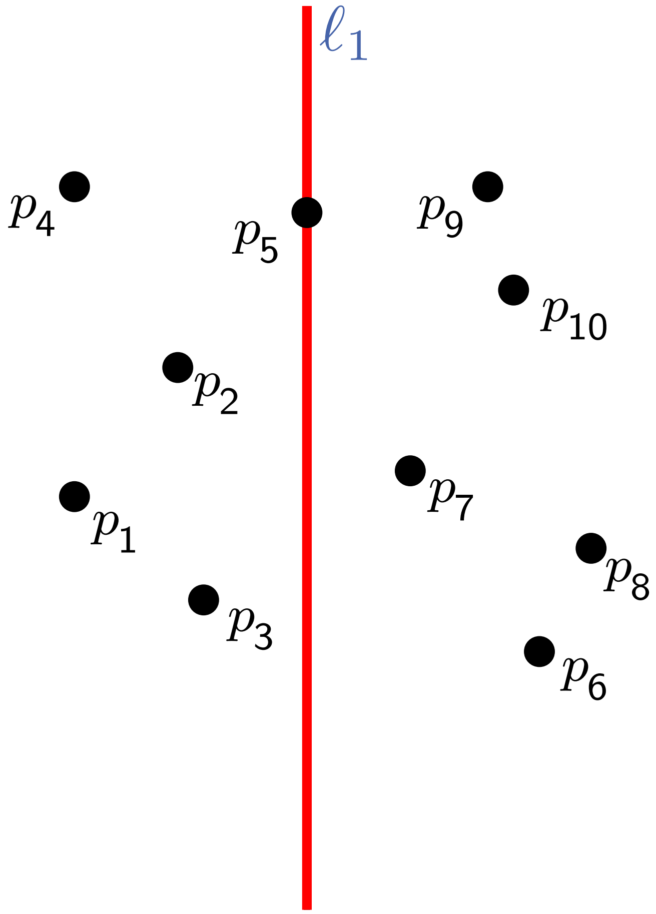

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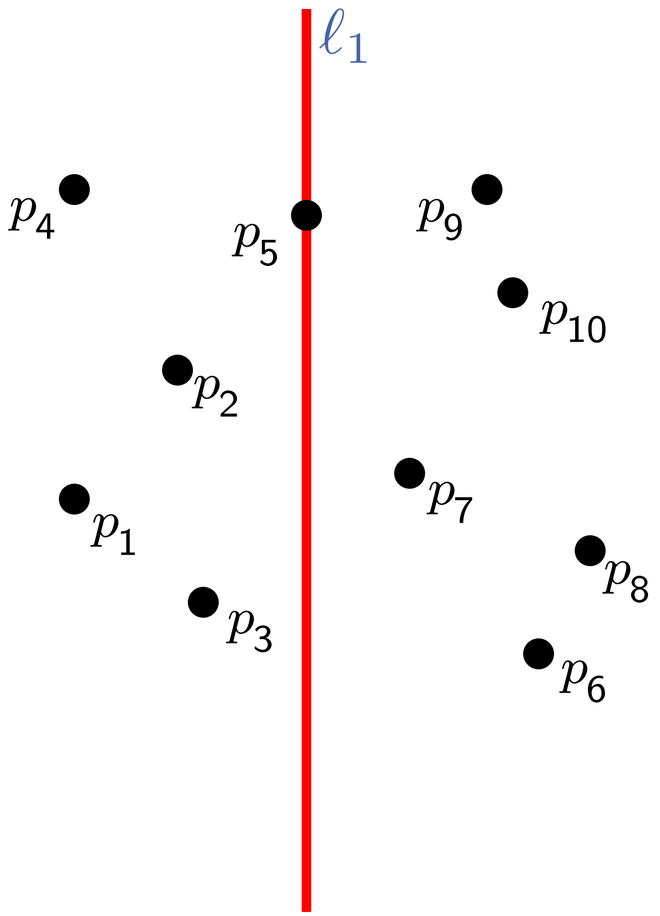

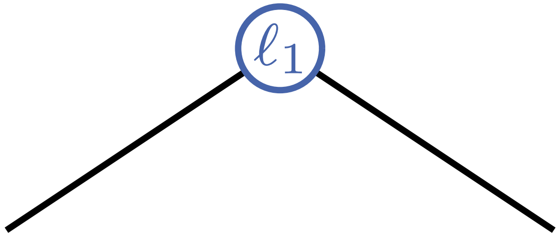

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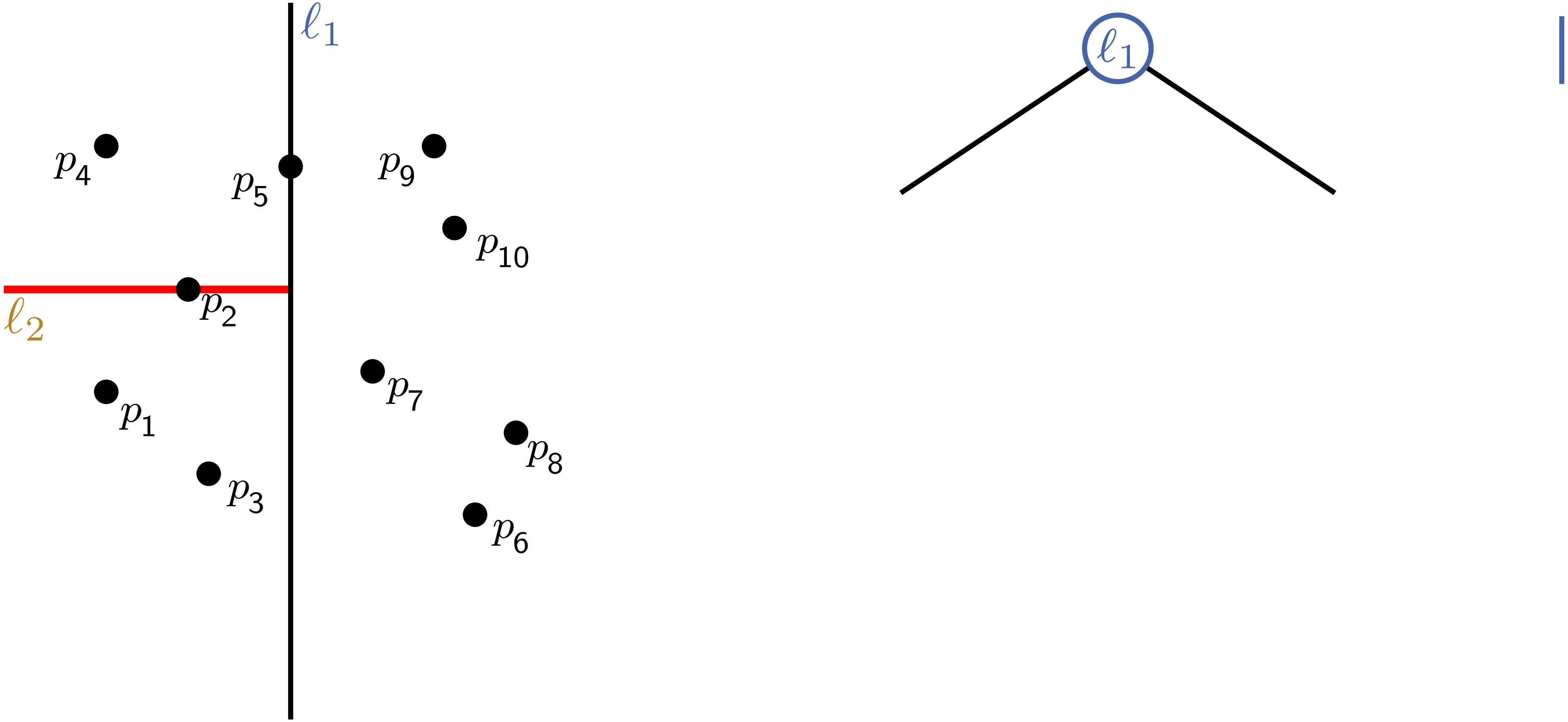

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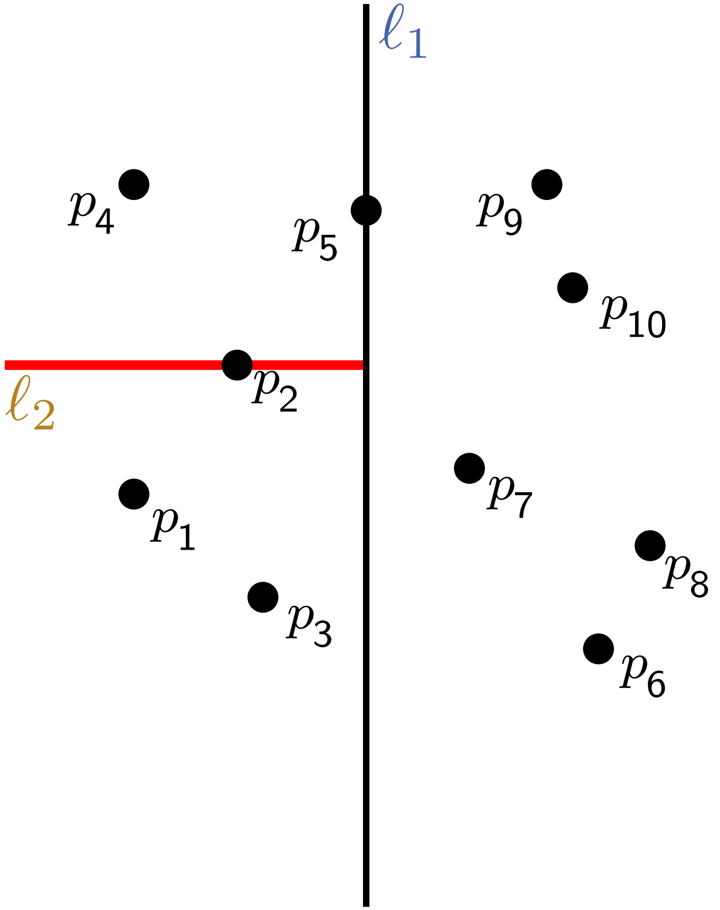

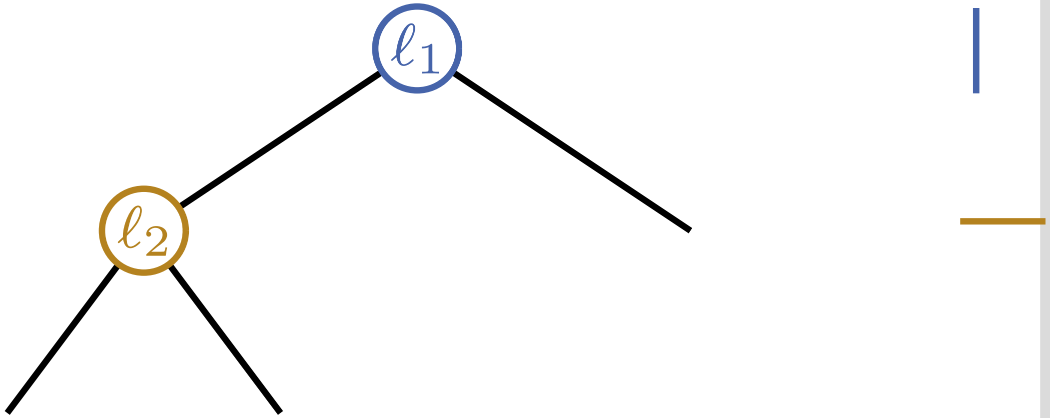

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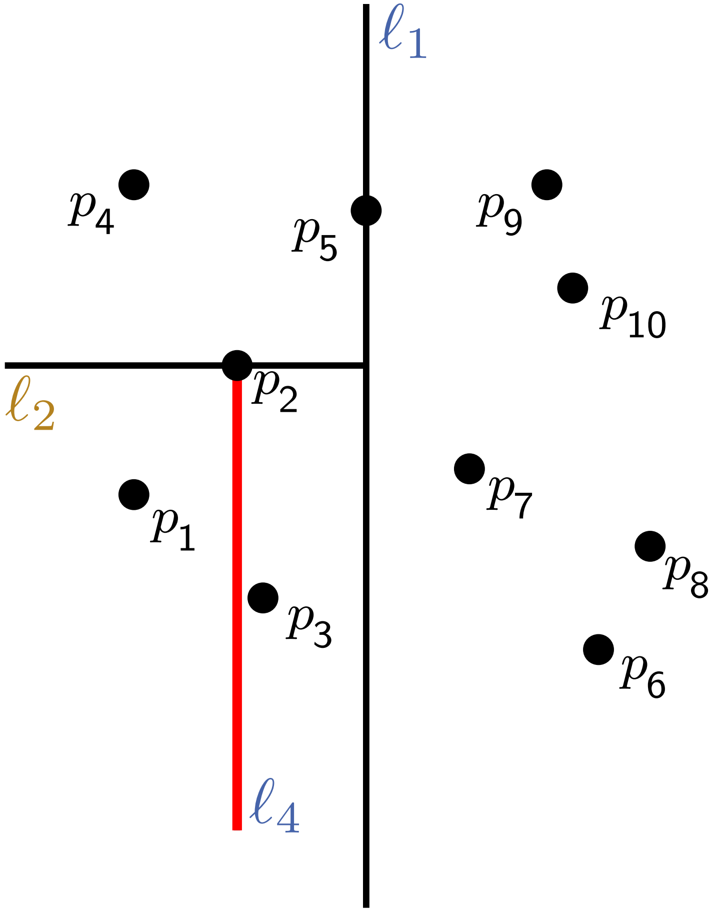

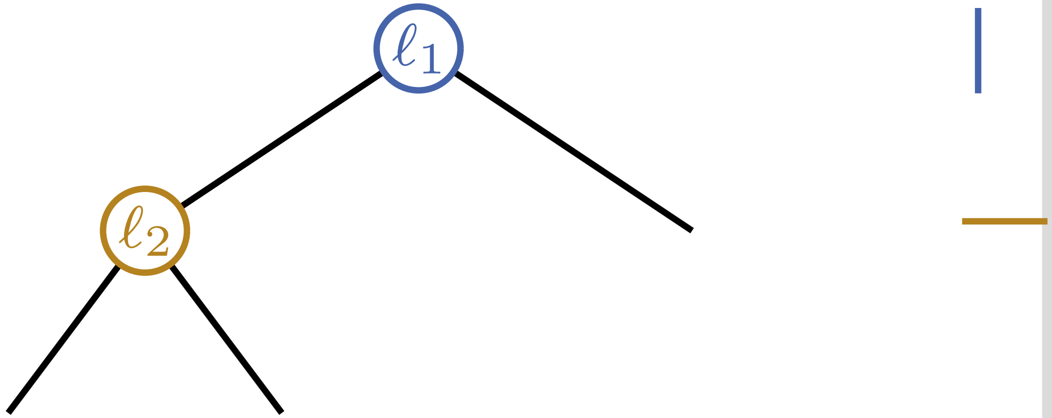

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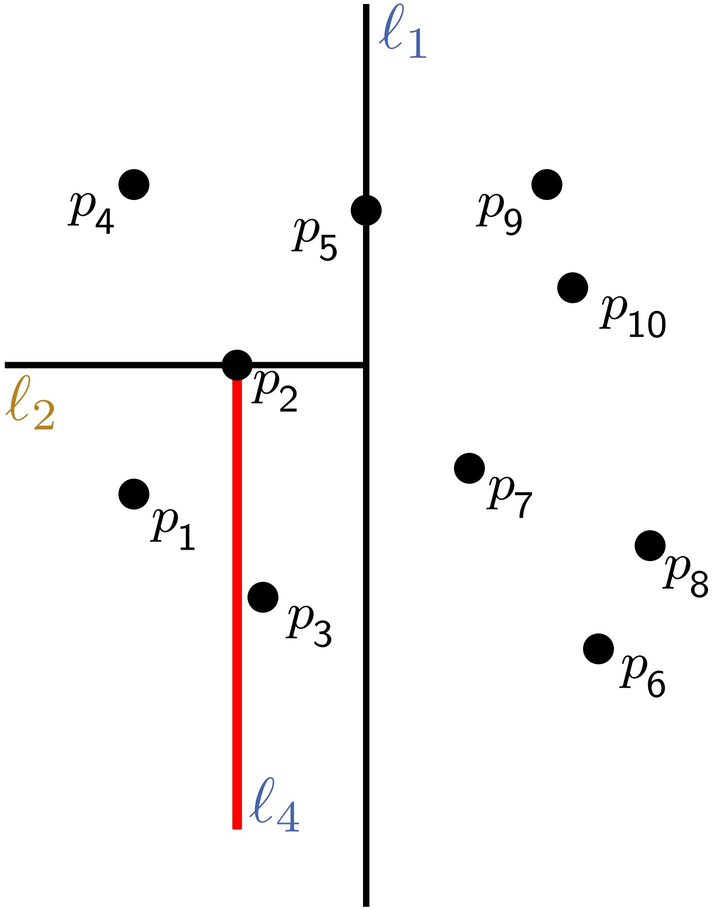

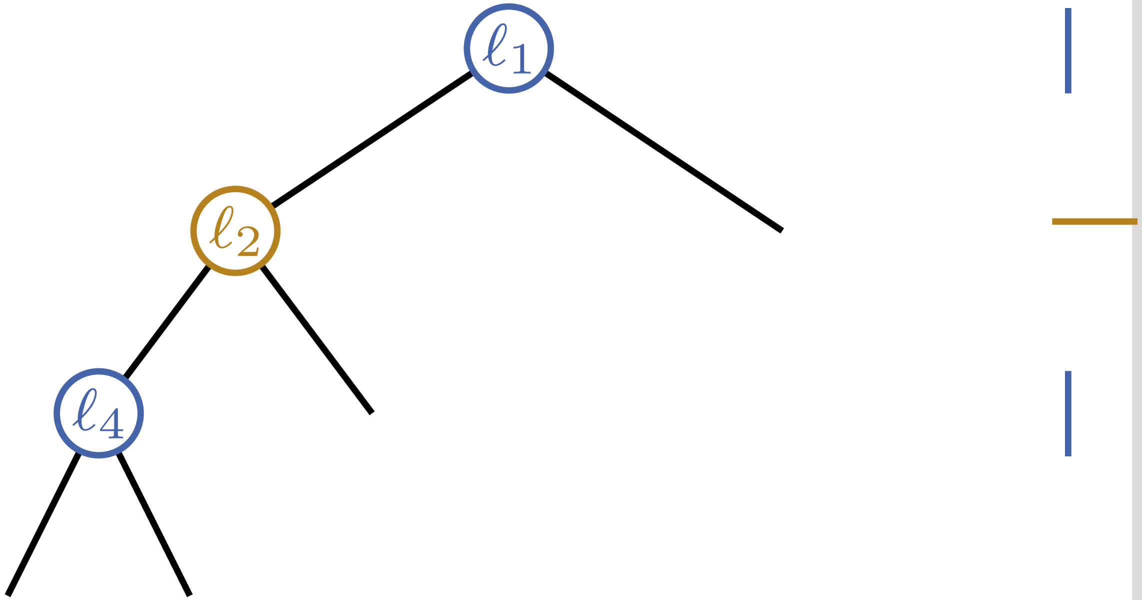

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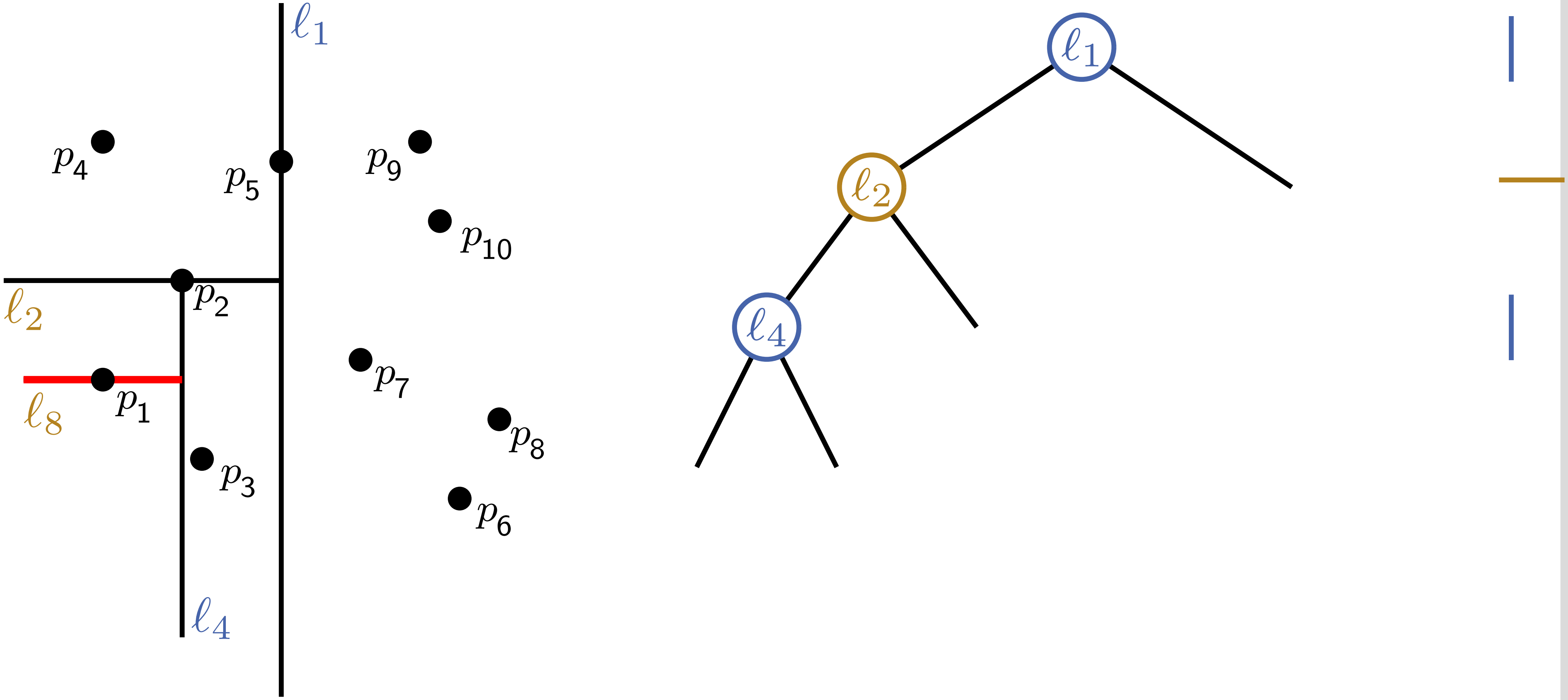

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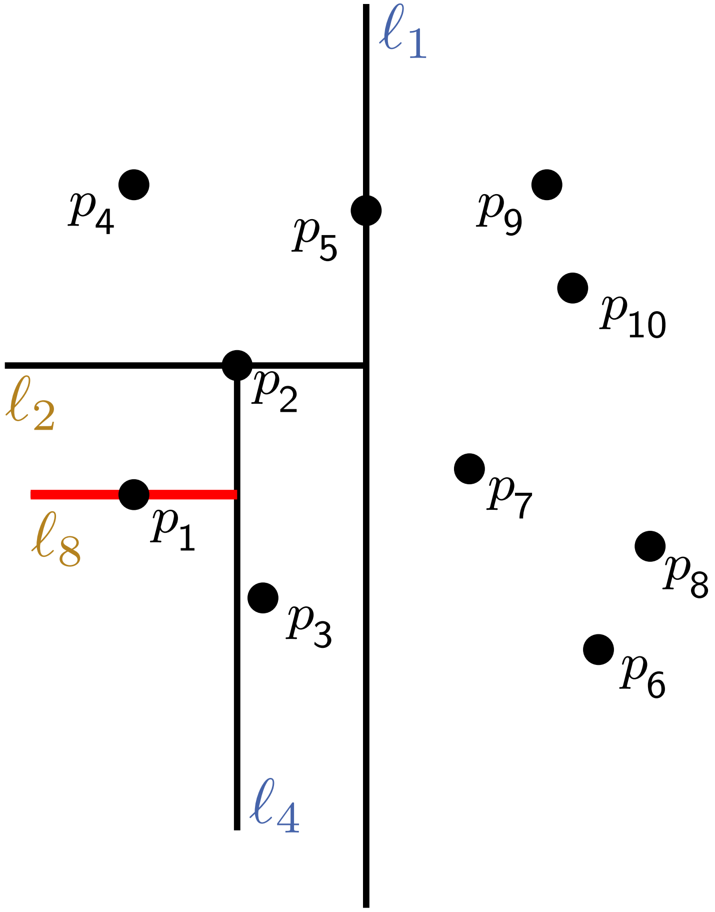

