Karlsruher Institut für Technologie

Algorithms for graph visualization

Data Structures for Planar Graph Embeddings

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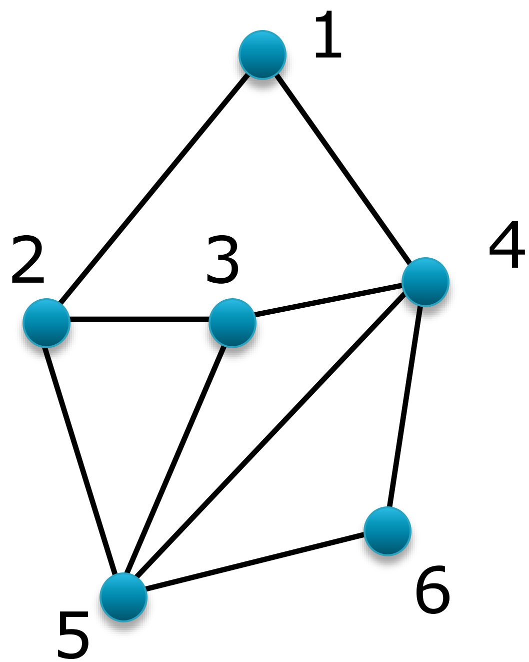
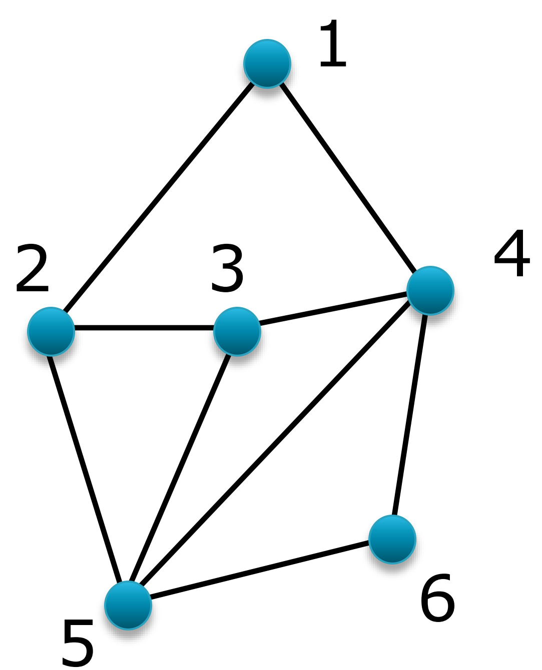
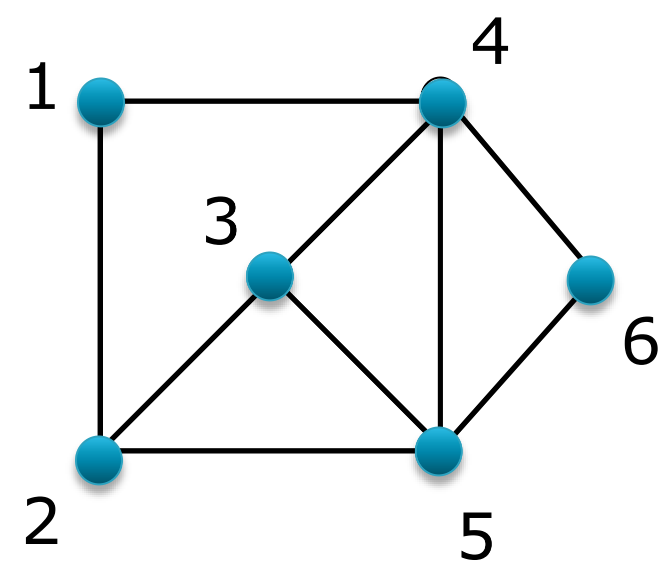
ROMA TRE UNIVERSITÀ DEGLI STUDI

A drawing of a graph



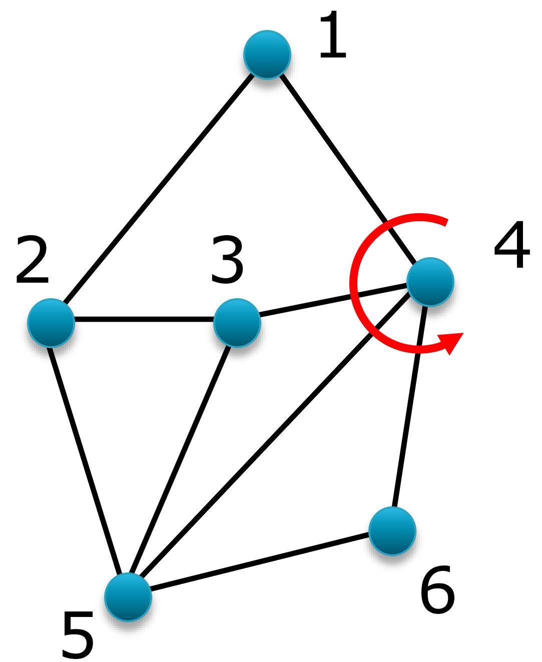
Two drawings of a graph

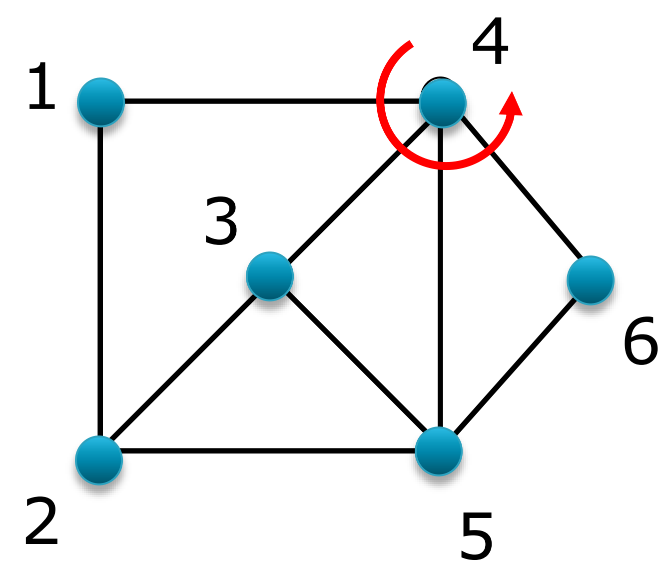




An embedding of a graph

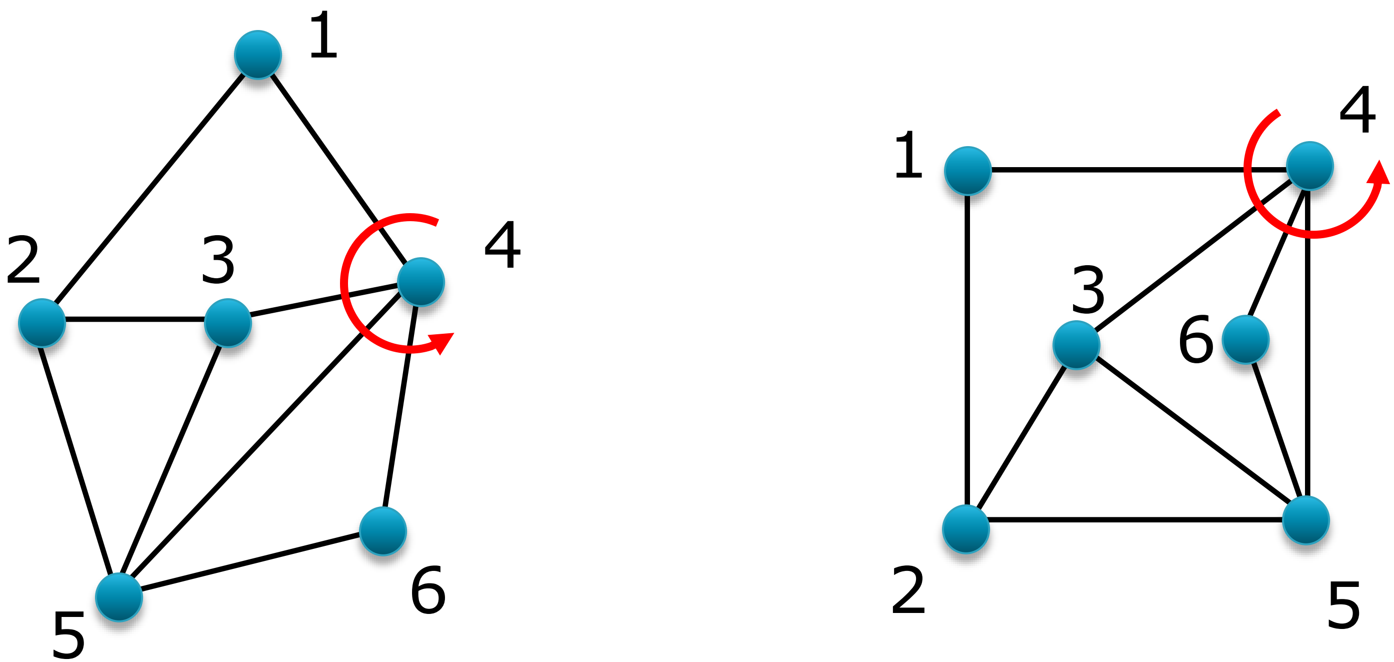
- Different geometry in the two drawings, but the ordering of the edges around each vertex is the same





Two embeddings of a graph

- Different “topology” in the two drawings



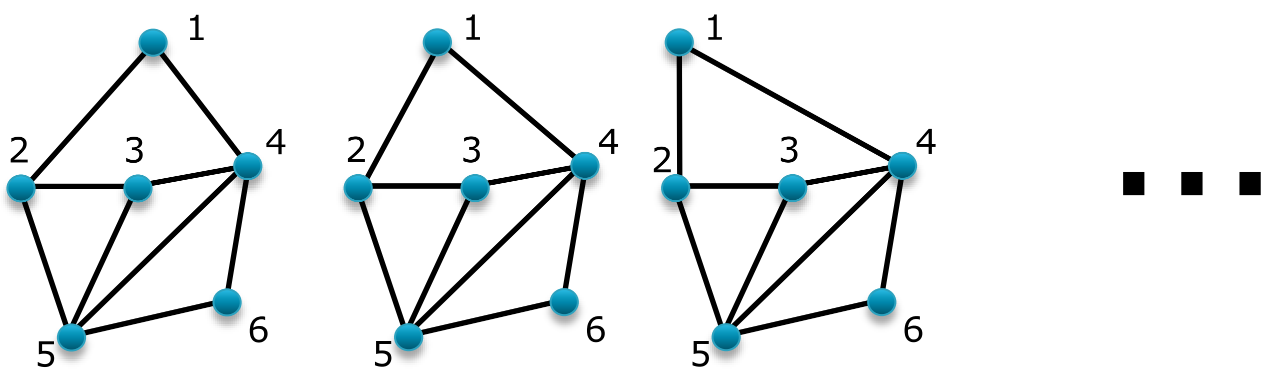
Properties of embeddings

- **Fàry's Theorem (1946)**:
  If a graph admits a planar drawing where edges are curves, than it also admits a straight-line planar drawing

- Planarity is a "topological" problem!
  - In order to say that a graph is planar, we need to test whether it admits a planar embedding
  - Forget about drawings
  - Well, it depends on the problem…

How many drawings/embeddings?

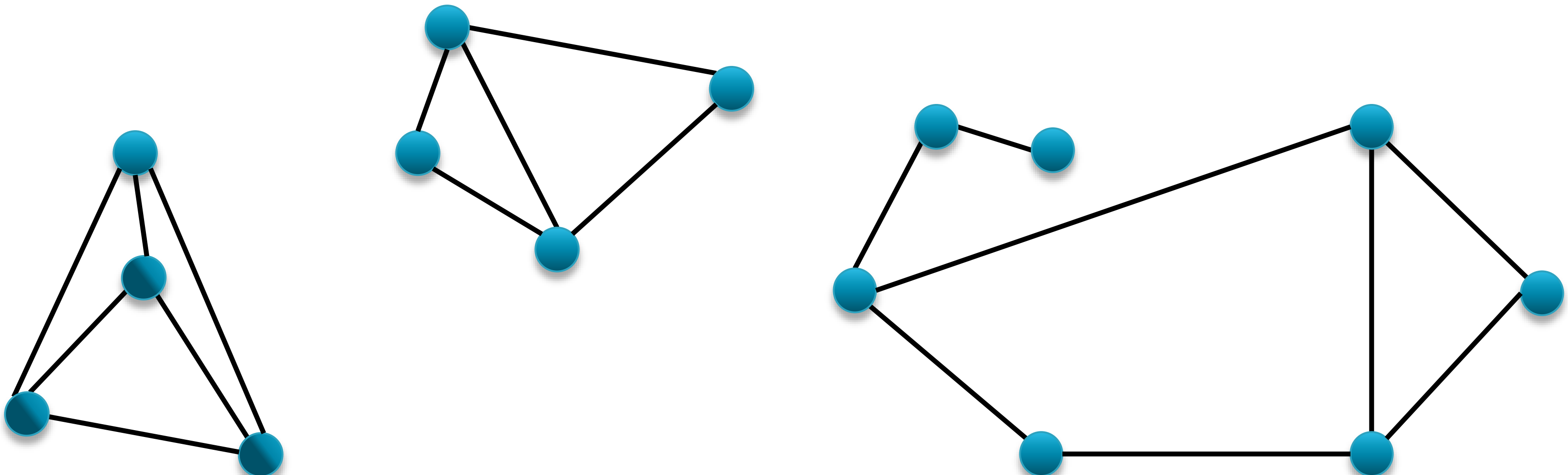
- Infinite number of drawings (Continuous space)



- Finite number of embeddings (Discrete space)
  - Planar embeddings are equivalence classes of planar drawings
  - So, how many planar embeddings?

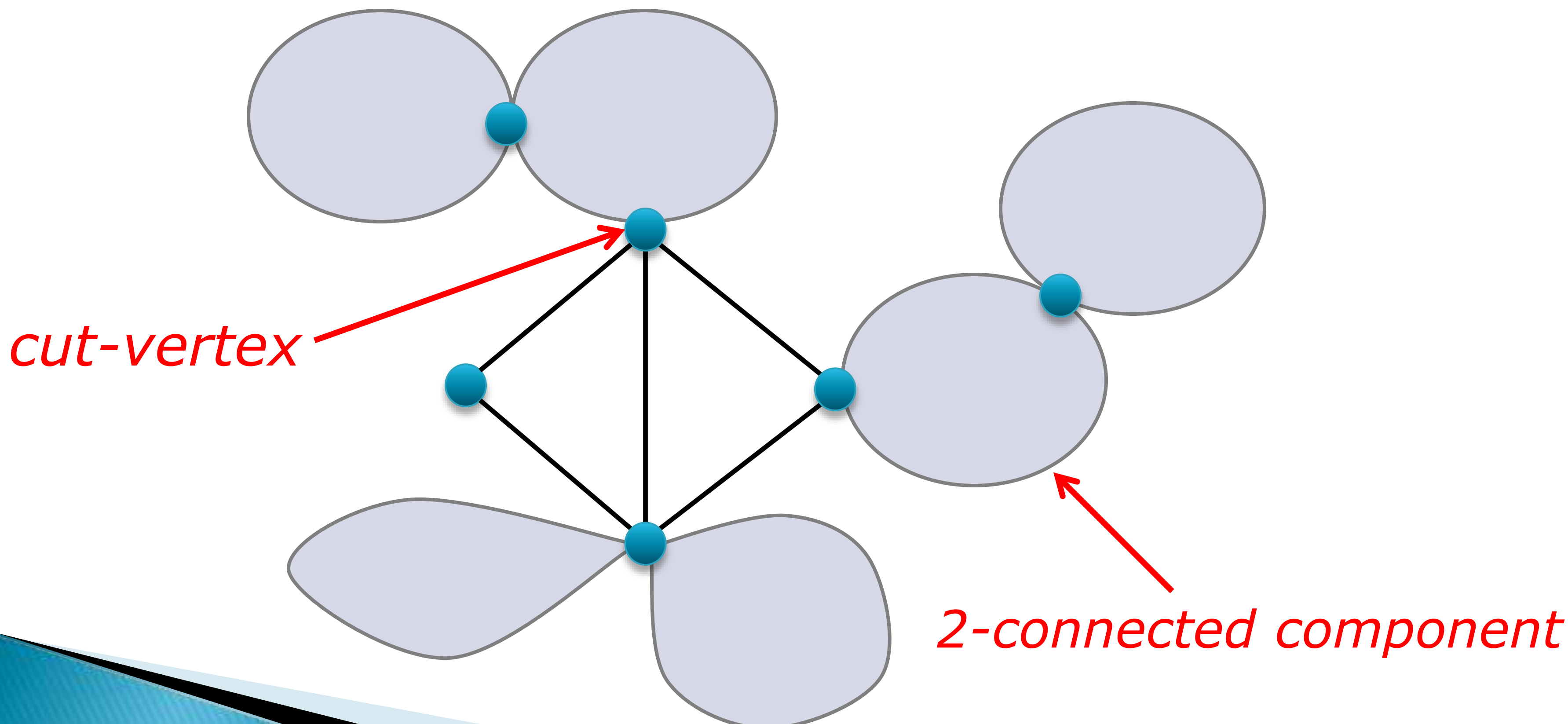
Connectivity

- A graph is connected if for every pair of vertices there exists a path connecting them
- A graph is k-connected if for every pair of vertices there exist *k disjoint* paths connecting them



