Algorithms for graph visualization

Contact representations of planar graphs.
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General idea for the construction of a contact representation of a planar graph using \( n \)-gons in worst case.
## Contact representation

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Rectangular Dual

Rectangular Subdivision System

Let $R$ be a rectangle. A **rectangular subdivision system** $\Phi$ of $R$ is a partition of $R$ into a set of non-intersecting smaller rectangles such that no four of them meet at the same point.
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Rectangular Dual

A rectangular dual of a graph $G = (V, E)$ is a rectangular subdivision system $\Phi$ and a one-to-one correspondence $f : V \to \Phi$ such that $(u, v) \in E$ if and only if the rectangles $f(u)$ and $f(v)$, corresponding to $u$ and $v$, share a common boundary.
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Rectangular Dual

- Which graphs have a rectangular dual?
Which graphs have a rectangular dual?

**Separating triangle**

Let $G$ be a graph. A triangle $C$ of $G$ whose removal results in at least two disconnected components is called a **separating triangle** of $G$. 

![Diagram of a graph with a separating triangle highlighted in red.](image-url)
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Does not have a rectangular dual!
(In order to enclose an area we need at least four boxes)
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Does not have a rectangular dual!

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No four rectangles meet at a point! Each face of $G$ must be a triangle!
Rectangular Dual

Necessary conditions for a planar graph $G$ to have a rectangular dual:

- $G$ must have at least 4 vertices on the outer face
- $G$ must have no separating triangle
- each internal face of $G$ must be a triangle
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A planar graph $G = (V, E)$ has a rectangular dual $R$ with four rectangles on the boundary of $R$ if and only if the following conditions hold:

- Every interior face of $G$ is a triangle and the exterior face of $G$ is a quadrangle;
- $G$ has no separating triangles
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Proper Triangular Planar Graph (PTP)
Rectangular Dual

In order to construct a rectangular dual we need to partition our edges on vertical and horizontal. Regular edge labeling (REL, for short) is a tool for that.
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Regular edge labeling

For each internal vertex:

For the boundary vertices:
Rectangular Dual

Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of $G$ $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph $G_{k-1}$ induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary $C_{k-1}$ of $G_{k-1}$ contains the edge $(v_S, v_W)$.
- $v_k$ is in exterior face of $G_{k-1}$, and its neighbors in $G_{k-1}$ form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k - 2$, $v_k$ has at least 2 neighbors in $G \setminus G_{k-1}$.
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Canonical ordering with extra condition on $v_k$!
Rectangular Dual

**Theorem (Refined canonical ordering)**

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of $G$ $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:
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Canonical ordering with extra condition on $v_k$!

Home task!
Rectangular Dual

Refined canonical ordering
Rectangular Dual

Refined canonical ordering
Rectangular Dual

Refined canonical ordering

1 $v_S$
2 $v_W$
3 $v_E$
$\Sigma$

$\Sigma_N$
Rectangular Dual

Refined canonical ordering
Rectangular Dual

Refined canonical ordering

$v_N$

$v_E$

$v_S$

$v_W$

1 2 3 4
Rectangular Dual

Refined canonical ordering
Rectangular Dual

Refined canonical ordering
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Rectangular Dual

Refined canonical ordering

\begin{tikzpicture}
  \node (vn) at (0,0) [circle, fill, inner sep=2pt] {$v_N$};
  \node (vw) at (-2,-3) [circle, fill, inner sep=2pt] {$v_W$};
  \node (ve) at (2,-3) [circle, fill, inner sep=2pt] {$v_E$};
  \node (v1) at (-1,-4) [circle, fill, inner sep=2pt] {$v_1$};
  \node (v2) at (-2,-5) [circle, fill, inner sep=2pt] {$v_2$};
  \node (v3) at (-1,-6) [circle, fill, inner sep=2pt] {$v_3$};
  \node (v4) at (-2,-7) [circle, fill, inner sep=2pt] {$v_4$};
  \node (v5) at (-1,-8) [circle, fill, inner sep=2pt] {$v_5$};
  \node (v6) at (-2,-9) [circle, fill, inner sep=2pt] {$v_6$};