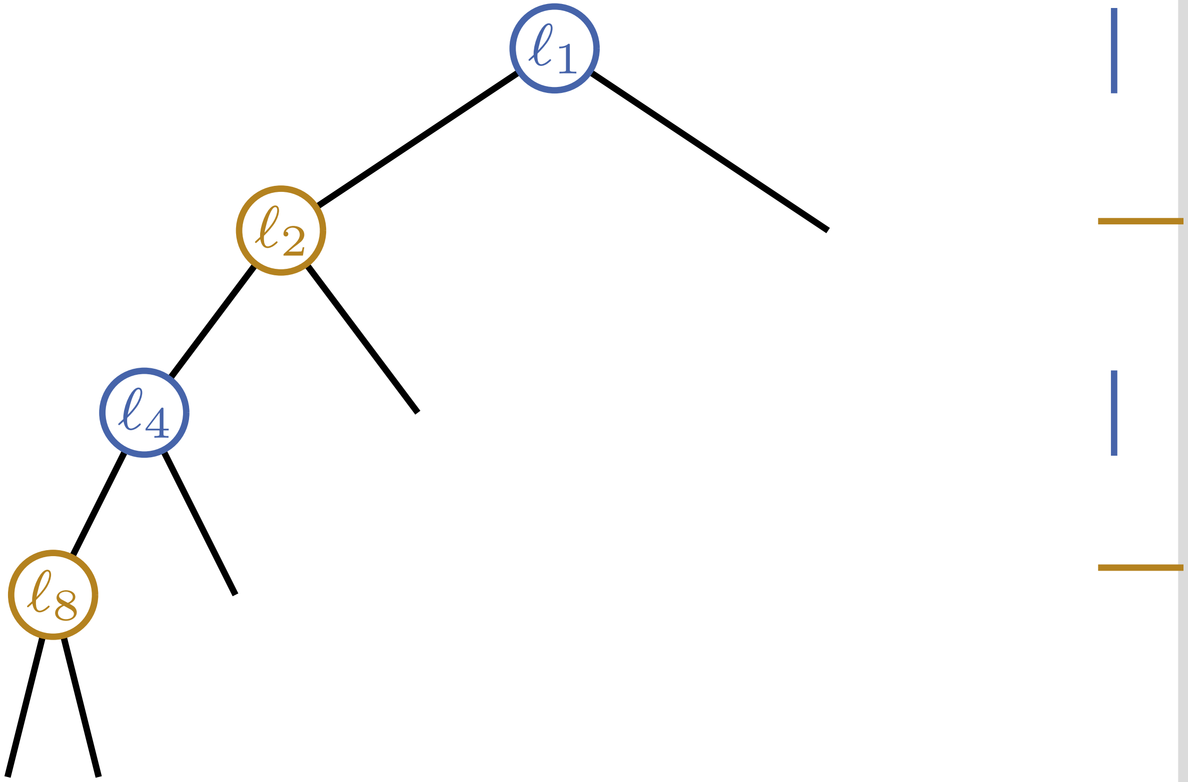

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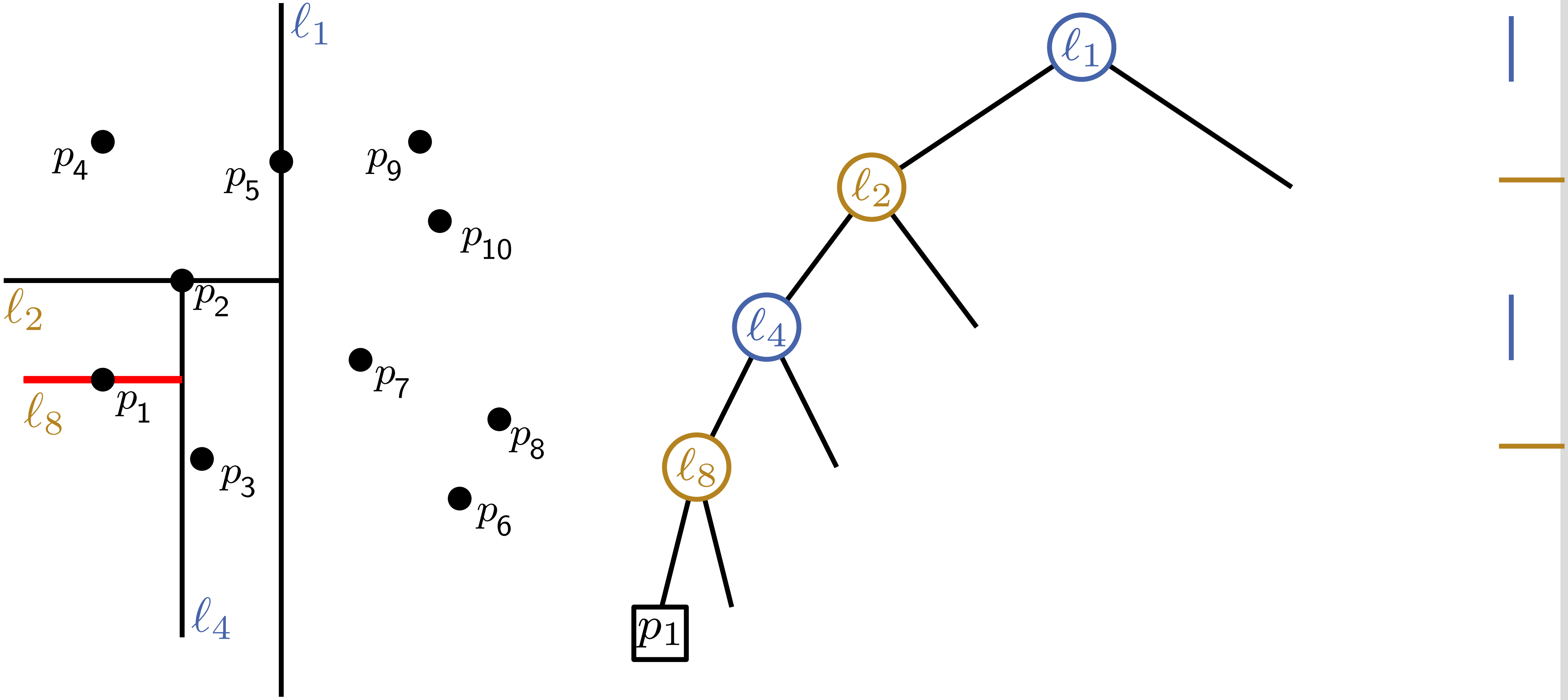

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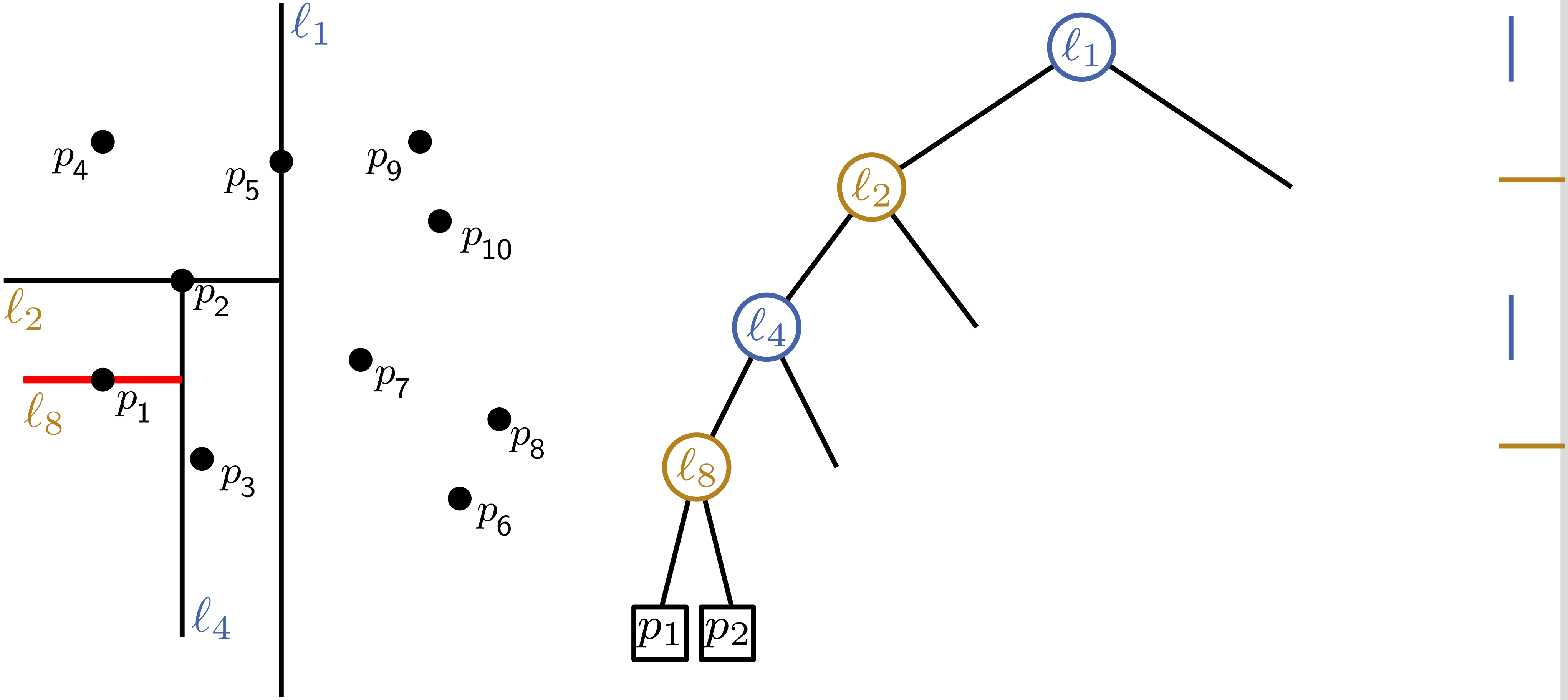

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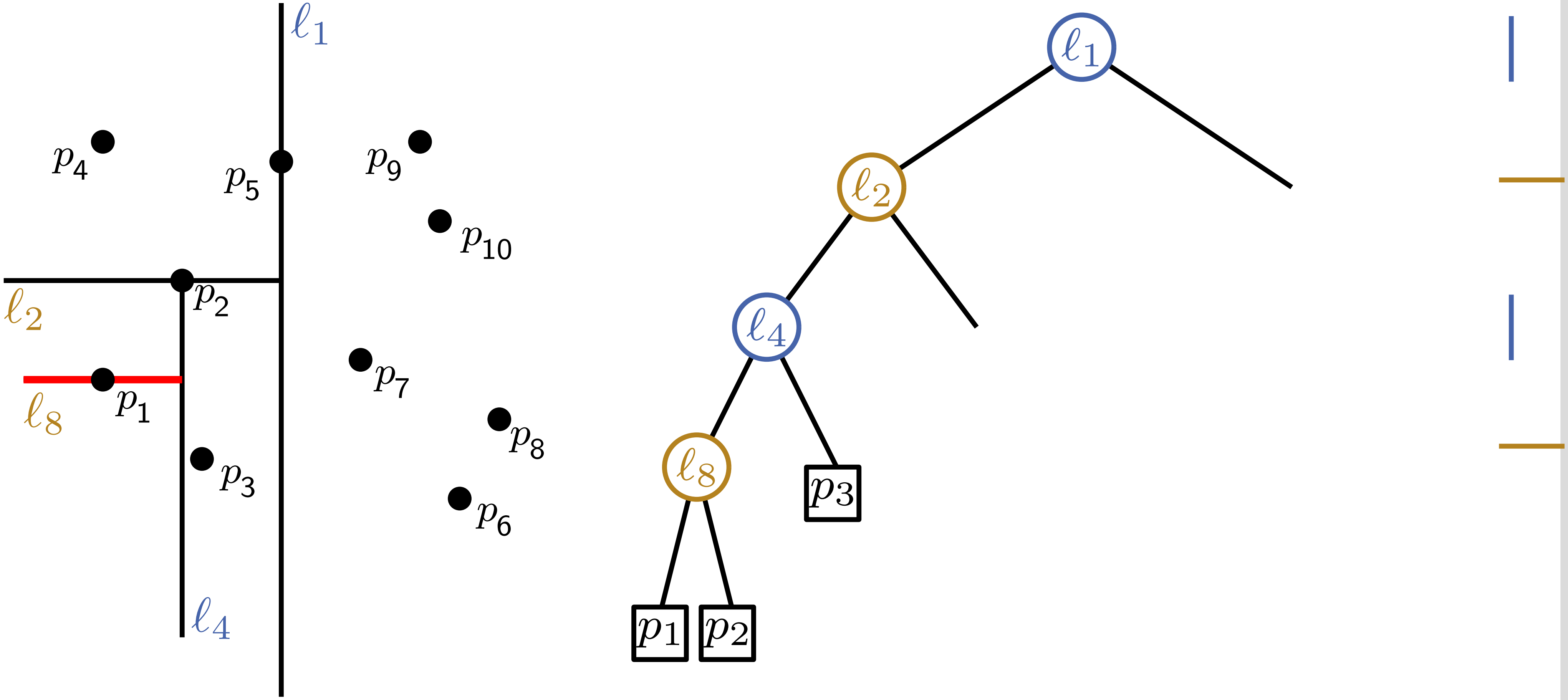

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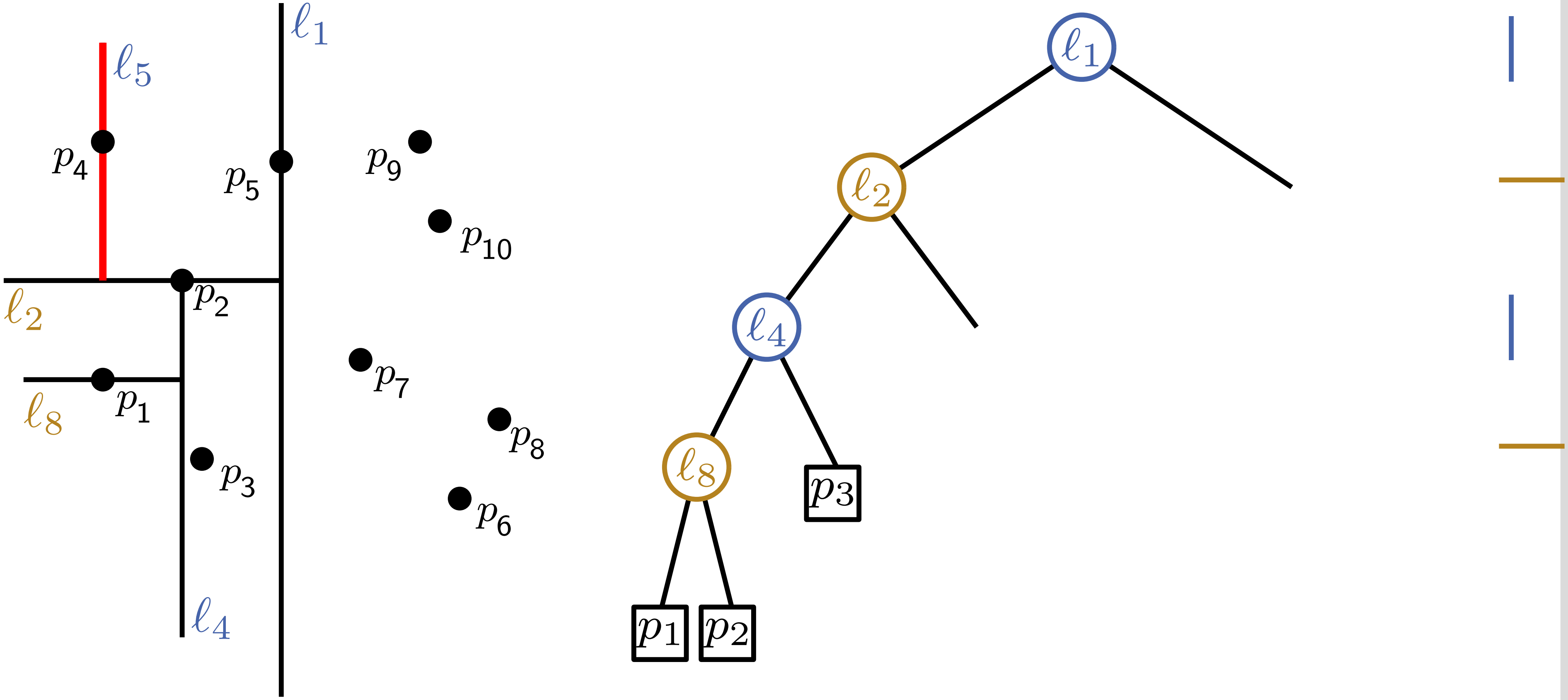

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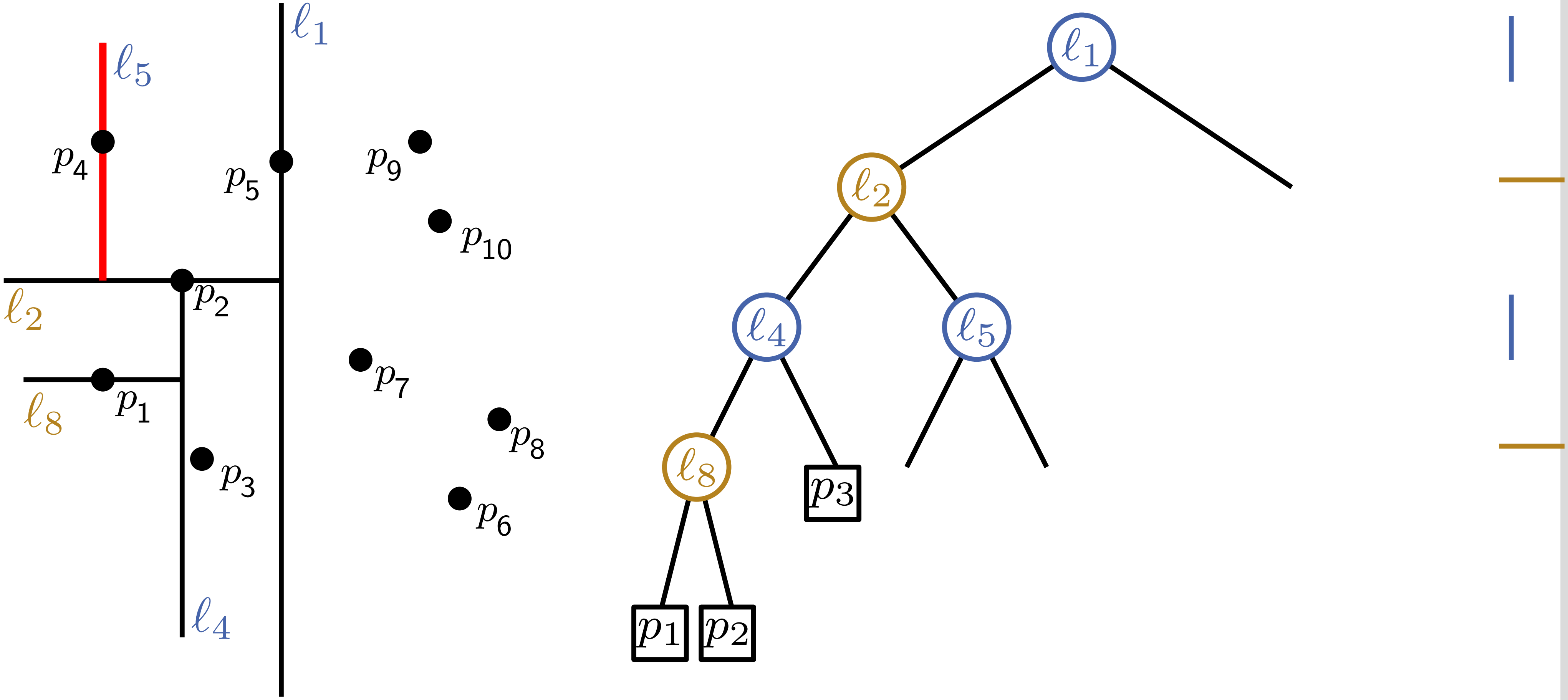

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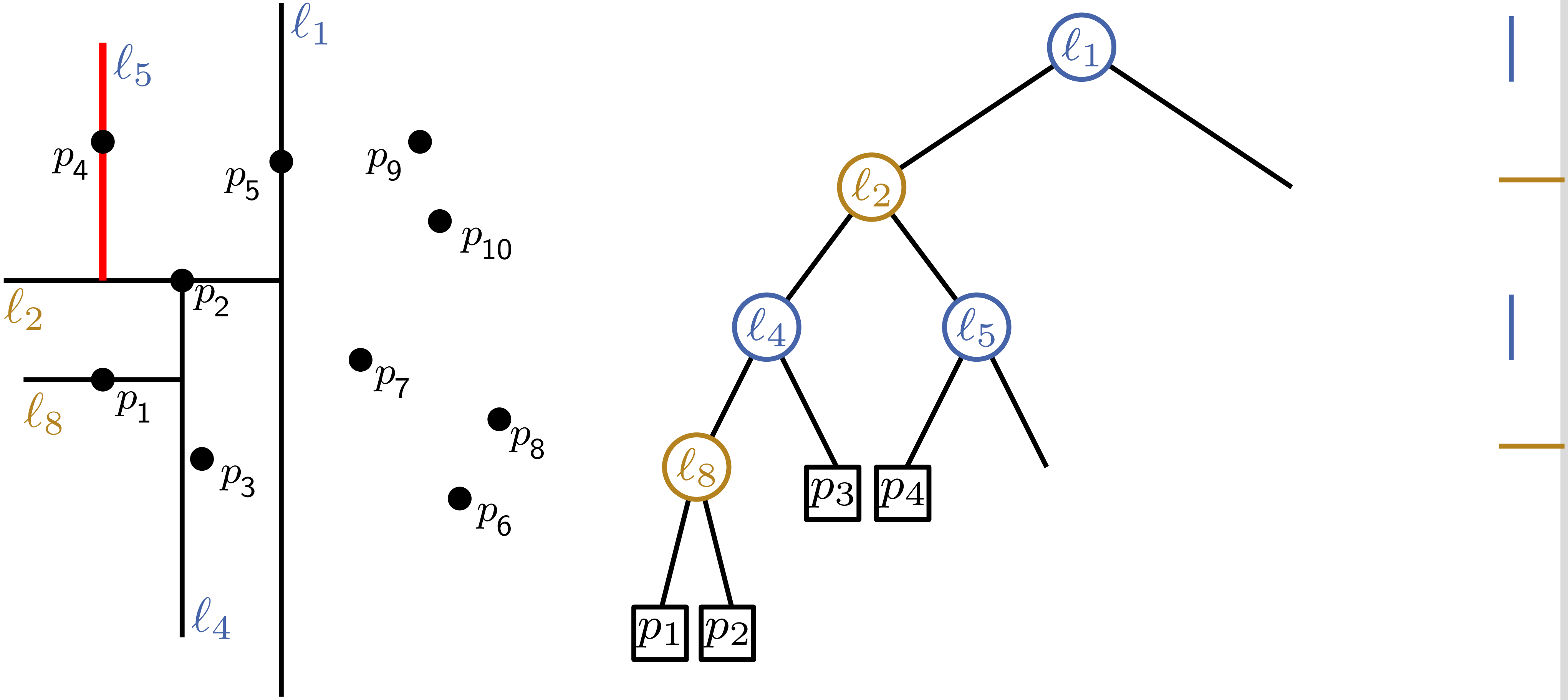

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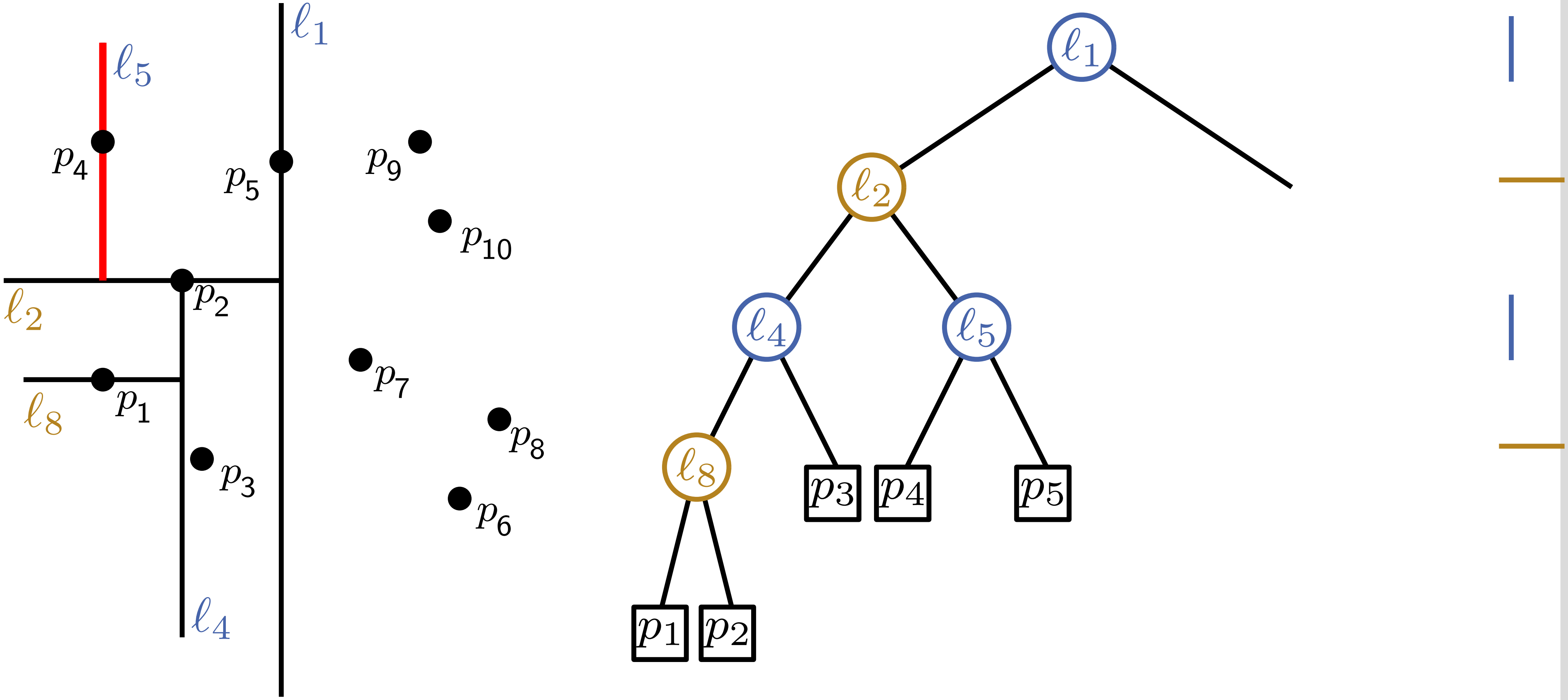

