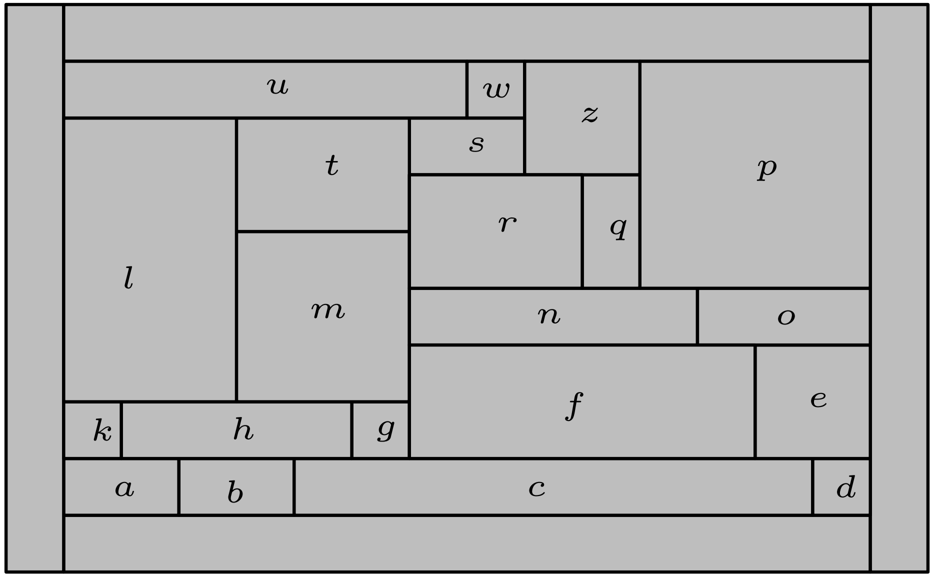
Algorithms for graph visualization

Contact representations of planar graphs.

WINTER SEMESTER 2014/2015

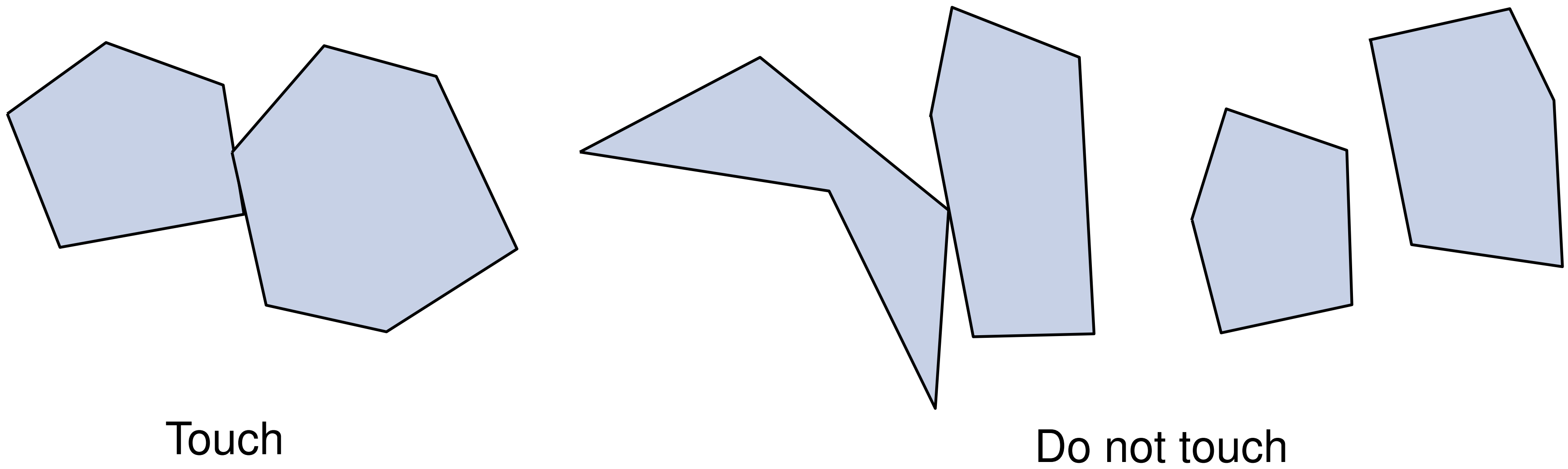
Tamara Mchedlidze – MARTIN NÖLLENBURG



Contact representation

Contact representation

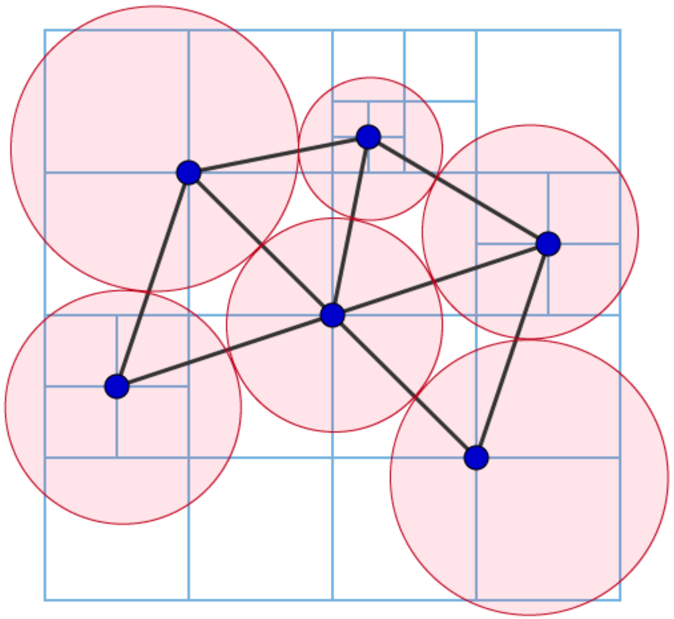
In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.



Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

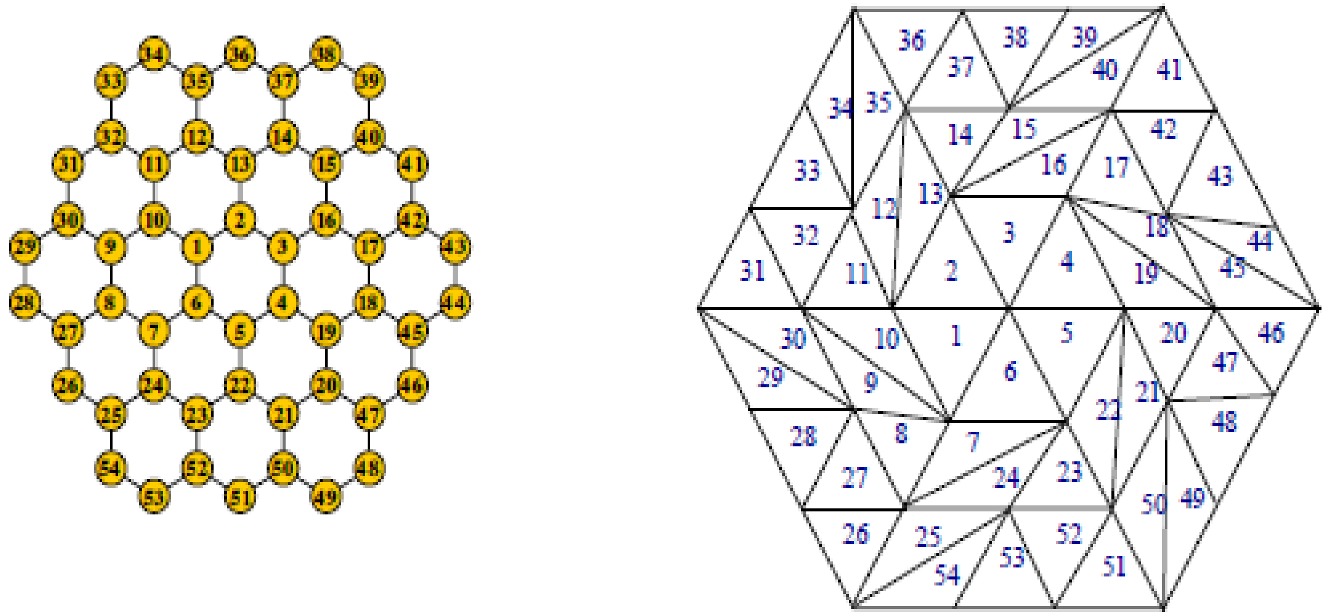


Touching disk representation

Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

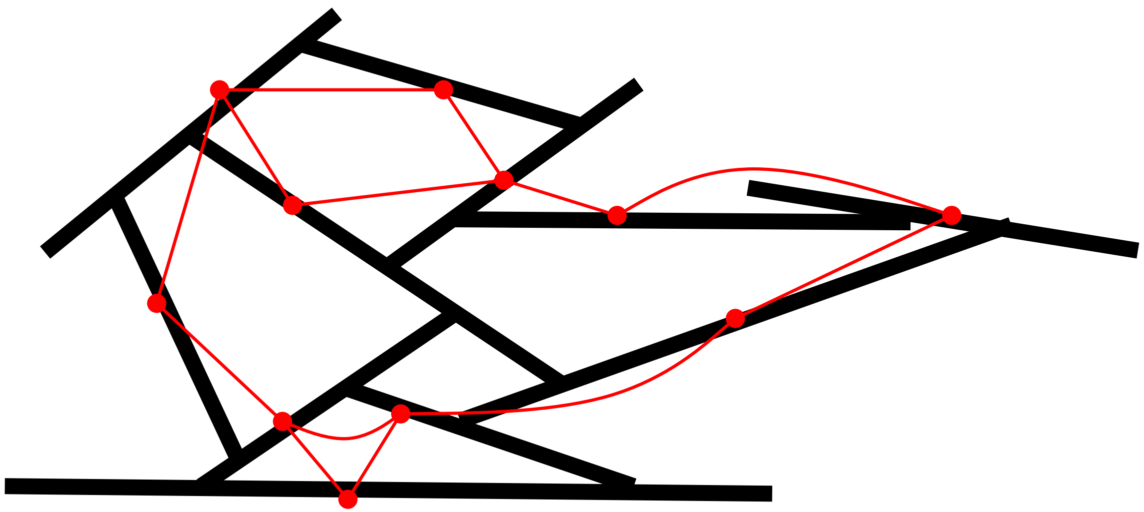


Touching triangle representation

Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

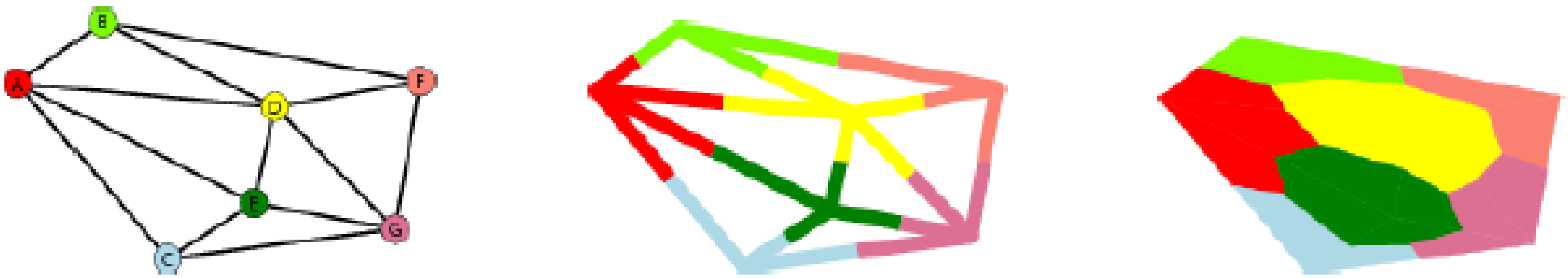


Touching segment representation

Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.