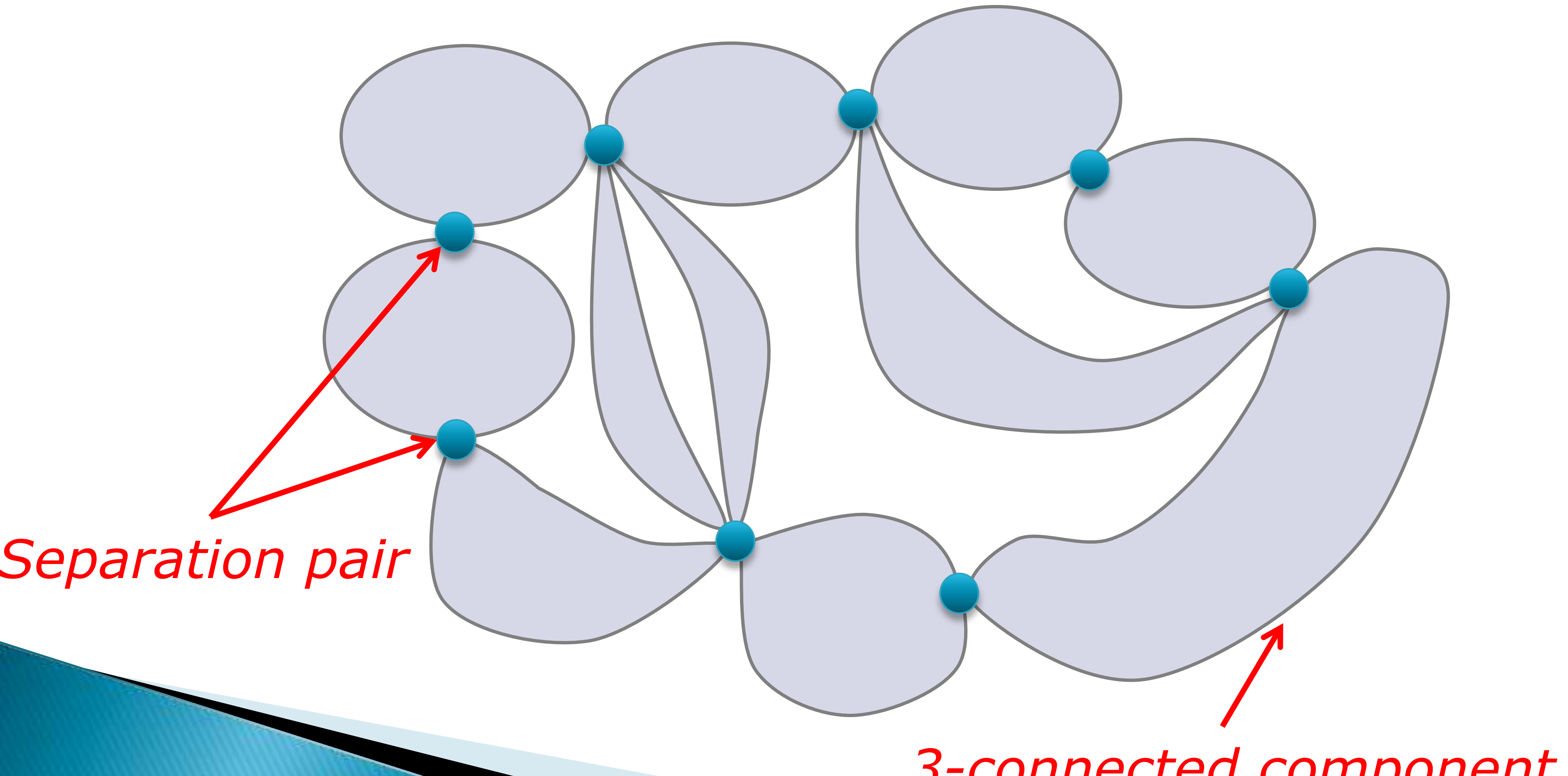
Connectivity

- ***k* = *1***: (simply) connected graph



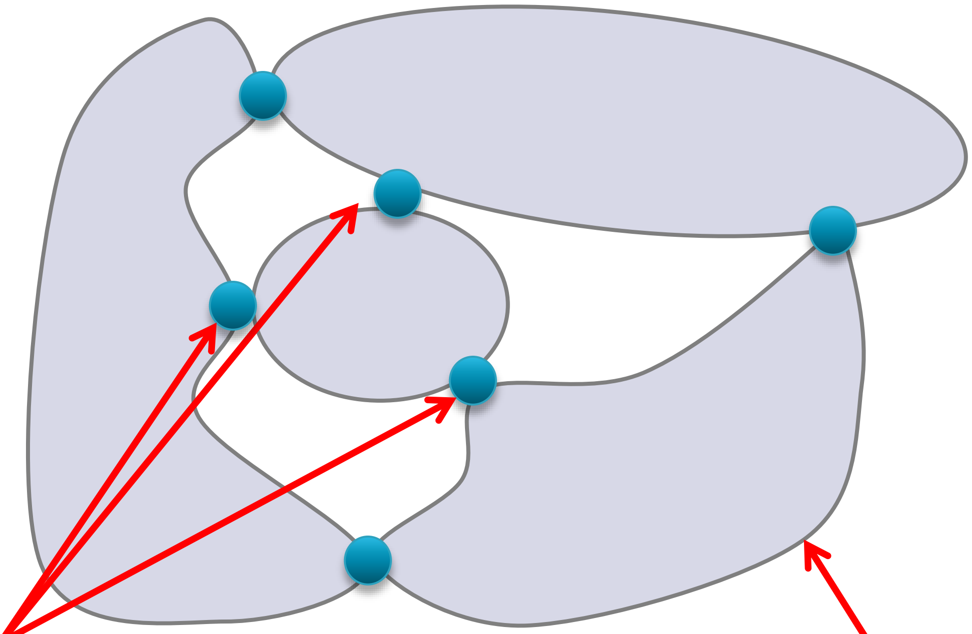
Connectivity

- ***k* = 2**: biconnected graph



Connectivity

- ***k* = *3***: triconnected graph



Separating triplet (triangle)

4-connected component

Question time

How connected can a planar graph be?

More formally, what is the largest value of k such that there exists a planar graph that is k-connected?

5

at most 3n-6 edges

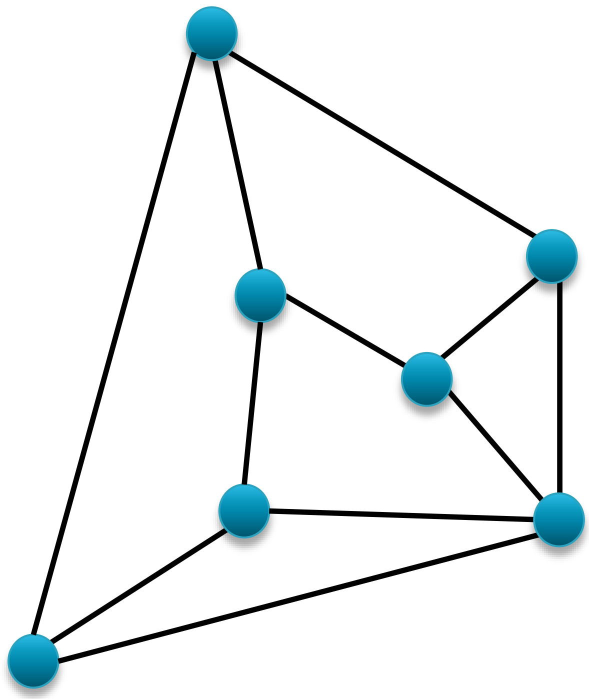
Question time

Why did we stop at $k = 3$?

Even more, why are we speaking about connectivity?

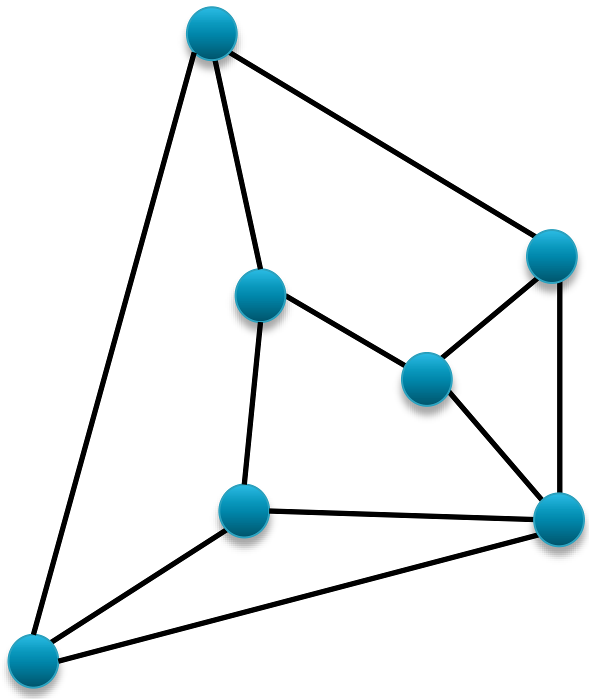
We'll answer both questions in a while

Connectivity – Embeddings



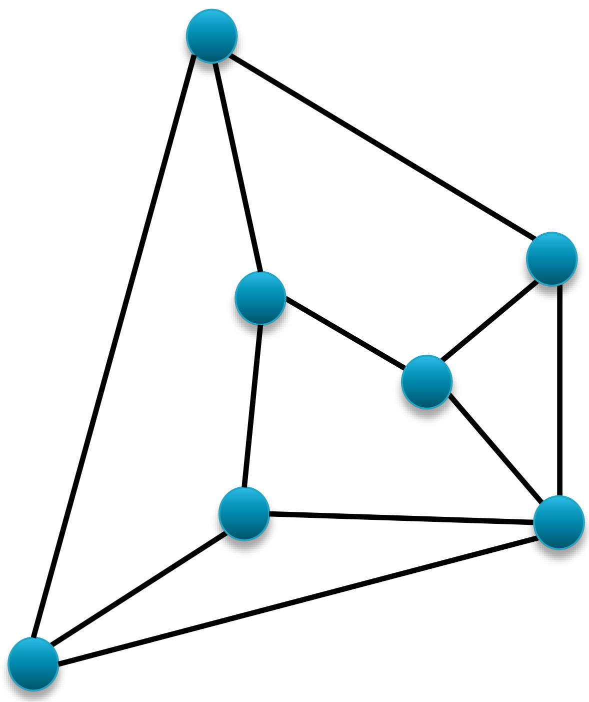
How connected is this graph?

Connectivity – Embeddings



How connected is this graph? 3

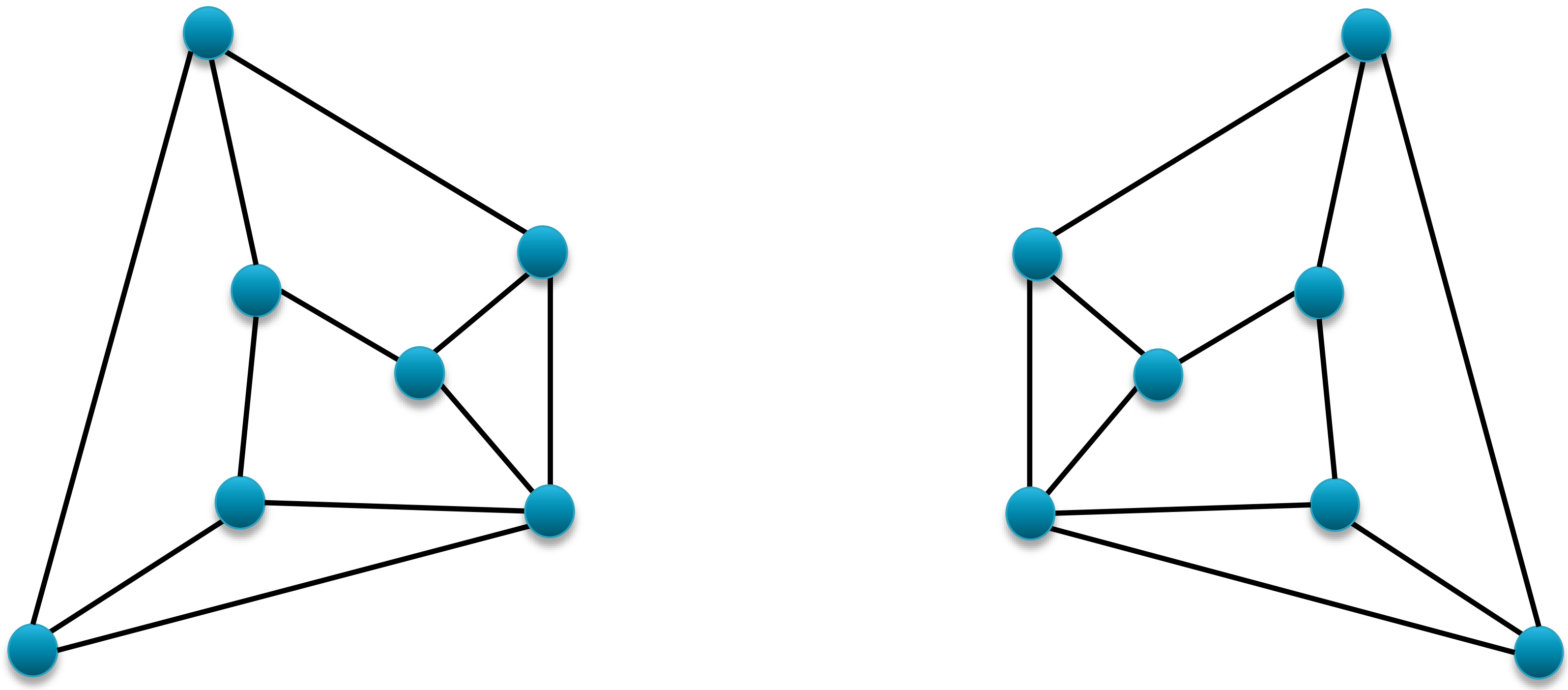
Connectivity – Embeddings



How connected is this graph? 3

How many planar embeddings?

Connectivity – Embeddings



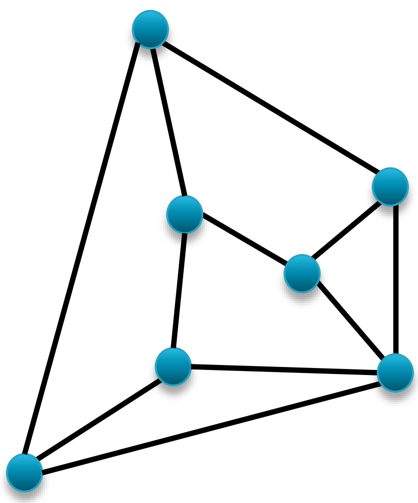
How connected is this graph? 3

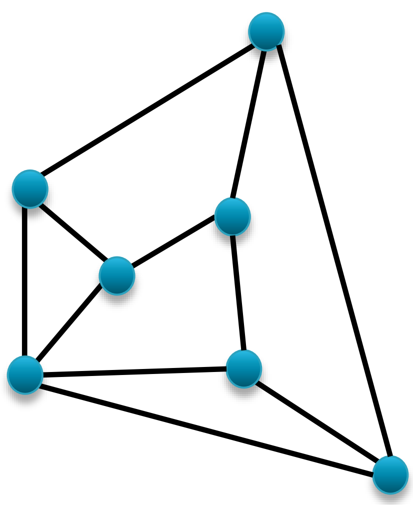
How many planar embeddings? 2

Connectivity – Embeddings

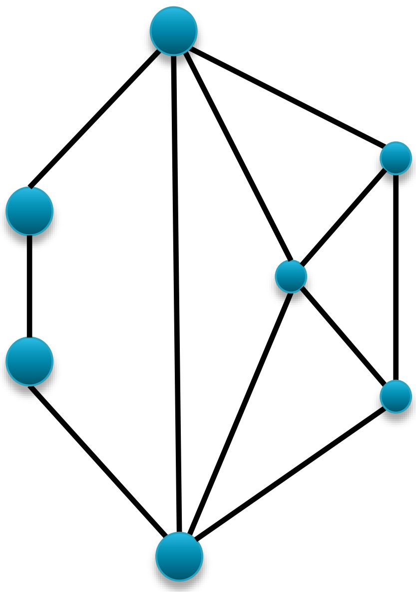
- **Theorem (Whitney, 1932)**:

  A 3-connected planar graph admits only two planar embeddings, which differ by a flip



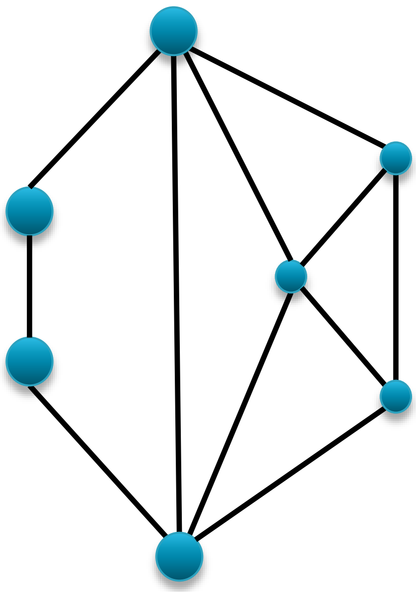


Connectivity – Embeddings



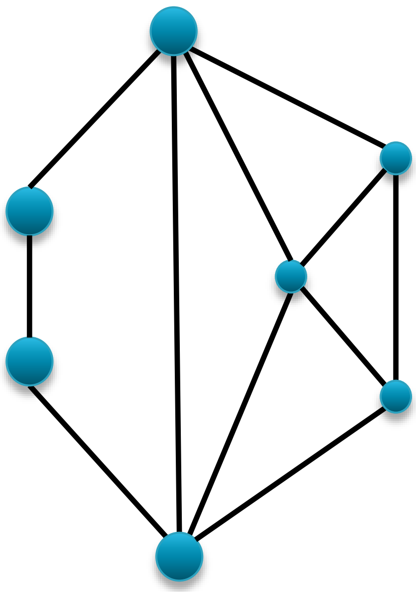
How connected is this graph?

Connectivity – Embeddings



How connected is this graph? 2

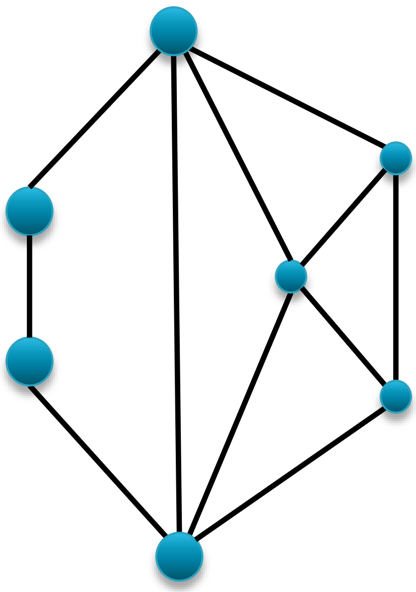
Connectivity – Embeddings

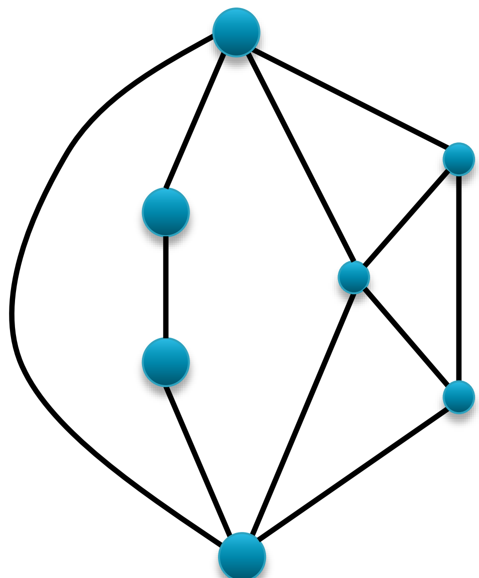


How connected is this graph? 2

How many planar embeddings?

Connectivity – Embeddings

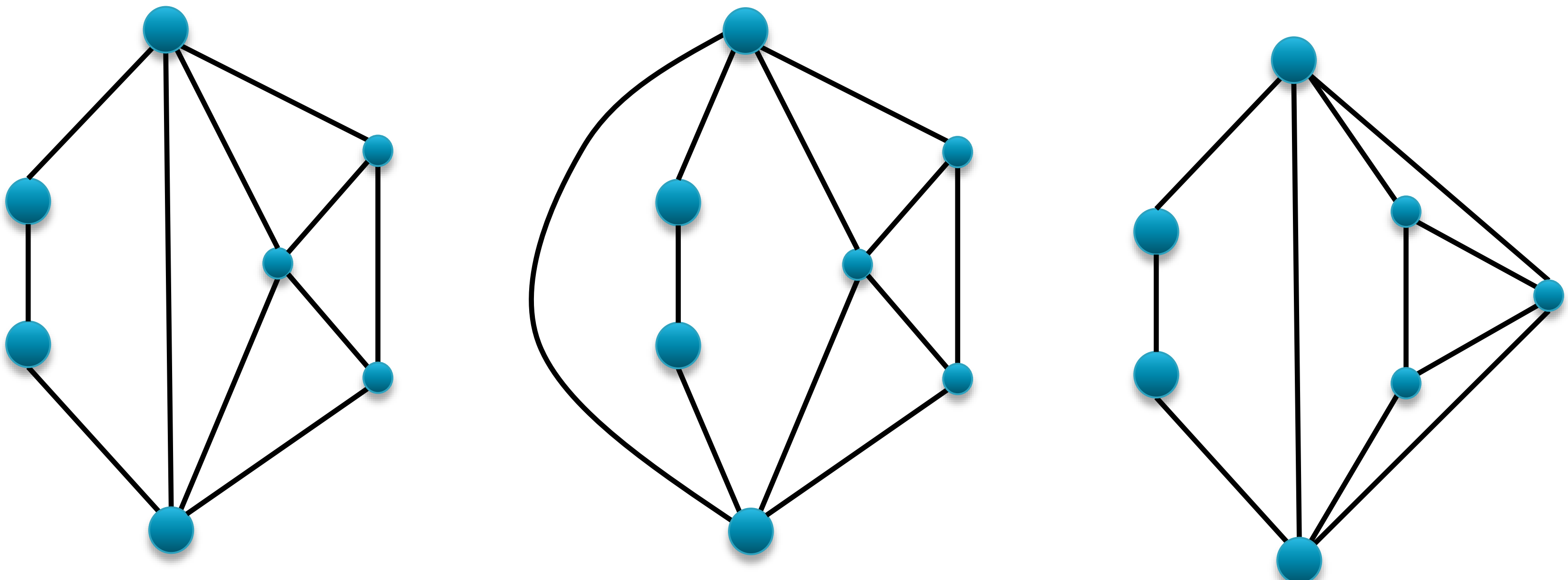




How connected is this graph? 2

How many planar embeddings?