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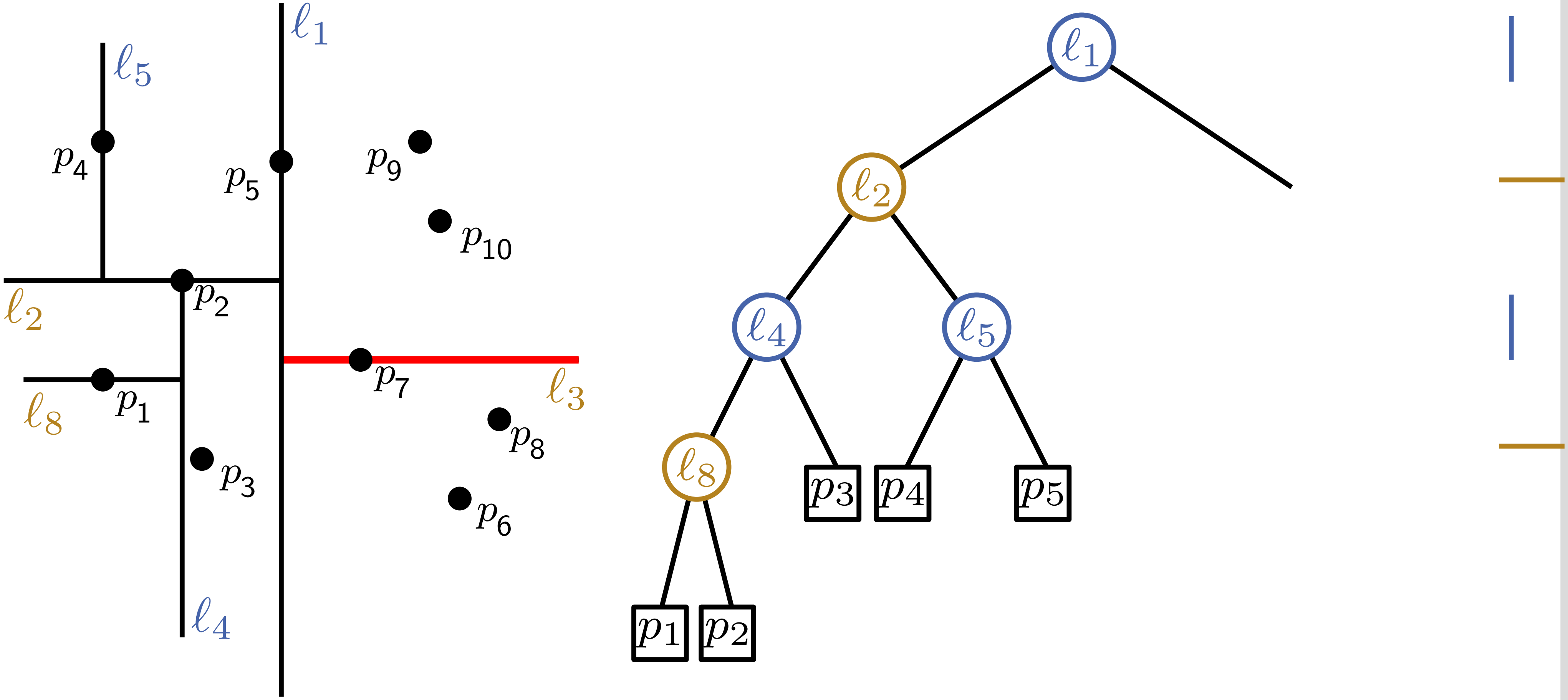

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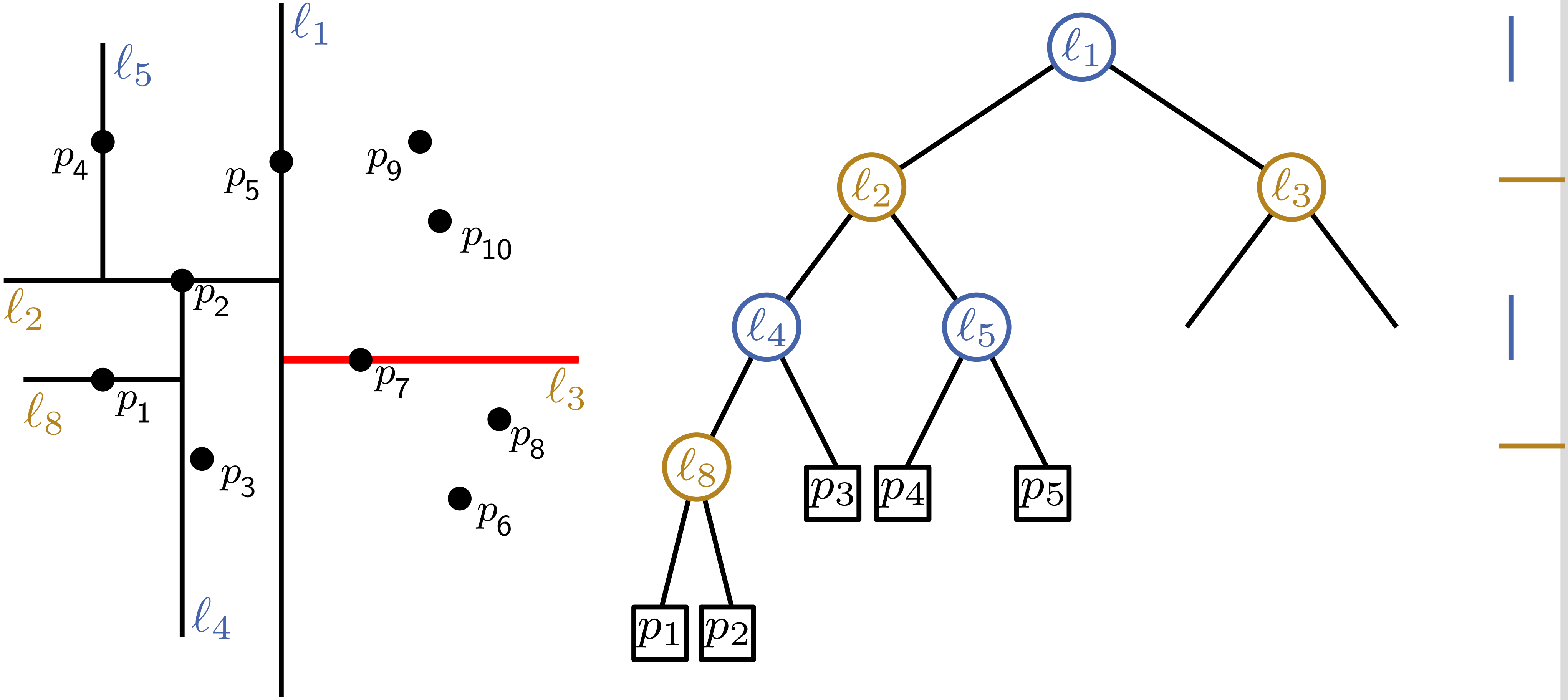

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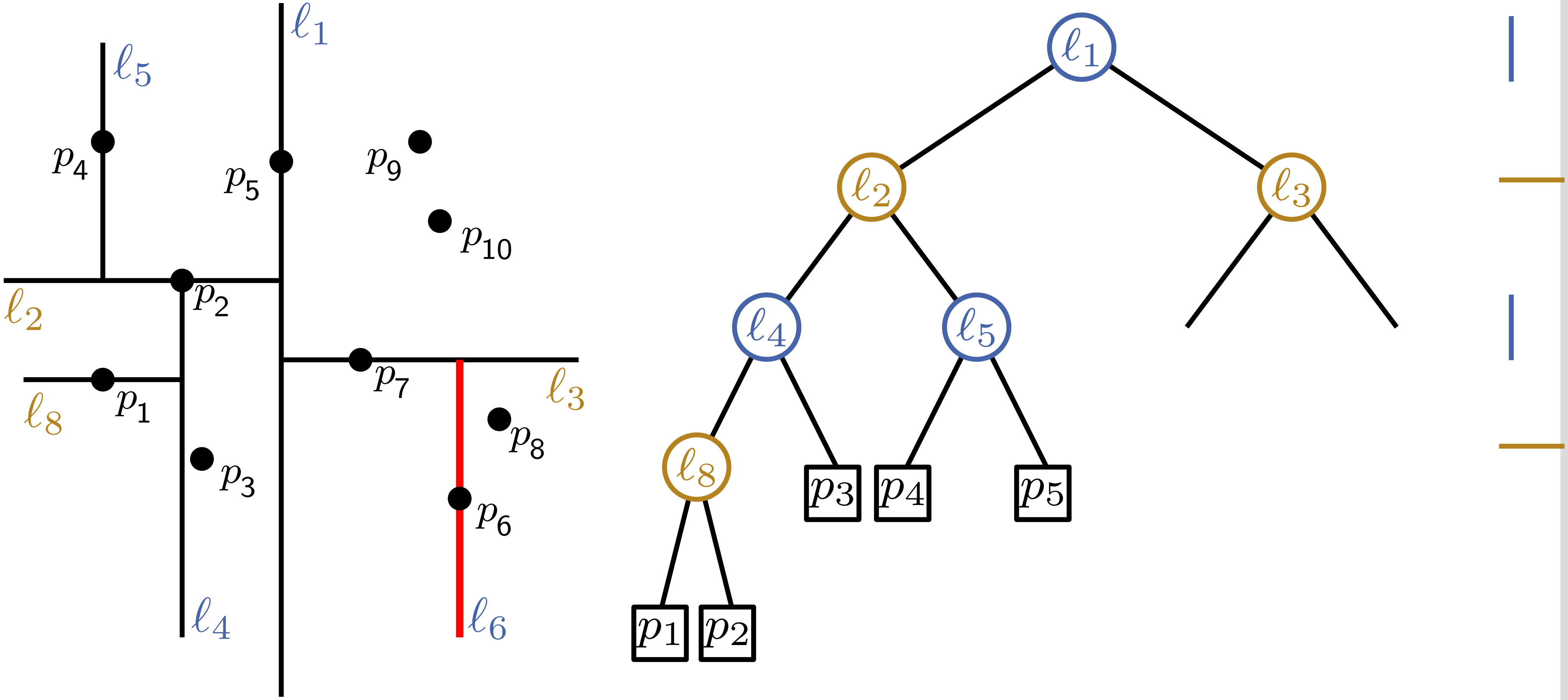

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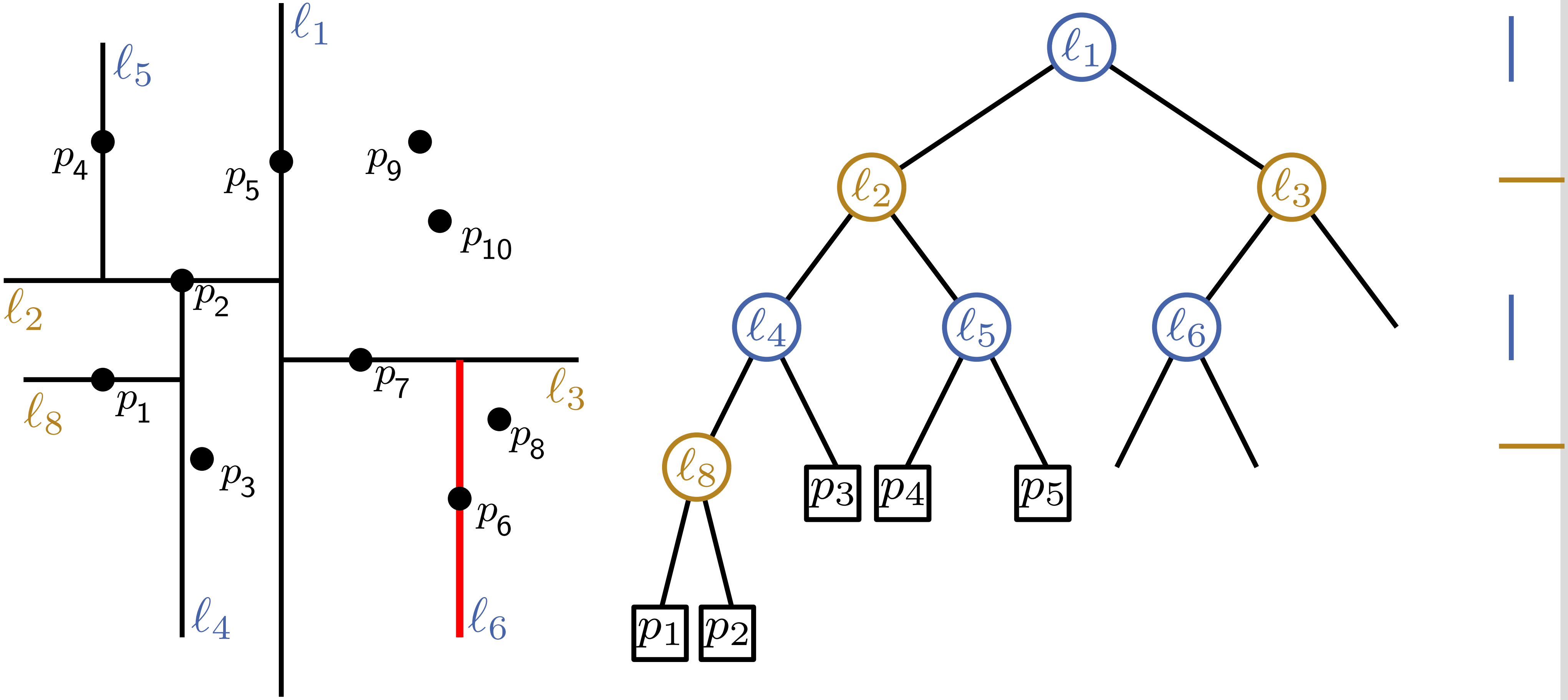

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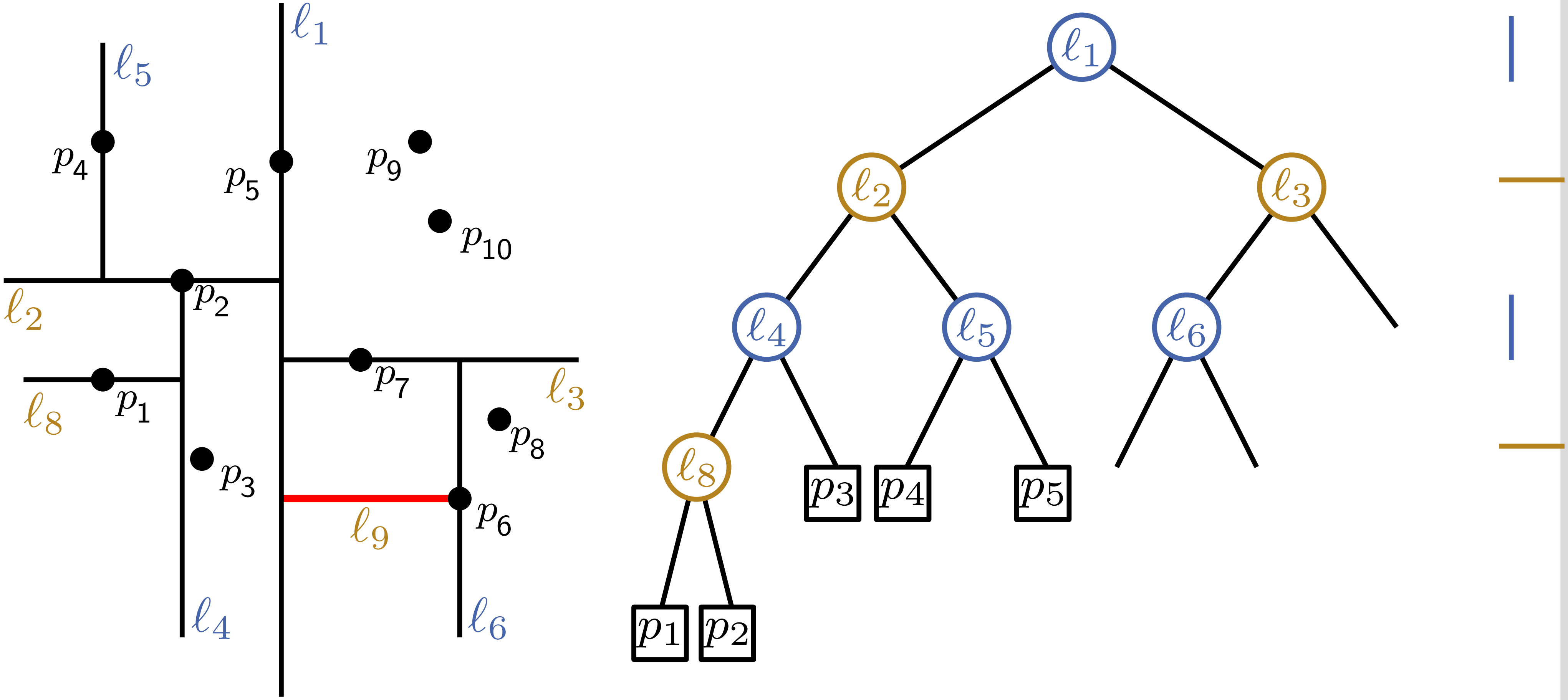

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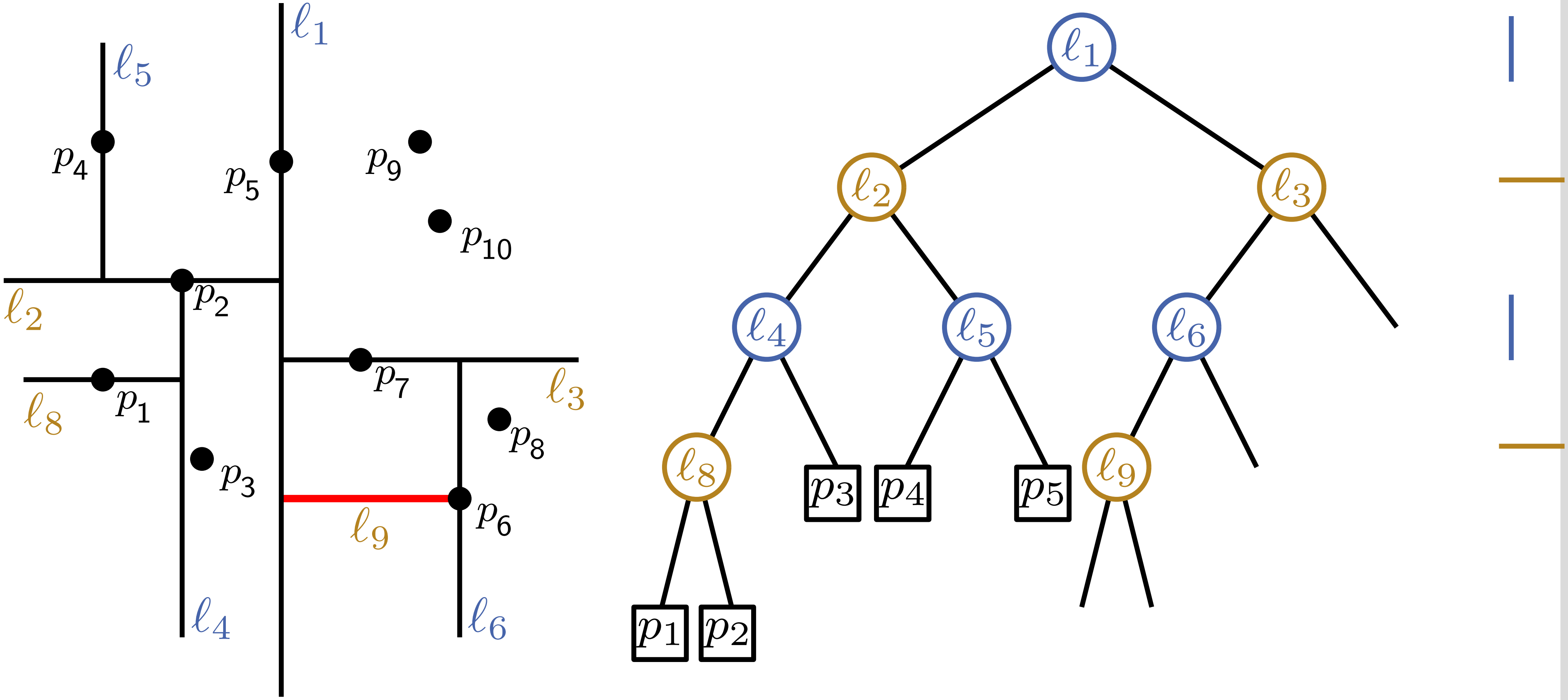

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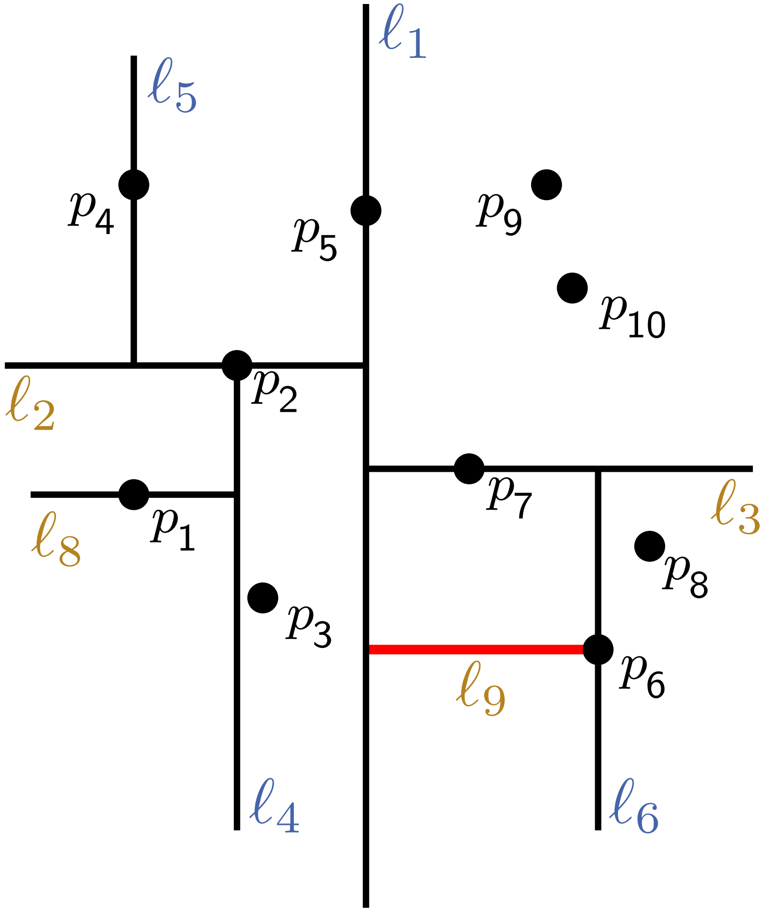