General idea for the construction of a contact representation of a planar graph using n-gons in worst case

Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

- Each planar graph has a touching disks representation (Koebe 1936)

Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

- Each planar graph has a touching disks representation (Koebe 1936)
- If we want to represent a planar graph as contact of k-gons, how high should be k?

Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

- Each planar graph has a touching disks representation (Koebe 1936)
- If we want to represent a planar graph as contact of k-gons, how high should be k?
- 6-gons are necessary and suffisient for planar graphs! (Gansner et. al. 2010)

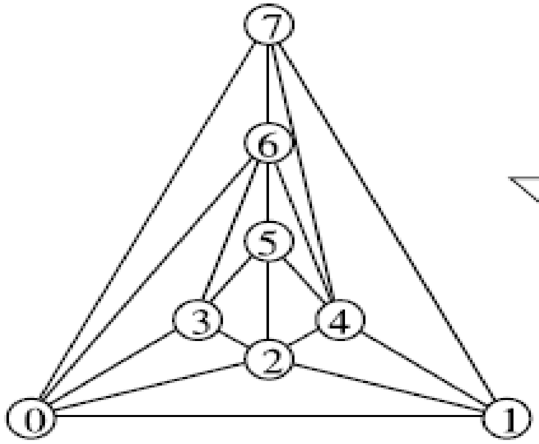
ɔhs!

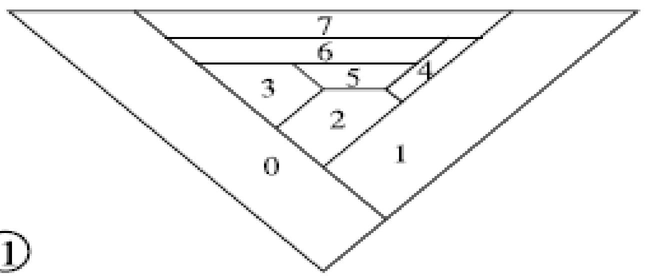
Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

- Each planar graph has a touching disks representation (Koebe 1936)
- If we want to represent a planar graph as contact of k-gons, how high should be k?
- 6-gons are necessary and suffisient for planar graphs! (Gansner et. al. 2010)



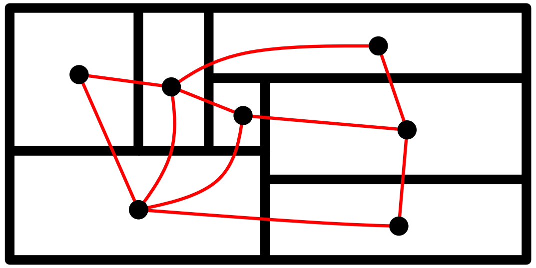


Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

- Each planar graph has a touching disks representation (Koebe 1936)
- If we want to represent a planar graph as contact of k-gons, how high should be k?
- 6-gons are necessary and suffisient for planar graphs! (Gansner et. al. 2010)
- Rectangles are sufficient for maximal 4-connected graphs!

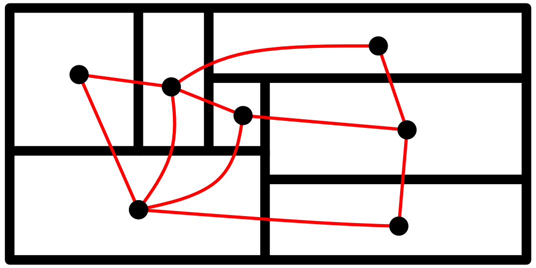


Contact representation

Contact representation

In a contact representation of a planar graph each vertex is repreented as a geometrical object such that two object thouch if and only if the corresponding vertices are connected by an edge.

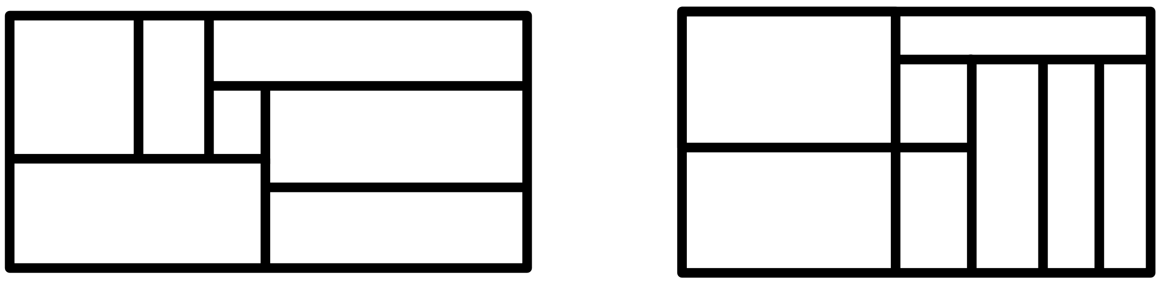
- Each planar graph has a touching disks representation (Koebe 1936)
- If we want to represent a planar graph as contact of k-gons, how high should be k?
- 6-gons are necessary and suffisient for planar graphs! (Gansner et. al. 2010)
- Rectangles are sufficient for maximal 4-connected graphs!