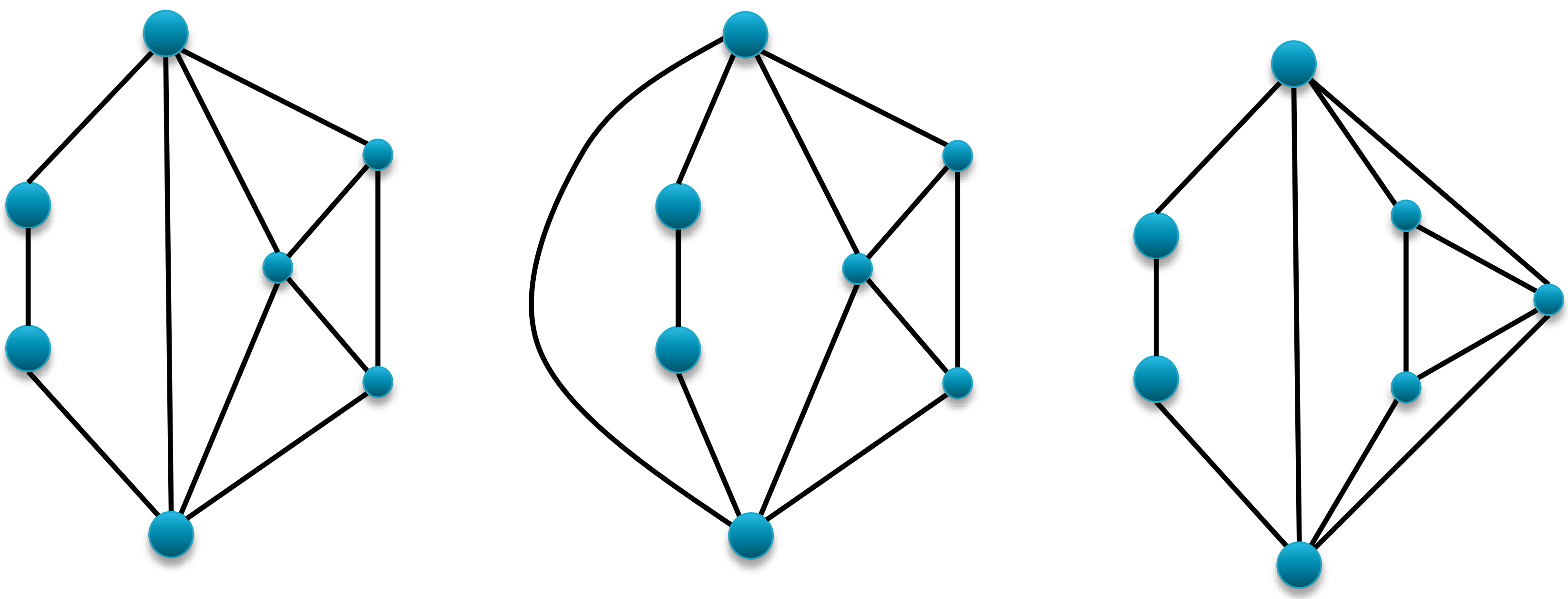
Connectivity – Embeddings



How connected is this graph? 2

How many planar embeddings?

Connectivity – Embeddings



- *Permutations of parallel subgraphs*
- *Flips of triconnected subgraphs*

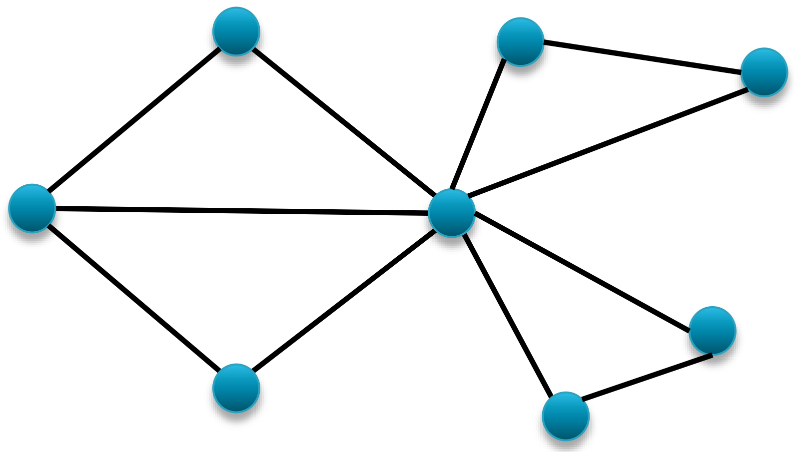
Connectivity – Embeddings

So, how many embeddings?

- *Permutations of k parallel subgraphs*
  - $k!$
- *Flips of k triconnected subgraphs*
  - 2^k

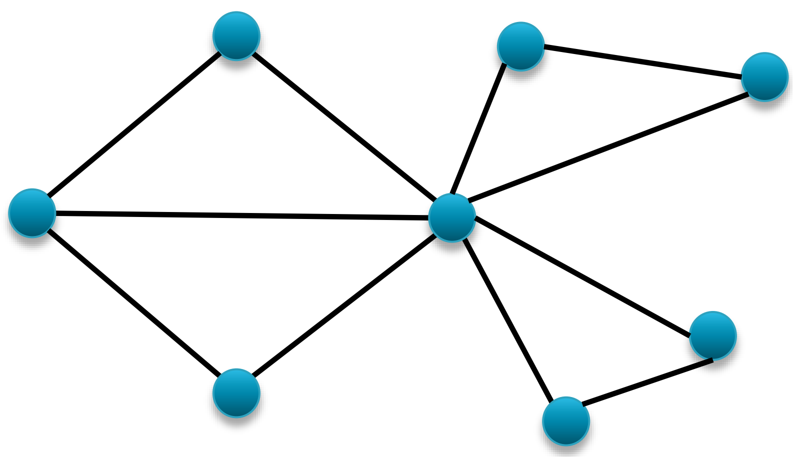
$O(n!2^n)$

Connectivity – Embeddings



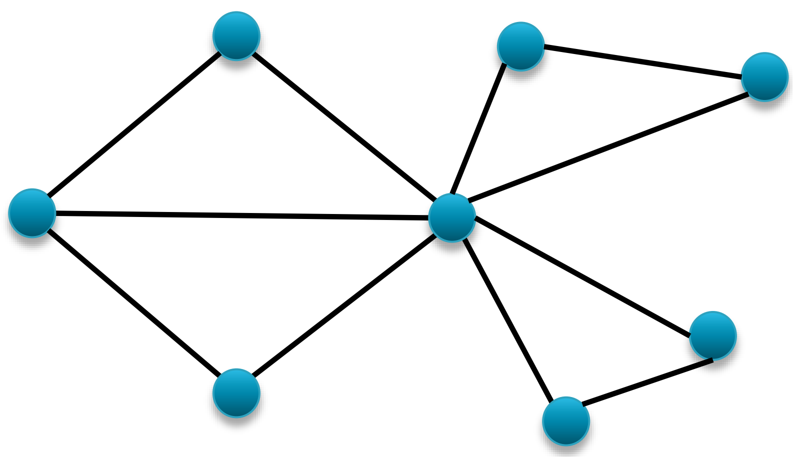
How connected is this graph?

Connectivity – Embeddings



How connected is this graph? 1

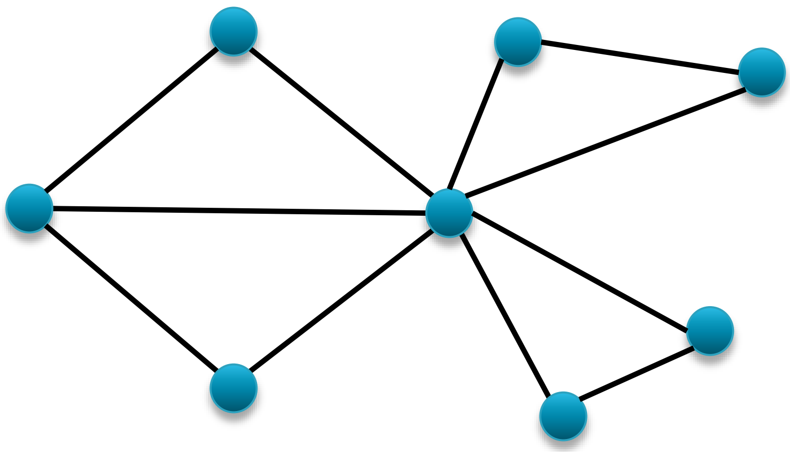
Connectivity – Embeddings

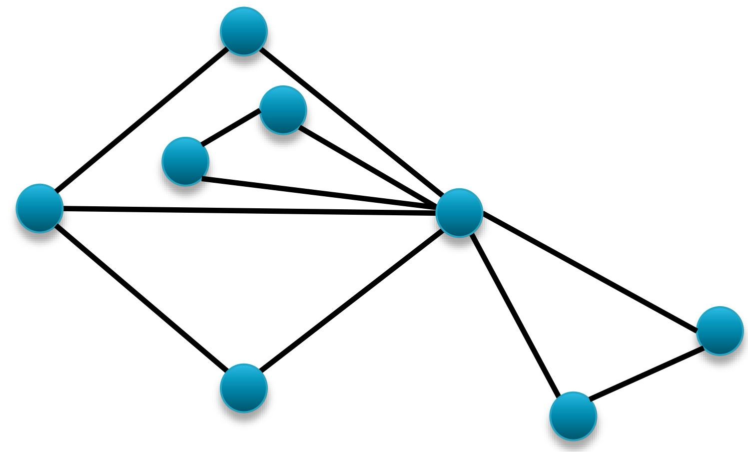


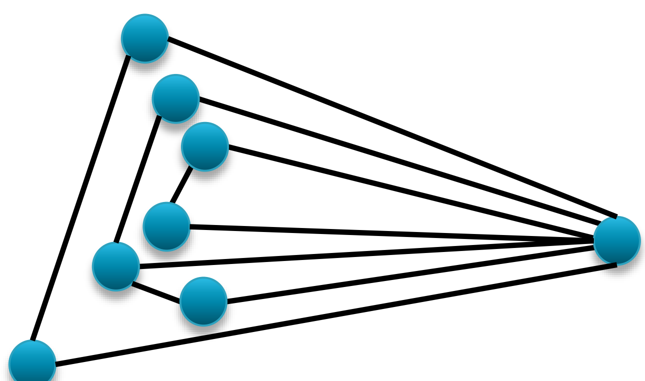
How connected is this graph? 1

How many planar embeddings?

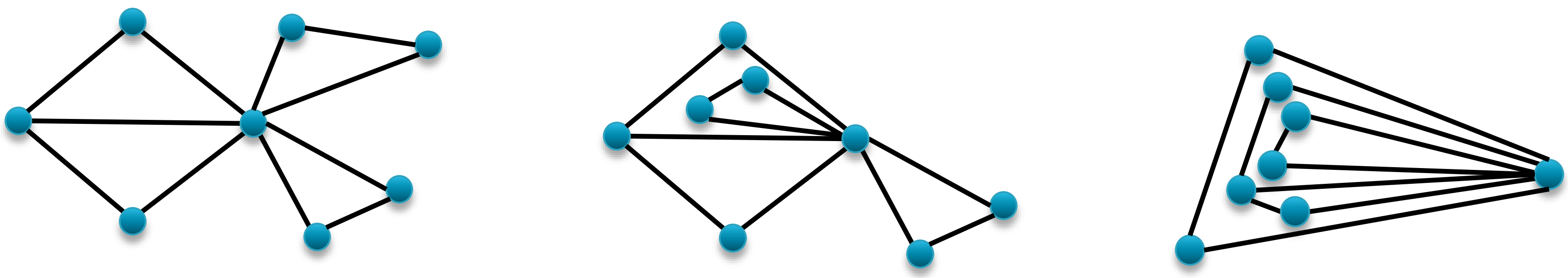
Connectivity – Embeddings







Connectivity – Embeddings



- *All possible nesting configurations*
- *Combined with all possible embeddings of the biconnected components*

Connectivity – Embeddings

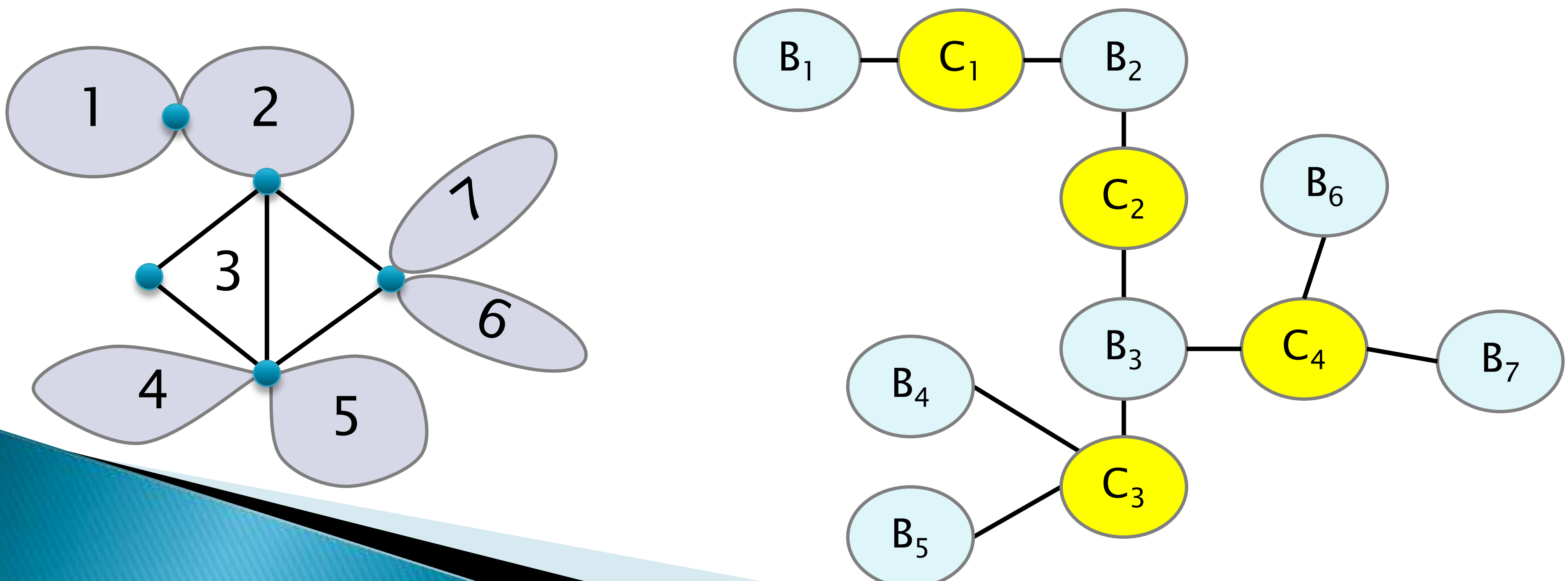
So, how many embeddings?

- *All possible nesting configurations*
- *Combined with all possible embeddings of the biconnected components*

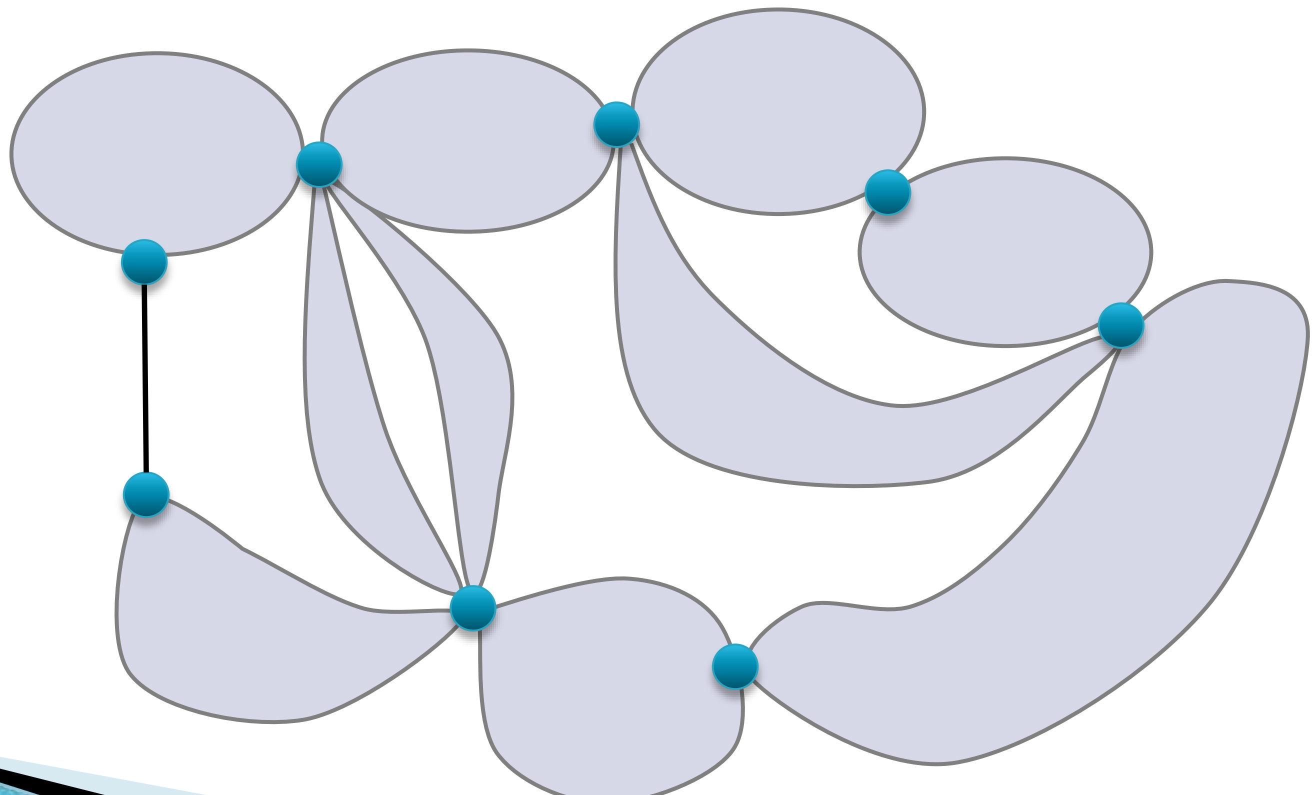
Quite a lot!