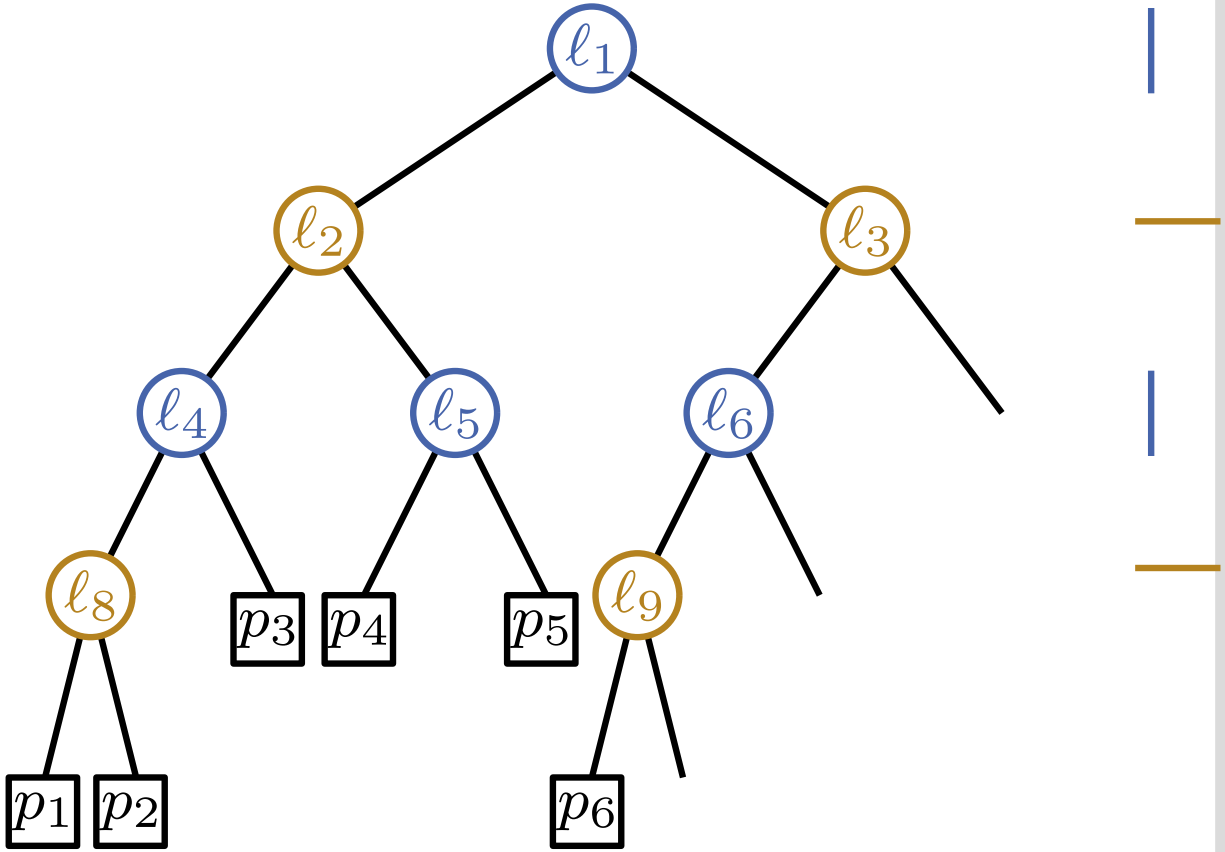

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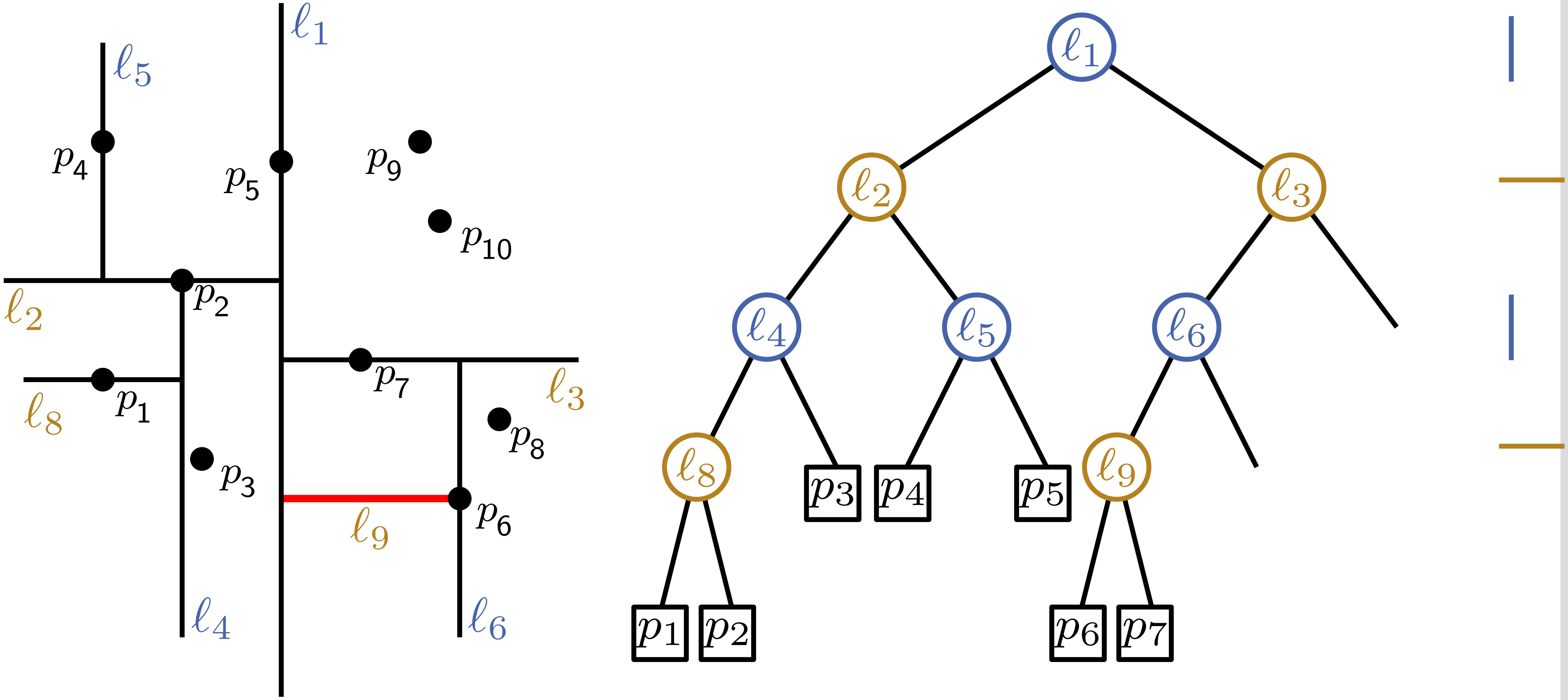

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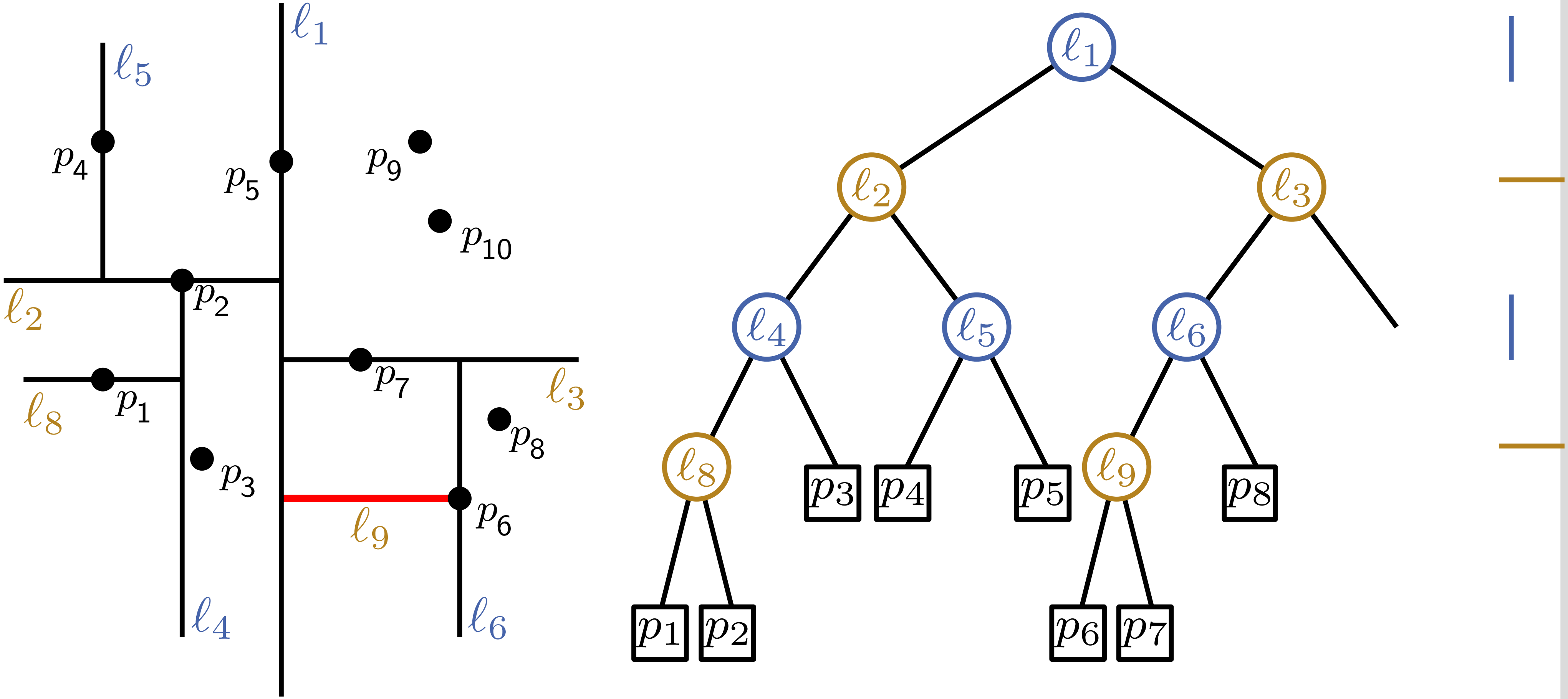

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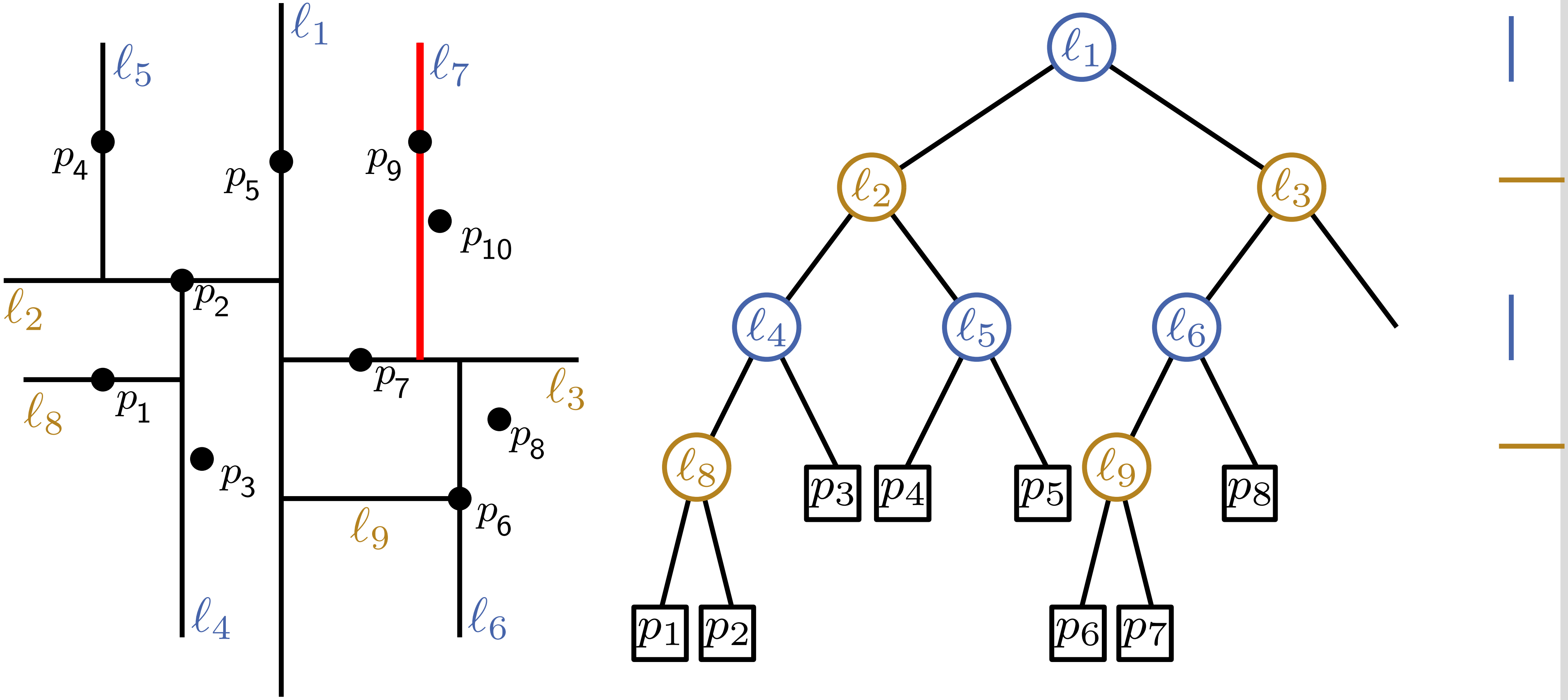

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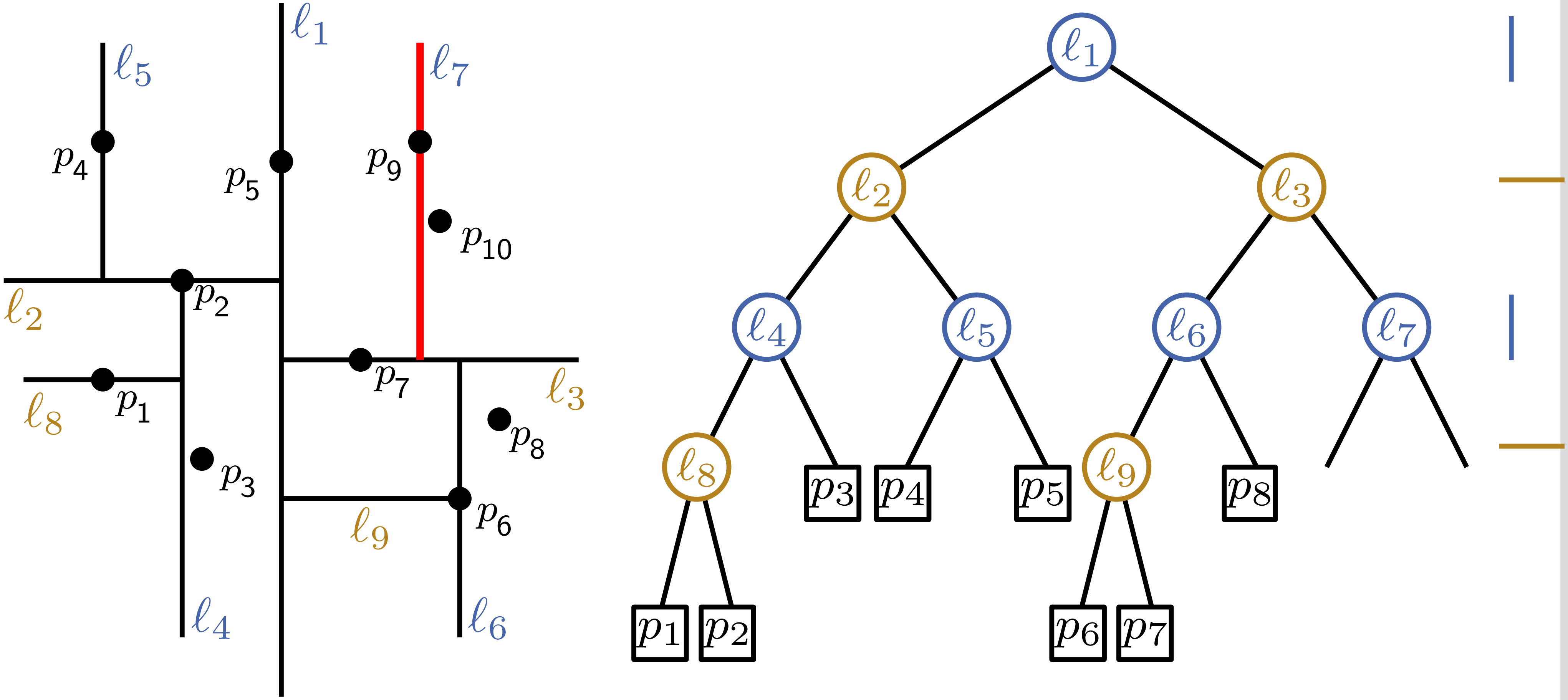

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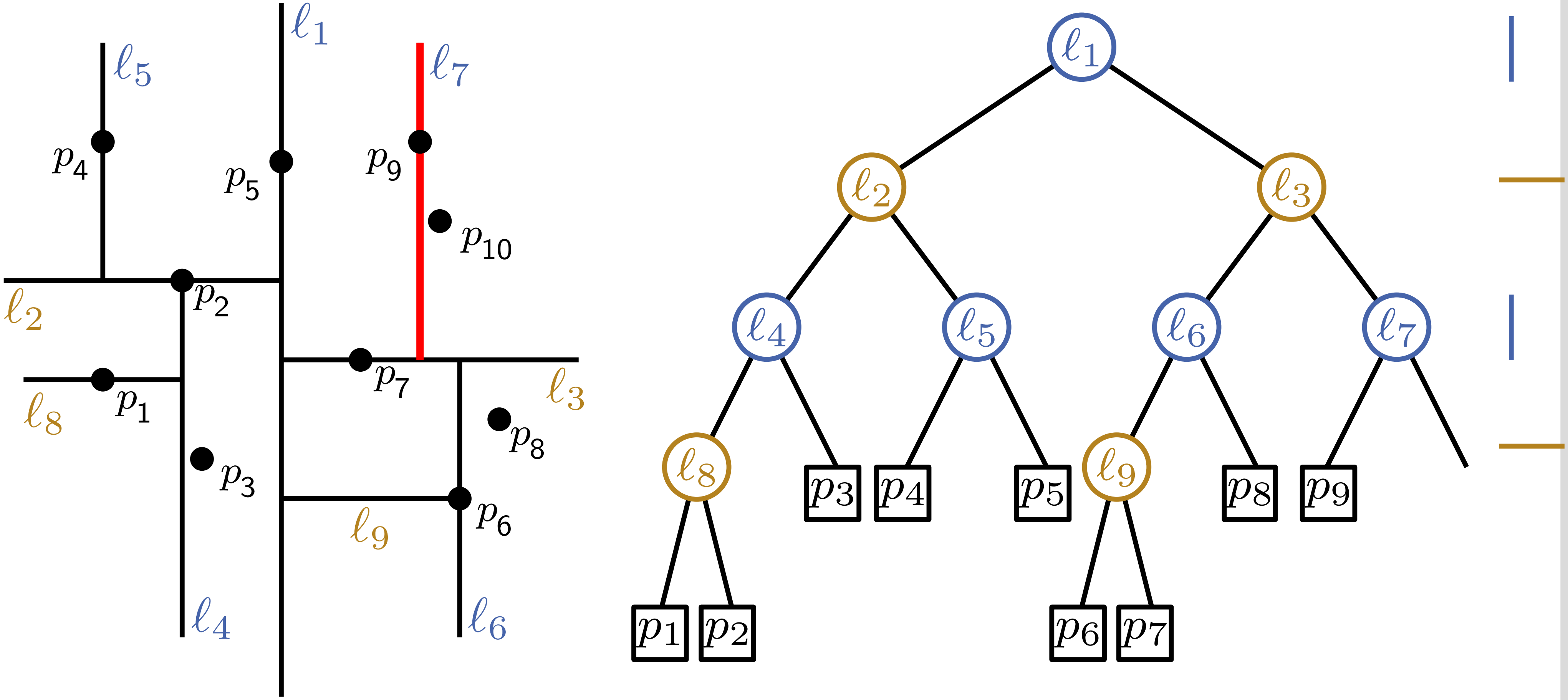

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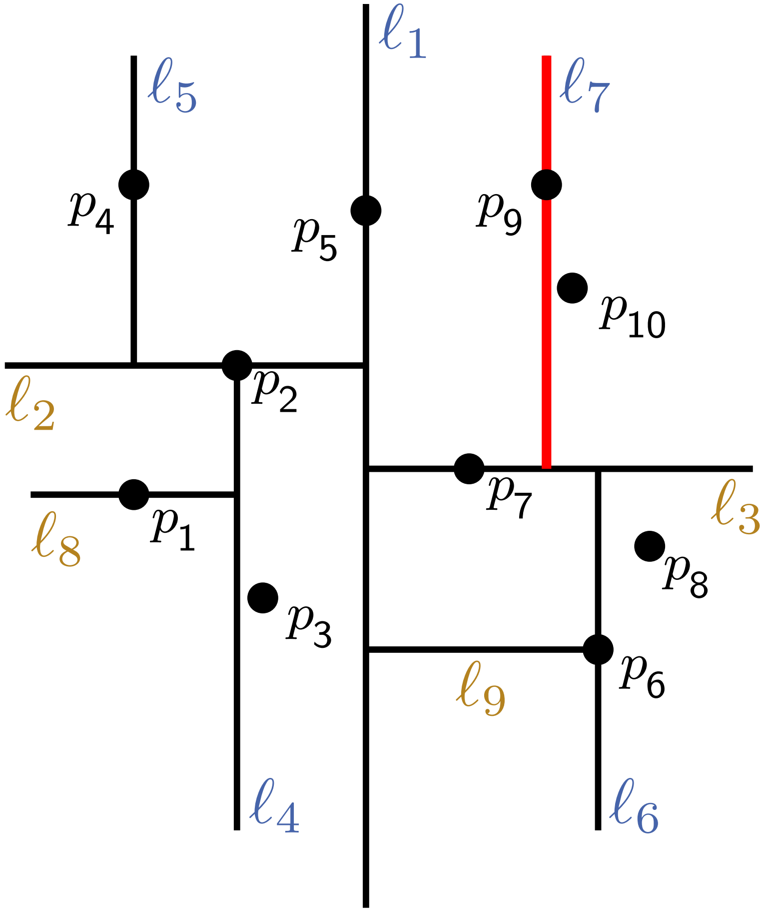
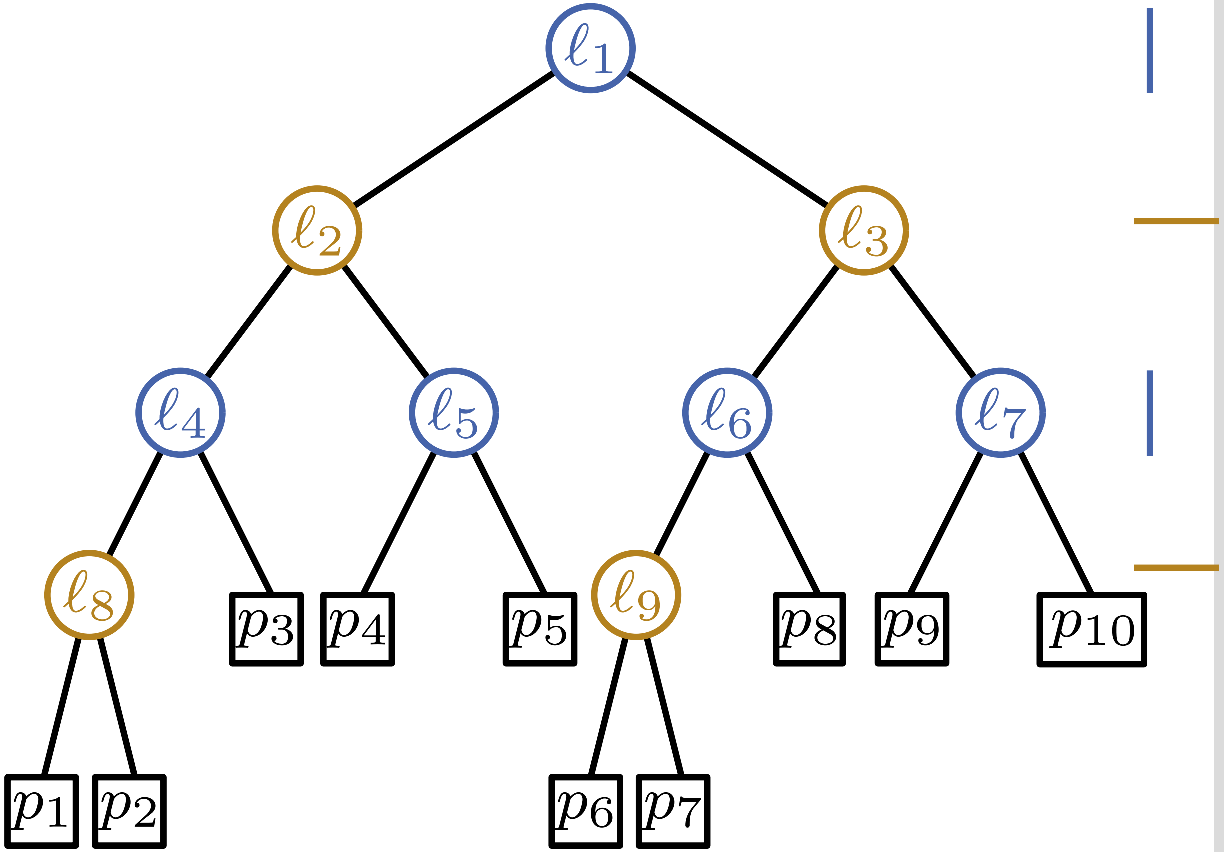
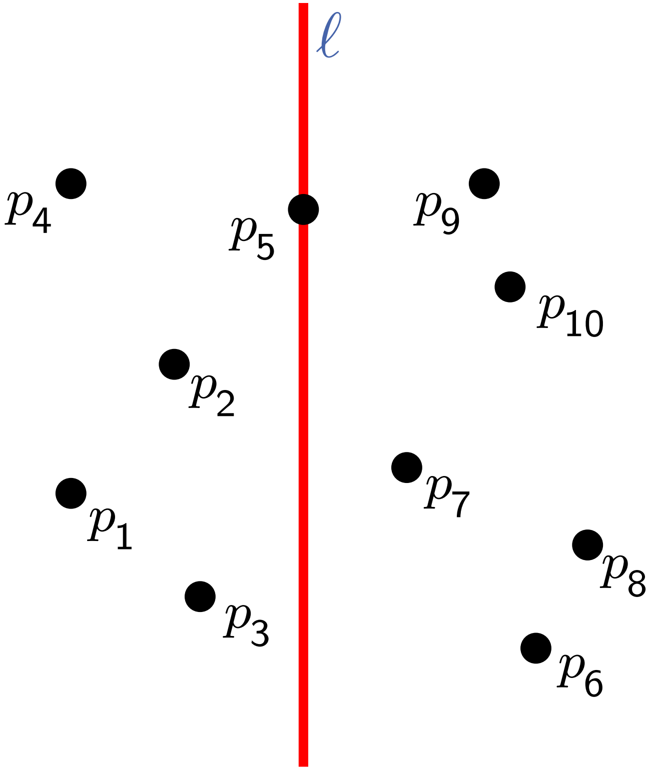




kd-Trees: Construction

BuildKdTree$(P, depth)$



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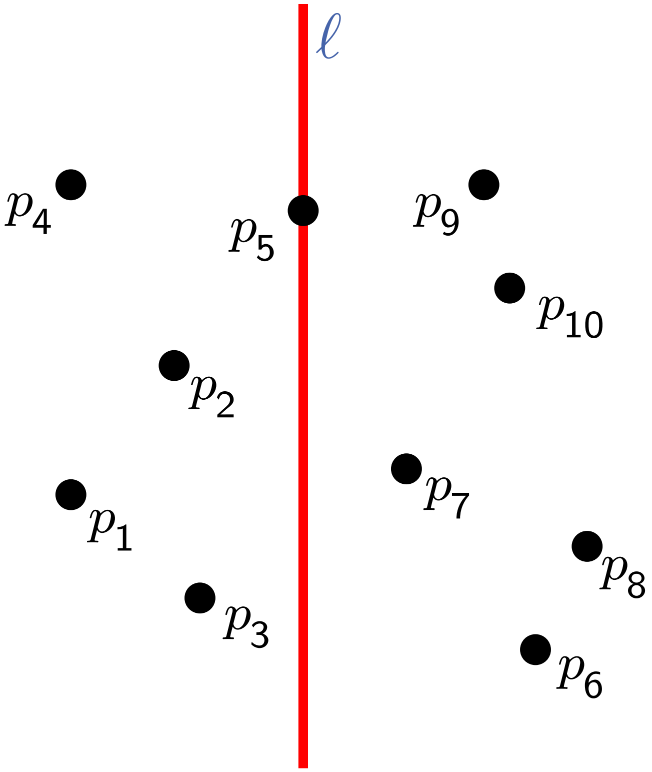