  \draw (vn) -- (vw);
  \draw (vn) -- (ve);
  \draw (vn) -- (v1);
  \draw (vn) -- (v2);
  \draw (vn) -- (v3);
  \draw (vn) -- (v4);
  \draw (vn) -- (v5);
  \draw (vn) -- (v6);

  \draw (vw) -- (v1);
  \draw (vw) -- (v2);
  \draw (vw) -- (v3);
  \draw (vw) -- (v4);
  \draw (vw) -- (v5);
  \draw (vw) -- (v6);

  \draw (ve) -- (v1);
  \draw (ve) -- (v2);
  \draw (ve) -- (v3);
  \draw (ve) -- (v4);
  \draw (ve) -- (v5);
  \draw (ve) -- (v6);

  \draw (v1) -- (v2);
  \draw (v1) -- (v3);
  \draw (v1) -- (v4);
  \draw (v1) -- (v5);
  \draw (v1) -- (v6);

  \draw (v2) -- (v3);
  \draw (v2) -- (v4);
  \draw (v2) -- (v5);
  \draw (v2) -- (v6);

  \draw (v3) -- (v4);
  \draw (v3) -- (v5);
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\end{tikzpicture}
Rectangular Dual

Refined canonical ordering
Rectangular Dual

Refined canonical ordering
Rectangular Dual

Given a refined canonical ordering of $G$ we construct a REL as follows:

- For each $(v_i, v_j)$ orient it from $v_i$ to $v_j$, for $i < j$;
- Base edge of $v_k$ is $(v_l, v_k)$, where $l < k$ is minimal.
- $v_k$ has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that $v_{t_1}$ is left point of $v_k$ and $v_{t_l}$ is right point of $v_k$.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of $v_k$, we call $(v_k, v_{k_1})$ left edge and $(v_k, v_{k_l})$ right edge.
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**Lemma 1**
Left edge or right edge can not be a base edge.
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Lemma 1
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Proof: Assume that left edge $(v_k, v_{k_1})$ is the base edge of $v_{k_1}$.
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- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of $v_k$, we call $(v_k, v_{k_1})$ left edge and $(v_k, v_{k_l})$ right edge.

**Lemma 2**

An edge is either a left edge, a right edge or a base edge.

**Proof:**

- The exclusive “or” follows from Lemma 1.
- Let $(v_{t_a}, v_k)$ be base edge of $v_k$.
- $v_{t_a}$ is right point of $v_{t_{a-1}}$, $v_{t_{a-1}}$ is right point of $v_{t_{a-2}}$, generally $v_{t_{i+1}}$ is right point of $v_{t_i}, 1 \leq i < a - 1$.
- Edges $(v_{t_i}, v_{k}), 1 \leq i < a - 1$, are right edges;
- Similarly we prove that edges $(v_{t_i}, v_{k}), a+1 \leq i < l$, are left edges;
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- Edges $(v_{t_i}, v_k)$, $1 \leq i < a - 1$, are right edges;
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Rectangular Dual

right edges

left edges

basis edge

$v_k$
Rectangular Dual
Rectangular Dual

right edges

left edges

basis edge

$v, k$
Rectangular Dual

right edges

left edges

basis edge

right edges

left edges

basis edge

basis edge

left edges
We call $T_b$ blue edges and $T_r$ red edges.
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**Lemma 3**

$\{T_r, T_b\}$ is a regular edge labeling.

**Proof:**

$k_l \geq 2$
We call $T_b$ blue edges and $T_r$ red edges.

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**Proof:**

$k_d = \max\{v_{k_1} \cdots v_{k_l}\}$

The base edges of $v_{k_2} \cdots v_{k_{l-1}}$ have $k_l \geq 2$. 

Left edge of $v_k$ 

Right edge of $v_k$
We call $T_b$ blue edges and $T_r$ red edges.

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**Proof:**

$k_d = \max\{v_{k_1}, \ldots, v_{k_l}\}$

$k_1 < k_2 < \cdots < k_d$ and $k_d > k_{d+1} > \cdots > k_l$

$\mathcal{U}_{k_1}$

left edge of $v_k$

$\mathcal{U}_k$

right edge of $v_k$

$k_l \geq 2$
We call $T_b$ blue edges and $T_r$ red edges.