Connected graphs: data structure

- Block Cut-vertex tree (BC-tree)
  - A B-node for each block
  - A C-node for each cut-vertex

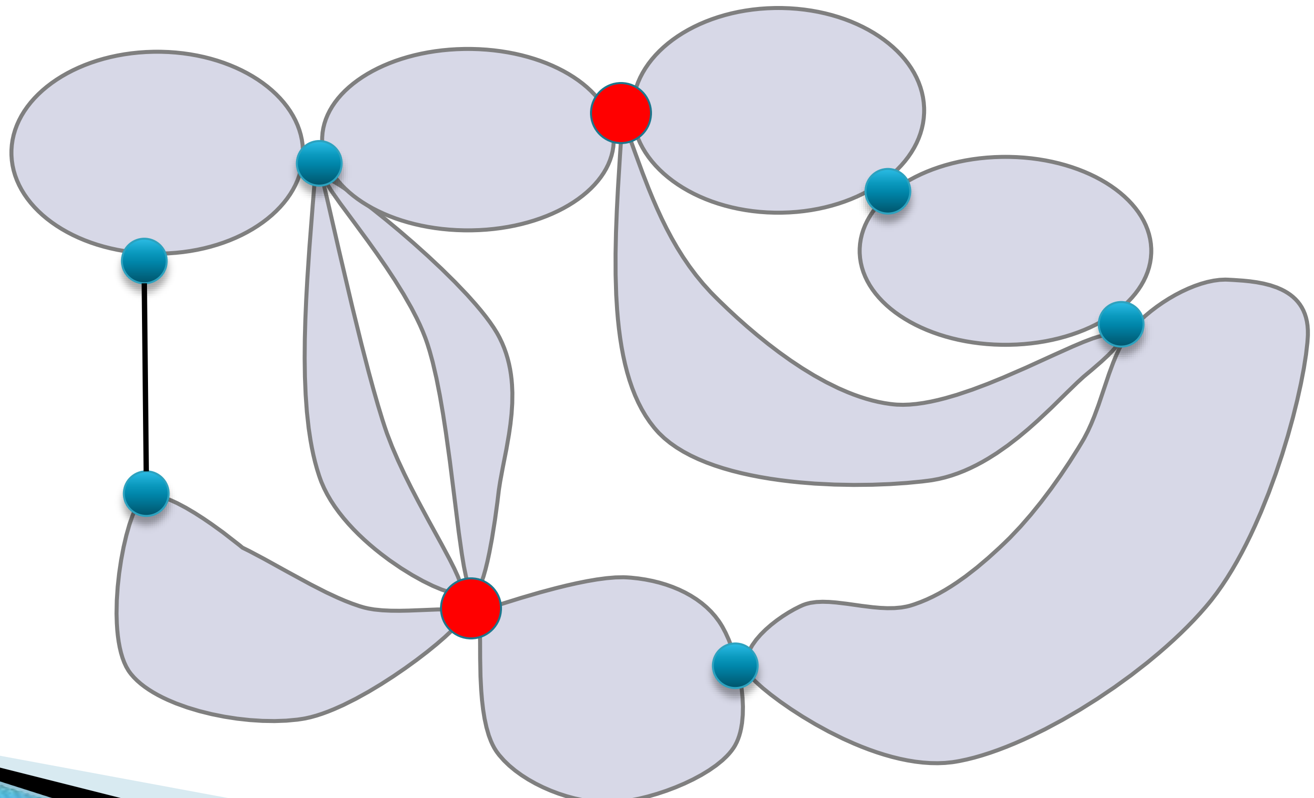


Biconnected: data structure



Biconnected: data structure

- There exist many separation pairs



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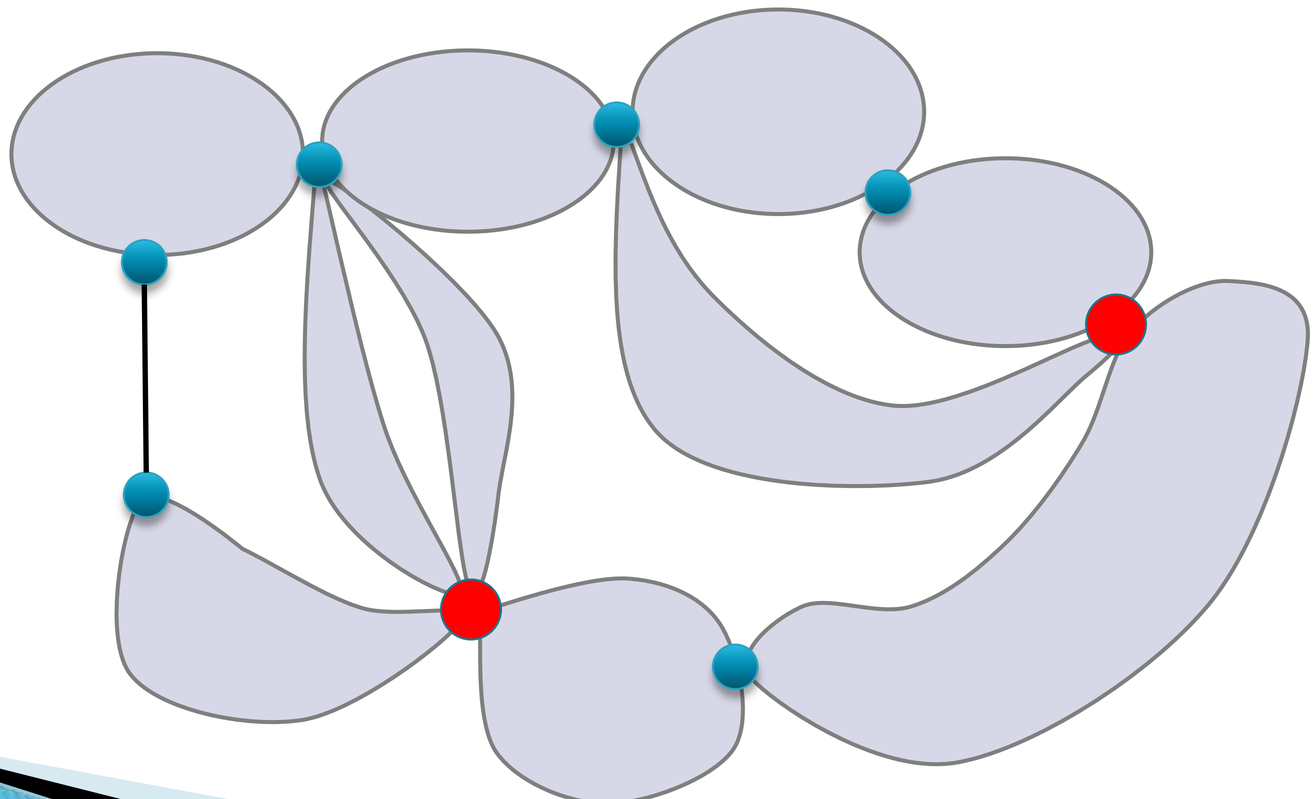

Biconnected: data structure

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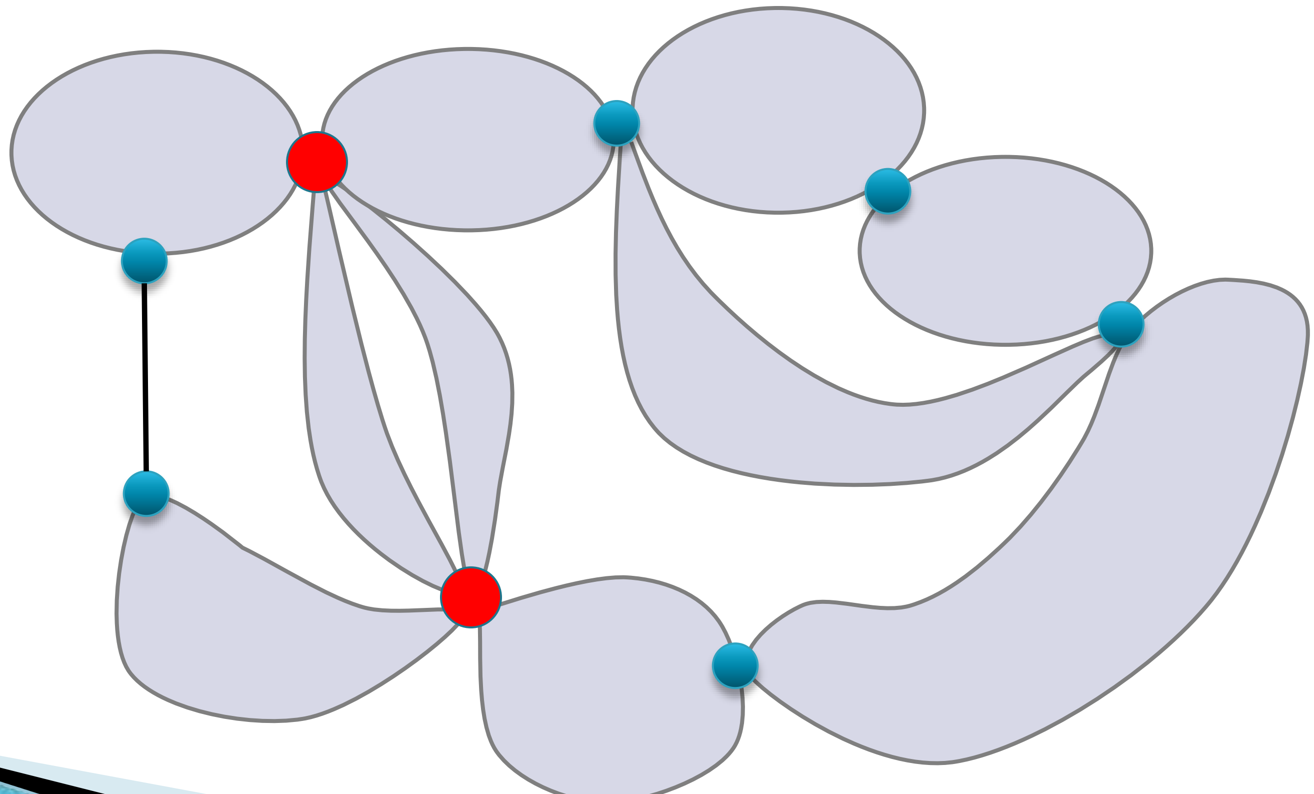

Biconnected: data structure

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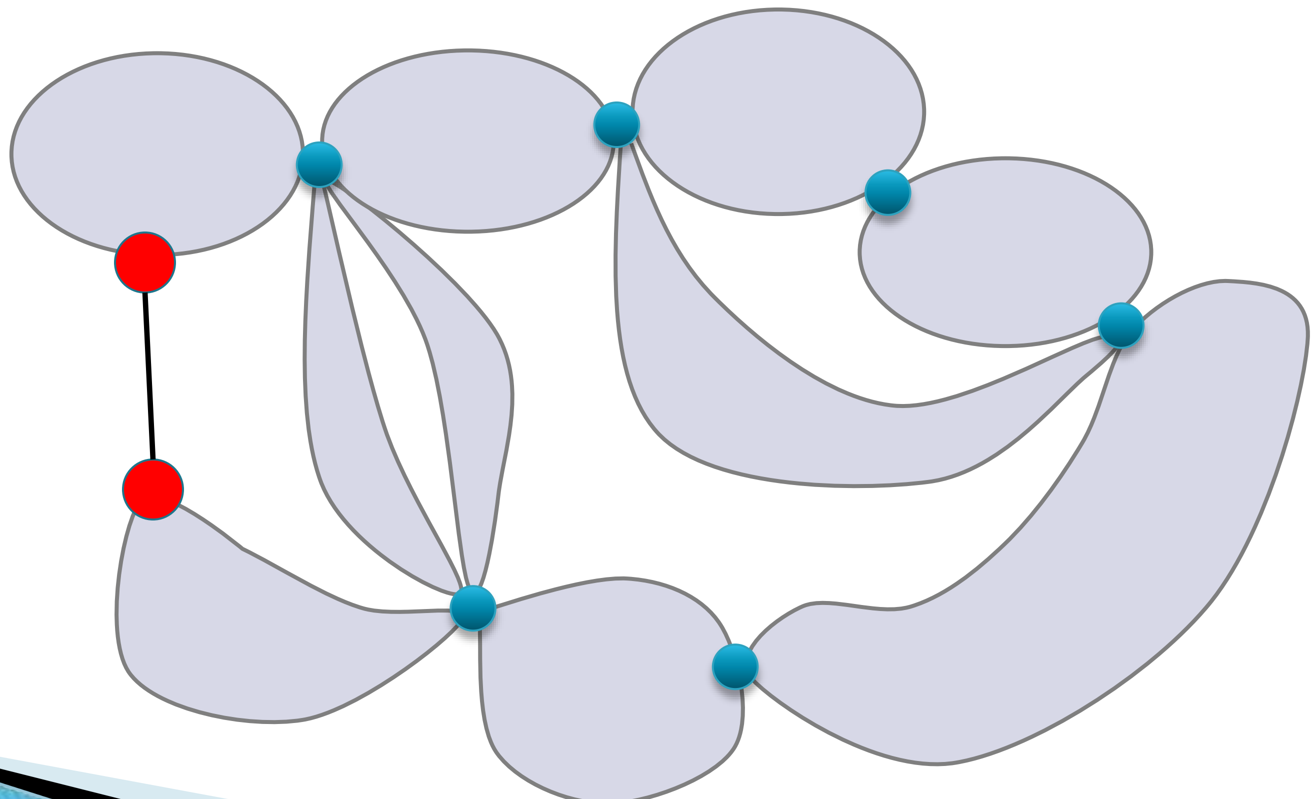
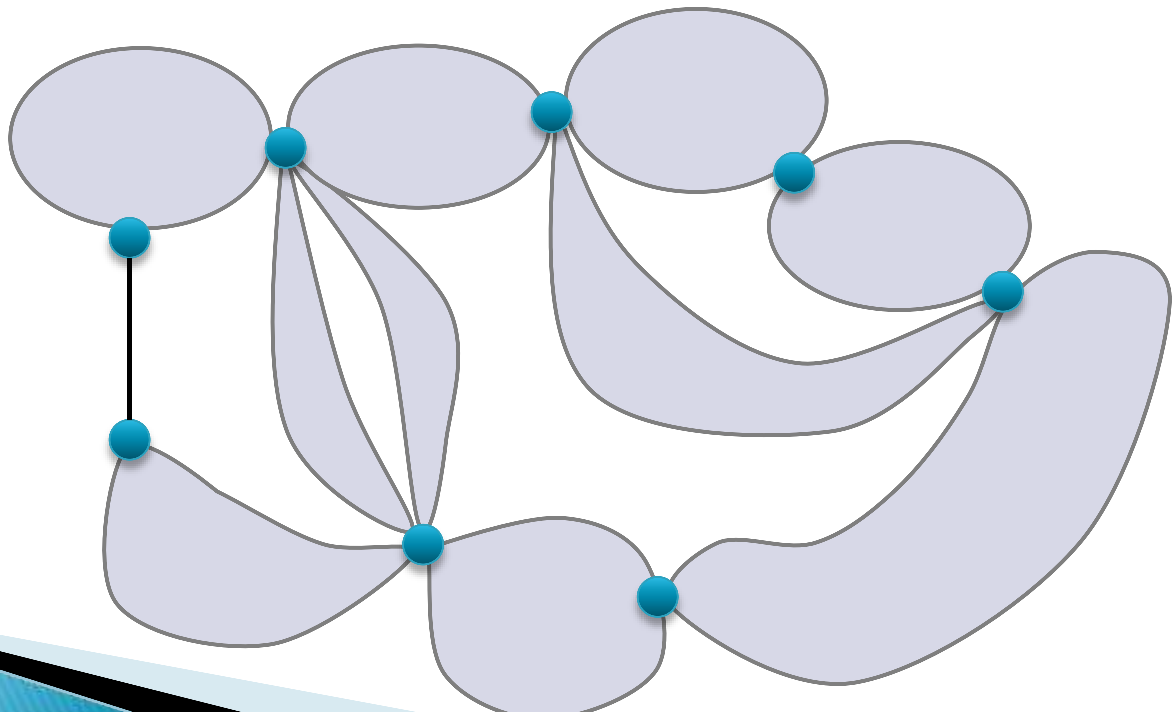


Biconnected: data structure

- We need a step-by-step decomposition
- At each step, we look at the graph from the point of view of a particular separation pair



SPQR-tree decomposition

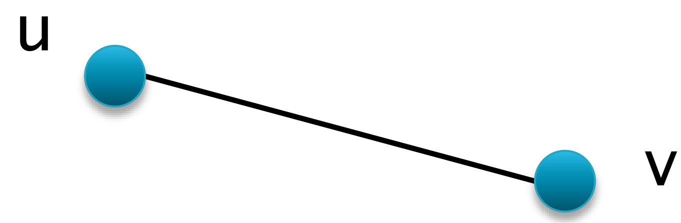
- A split pair {u,v} is a pair of vertices such that:
  - {u,v} is a separation pair, or
  - (u,v) is an edge
- An SPQR-tree is a rooted tree whose nodes are of 4 types:
  - Series-nodes
  - Parallel-nodes
  - Q-nodes
  - Rigid-nodes

SPQR-trees

- We first select any edge as the root
- At each step we consider a split pair {u,v} and add a node whose type depends on how the graph looks like from the point of view of {u,v}
- Each node of the SPQR-tree is associated with a multigraph, called *skeleton*, describing how the children of the node are arranged
  - Each child corresponds to an edge of the skeleton, called *virtual edge*

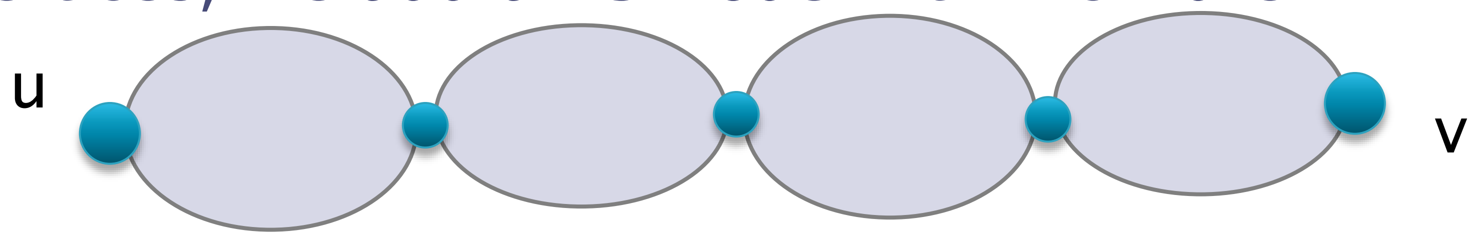
Q-node

- If the graph between the split pair {u,v} is an edge, we add a Q-node
- The skeleton of the Q-node is just an edge



S-node

- If the graph between the split pair {u,v} is a chain (series) of k components separated by cut-vertices, we add an S-node with k children



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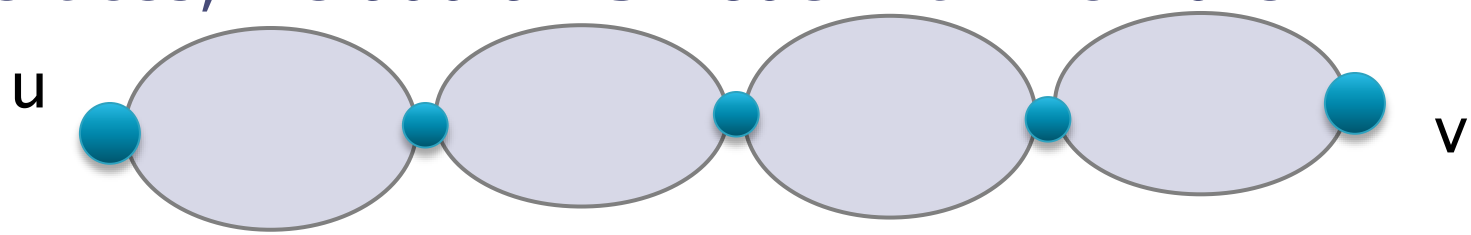
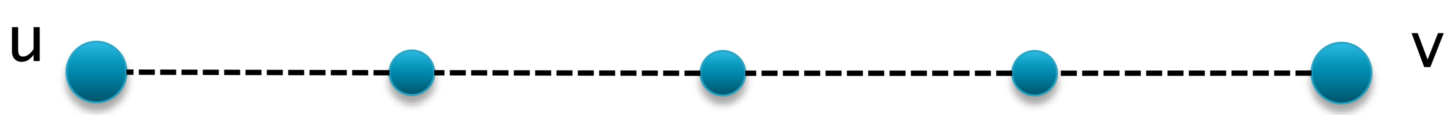


- The skeleton of the S-node is a path between u and v whose internal vertices are the cut-vertices