Rectangular Dual

Rectangular Subdivision System

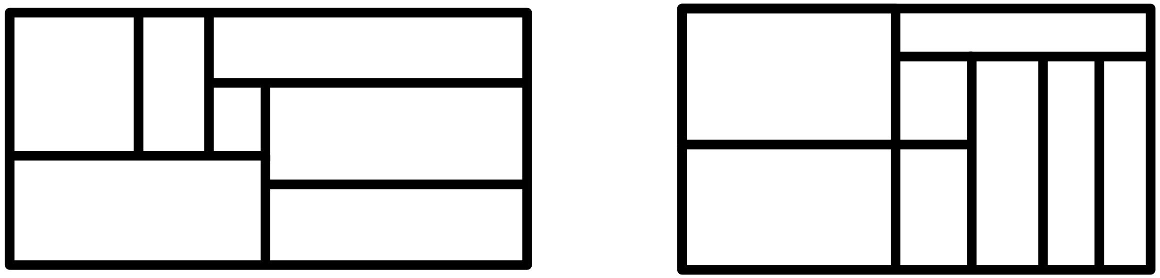
Let R be a rectangle. A rectangular subdivision system Φ of R is a partition of R into a set of non-intersecting smaller rectangles such that no four of them meet at the same point.



Rectangular Dual

Rectangular Subdivision System

Let R be a rectangle. A rectangular subdivision system Φ of R is a partition of R into a set of non-intersecting smaller rectangles such that no four of them meet at the same point.



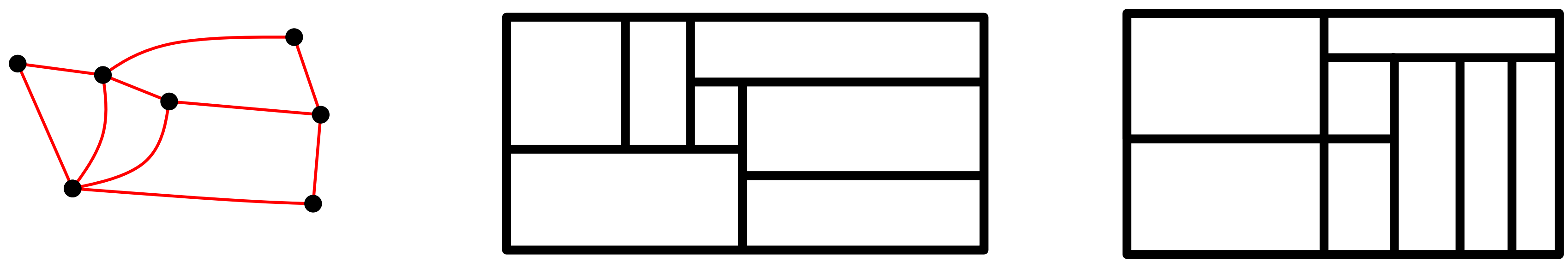
Rectangular Dual

A rectangular dual of a graph $G = (V, E)$ is a rectangular subdivision system Φ and a one-to-one correspondence $f : V \to \Phi$ such that $(u, v) \in E$ if and only if the rectangles $f(u)$ and $f(v)$, corresponding to u and v, share a common boundary.

Rectangular Dual

Rectangular Subdivision System

Let R be a rectangle. A rectangular subdivision system Φ of R is a partition of R into a set of non-intersecting smaller rectangles such that no four of them meet at the same point.



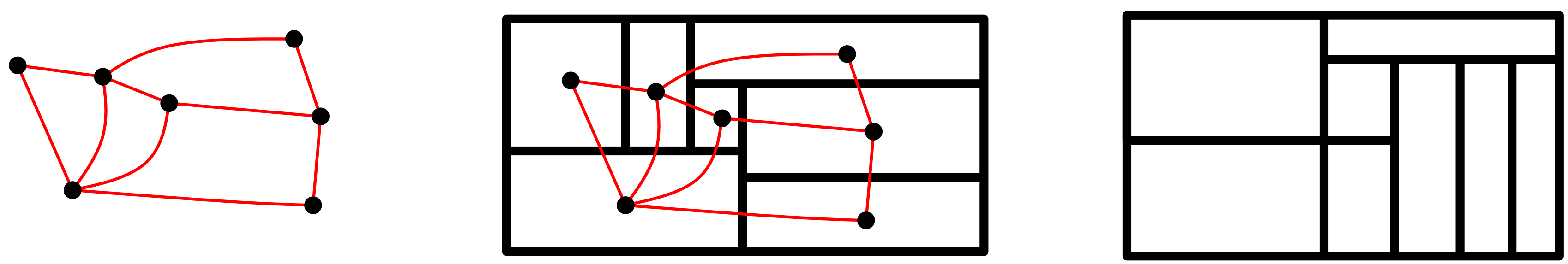
Rectangular Dual

A rectangular dual of a graph $G = (V, E)$ is a rectangular subdivision system Φ and a one-to-one correspondence $f : V \rightarrow \Phi$ such that $(u, v) \in E$ if and only if the rectangles $f(u)$ and $f(v)$, corresponding to u and v, share a common boundary.

Rectangular Dual

Rectangular Subdivision System

Let R be a rectangle. A rectangular subdivision system Φ of R is a partition of R into a set of non-intersecting smaller rectangles such that no four of them meet at the same point.



Rectangular Dual

A rectangular dual of a graph $G = (V, E)$ is a rectangular subdivision system Φ and a one-to-one correspondence $f : V \to \Phi$ such that $(u, v) \in E$ if and only if the rectangles $f(u)$ and $f(v)$, corresponding to u and v, share a common boundary.

Rectangular Dual

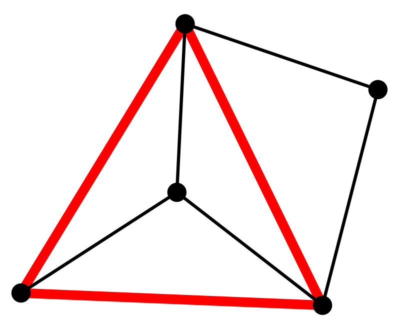
- Which graphs have a rectangular dual?

Rectangular Dual

- Which graphs have a rectangular dual?

Separating triangle

Let G be a graph. A triangle C of G whose removal results in at least two disconnected components is called a separating triangle of G.