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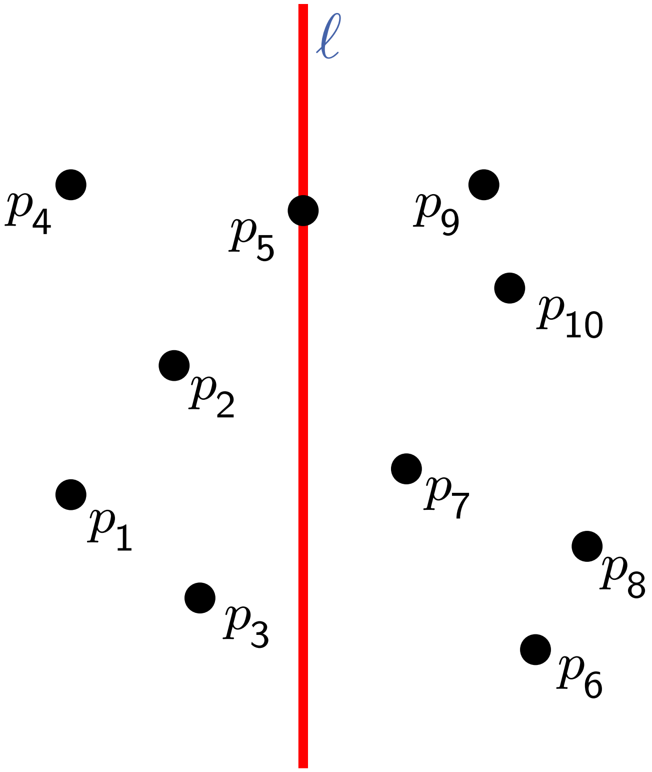
BuildKdTree($P$, $depth$)
  if $|P| = 1$ then
      return leaf with a point in $P$
  else
```

kd-Trees: Construction



```
BuildKdTree($P$, $depth$)
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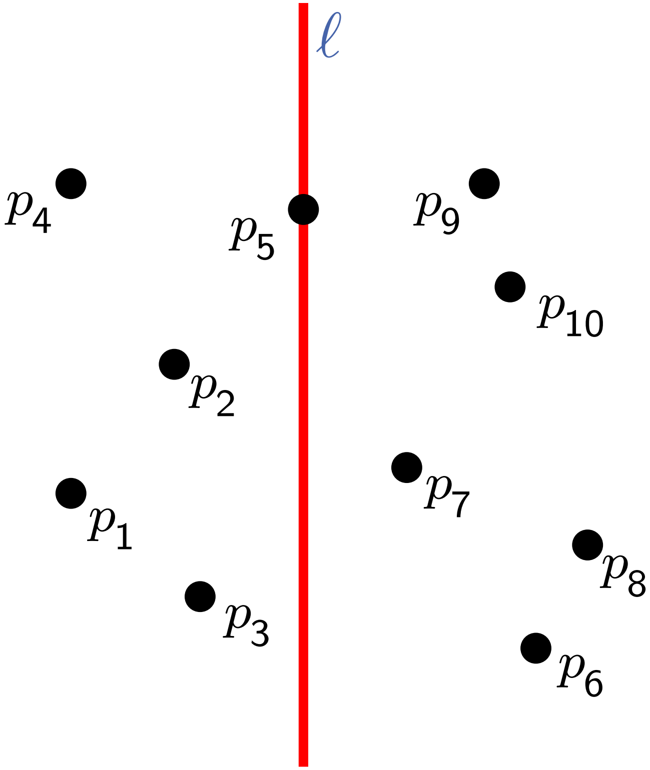
  else
      if $depth$ even then

      else

```

kd-Trees: Construction



```
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  if $|P| = 1$ then
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  else
      if $depth$ even then
          divide $P$ vertically at
          $\ell : x = x_{\text{median}(P)}$ in
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      else
```

point $\lceil |P|/2 \rceil$

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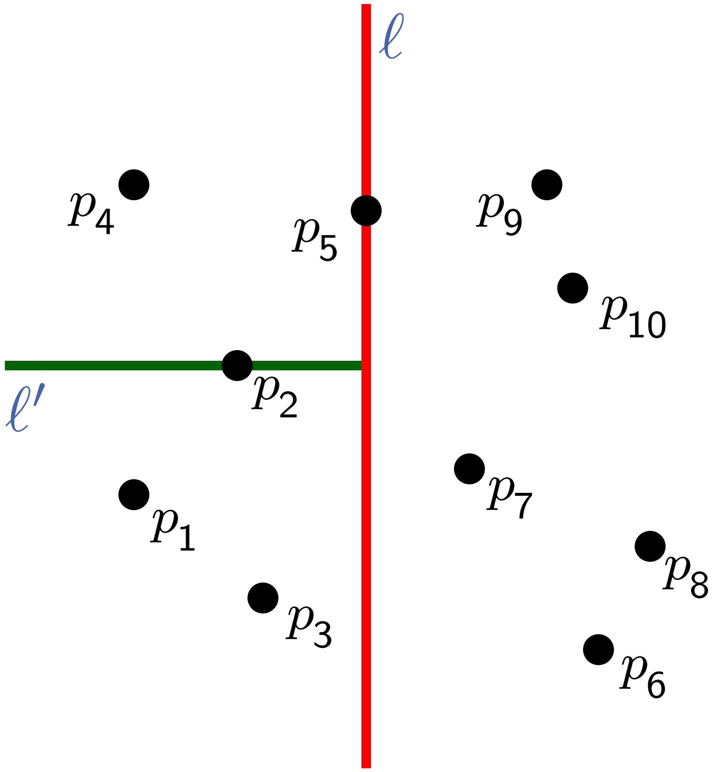

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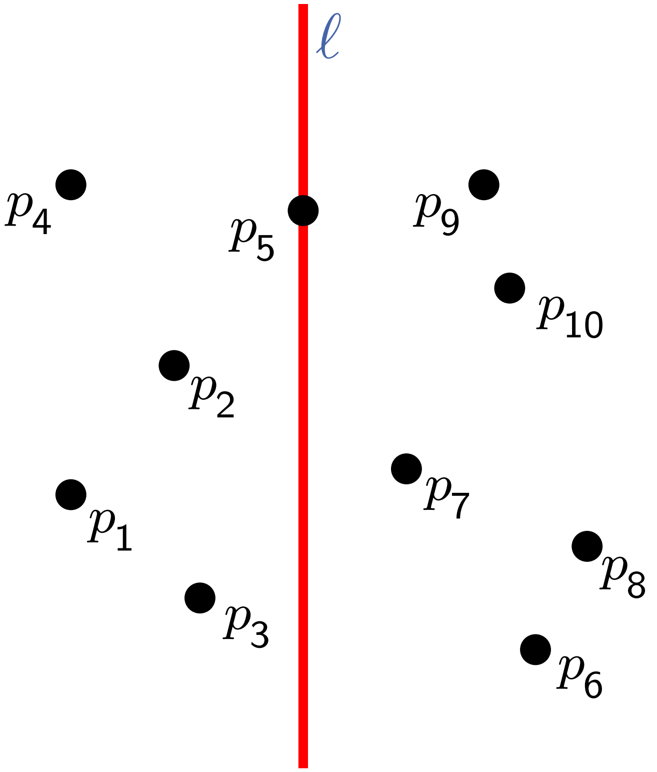

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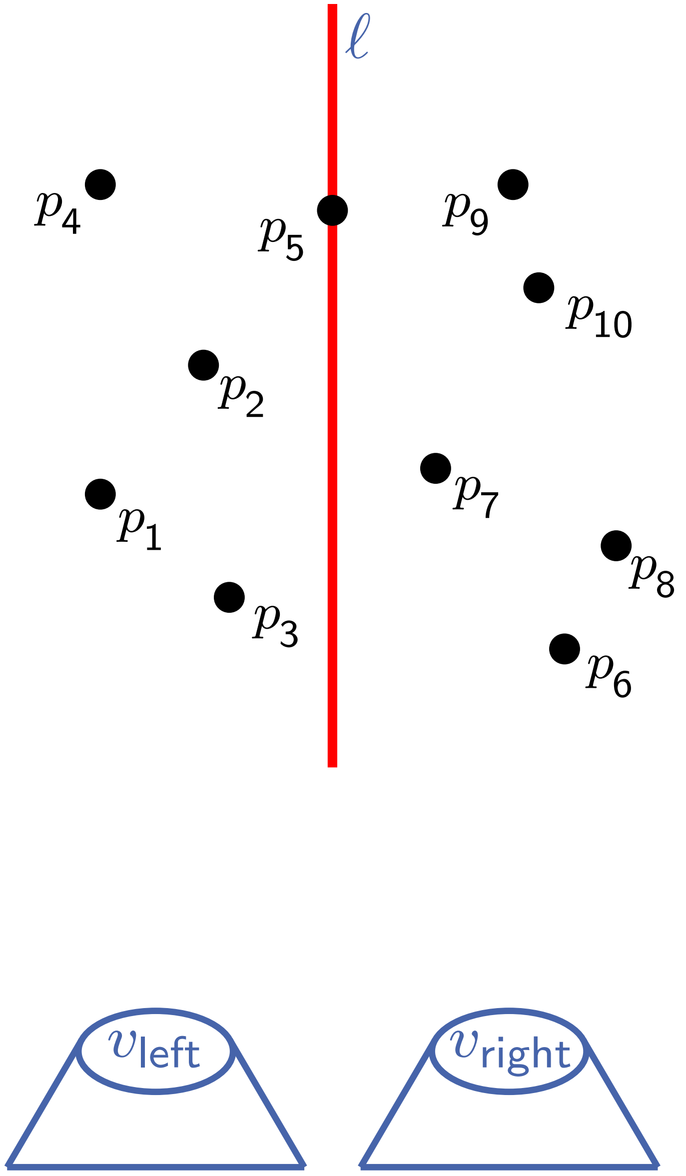

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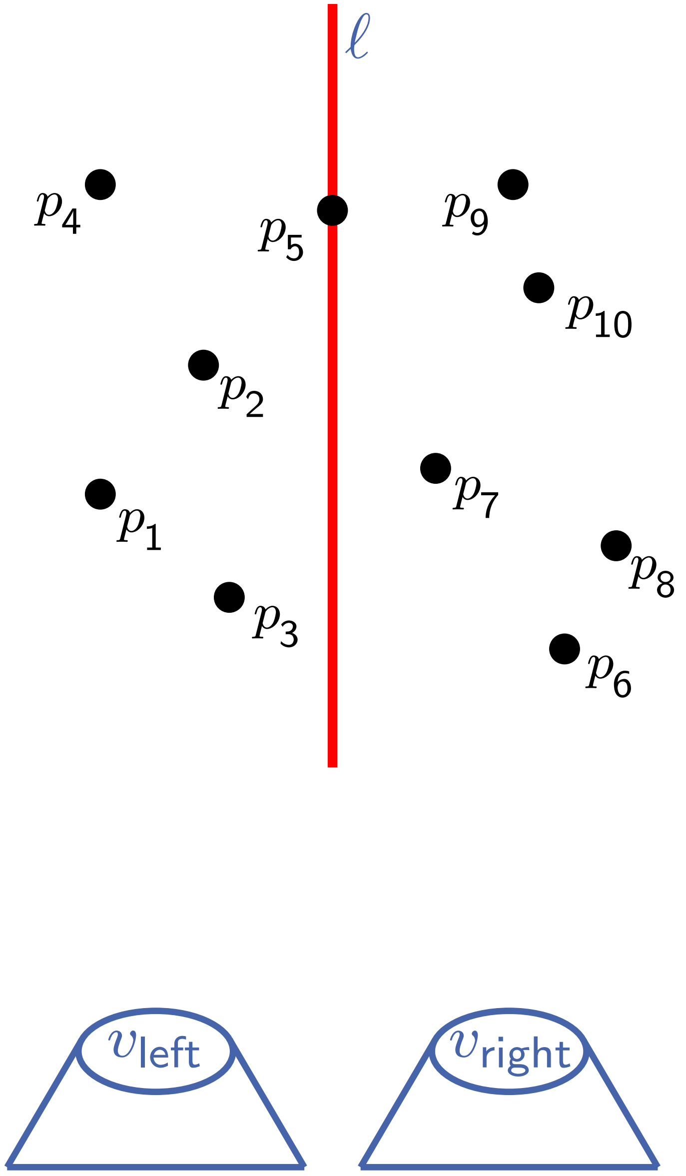

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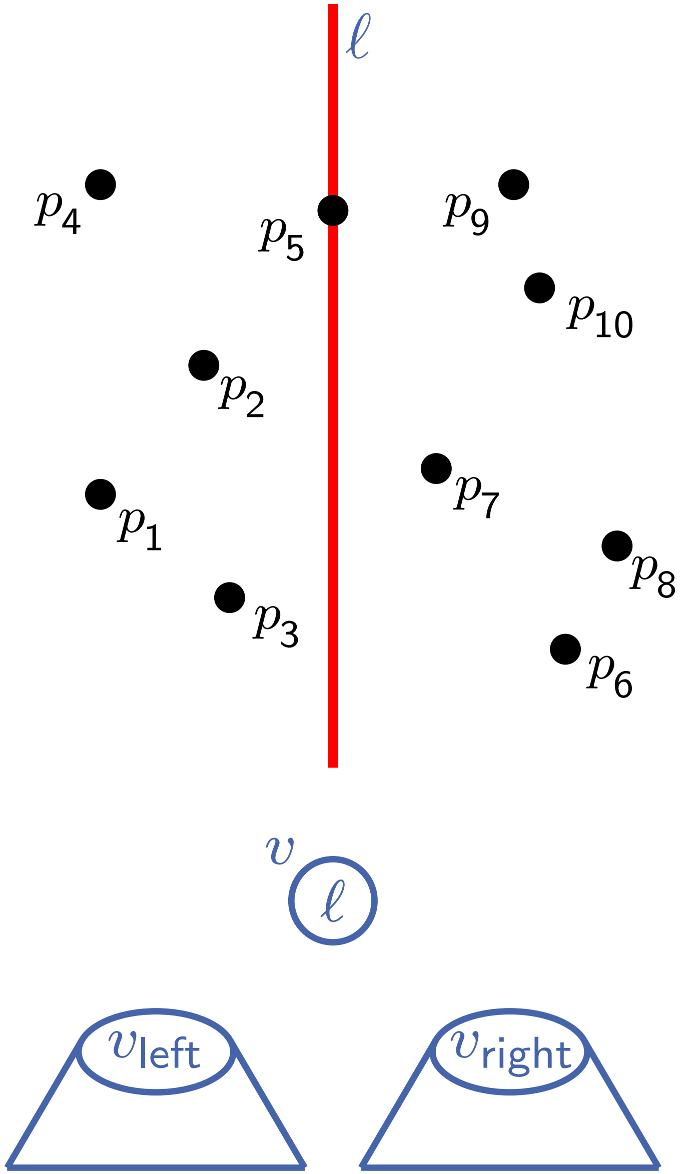

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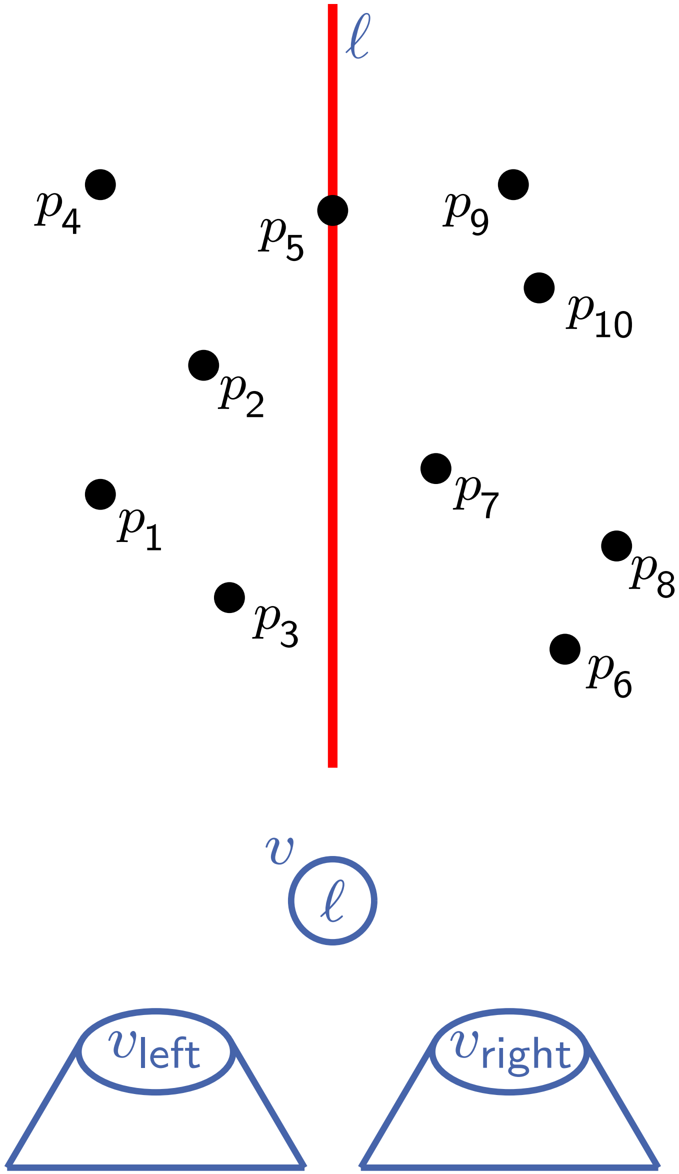

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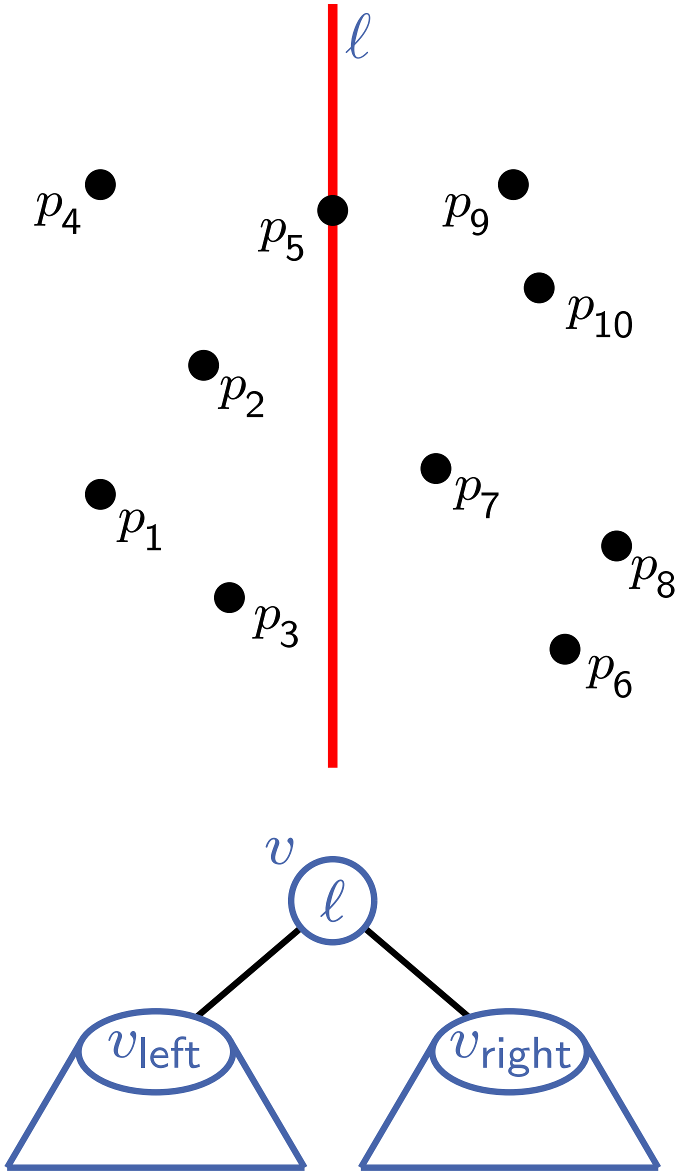

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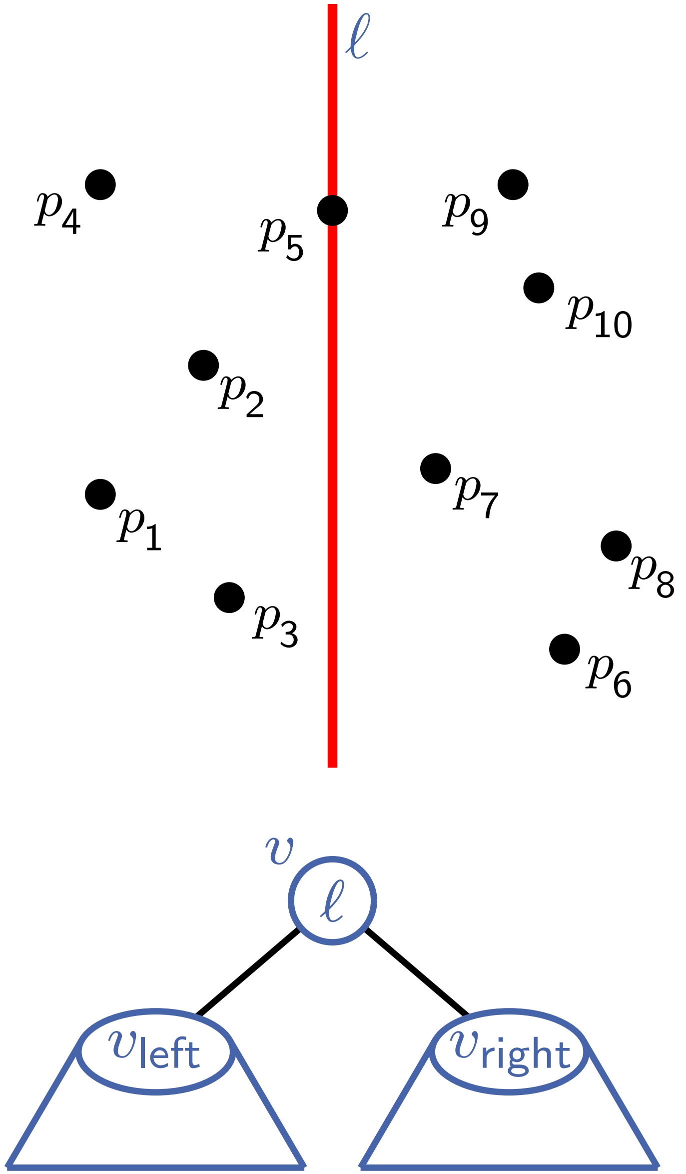

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        return $v$
```

Analysis of kd-Tree Construction

Lemma 1: A kd-tree for n points in $\mathbb{R}^2$ can be constructed in $O(n \log n)$ time, using $O(n)$ space.