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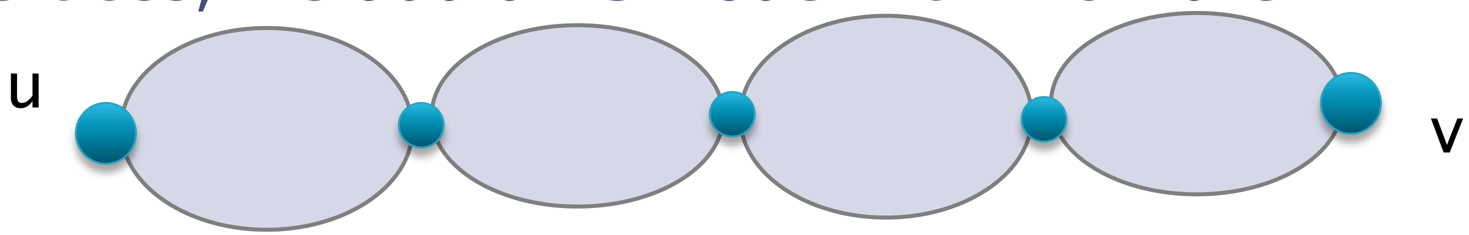
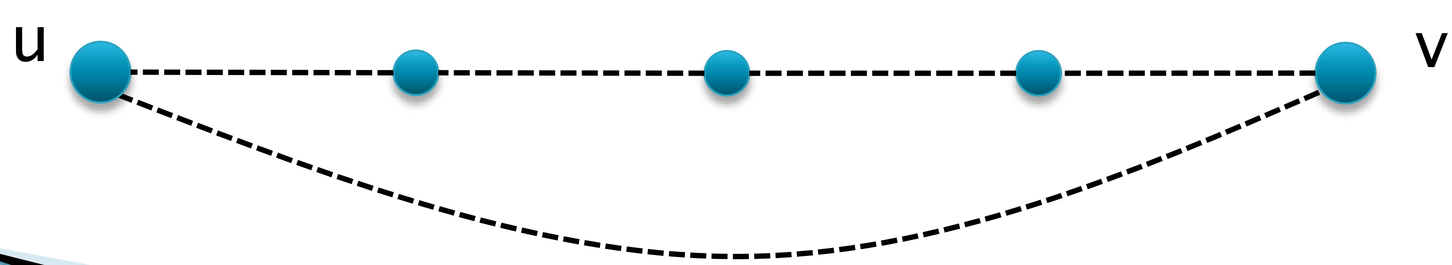
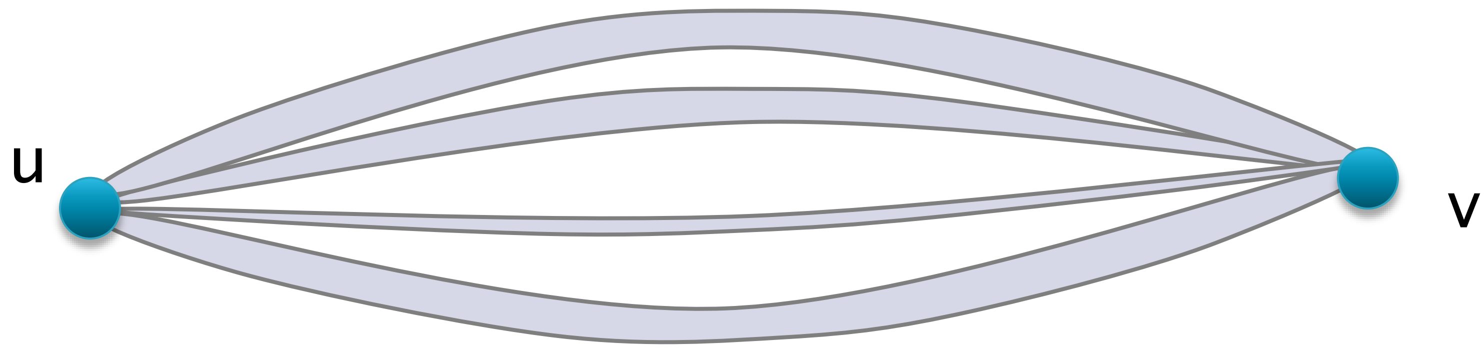


- We add a (virtual) edge between u and v that represents the "rest of the graph"



P-node

- If the graph between the split pair {u,v} is a composition of k parallel components, we add a P-node with k children



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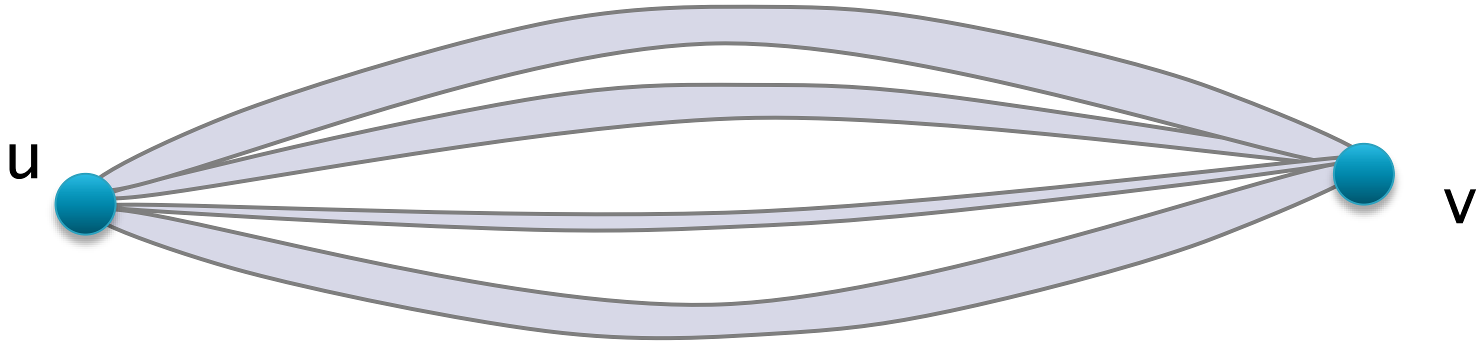
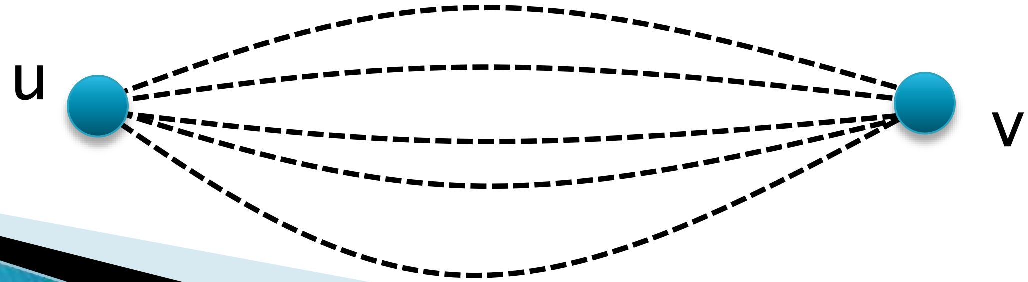
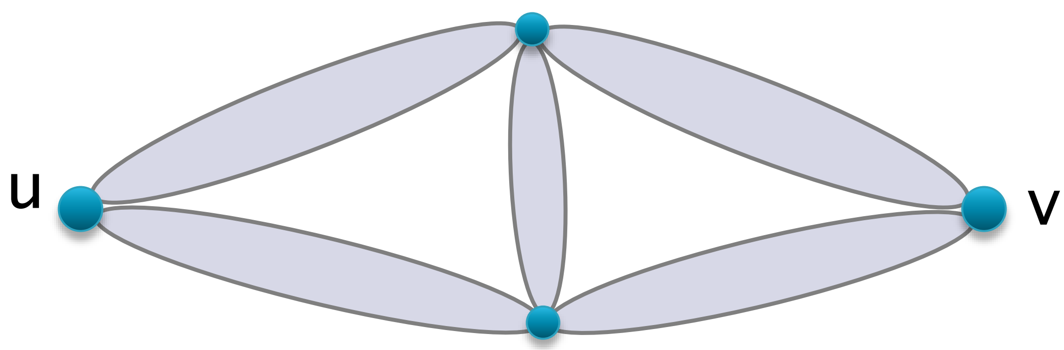


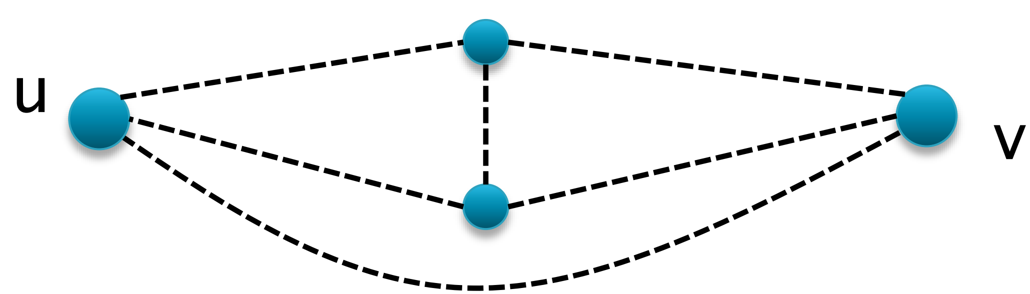
- The skeleton is composed of k+1 edges between u and v (one is for the rest of the graph)



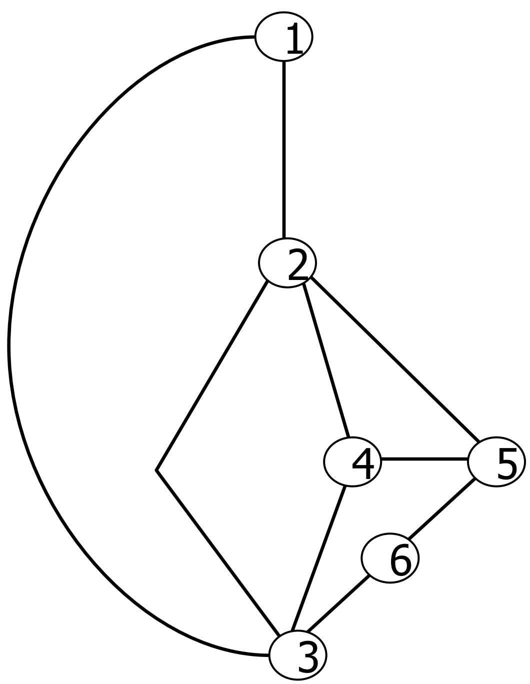
R-node

- In all the other cases, we add an R-node whose skeleton (including the edge between u and v) is a triconnected graph, and add a child for each edge of the skeleton (except for one)



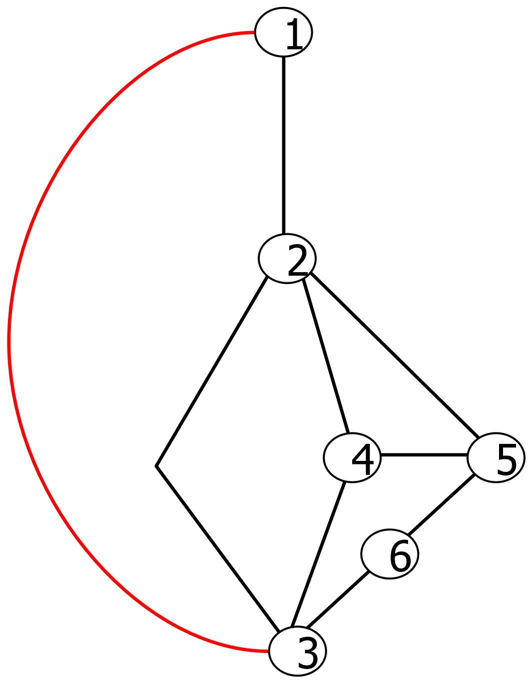


SPQR-tree: an example

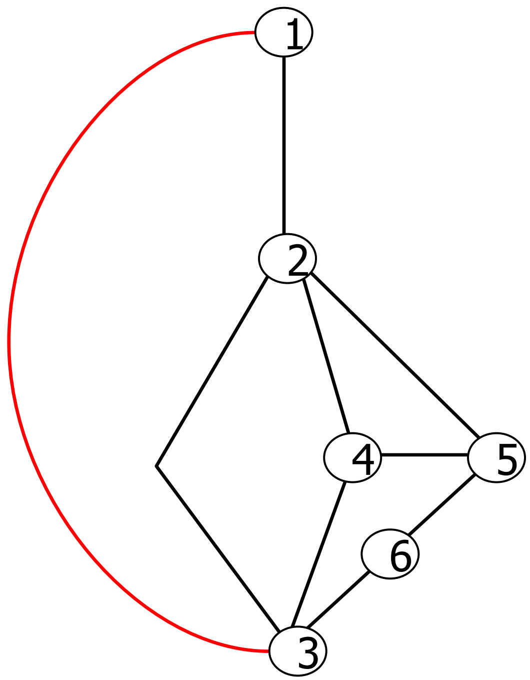


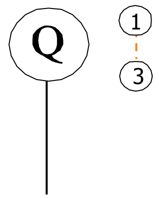
SPQR-tree: an example

Reference (root) edge



SPQR-tree: an example





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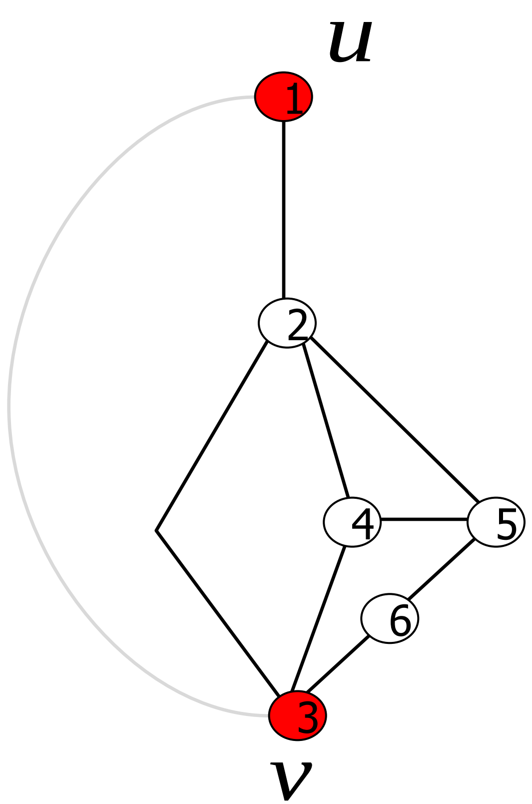
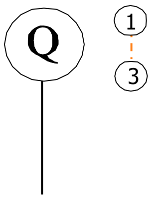
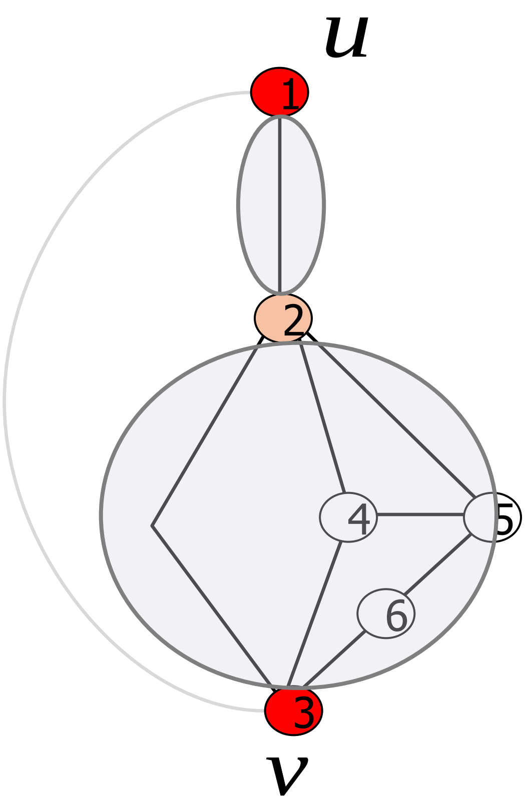
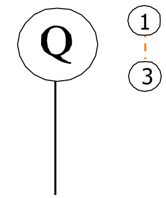




SPQR-tree: an example





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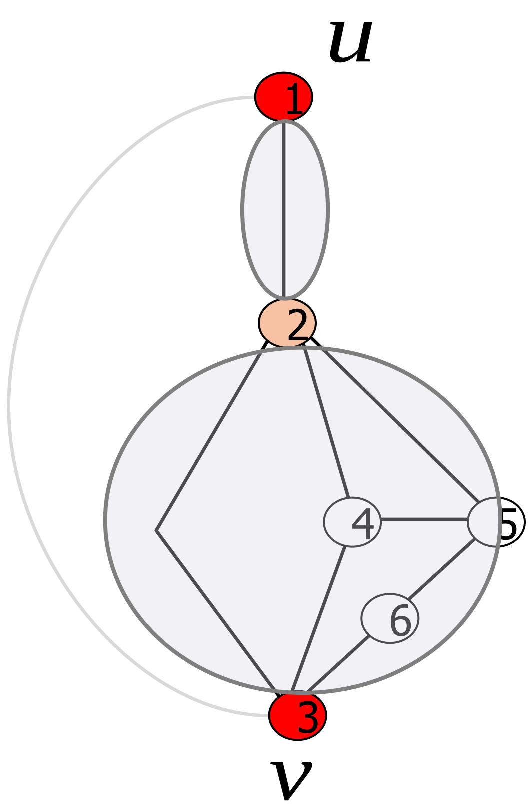
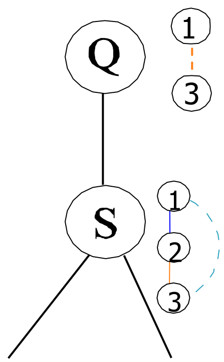
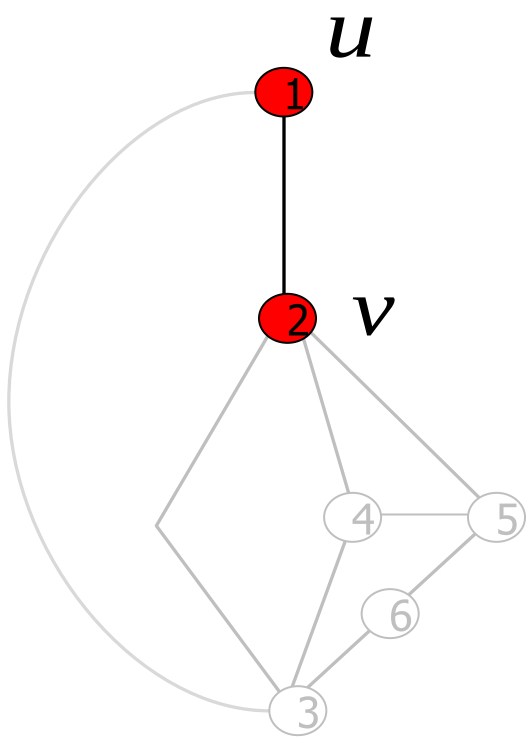
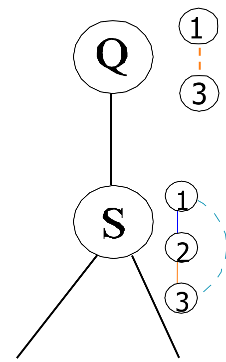