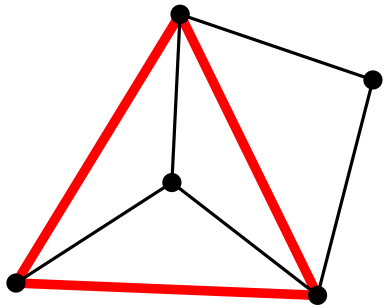


Rectangular Dual

- Which graphs have a rectangular dual?

Separating triangle

Let G be a graph. A triangle C of G whose removal results in at least two disconnected components is called a separating triangle of G.



Does not have a rectangular dual!

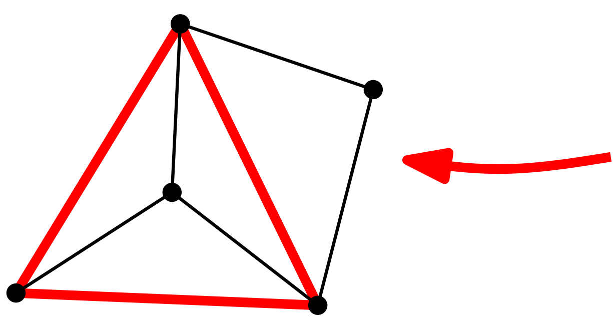
(In order to enclose an area we need at least four boxes)

Rectangular Dual

- Which graphs have a rectangular dual?

Separating triangle

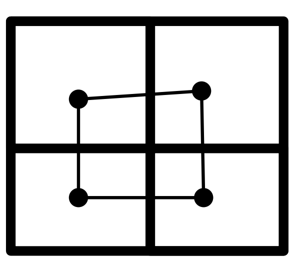
Let G be a graph. A triangle C of G whose removal results in at least two disconnected components is called a separating triangle of G.



Does not have a rectangular dual!

(In order to enclose an area we need at least four boxes)

No four rectangles meet a a point!

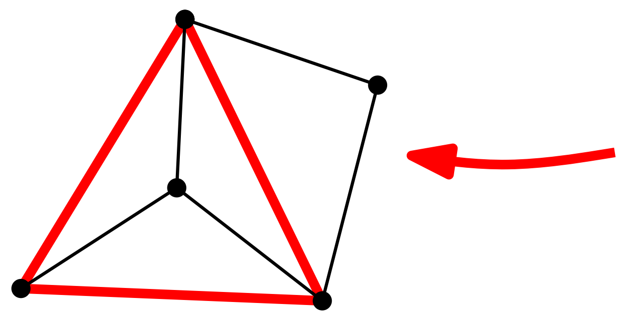


Rectangular Dual

- Which graphs have a rectangular dual?

Separating triangle

Let G be a graph. A triangle C of G whose removal results in at least two disconnected components is called a separating triangle of G.

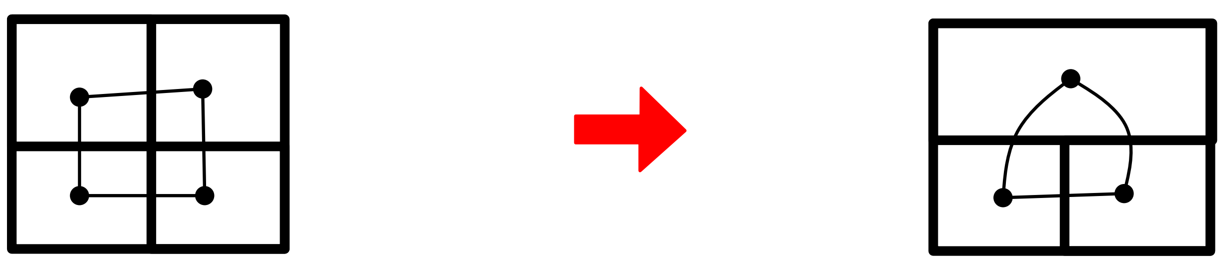


Does not have a rectangular dual!

(In order to enclose an area we need at least four boxes)

No four rectangles meet a a point!

Each face of G must be a triangle!



Rectangular Dual

Necessary conditions for a planar graph G to have a rectangular dual:

- G must have at least 4 vertices on the outer face
- G must have no separating triangle
- each internal face of G must be a triangle

Rectangular Dual

Necessary conditions for a planar graph G to have a rectangular dual:

- G must have at least 4 vertices on the outer face
- G must have no separating triangle
- each internal face of G must be a triangle

We will prove that these conditions are sufficient!

Rectangular Dual

Necessary conditions for a planar graph G to have a rectangular dual:

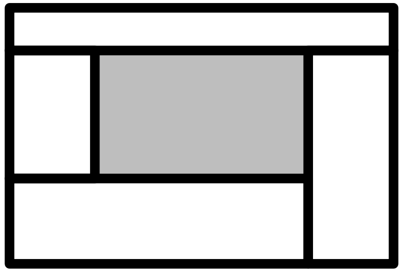
- G must have at least 4 vertices on the outer face
- G must have no separating triangle
- each internal face of G must be a triangle

We will prove that these conditions are sufficient!

Theorem [1985,1986,1987,1988,1990,1993 by Xin He]

A planar graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following conditions hold:

- Every interior face of G is a triangle and the exterior face of G is a quadrangle;
- G has no separating triangles



Rectangular Dual

Necessary conditions for a planar graph G to have a rectangular dual:

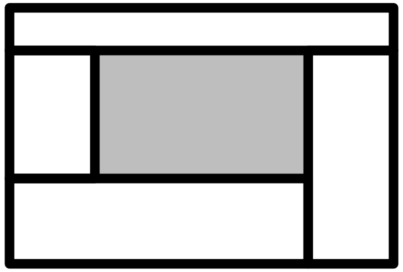
- G must have at least 4 vertices on the outer face
- G must have no separating triangle
- each internal face of G must be a triangle

We will prove that these conditions are sufficient!