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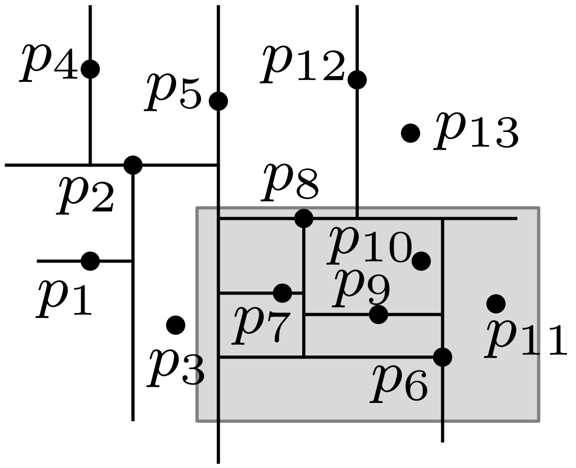
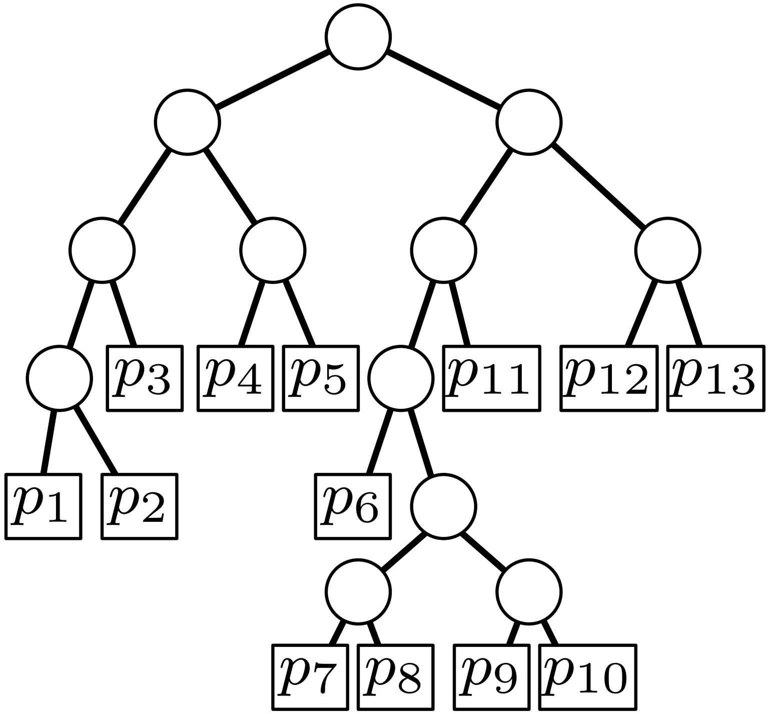
- Linear space, since we are using a binary tree with n leaves.

Range Queries in a kd-Tree





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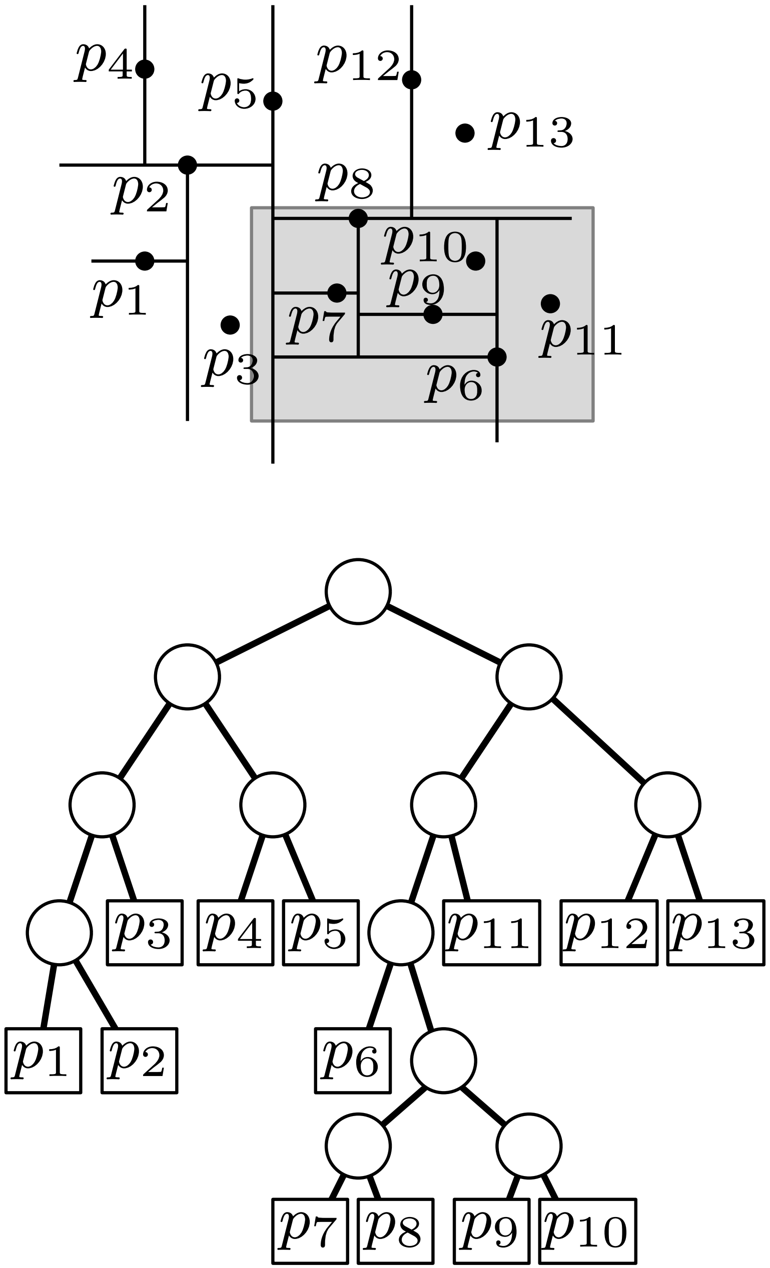

```
SearchKdTree($v, R$)
  if $v$ leaf then
    report point $p$ in $v$ when $p \in R$
  else
    if region(lc($v$)) $\subseteq R$ then
      ReportSubtree(lc($v$))
    else
      if region(lc($v$))$\cap R \neq \emptyset$ then
      SearchKdTree(lc($v$), $R$)
    if region(rc($v$)) $\subseteq R$ then
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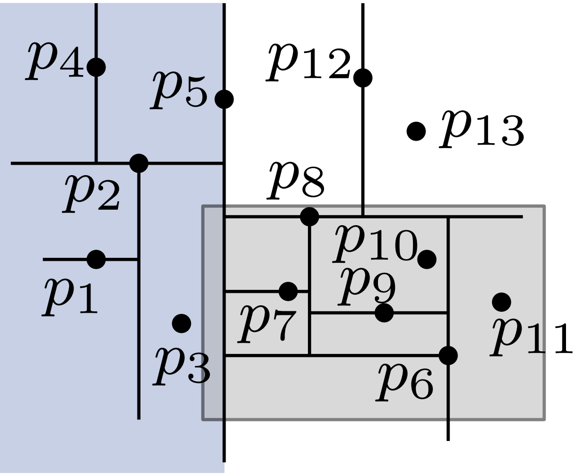
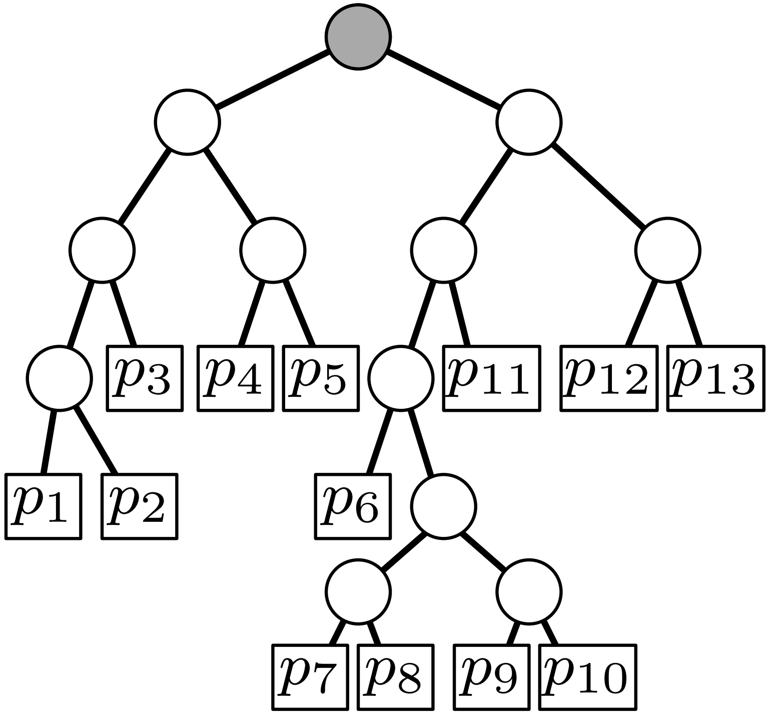
```

Range Queries in a kd-Tree





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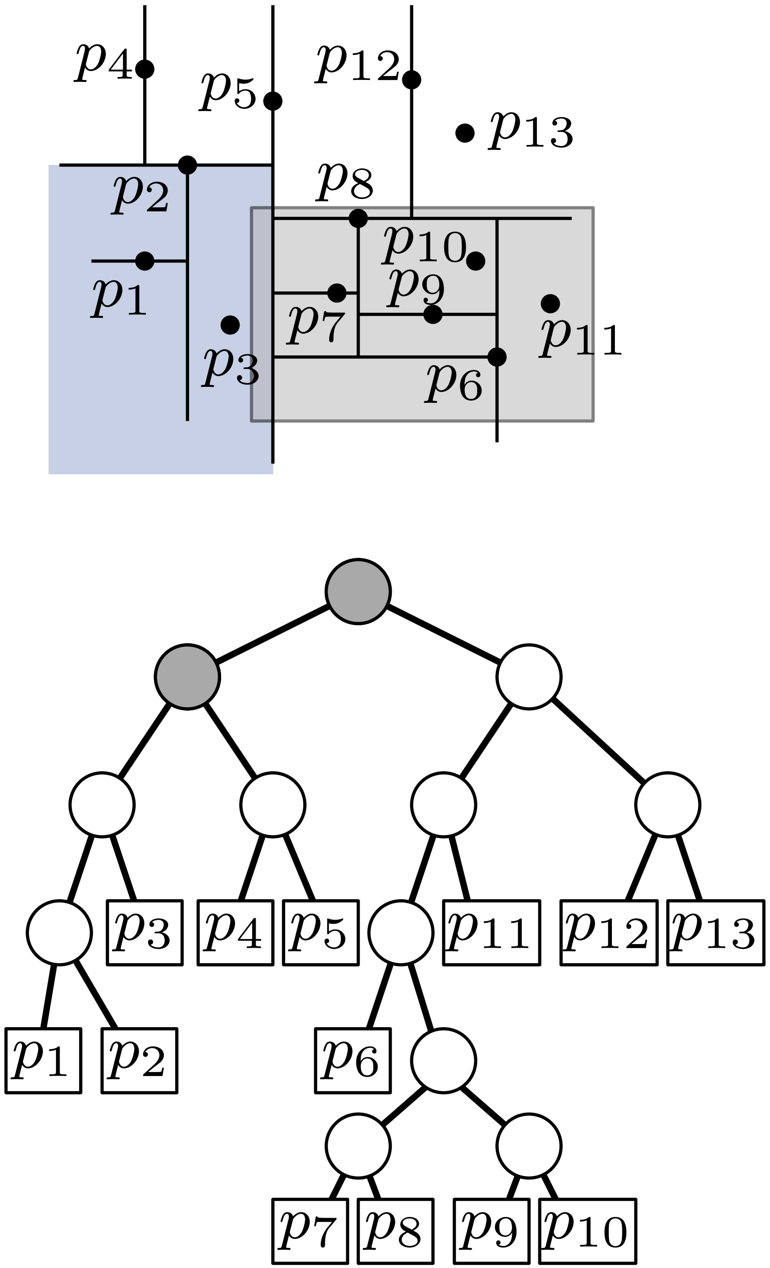

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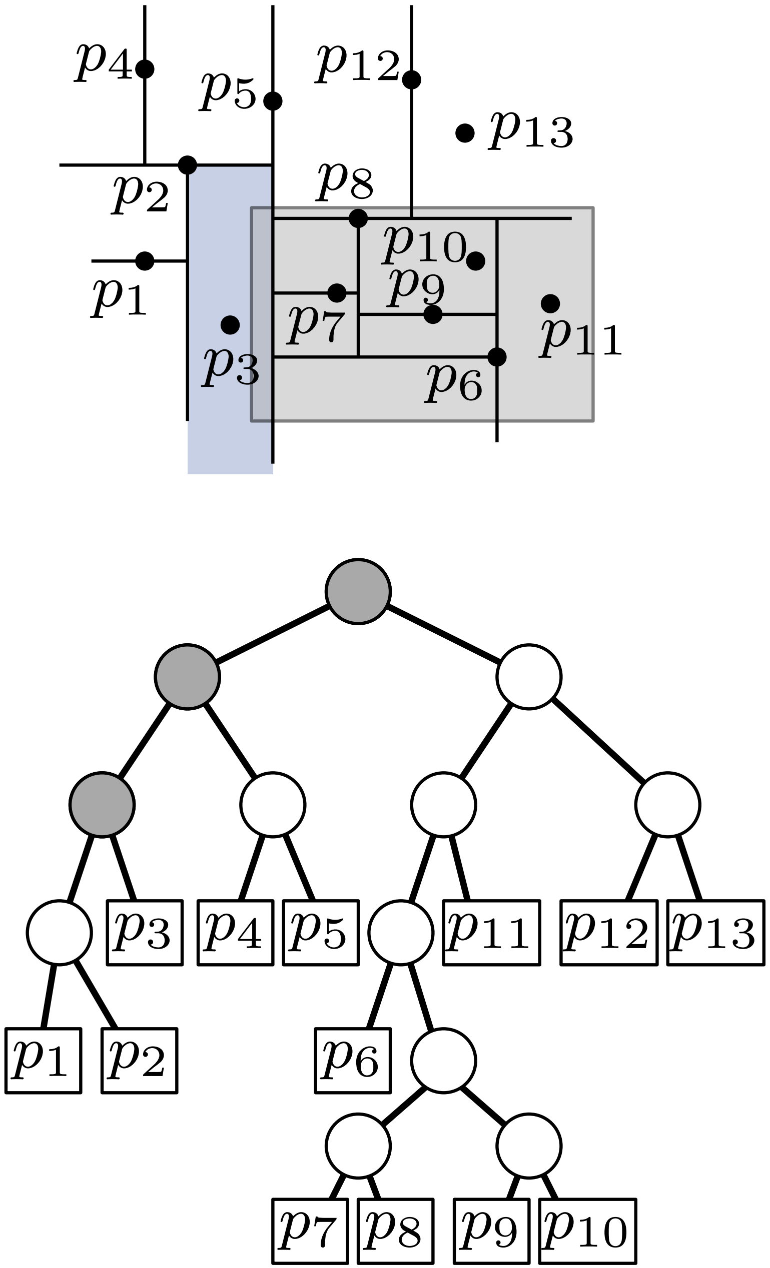
Range Queries in a *kd*-Tree



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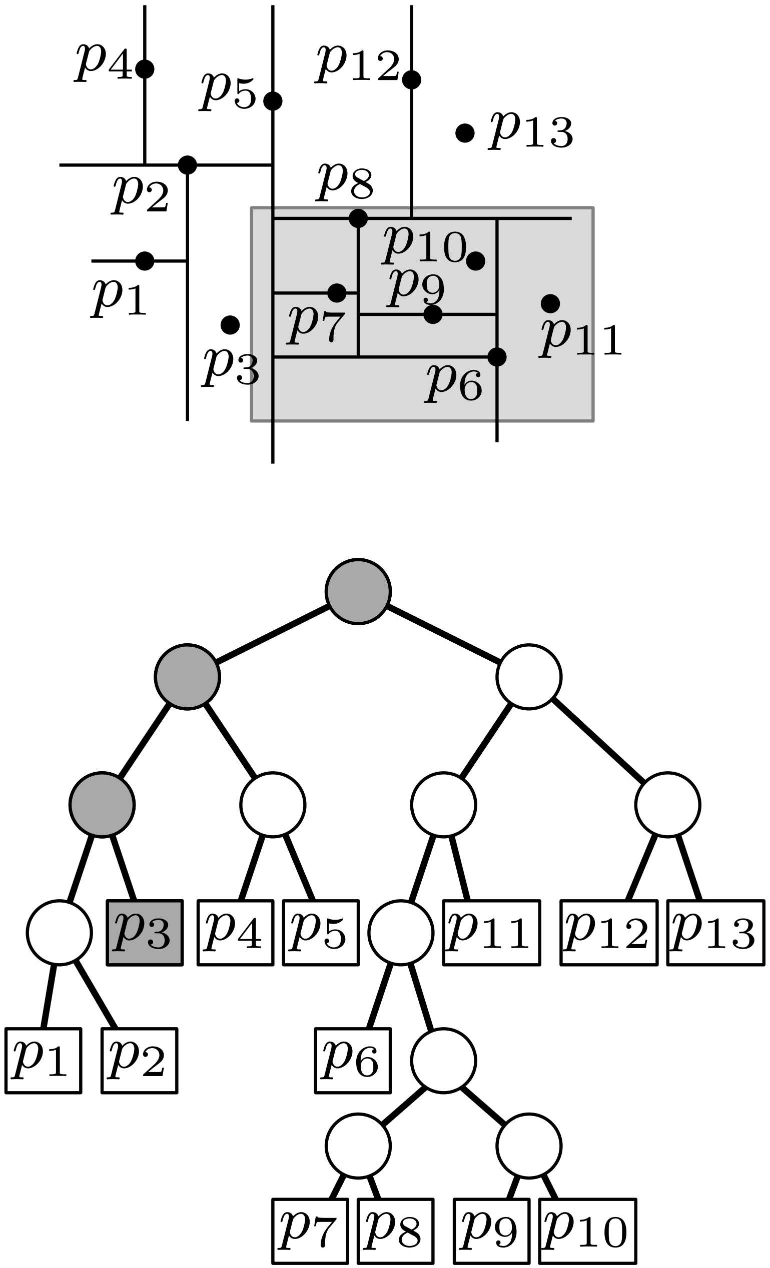

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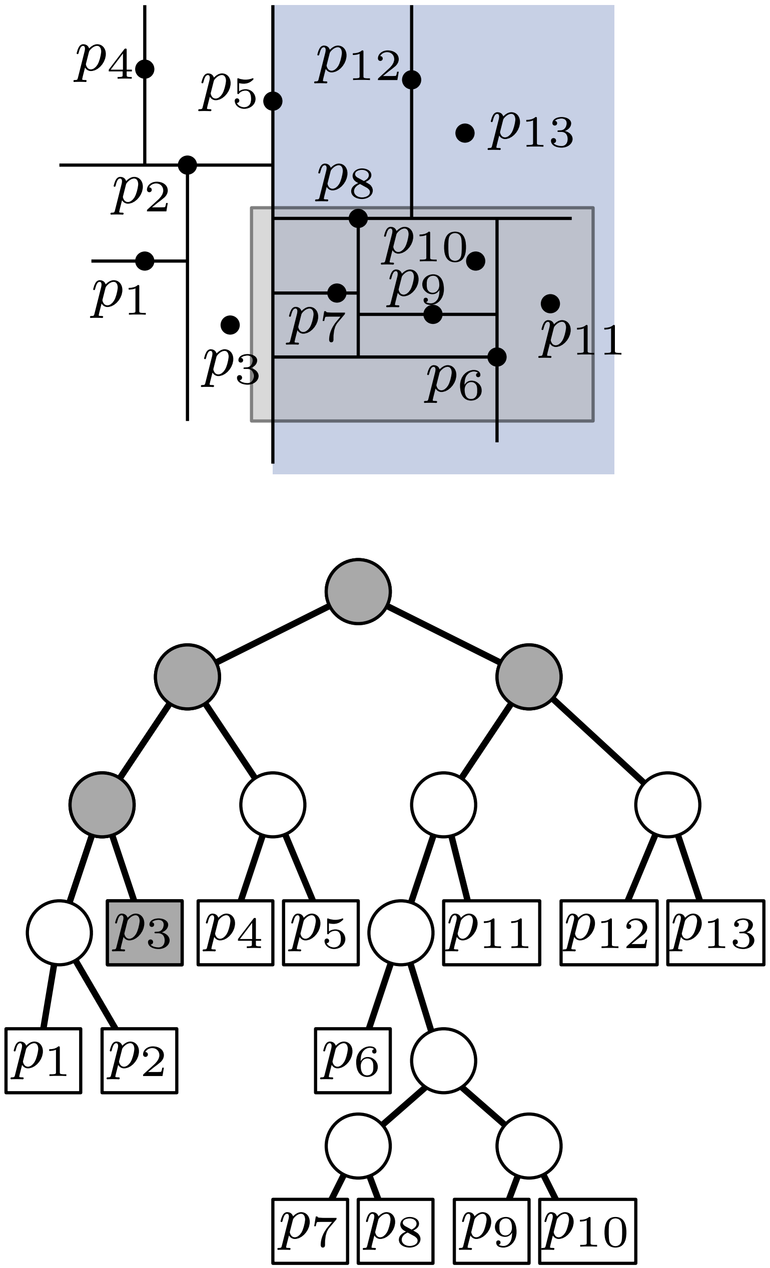

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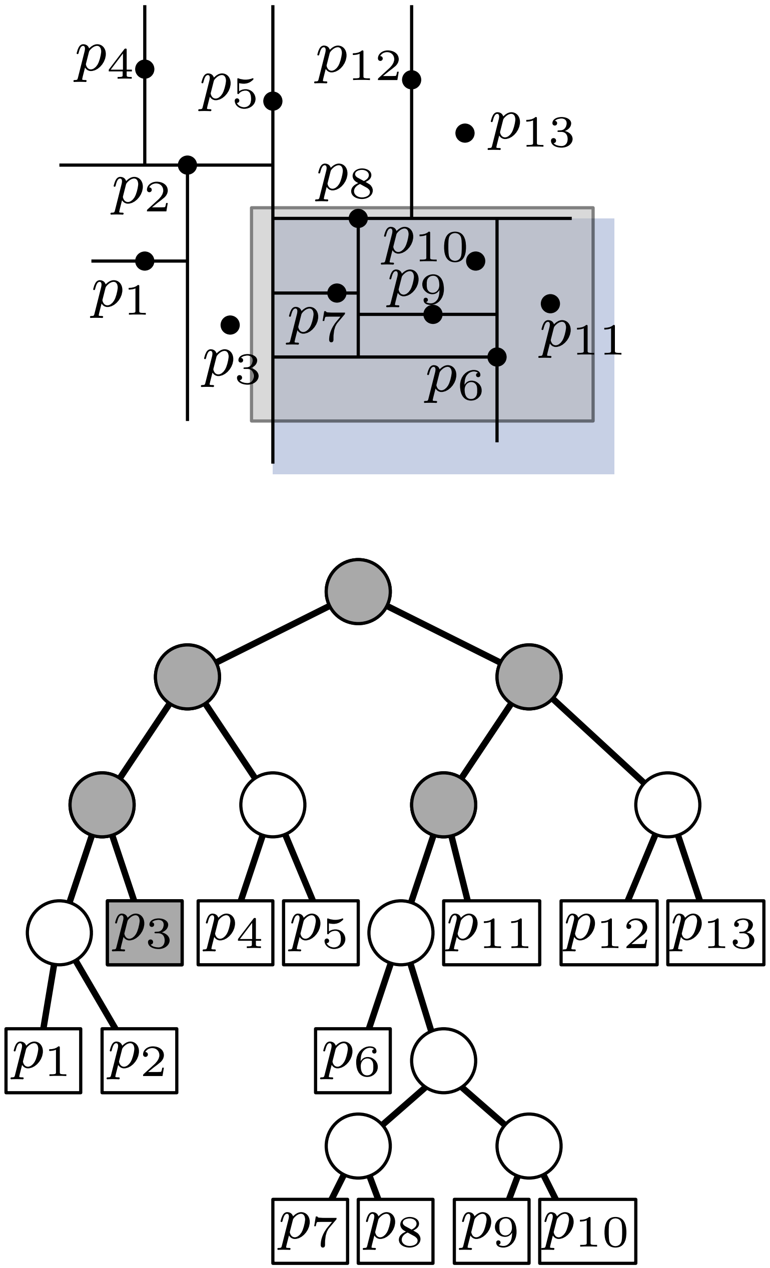