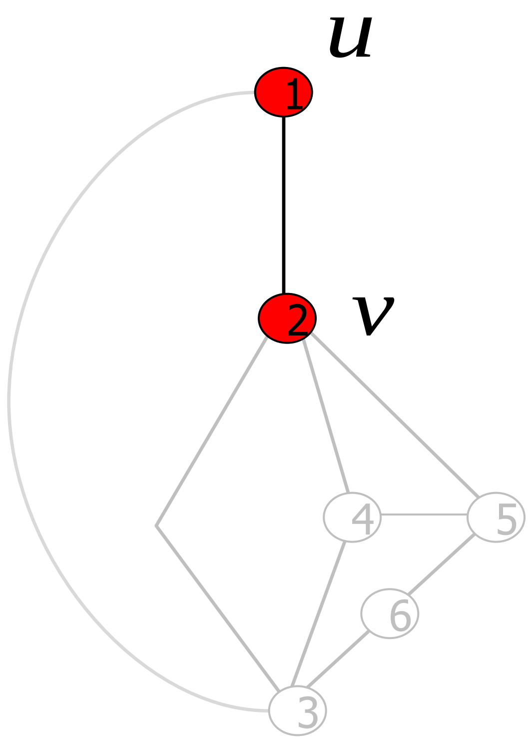


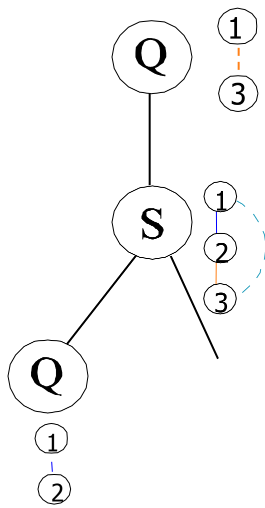
SPQR-tree: an example



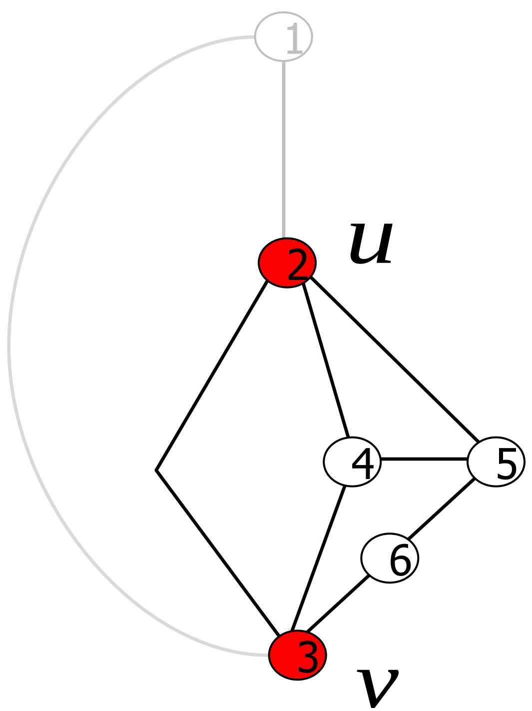


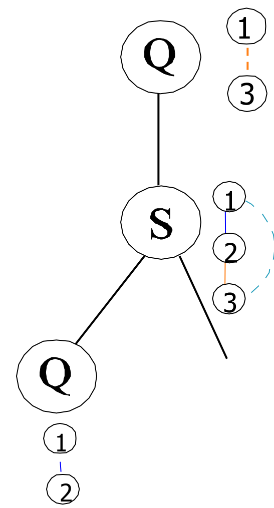
SPQR-tree: an example



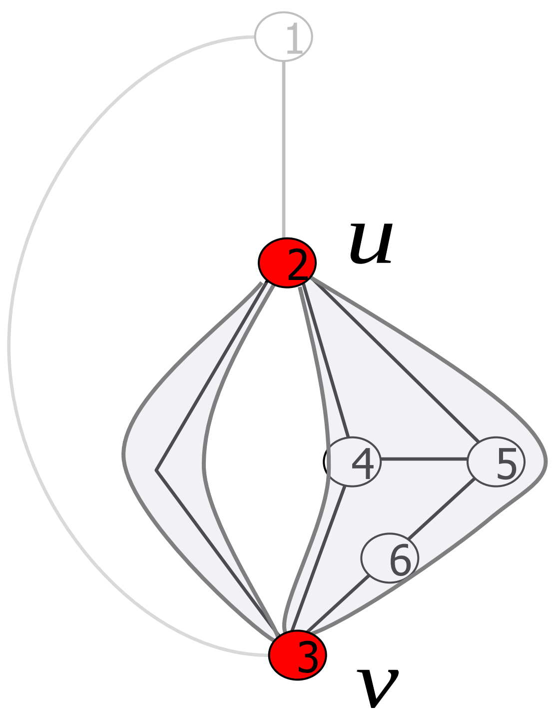


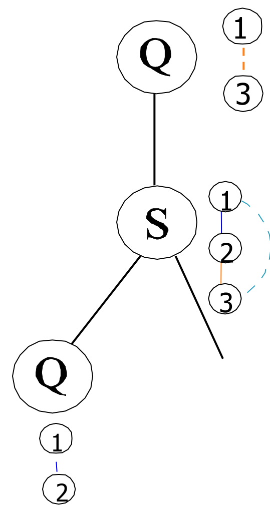
SPQR-tree: an example



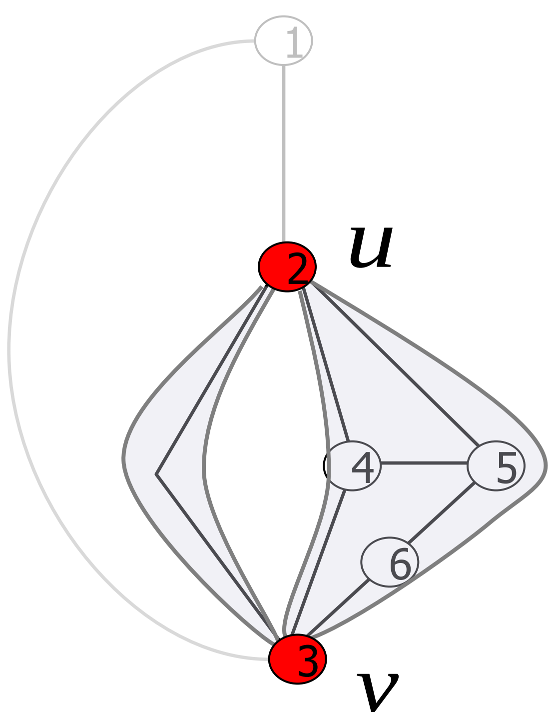


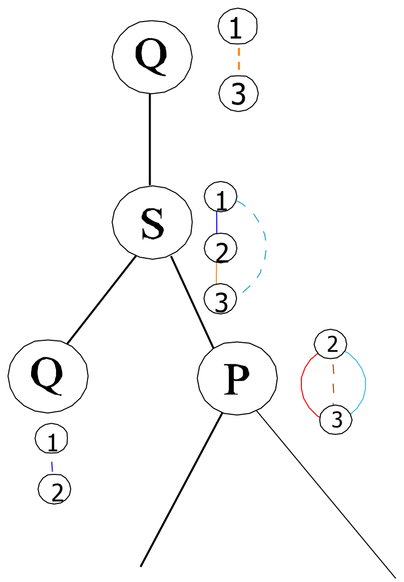
SPQR-tree: an example



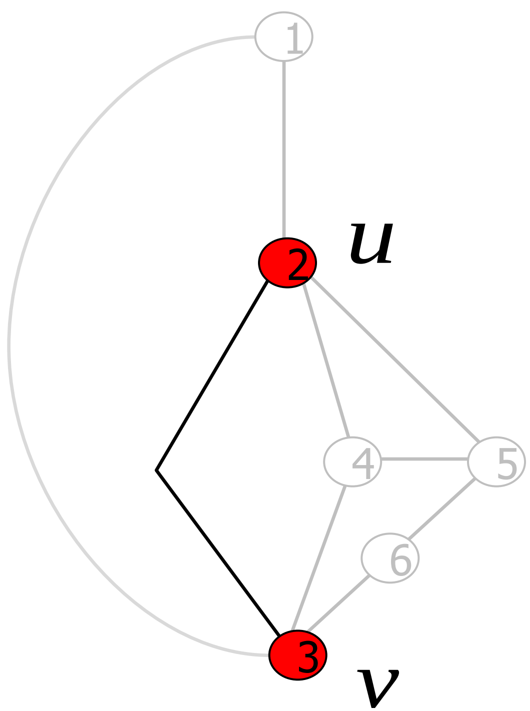


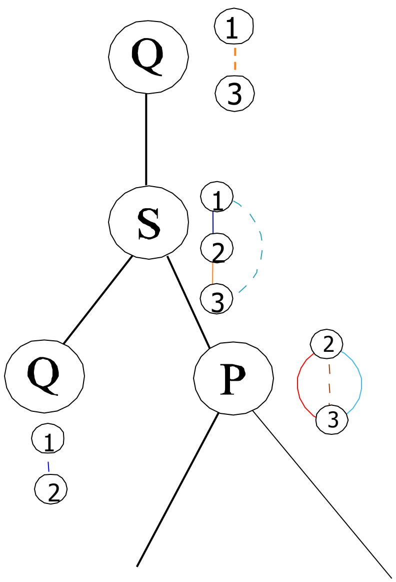
SPQR-tree: an example



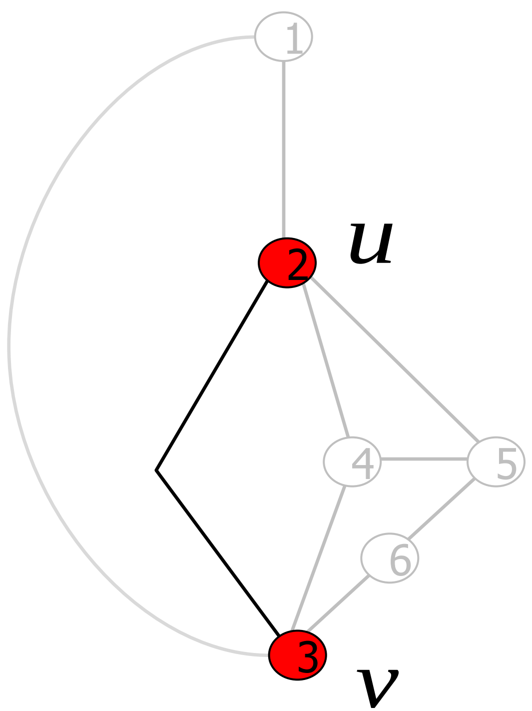


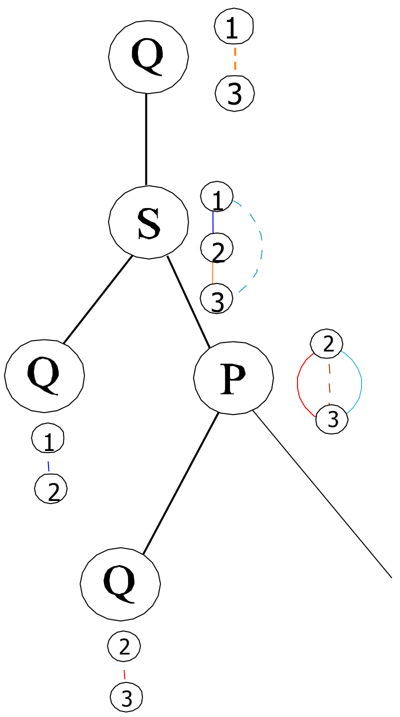
SPQR-tree: an example



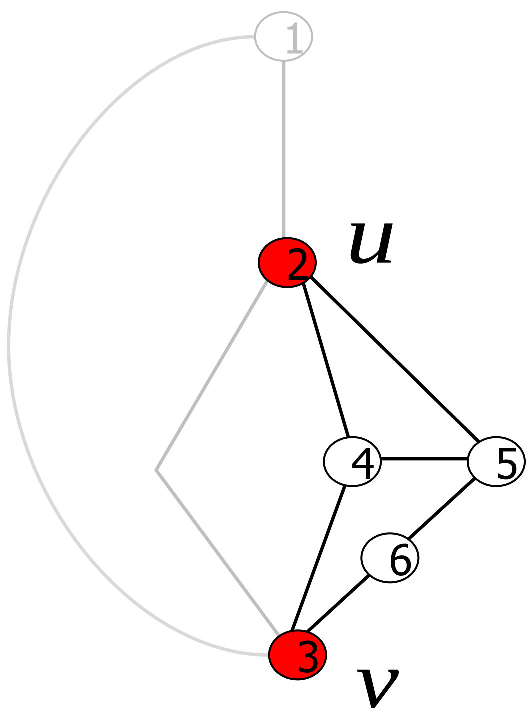


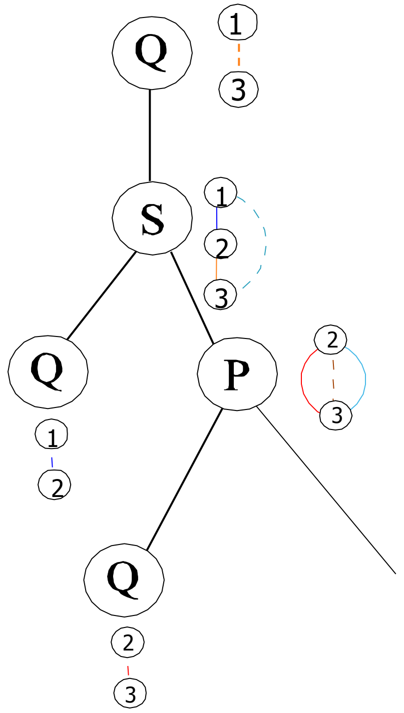
SPQR-tree: an example



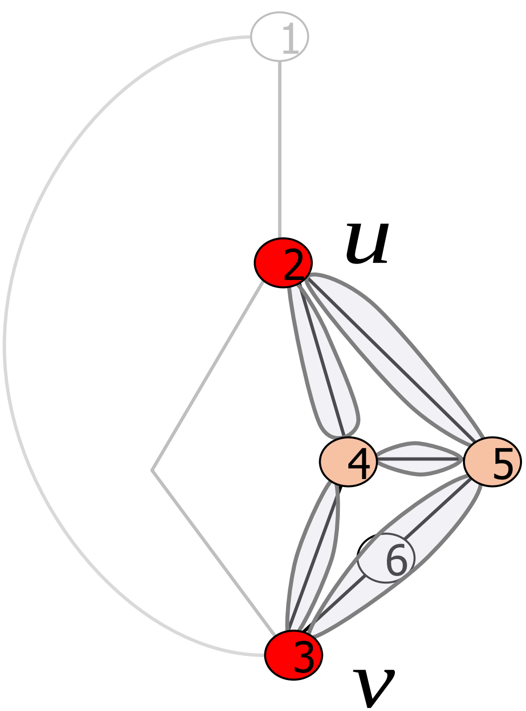


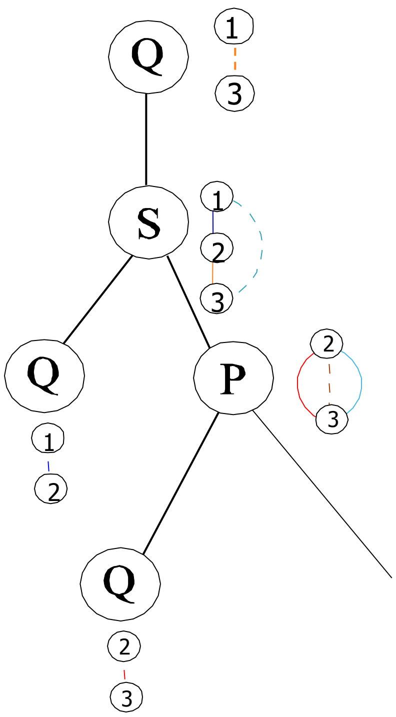
SPQR-tree: an example





SPQR-tree: an example





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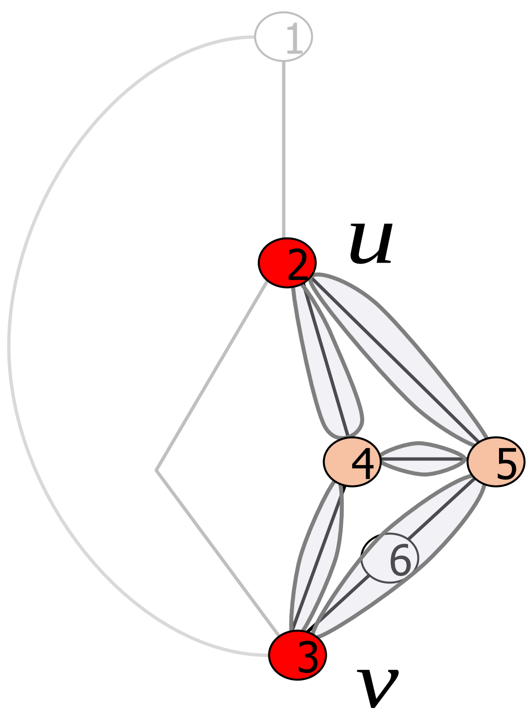