Theorem [1985,1986,1987,1988,1990,1993 by Xin He]

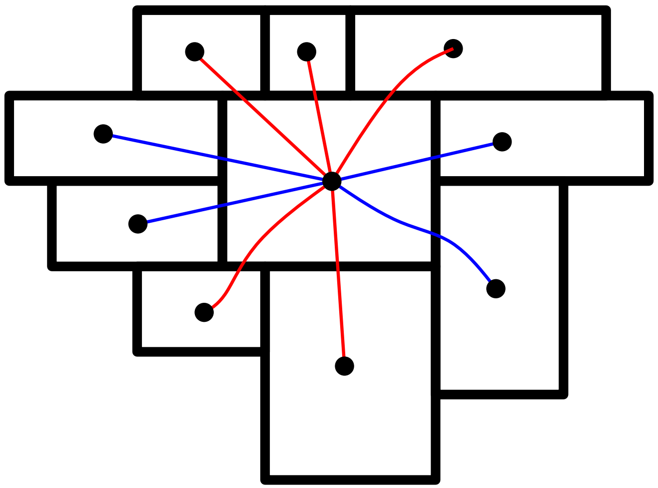
A planar graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following conditions hold:

- Every interior face of G is a triangle and the exterior face of G is a quadrangle;
- G has no separating triangles



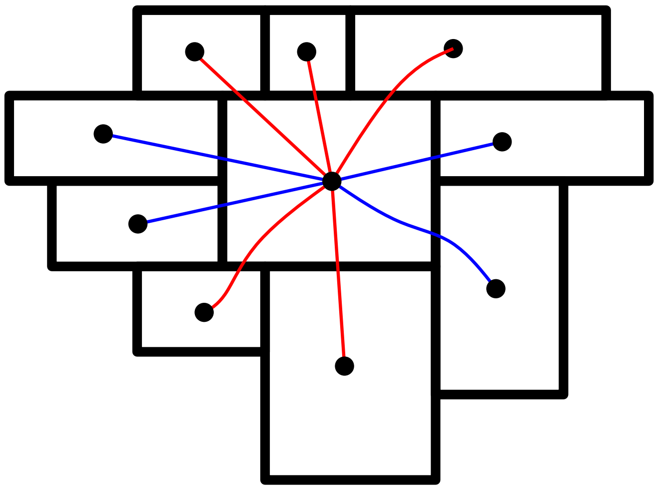
Proper Triangular Planar Graph (PTP)

Rectangular Dual



In order to construct a rectangular dual we need to partition our edges on vertical and horizontal. Regular edge labeling (REL, for short) is a tool for that.

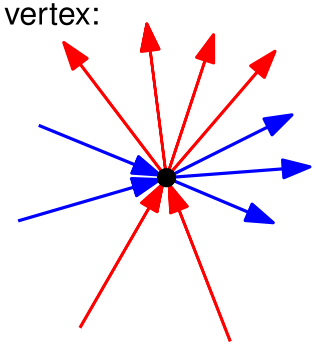
Rectangular Dual



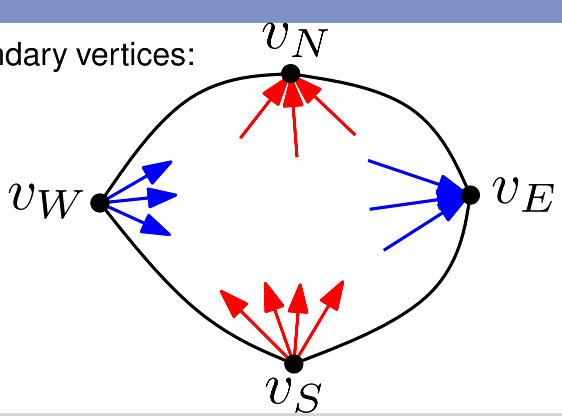
In order to construct a rectangular dual we need to partition our edges on vertical and horizontal. Regular edge labeling (REL, for short) is a tool for that.

Regular edge labeling

For each internal vertex:



For the boundary vertices:

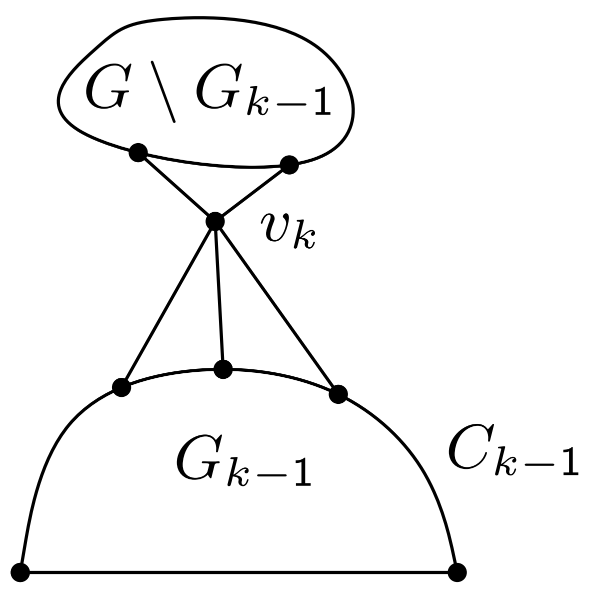


Rectangular Dual

Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k - 2$, v_k has at least 2 neighbors in $G \setminus G_{k-1}$