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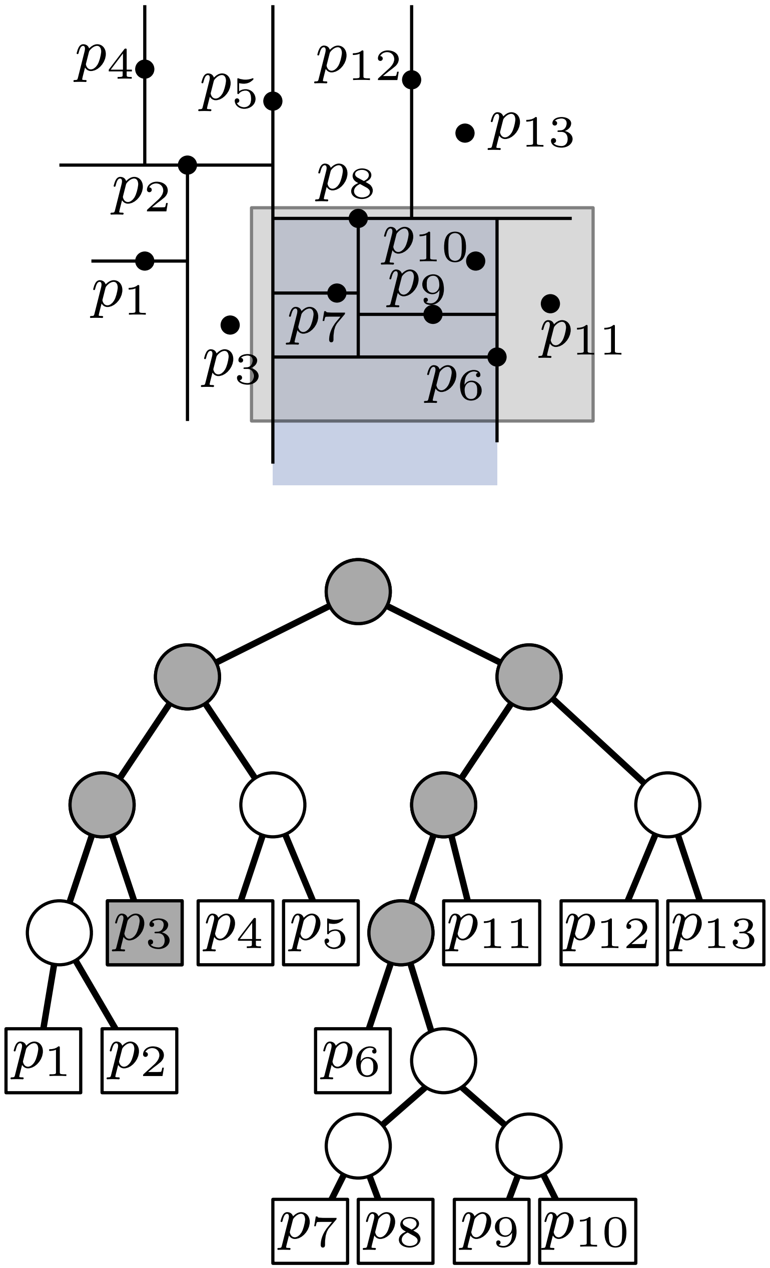

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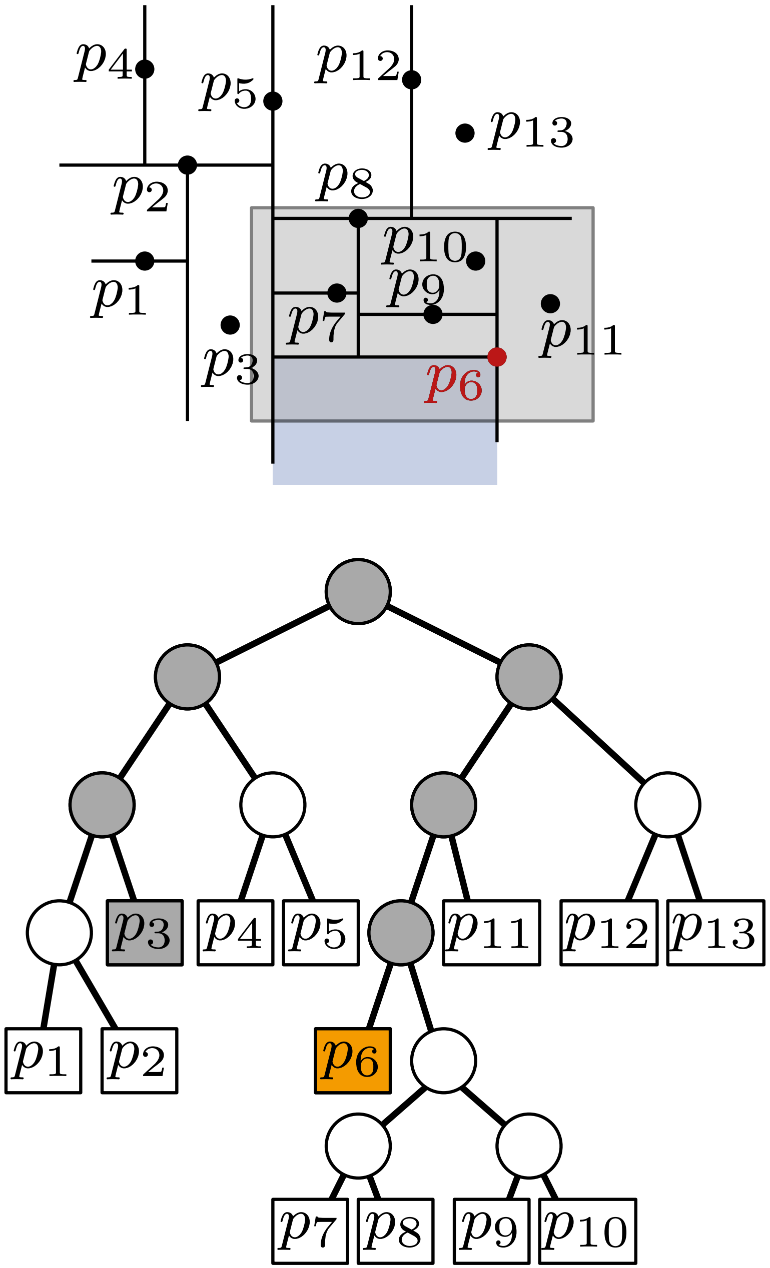

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            ReportSubtree(lc($v$))
        else
            if region(lc($v$))$\cap R \neq \emptyset$ then
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        if region(rc($v$)) $\subseteq R$ then
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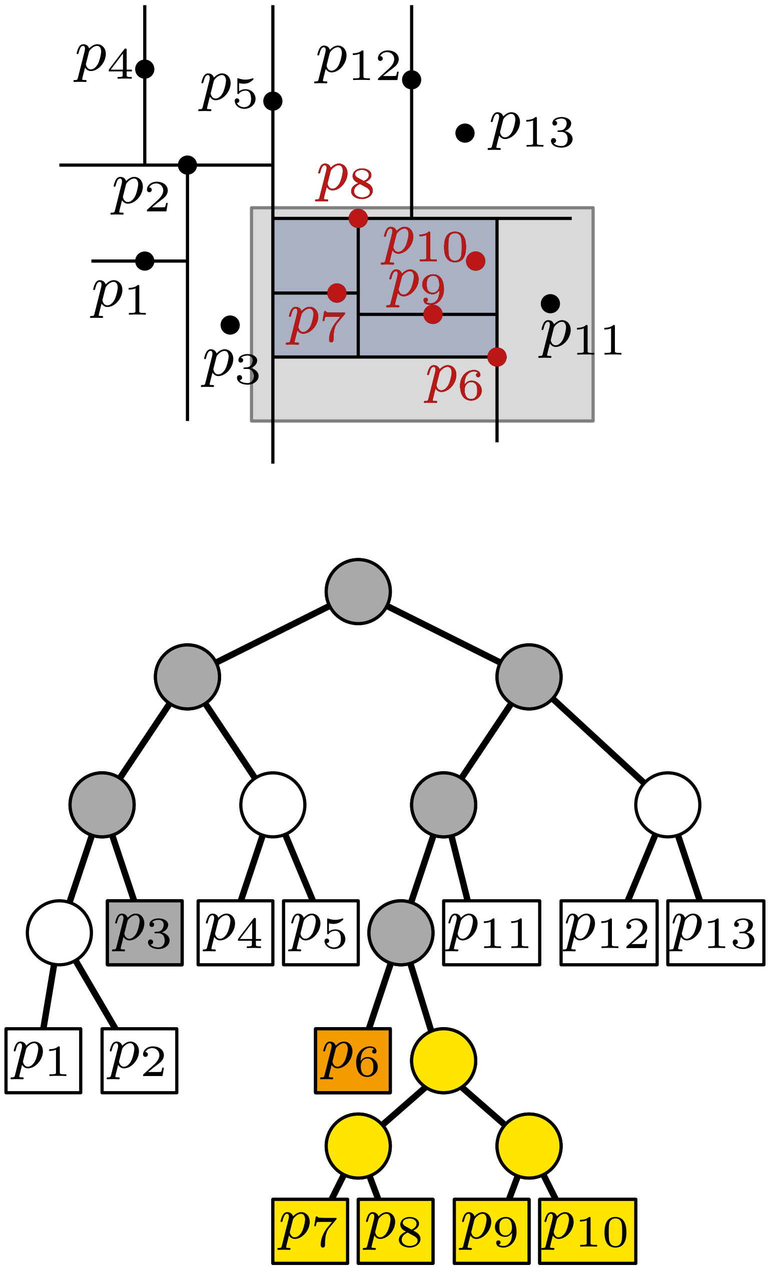
```

Range Queries in a *kd*-Tree



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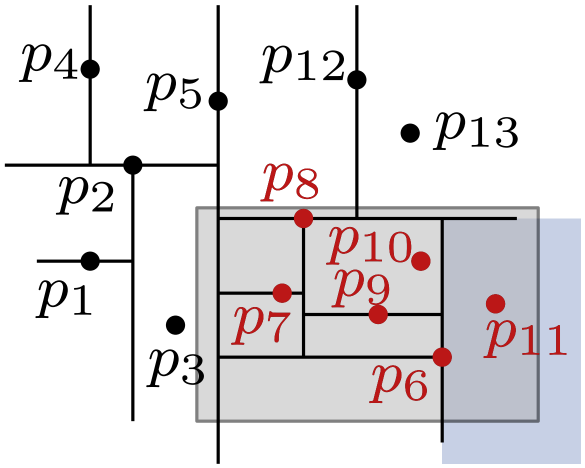

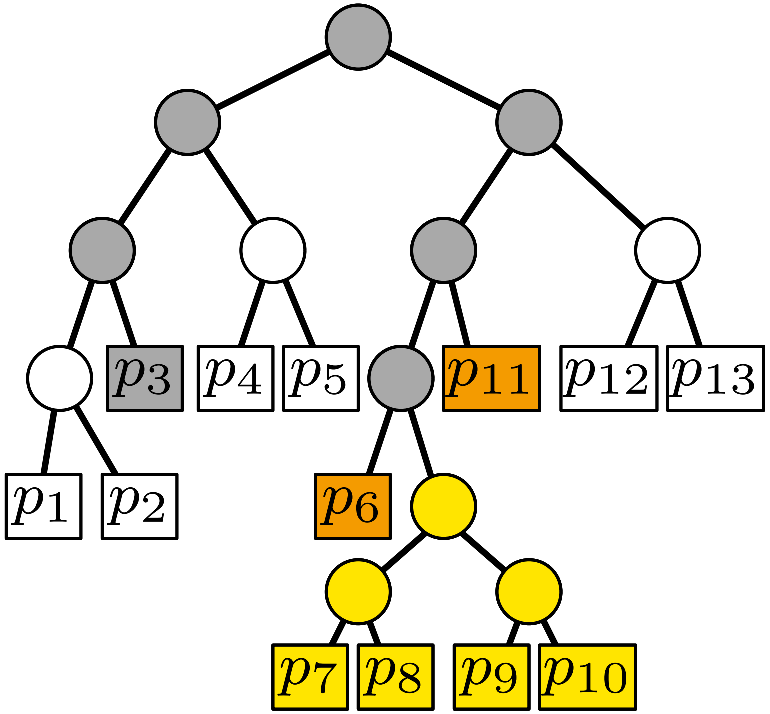

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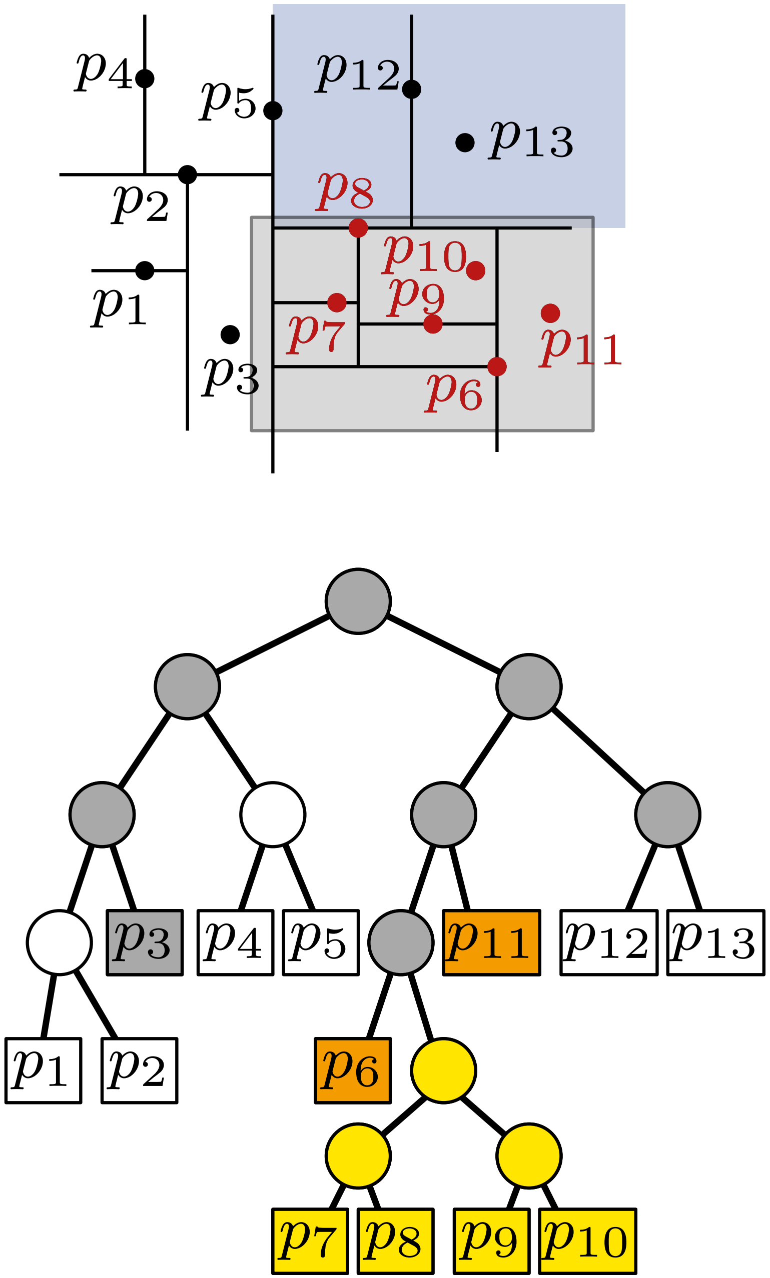

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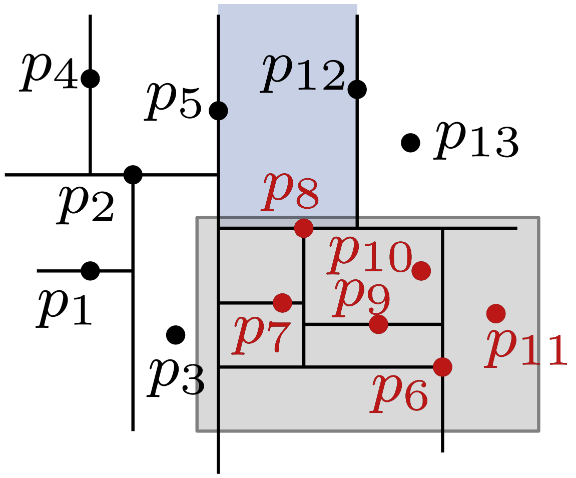

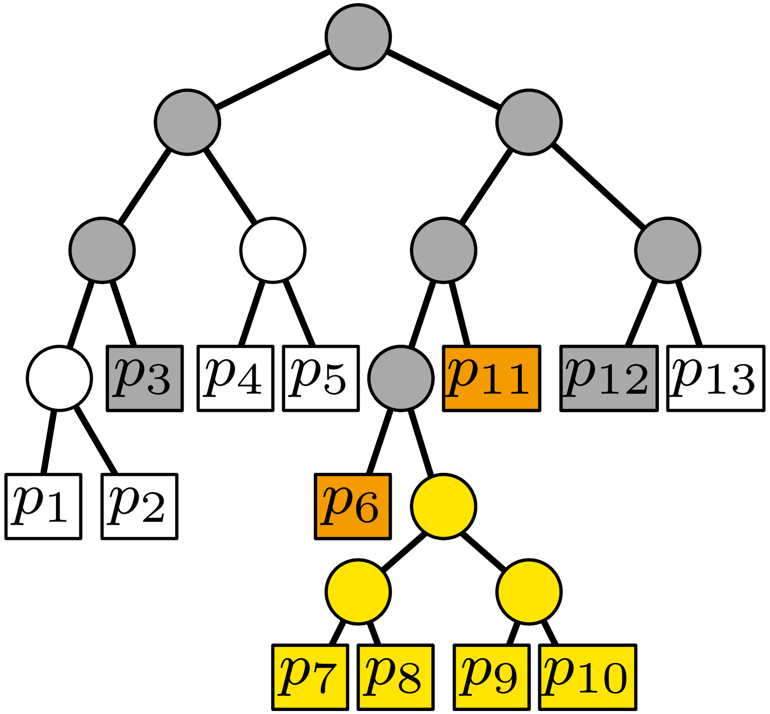

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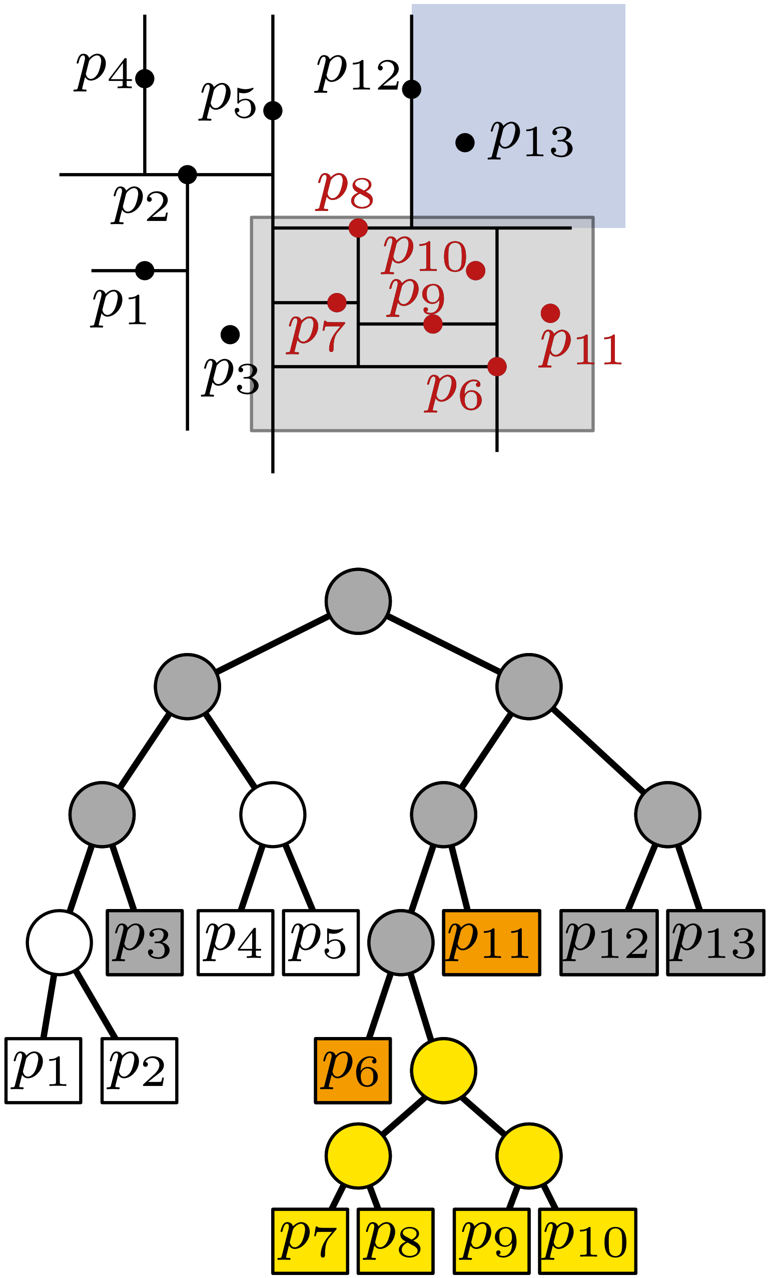

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Analysis of Queries in kd-Trees

Lemma 2: A range query with an axis-aligned rectangle R in a kd-tree on n points may use $O(\sqrt{n} + k)$ time, where k is the number of reported points.

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Proof sketch:

- Calls to ReportSubtree take $O(k)$ time in total

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Proof sketch:

- Calls to ReportSubtree take $O(k)$ time in total
- Still missing:
  Number of remaining nodes visited
  → Exercise

Orthogonal Range Queries for $d = 2$

Given: Set P of n points in $\mathbb{R}^2$

Goal: A data structure to efficiently answer range queries of the form $R = [x, x'] \times [y, y']$

Ideas for generalizing the 1d case?

Solutions:

- *one* search tree, alternate search for x and y coordinates

  → kd-**Tree** ✓

- *primary* search tree on x-coordinates,
  several *secondary* search trees on y-coordinates

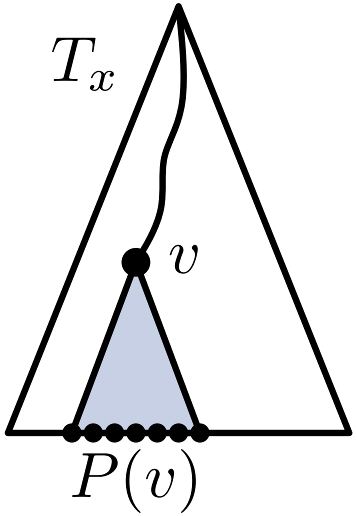
  → **Range Tree**

Temporary assumption: general position, that is no two points have the same x- or y-coordinates

Range Trees

Idea: Use 1-dimensional search trees on two levels:

- a 1d search tree T_x on x-coordinates



Range Trees

Idea: Use 1-dimensional search trees on two levels:

- a 1d search tree T_x on x-coordinates
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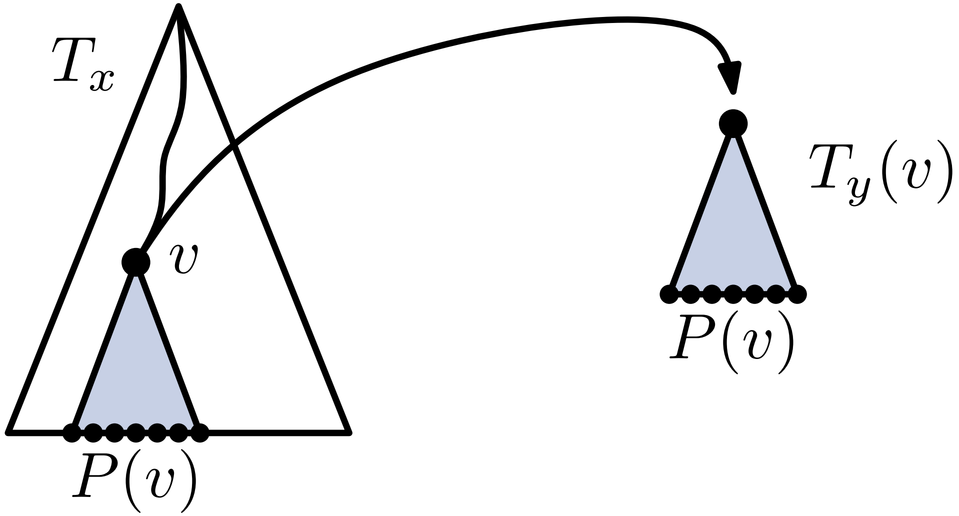

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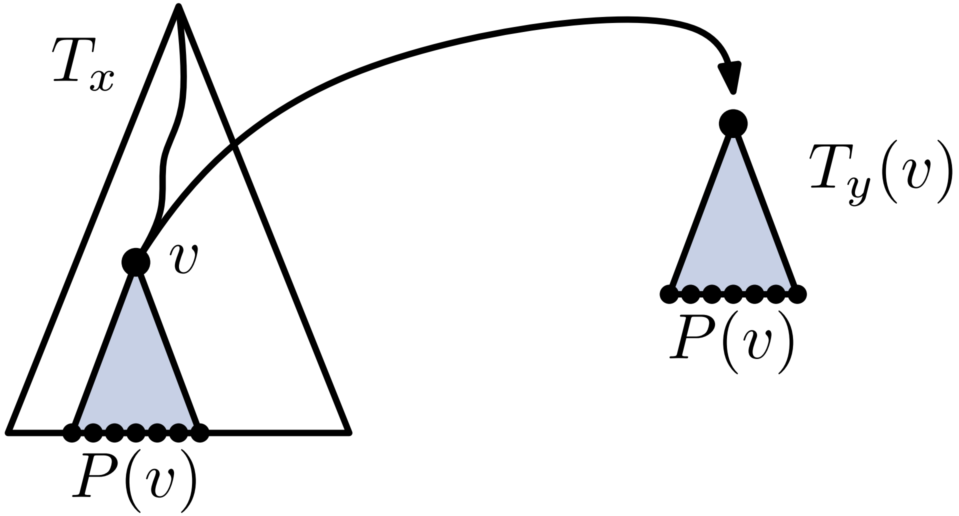
- a 1d search tree T_x on x-coordinates
- in each node v of T_x a 1d search tree $T_y(v)$ stores the canonical subset $P(v)$ on y-coordinates
- compute the points by x-query in T_x and subsequent y-queries in the auxiliary structures T_y for the subtrees in T_x



Range Trees: Construction