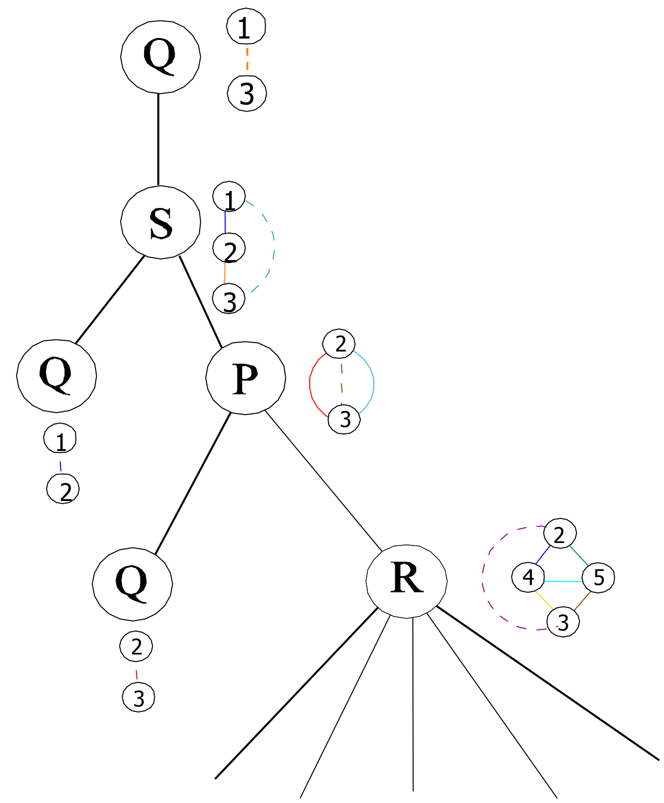

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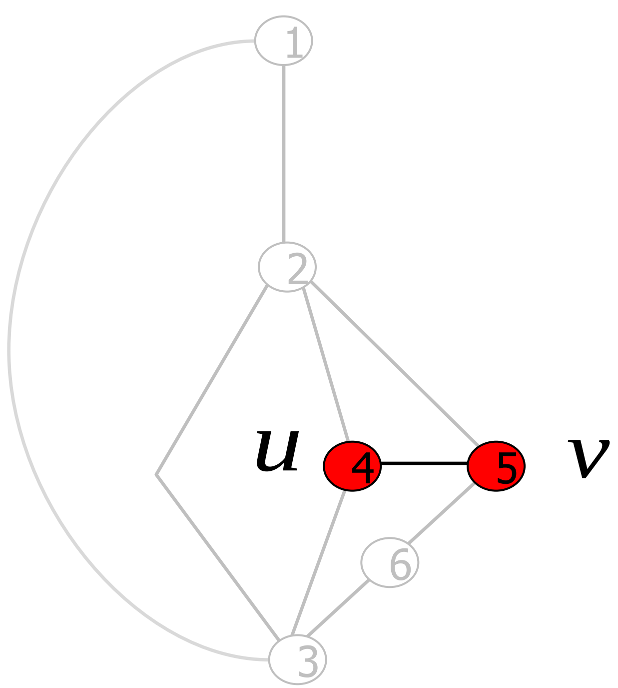
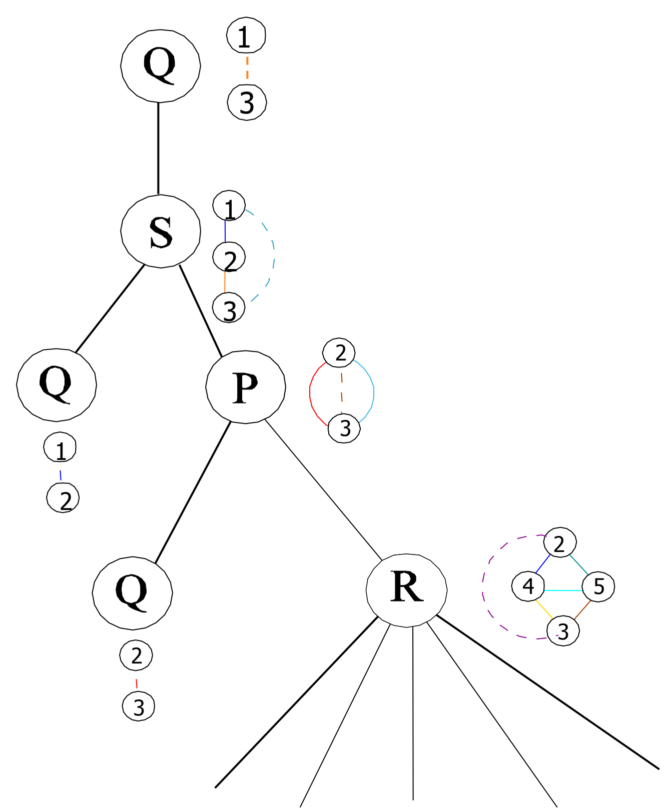
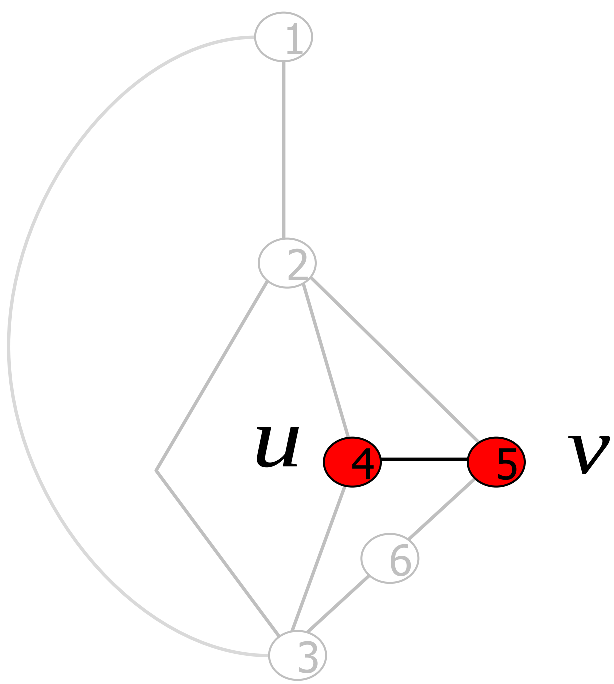
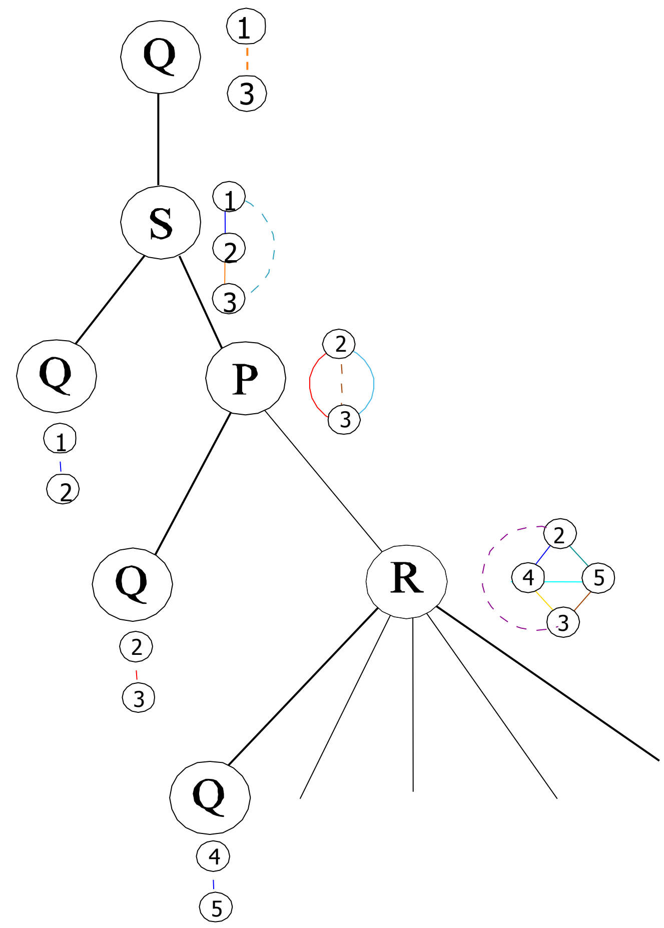


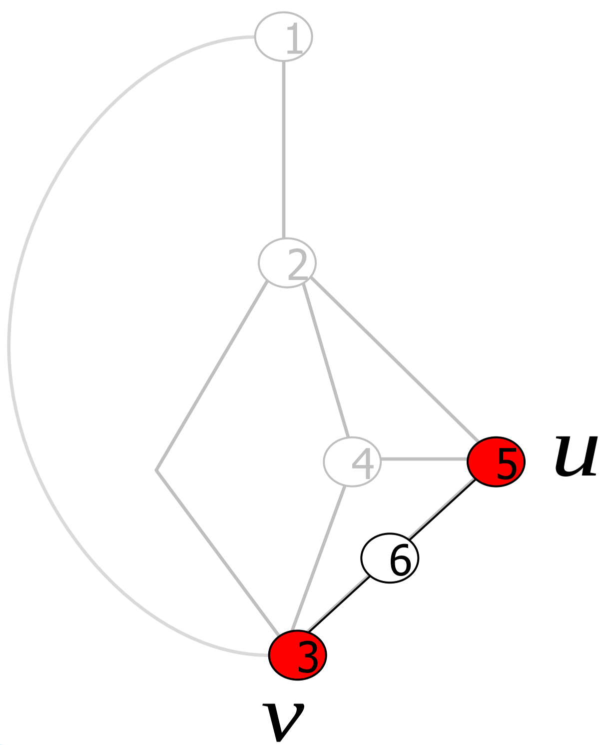


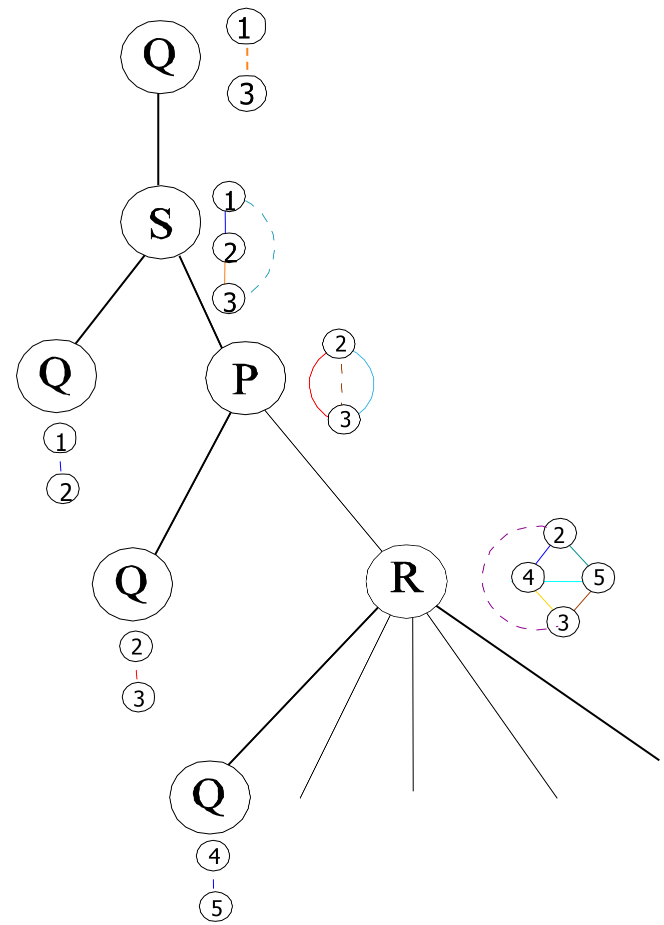
SPQR-tree: an example



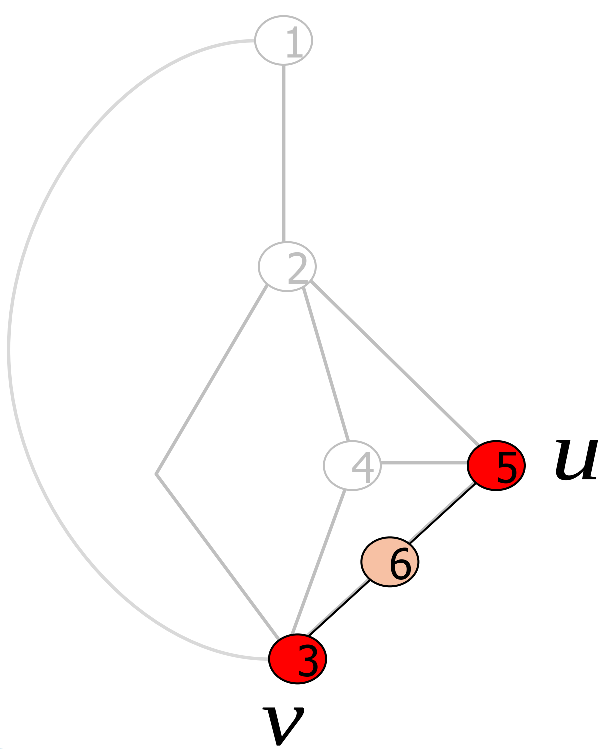


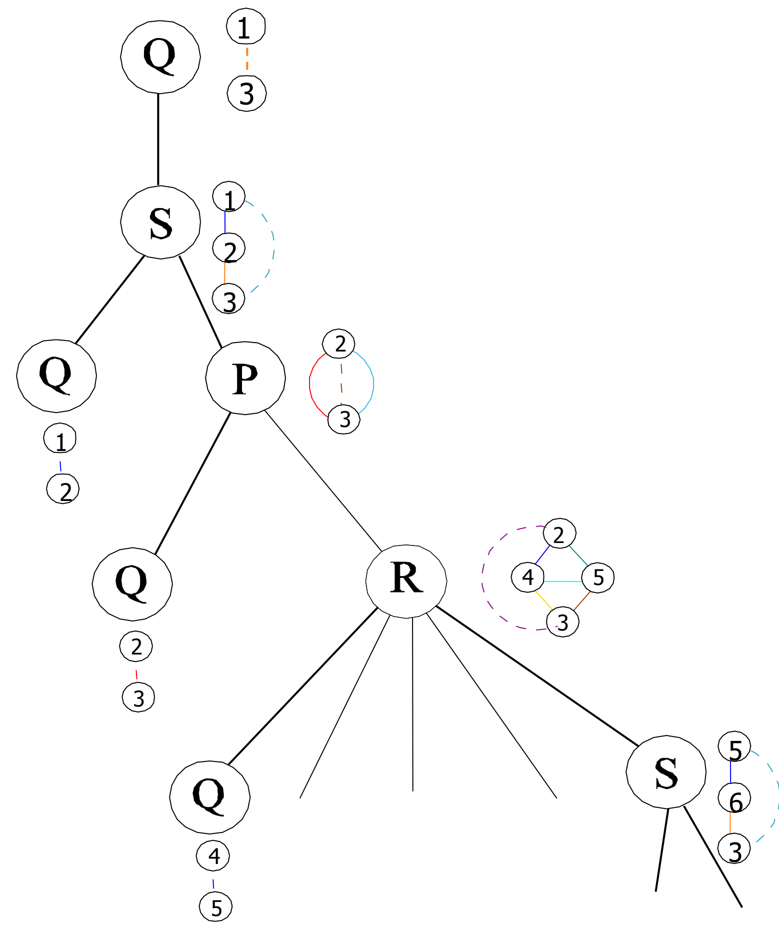
SPQR-tree: an example



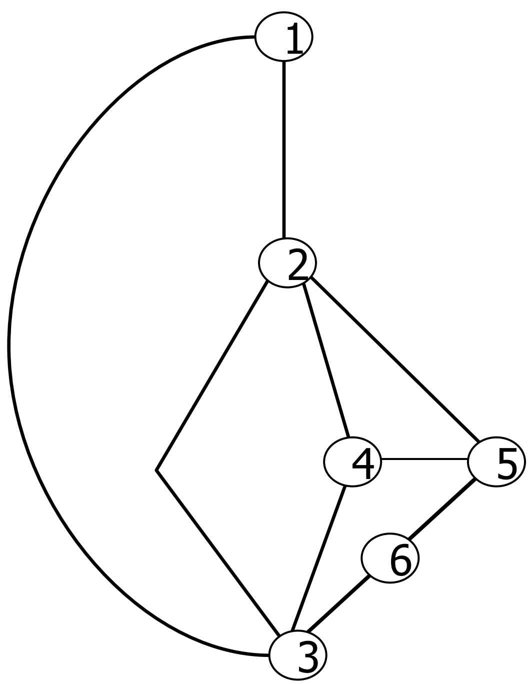


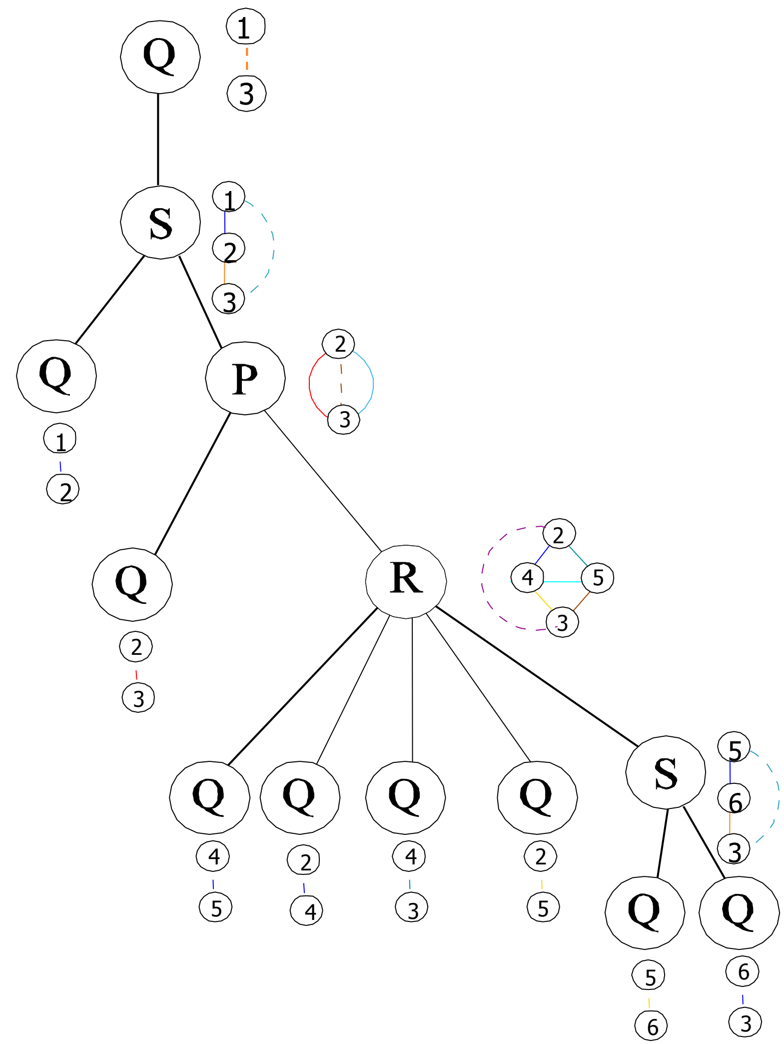
SPQR-tree: an example





SPQR-tree: an example





SPQR-trees – Embeddings

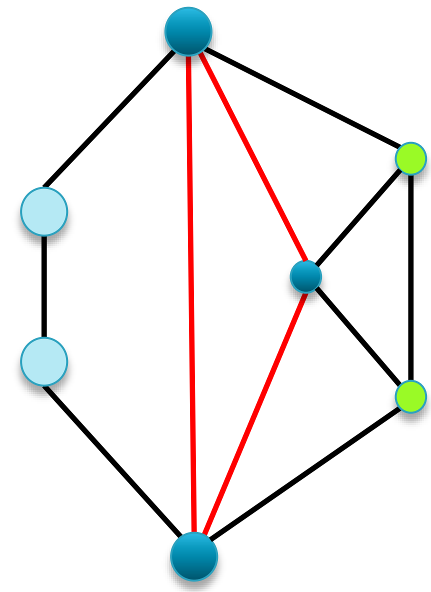
- To describe an embedding:
  - Flip of R-nodes skeletons
  - Ordering of multi-edges of P-nodes skeletons
- An SPQR-tree rooted at a reference edge *e* represents all the embeddings of the graph in which *e* is on the outer face
- If you choose a different reference edge, the resulting SPQR-tree is the same (up to a re-rooting)

SPQR-trees

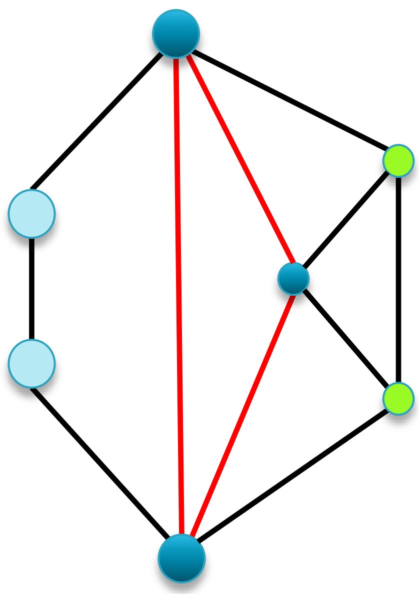
- G. Di Battista and R. Tamassia. *Incremental Planarity Testing*. *FOCS 1989*
- G. Di Battista and R. Tamassia. *On-Line Planarity Testing*. *SIAM Journal on Computing, 1996*
- Applications:
    - (Dynamic) Planarity testing
    - Navigating the graph (recursively) to compute embeddings, drawings, colorings, …
    - Computing an embedding that has some property or that is optimal with respect to some measure

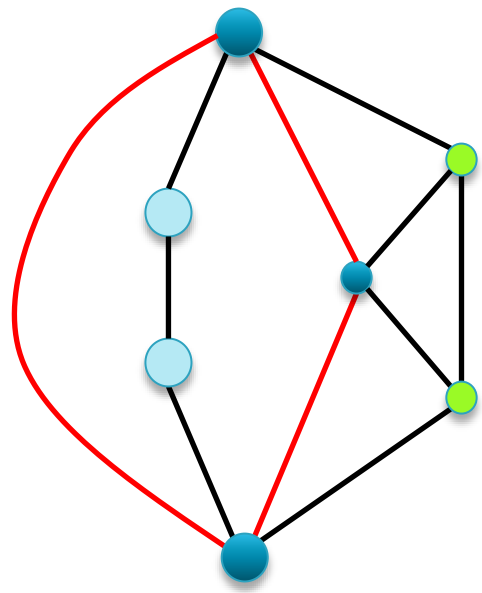
An application

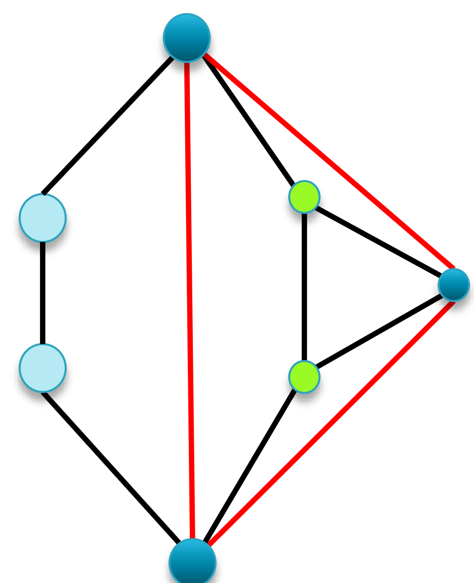
- Input:
  - A biconnected planar graph G;
  - A simple cycle C of G;
  - A partition of the vertices of G/C in two sets V1 and V2
- Output:
  - An embedding of G such that the vertices of V1 and those of V2 are separated by C
    - vertices of V1 are inside C and those of V2 are outside, or vice versa



An application

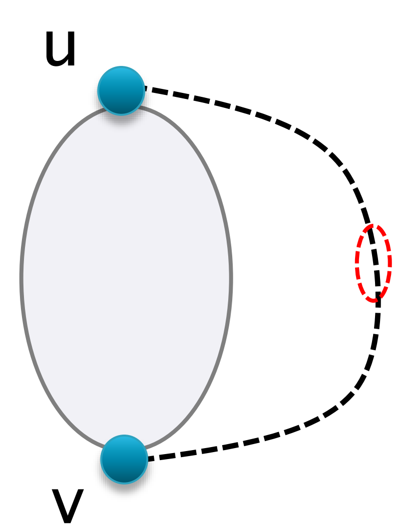






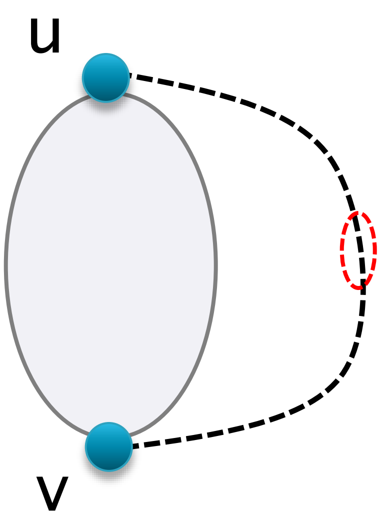
An application

- Given a split pair and a component with respect to it, there exist 3 possibilities:
  1. No vertex of C belongs to the component