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**Proof:**

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$k_1 < k_2 < \cdots < k_d$ and $k_d > k_{d+1} > \cdots > k_l$

$(v_k, v_{k_i}), 2 \leq i \leq d - 1$ are red

$(v_k, v_{k_i}), d + 1 \leq i \leq l - 1$ are blue

edge $(v_k, v_{k_d})$ is either red or blue
We call $T_b$ blue edges and $T_r$ red edges.

**Lemma 3**

\(\{T_r, T_b\}\) is a regular edge labeling.

**Proof:**

\[k_d = \max\{ v_{k_1} \ldots v_{k_l} \}\]

\[k_1 < k_2 < \cdots < k_d \text{ and } k_d > k_{d+1} > \cdots > k_l\]

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\[(v_{k}, v_{k_i}), d + 1 \leq i \leq l - 1 \text{ are blue}\]

$\text{edge } (v_{k_i}, v_{k_d}) \text{ is either red or blue}$
Rectangular Dual
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S-N net $G_{S-N}$
Rectangular Dual

W-E net $G_{W-E}$
Rectangular Dual

S-N net $G_{S-N}$
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S-N net $G_{S-N}$
Rectangular Dual
Rectangular Dual
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Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL $T_r, T_b$ of $G$;
- Construct a S-N net $G_{S-N}$ of $G$ (consists of $T_r$ plus outer edges)
- Construct the dual $G_{S-N}^*$ of $G_{S-N}$ and compute a topological ordering $f_{sn}$ of $G_{S-N}^*$
- For each vertex $v \in V$, let $f$ and $g$ be the face on the left and face on the right of $v$. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$
Algorithm Rectangular dual

Input: A PTP graph $G = (V, E)$

- Find a REL $T_r, T_b$ of $G$;
- Construct a S-N net $G_{S-N}$ of $G$ (consists of $T_r$ plus outer edges);
- Construct the dual $G^\star_{S-N}$ of $G_{S-N}$ and compute a topological ordering $f_{sn}$ of $G^\star_{S-N}$;
- For each vertex $v \in V$, let $f$ and $g$ be the face and face $v$. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_{N}) = x_1(v_{S}) = 1$ and $x_2(v_{N}) = x_2(v_{S}) = \max_{f_{sn}} - 1$. 

Rectangular Dual

Algorithm Rectangular dual

Input: A PTP graph $G = (V, E)$

1. Find a REL $T_r, T_b$ of $G$;
2. Construct a $W$-net $G_{W-E}$ of $G$ (consists of $T_b$ plus outer edges);
3. Construct the dual $G_{SN}$ of $G_{SN}$ and compute a topological ordering $f_{SN}$ of $G_{SN}$;
4. For each vertex $v \in V$, let $f$ and $g$ be the face of $v$ and face through $v$. Set $x_1(v) = f_{SN}(f)$ and $x_2(v) = f_{SN}(g)$.
5. Define and
Algorithm Rectangular dual

Input: A PTP graph $G = (V, E)$

- Find a REL $T_r, T_b$ of $G$;
- Construct a W-E net $G_{W-E}$ of $G$ (consists of $T_b$ plus outer edges);
- Construct the dual $G^*_{W-E}$ and compute a topological ordering $f_{we}$ of $G^*_{W-E}$;
- For each vertex $v \in V$, let $f$ and $g$ be the face and face
  of $v$. Set $f = f_{sn}(f)$ and $g = f_{sn}(g)$.
- Define and
Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph \( G = (V, E) \)

- Find a REL \( T_r, T_b \) of \( G \);
- Construct a W-E net of \( G \) (consists of \( T_b \) plus outer edges)
- Construct the dual of \( G^* \) and compute a topological ordering \( f_{we} \) of \( G^*_W-E \)
- For each vertex \( v \in V \), let \( f \) and \( g \) be the face below and face above \( v \). Set \( y_1(v) = f_{SN}(f) \) and \( y_2(v) = f_{SN}(g) \).
- Define and
Algorithm Rectangular dual