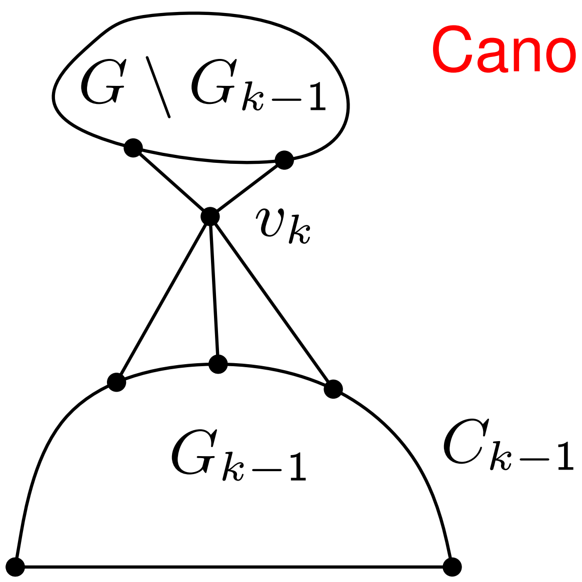


Rectangular Dual

Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k-2$, v_k has at least 2 neighbors in $G \setminus G_{k-1}$



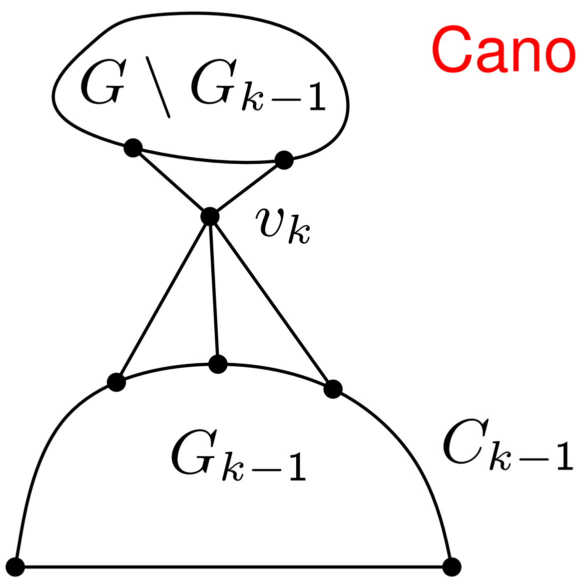
Canonical ordering with extra condition on v_k!

Rectangular Dual

Theorem (Refined canonical ordering)

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k-2$, v_k has at least 2 neighbors in $G \setminus G_{k-1}$



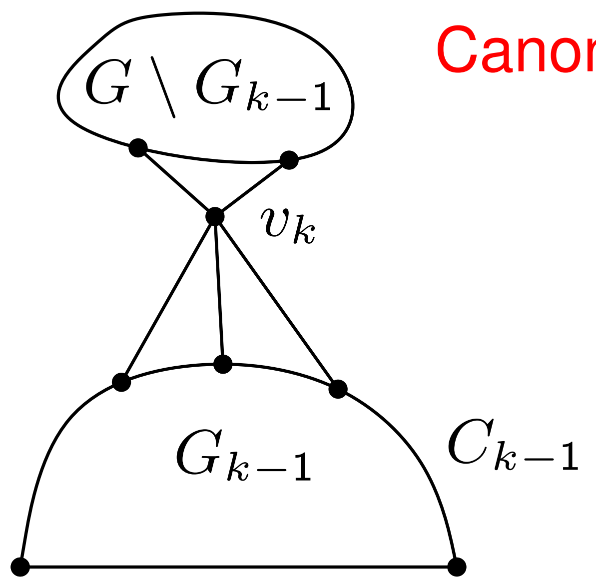
Canonical ordering with extra condition on v_k!

Rectangular Dual

Theorem (Refined canonical ordering)

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \ldots, v_n = v_N$ such that for every $4 \leq k \leq n$:

- The subgraph G_{k-1} induced by $v_1, \ldots, v_{k-1}$ is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W).
- v_k is in exterior face of G_{k-1}, and its neighbors in G_{k-1} form (at least 2-element) subinterval of the path $C_{k-1} \setminus (v_S, v_W)$. If $k \leq k-2$, v_k has at least 2 neighbors in $G \setminus G_{k-1}$