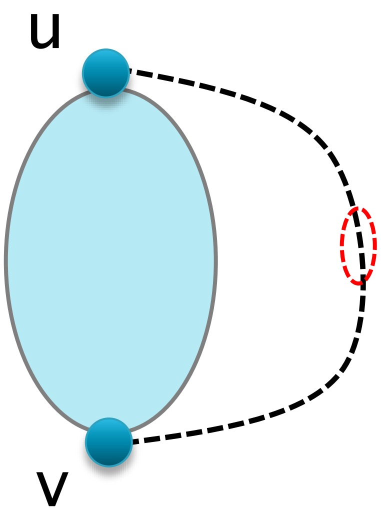
An application

- Given a split pair and a component with respect to it, there exist 3 possibilities:
  1. No vertex of C belongs to the component
     - All the vertices of the component must belong to the same set



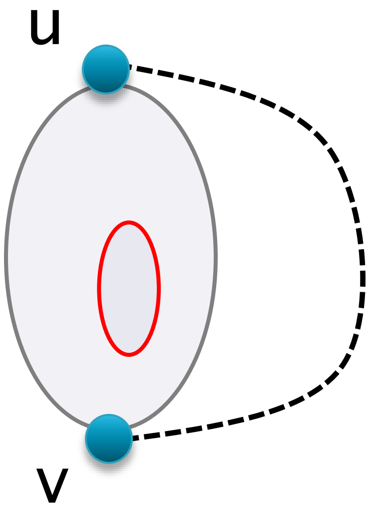
An application

- Given a split pair and a component with respect to it, there exist 3 possibilities:
  1. No vertex of C belongs to the component
     - All the vertices of the component must belong to the same set
     - The node is *1-colored*



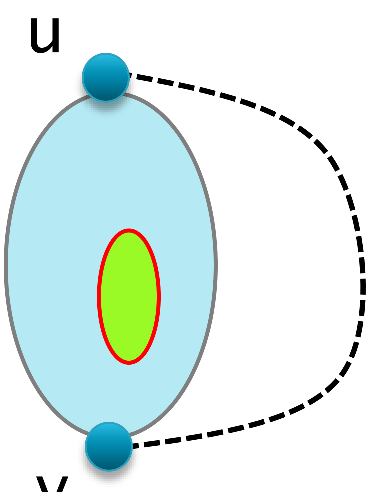
An application

2. All the vertices of C belong to the component
   - The node *contains* C



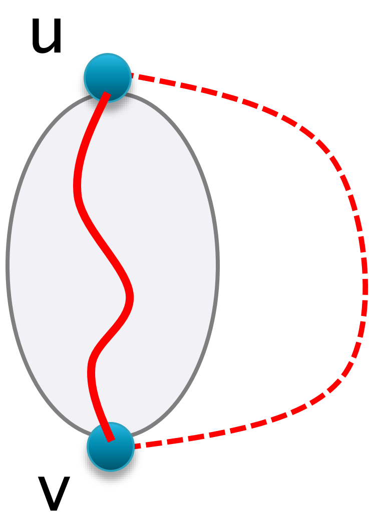
An application

2. All the vertices of C belong to the component
    - The node *contains* C
    - Vertices must be ‘‘correctly placed’’ inside/outside C



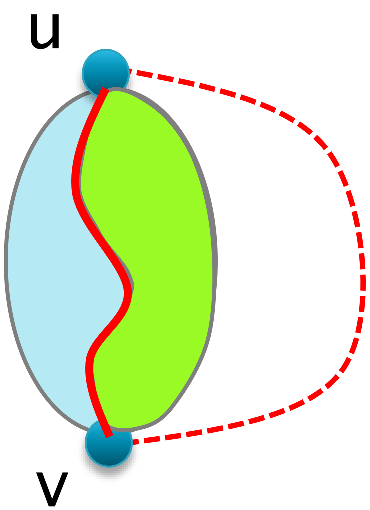
An application

3. Some (but not all) of the vertices of C belong to the component
   - The node is *traversed*



An application

3. Some (but not all) of the vertices of C belong to the component
   - The node is *traversed*
   - Vertices of the component that are separated by the path of C between u and v must belong to different sets
     - The node is *well-separated*

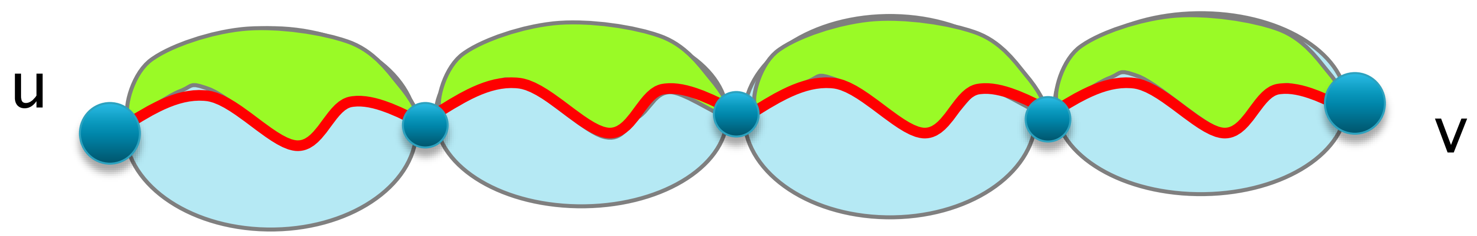


Algorithm

- Compute the SPQR-tree T of G rooted at any reference edge
- Perform a bottom-up visit of T
  - At each step, consider a node of T and test whether there exists an embedding of the skeleton of the node that satisfies the properties with respect to the cycle
  - The test is based on the fact that the all the children of the node in T have already been tested (and embedded)
  - Depending on the type of the node, the test and the embedding algorithm is different

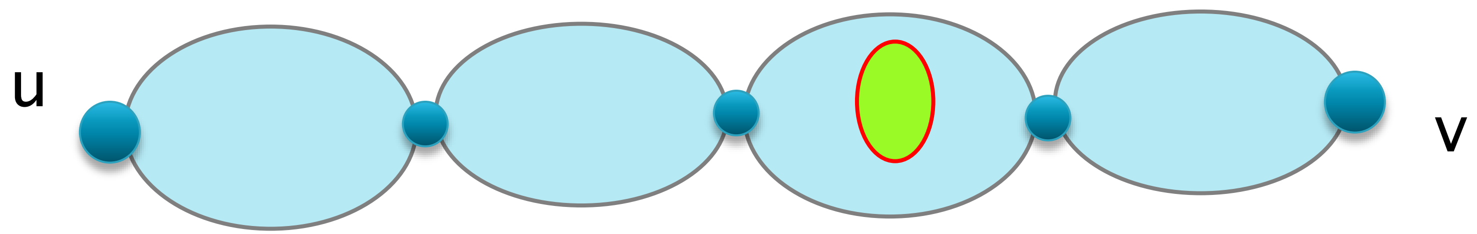
Algorithm: S-node

- If one of the children of the node is traversed by the cycle, then all the children are traversed (and the node itself is traversed)
  - Since all the children are well-separated by induction, the S-node can be made well-separated by flipping the children in such a way that elements of the same set are on the same side



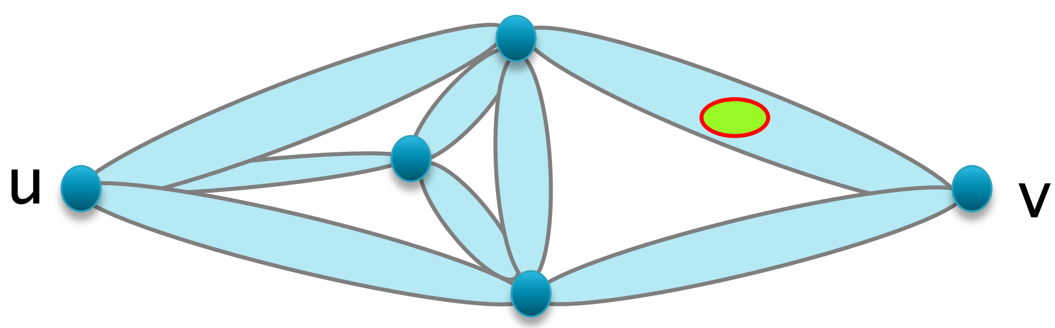
Algorithm: S-node

- If none of the children is traversed, then all of them (possibly except one) are 1-colored
  - Just check that the color is the same!
  - If one of the children contains C, then check if the color outside C is the same as the color of the others



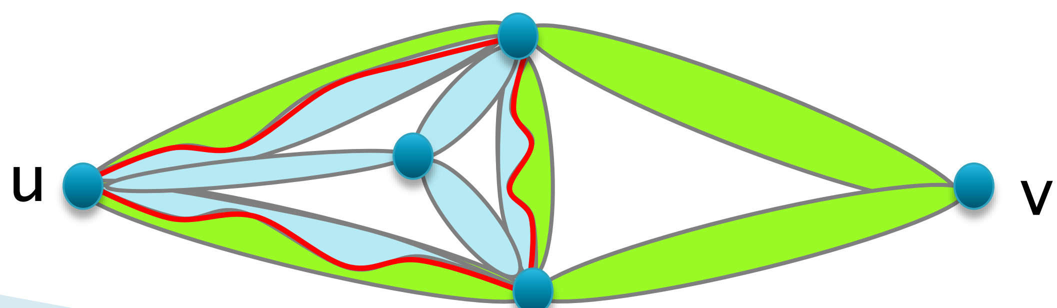
Algorithm: R-node

- If none of the children is traversed, then all of them (possibly except one) are 1-colored
  - Just check that the color is the same!
  - If one of the children contains C, then check if the color outside C is the same as the color of the others



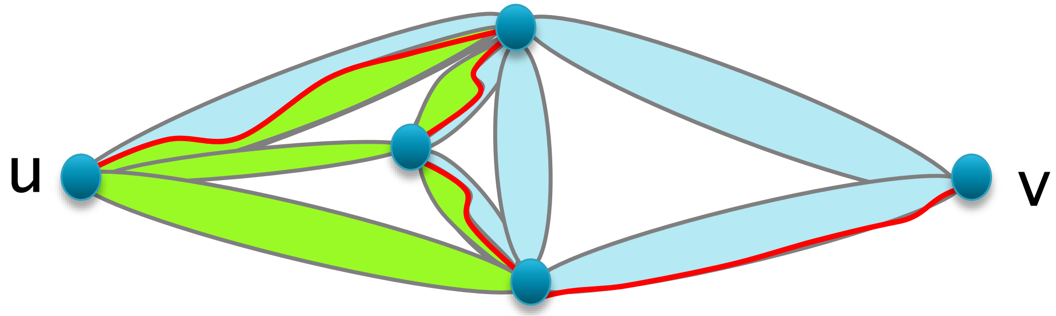
Algorithm: R-node

- If one of the children is traversed, then some of the others are traversed. Two cases:
  - If the node contains the whole cycle
    - All the not-traversed children are 1-colored
      - just check whether the color is the correct one
    - All the traversed children are well-separated
      - choose the correct flip



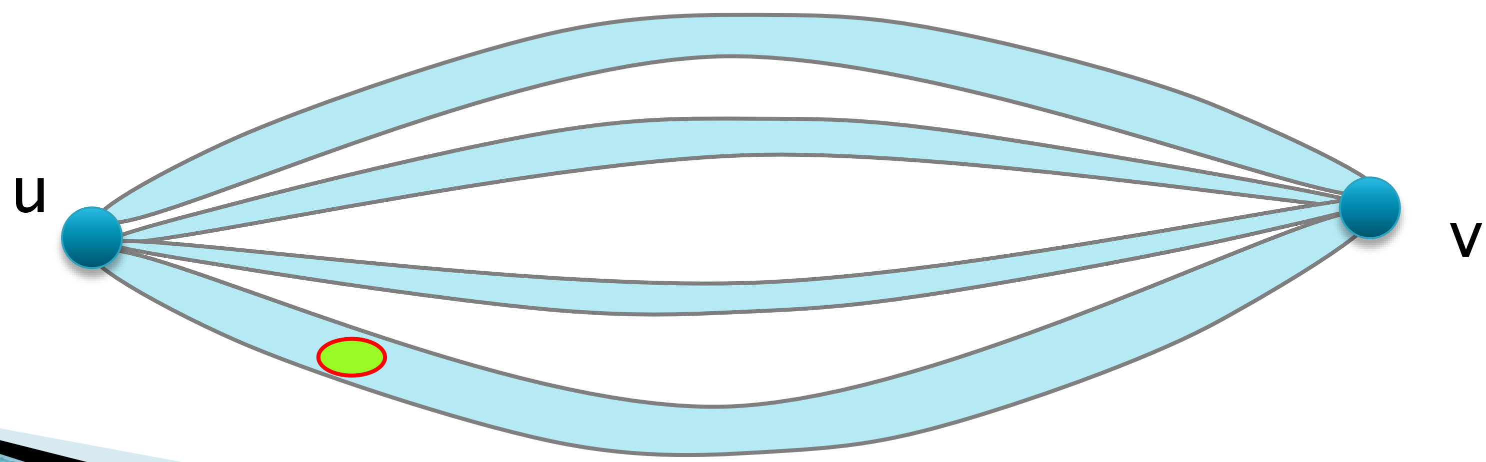
Algorithm: R-node

- If the node contains part of the cycle (the node is traversed)
  - All the not-traversed children are 1-colored
    - just check whether the color is the correct one to make the node well-separated
  - All the traversed children are well-separated
    - choose the correct flip



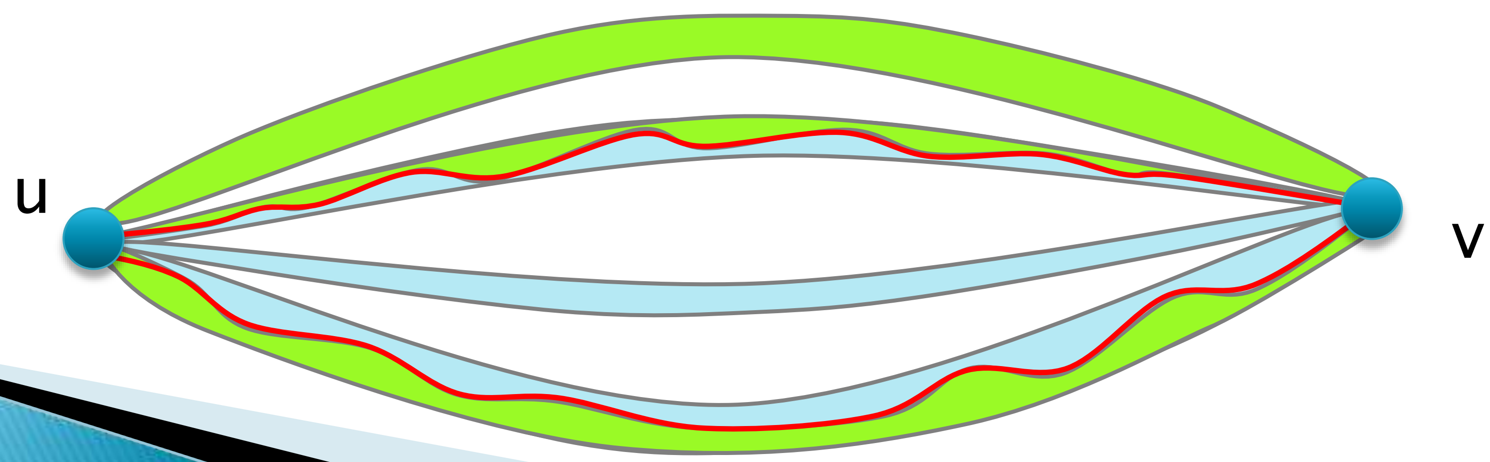
Algorithm: P-node

- If none of the children is traversed, then all of them (possibly except one) are 1-colored
  - Just check that the color is the same!
  - If one of the children contains C, then check if the color outside C is the same as the color of the others



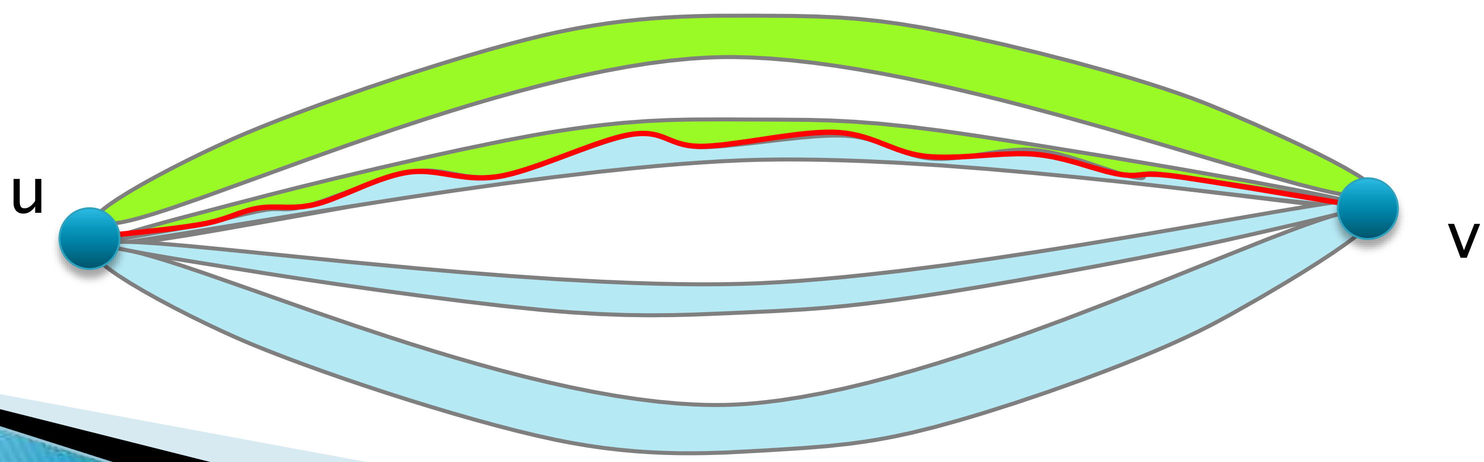
Algorithm: P-node

- At most 2 children are traversed
- If they are 2, the node contains the cycle
  - Order (permute) the children so that the 1-colored children are correctly placed inside/outside C
    - To choose the inside/outside, look at any vertex in the rest of the graph
  - Flip the 2 traversed children correctly



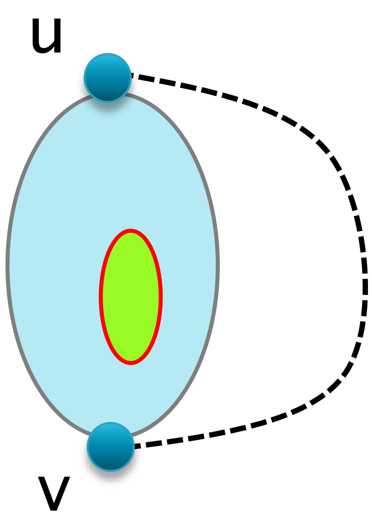
Algoritmo: P-node

- If there is 1 traversed child, the node is traversed
  - Order (permute) the children so that the 1-colored children are on different sides of the traversed child
    - Any left/right subdivision is good, we can flip the whole component later, if needed
  - Flip the traversed child correctly



Algorithm

- If the conditions are satisfied for every node, and in particular for the unique child of the root (that is considered at the last step of the bottom-up visit), the test is positive



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