Input: A PTP graph $G = (V, E)$

- Find a REL $T_r, T_b$ of $G$;
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- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$.
Rectangular Dual

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- For each vertex $v \in V$, let $f$ and $g$ be the face below and face above $v$. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$.
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$
- For each $v \in V$, assign a rectangle $R(v)$ bounded by $x$-coordinates $x_1(v), x_2(v)$ and $y$-coordinates $y_1(v), y_2(v)$. 
Rectangular Dual
Algorithmen zur Visualisierung von Graphen

Tamara Mchedlidze

Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

Rectangular Dual

$x_1(v_N) = 1, \ x_2(v_N) = 15$
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$x_1(a) = 1, \ x_2(a) = 3$
$x_1(b) = 3, \ x_2(b) = 5$
$x_1(c) = 5, \ x_2(c) = 14$
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$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0, y_2(v_E) = 10$
$y_1(v_N) = 9, y_2(v_N) = 10$
$y_1(v_S) = 0, y_2(v_S) = 1$
$y_1(a) = 1, y_2(a) = 2$
$y_1(b) = 1, y_2(b) = 2$
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$y_1(d) = 1, y_2(d) = 2$
$y_1(e) = 2, y_2(e) = 3$
### Rectangular Dual

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Rectangular Dual

In the following we prove that presented algorithm constructs a rectangular dual of $G$.


- Let $f_1, \ldots, f_k$ be the faces of $G^*_{S-N}$ (resp. $G^*_{W-E}$), enumerated according to st-numbering $f_{sn}$ (resp. $f_{we}$).

- Let $G^i_{S-N}$ (resp. $G^i_{W-E}$) denote the subgraph of $G$ that is induced by vertices and edges of $f_1, \ldots, f_i$.

- We denote $P_i$ (resp. $Q_i$) the right (resp. top) boundary of $G^i_{S-N}$ (resp. $G^i_{W-E}$).
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

S-N net $G_{S-N}$

$P_6$
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

S-N net $G_{S-N}$

$P_{13}$
Rectangular Dual

S-N net $G_{S-N}$
Rectangular Dual

- Paths $P_i$ and $Q_j$ for any $i, j$ (except for (a) $i = 0, j = 0$, (b)
  $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d)
  $i = \max f_{sn} - 1, j = \max f_{we} - 1$) cross at exactly one vertex.
Rectangular Dual

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Lemma 4

Let $v \in V$, $f$ and $g$ are the left and the right face of $v$. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex $v$ belongs to path $P_i$ if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Proof...
Rectangular Dual

- Paths $P_i$ and $Q_j$ for any $i, j$ (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) cross at exactly one vertex.

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Proof...

Lemma 5

Let $v \in V$, $f$ and $g$ are the faces below and above $v$ in $G_{W-E}$. Let $y_1(v) = f_{we}(f)$ and $y_2(v) = f_{we}(g)$. Vertex $v$ belongs to path $Q_j$ if and only if $y_1(v) \leq j \leq y_2(v) - 1$.

Proof (identical)
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.
Rectangular Dual

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Proof:
Rectangular Dual

Lemma 6

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Proof: Show that there exists a vertex over this box: \( u \in P_i \cap Q_j \)
Lemma 6

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Proof: Show that there is at most one vertex over this box.
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Proof: Show that there is at most one vertex over this box

\[ x_1(u) \leq i \text{ and } i+1 \leq x_2(u) \]
Rectangular Dual

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(Lemma 4)

\( u \) belongs to \( P_i \)
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\[ u \text{ belongs to } P_i \]

Similarly: \( v \in P_i, u \in Q_j, v \in Q_j \).
**Lemma 6**

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

**Proof:** Show that there is **at most one vertex** over this box.

Paths $P_i$ and $Q_j$ intersect at two vertices $u$ and $v$.

\[
x_1(u) \leq i \text{ and } i + 1 \leq x_2(u)
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(Lemma 4)

$u$ belongs to $P_i$

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$. 

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\[u \text{ belongs to } P_i\]

Similarly: \(v \in P_i, u \in Q_j, v \in Q_j\).

Paths \(P_i\) and \(Q_j\) intersect at two vertices \(u\) and \(v\).