```
BuildRangeTree($P$)
  if $|P| = 1$ then
    Create leaf $v$ for the point in $P$
  else
    Split $P$ at $x_{\text{median}}$ into $P_1 = \{p \in P \mid p_x \leq x_{\text{median}}\}$, $P_2 = P \setminus P_1$
    $v_{\text{left}} \leftarrow$ BuildRangeTree($P_1$)
    $v_{\text{right}} \leftarrow$ BuildRangeTree($P_2$)
    Create node $v$ with pivot $x_{\text{median}}$ and children $v_{\text{left}}$ and $v_{\text{right}}$
  $T_y(v) \leftarrow$ binary search tree for $P$ w.r.t $y$-coordinates
  return $v$
```

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```

Problem: How much space and runtime does BuildRangeTree use?

Lemma 3: A Range Tree for n points in $\mathbb{R}^2$ uses $O(n \log n)$ space and can be constructed in $O(n \log n)$ time.

Range Queries in a Range Tree

Reminder:

```
**1dRangeQuery**$(T, x, x')$
    $v_{\text{split}} \leftarrow$ FindSplitNode$(T, x, x')$
    if $v_{\text{split}}$ is leaf then report $v_{\text{split}}$
    else
        $v \leftarrow$ lc$(v_{\text{split}})$
        while $v$ not leaf do
            if $x \leq x_v$ then
                ReportSubtree(rc$(v)$)
                $v \leftarrow$ lc$(v)$
            else  $v \leftarrow$ rc$(v)$
        report $v$
        // analogous for $x'$ and rc$(v_{\text{split}})$
```

Range Queries in a Range Tree

Reminder:

```
~~1dRangeQuery($(T, x, x')$)~~ 2dRangeQuery($T, [x, x'] \times [y, y']$)
  $v_{\text{split}} \leftarrow$ FindSplitNode($T, x, x'$)
  if $v_{\text{split}}$ is leaf then report $v_{\text{split}}$
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      report $v$
      // analogous for $x'$ and rc($(v_{\text{split}})$)
```

Lemma 4: A range query in a Range Tree takes $O(\log^2 n + k)$ time, where k is the number of reported points.

Range Queries with Fractional Cascading

Observation: Range queries in a Range Tree perform $O(\log n)$ 1d queries, each taking $O(\log n + k_v)$ time. The query interval $[y, y']$ is always the same!

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Example: Two sets $B \subseteq A \subseteq \mathbb{R}$ in sorted arrays

A	3	10	19	23	30	37	59	62	70	80	100	105

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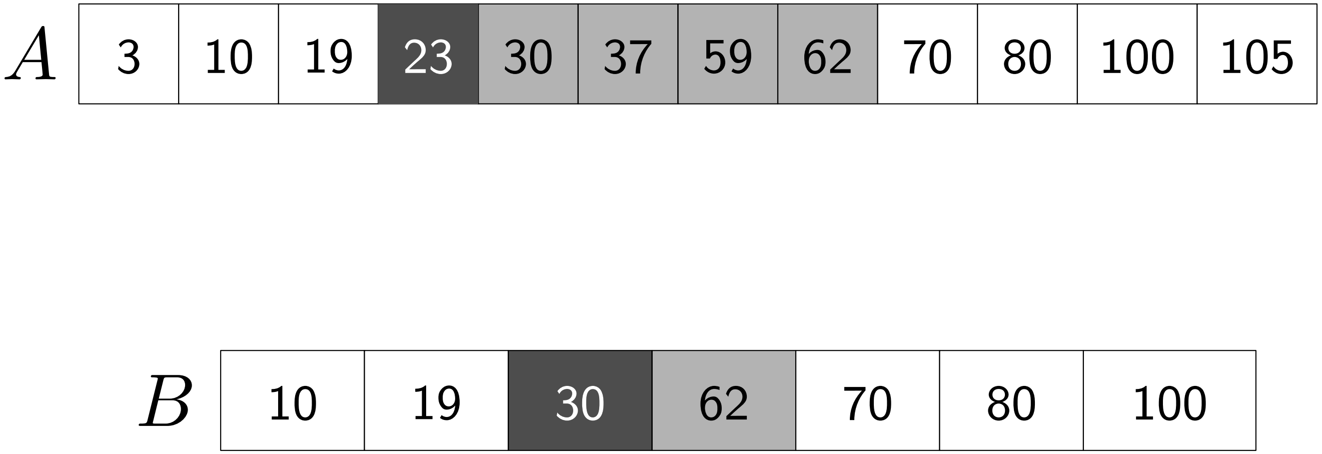

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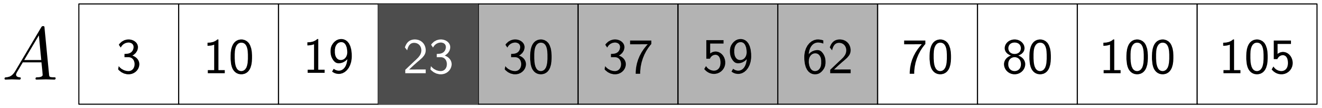

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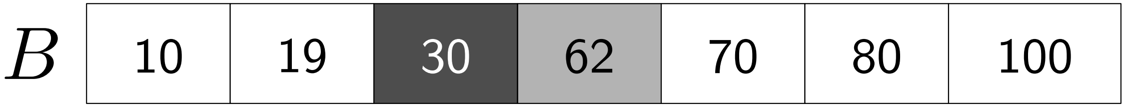
Can we do better than two binary searches?



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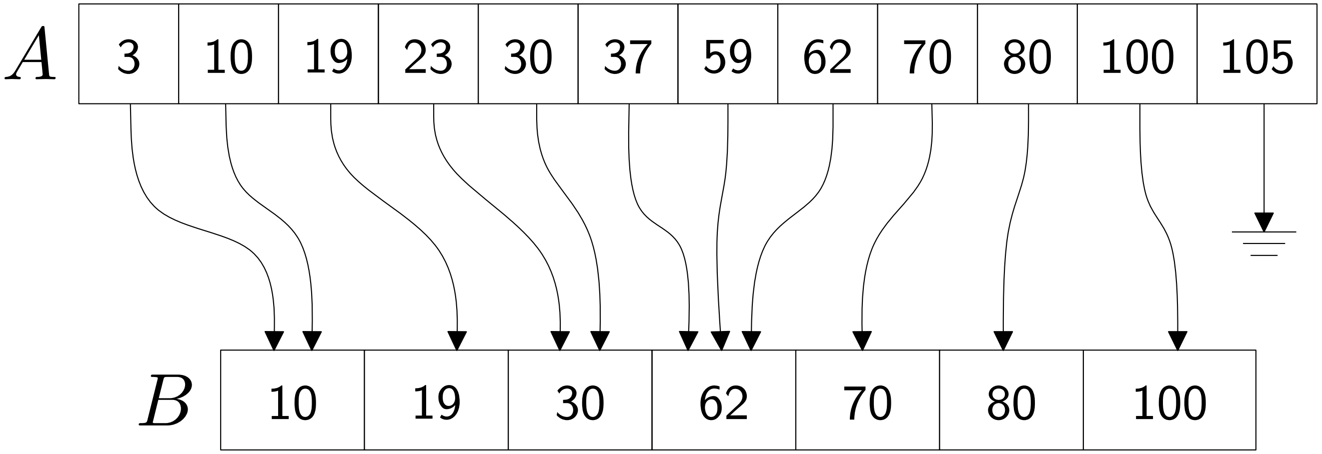

link $a \in A$ with smallest $b \geq a$ in B

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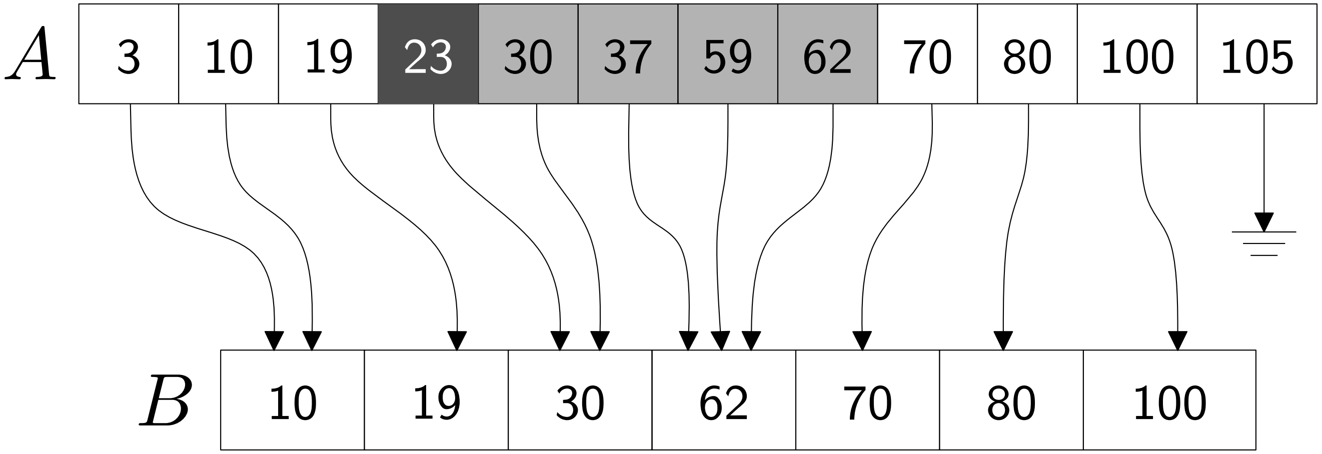

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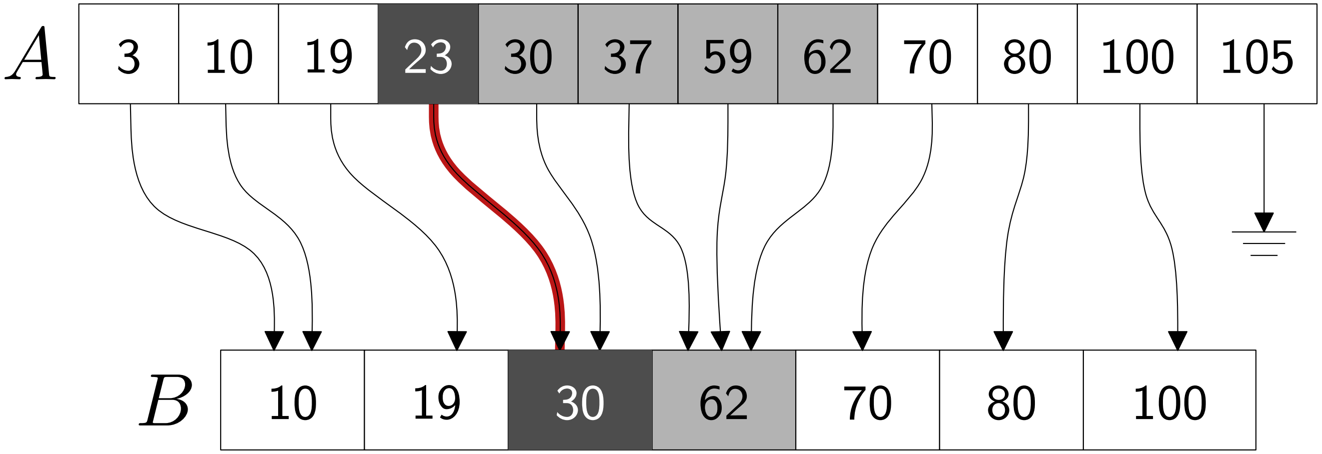

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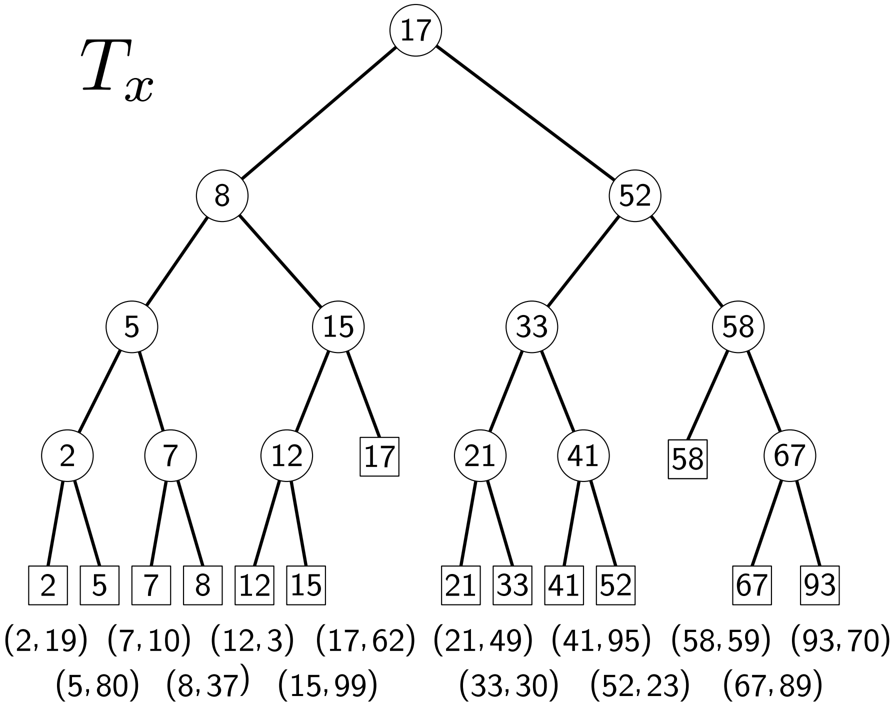
Pointer yields starting point for second search in $O(1)$ time

Speed-up with Fractional Cascading

- In Range Trees we have $P(\mathrm{lc}(v)) \subseteq P(v)$ and $P(\mathrm{rc}(v)) \subseteq P(v)$ as the canonical sets.



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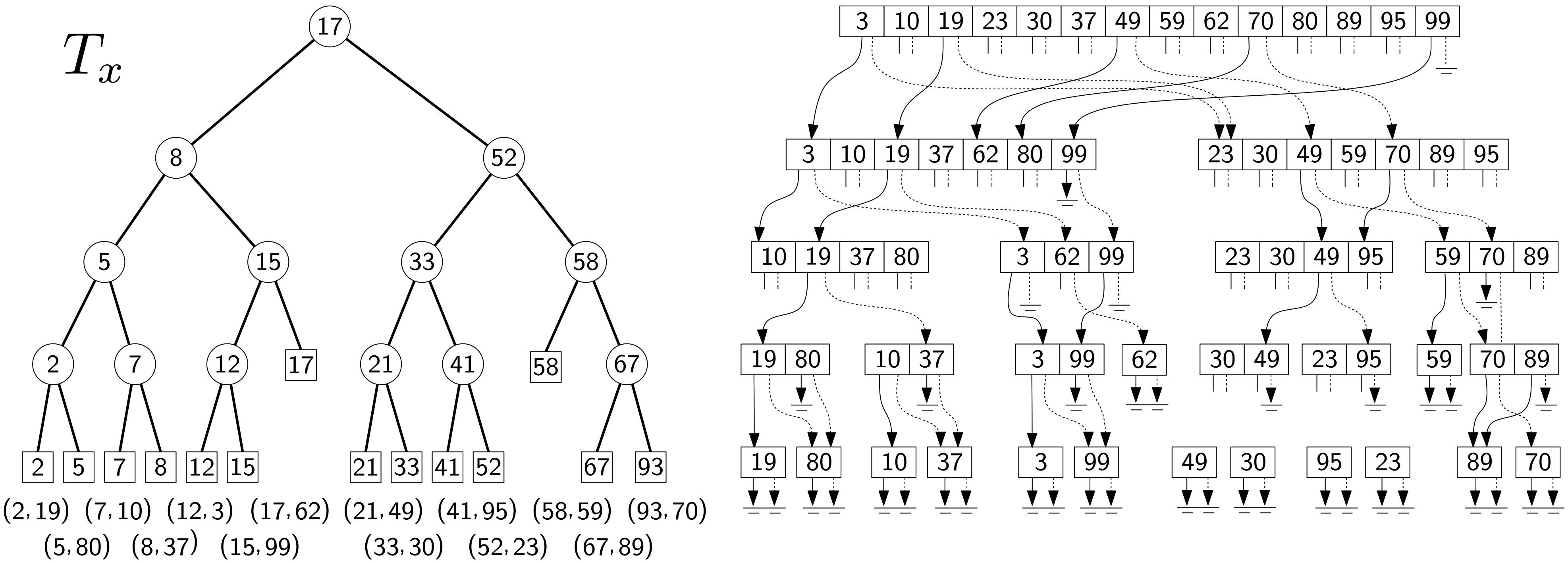

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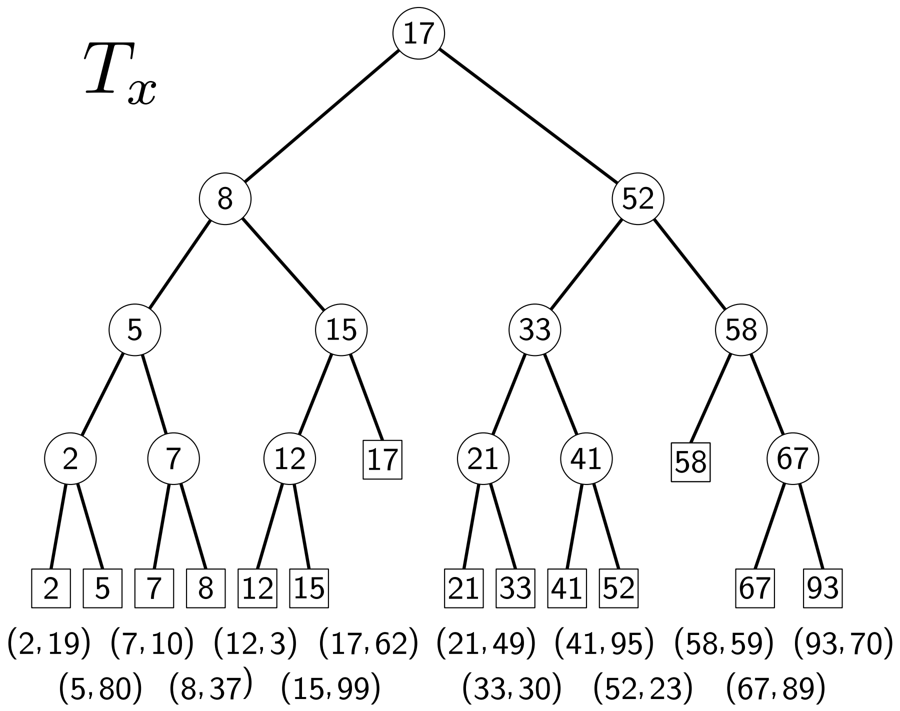

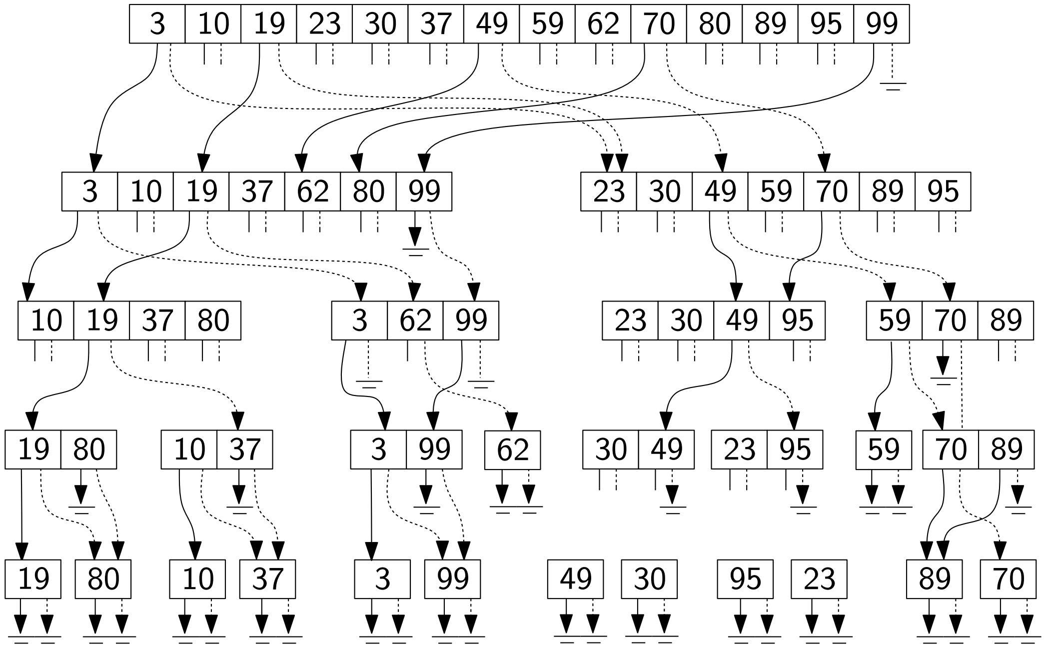

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Theorem 2: A Layered Range Tree on n points in $\mathbb{R}^2$ can be constructed in $O(n \log n)$ time and space. Range queries take $O(\log n + k)$ time, where k is the number of reported points.

Arbitrary Point Sets

So far: Points in general position, where no two points have the same x- or y-coordinate

Idea: Instead of $\mathbb{R}$, use pairs of numbers $(a|b)$ with total order ↔ lexicographic order

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$$\big\downarrow$$

$$\hat{R} = [(x| -\infty), (x'| +\infty)] \times [(y| -\infty), (y'| +\infty)]$$

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$$\downarrow$$

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Then:

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→ unique coord.

Rectangle $R = [x, x'] \times [y, y']$

$$\big\downarrow$$

$$\hat{R} = [(x|-\infty), (x'|+\infty)] \times [(y|-\infty), (y'|+\infty)]$$

Then: $p \in R \;\Leftrightarrow\; \hat{p} \in \hat{R}$

Summary

Given: Set P of n points in $\mathbb{R}^2$

Construct: Data structures with efficient range queries of the form $R = [x, x'] \times [y, y']$

→ We have seen two alternatives

	kd-Tree	Range Tree
Preprocessing		
Space		
Query time		

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Space	$O(n)$	$O(n \log n)$
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Given: Set P of n points in $\mathbb{R}^2$

Construct: Data structures with efficient range queries of the form $R = [x, x'] \times [y, y']$

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Yes, we can transform any polygon into a point in 4d space (exercise) or we can use windowing queries (comes in a later lecture).

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2) **Augmented dynamic range trees** [Mehlhorn, Näher '90]
support updates in $O(\log n \log \log n)$ time and queries in $O(\log n \log \log n + k)$ time