Which is a contradiction to the property of paths \(P_i, Q_j\) except for the cases when:

(a) \(i = 0, j = 0\), (b) \(i = \max f_{sn} - 1, j = 0\), (c) \(i = 0, j = \max f_{we} - 1\), (d) \(i = \max f_{sn} - 1, j = \max f_{we} - 1\) (corner boxes).
Lemma 7

Let $G_{S-N}$ and $G_{W-E}$. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{W-E}$ containing at least two edges, then $x_2(u) < x_1(v)$;
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from $u$ to $v$ in $G_{S-N}$ containing at least two edges, then $y_2(u) < y_1(v)$.

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- If there exist a directed path from $u$ to $v$ in $G_{S-N}$ containing at least two edges, then $y_2(u) < y_1(v)$.

Proof...

Lemma 8

The assignment provided by the algorithm has the following property: rectangles assigned to vertices $u$ and $v$ have a common segment if and only if there exists edge $(u, v)$ in the graph.

Proof:
Assume $R(u)$ and $R(v)$ have a common boundary.
Assume $R(u)$ and $R(v)$ have a common boundary.

$x_1(v) \leq i$, $i + 1 \leq x_2(v)$ and $x_1(u) \leq i$, $i + 1 \leq x_2(u)$
Assume $R(u)$ and $R(v)$ have a common boundary.

\[ x_1(v) \leq i, \quad i + 1 \leq x_2(v) \quad \text{and} \quad x_1(u) \leq i, \quad i + 1 \leq x_2(u) \]

(Lemma 4)

$u, v$ belong to $P_i$
Assume $R(u)$ and $R(v)$ have a common boundary.

\[
x_1(v) \leq i, \ i + 1 \leq x_2(v) \text{ and } x_1(u) \leq i, \ i + 1 \leq x_2(u)
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(Lemma 4)

\[u, v \text{ belong to } P_i\]

If path between $u$ and $v$ has at least 2 edges, then by Lemma 7,
\[y_2(u) < y_1(v)\]
Rectangular Dual

Assume $R(u)$ and $R(v)$ have a common boundary.

If path between $u$ and $v$ has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

A contradiction to the hypothesis!
Rectangular Dual

- Assume there exists an edge \((u, v) \in G_{W-E}\).

- Let \(Q_j\) be the path of \(G_{W-E}\) where \((u, v)\) belongs. By Lemma 5, \(y_1(u) \leq j\), \(j + 1 \leq y_2(u)\) and \(y_1(v) \leq j\), \(j + 1 \leq y_2(v)\).

- By Lemma 7, \(x_2(u) = x_1(v)\).
Assume there exists an edge \((u, v) \in G_{W-E}\).

Let \(Q_j\) be the path of \(G_{W-E}\) where \((u, v)\) belongs. By Lemma 5, \(y_1(u) \leq j\), \(j + 1 \leq y_2(u)\) and \(y_1(v) \leq j\), \(j + 1 \leq y_2(v)\).

By Lemma 7, \(x_2(u) = x_1(v)\).
Assume there exists an edge \((u, v) \in G_{W-E}\).

Let \(Q_j\) be the path of \(G_{W-E}\) where \((u, v)\) belongs. By Lemma 5, \(y_1(u) \leq j, j + 1 \leq y_2(u)\) and \(y_1(v) \leq j, j + 1 \leq y_2(v)\).

By Lemma 7, \(x_2(u) = x_1(v)\).

Lemma 8 is proved!
Rectangular Dual

**Theorem**

Every PTP graph $G$ has a rectangular dual which can be computed in linear time.
Theorem

Every PTP graph $G$ has a rectangular dual which can be computed in linear time.

- Compute a planar embedding of $G$
- Compute a revised canonical ordering of $G$
- Traverse the graph and color the edges, construct $G_{S-N}$ and $G_{E-W}$
- Construct the duals $G^*_{S-N}$ and $G^*_{E-W}$ of $G_{S-N}$ and $G_{E-W}$, respectively
- Compute a topological ordering of $G^*_{S-N}$ and $G^*_{E-W}$
- Assign coordinates to the rectangles representing vertices.