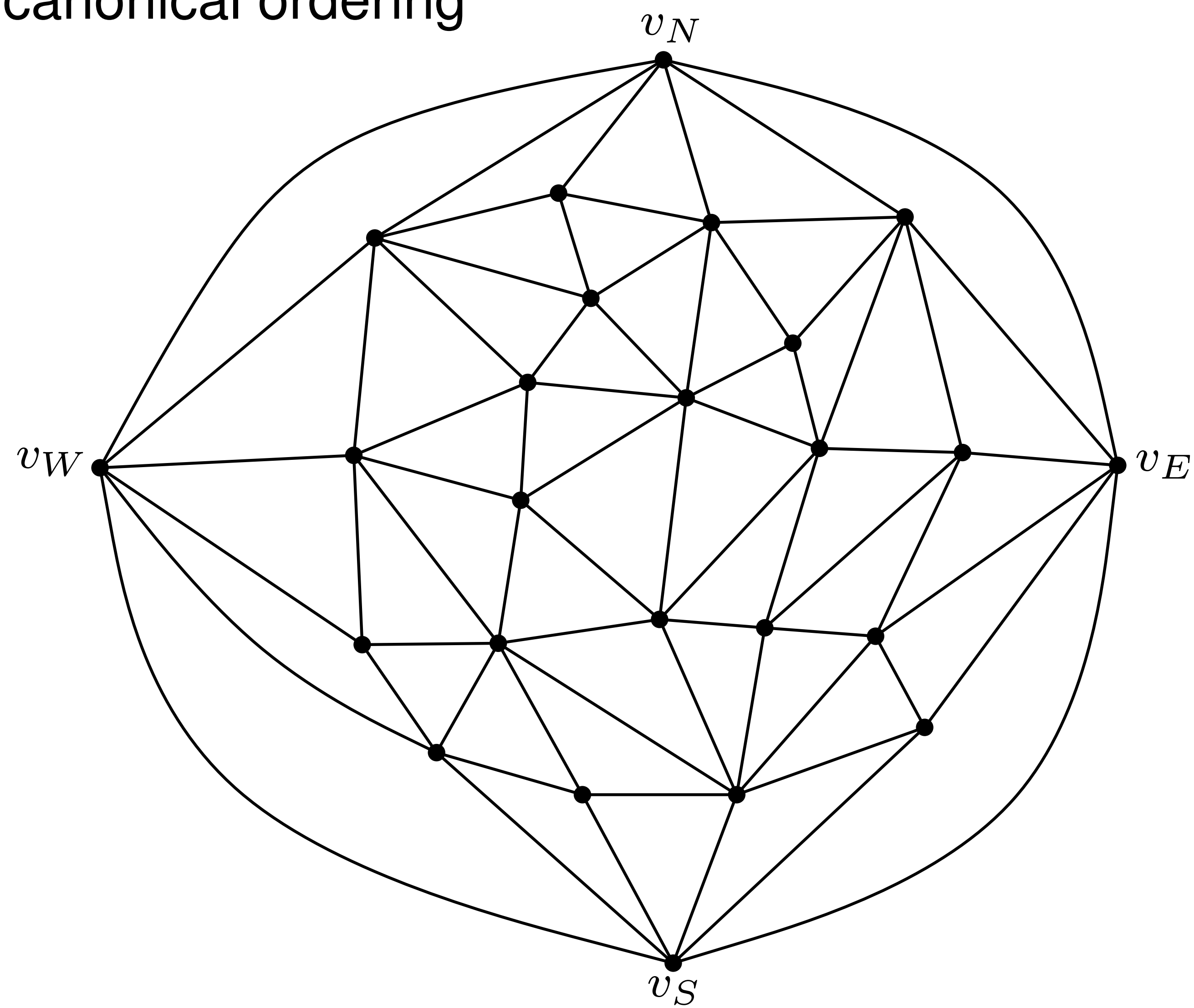


Canonical ordering with extra condition on v_k!

Home task!

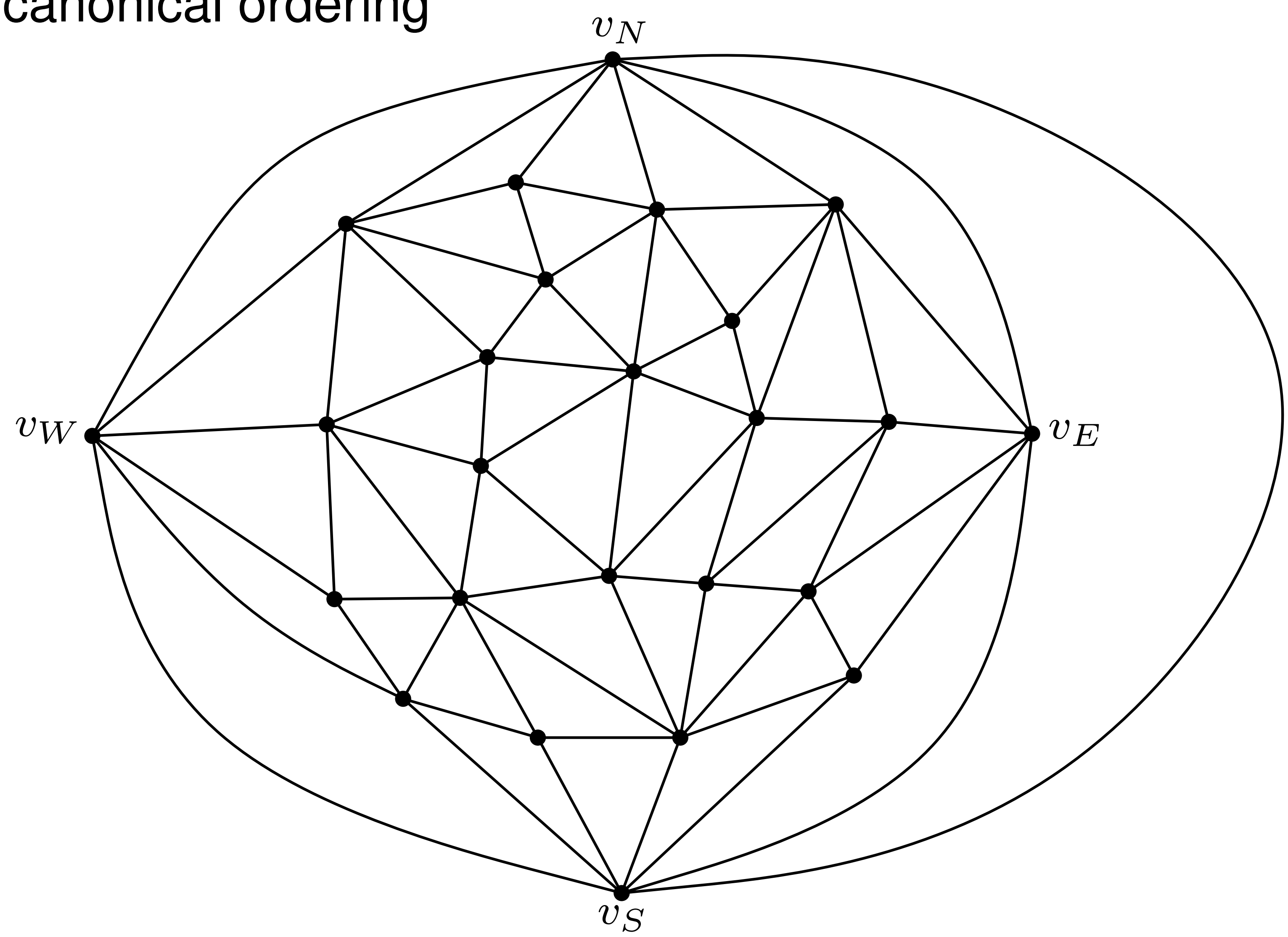
Rectangular Dual

Refined canonical ordering



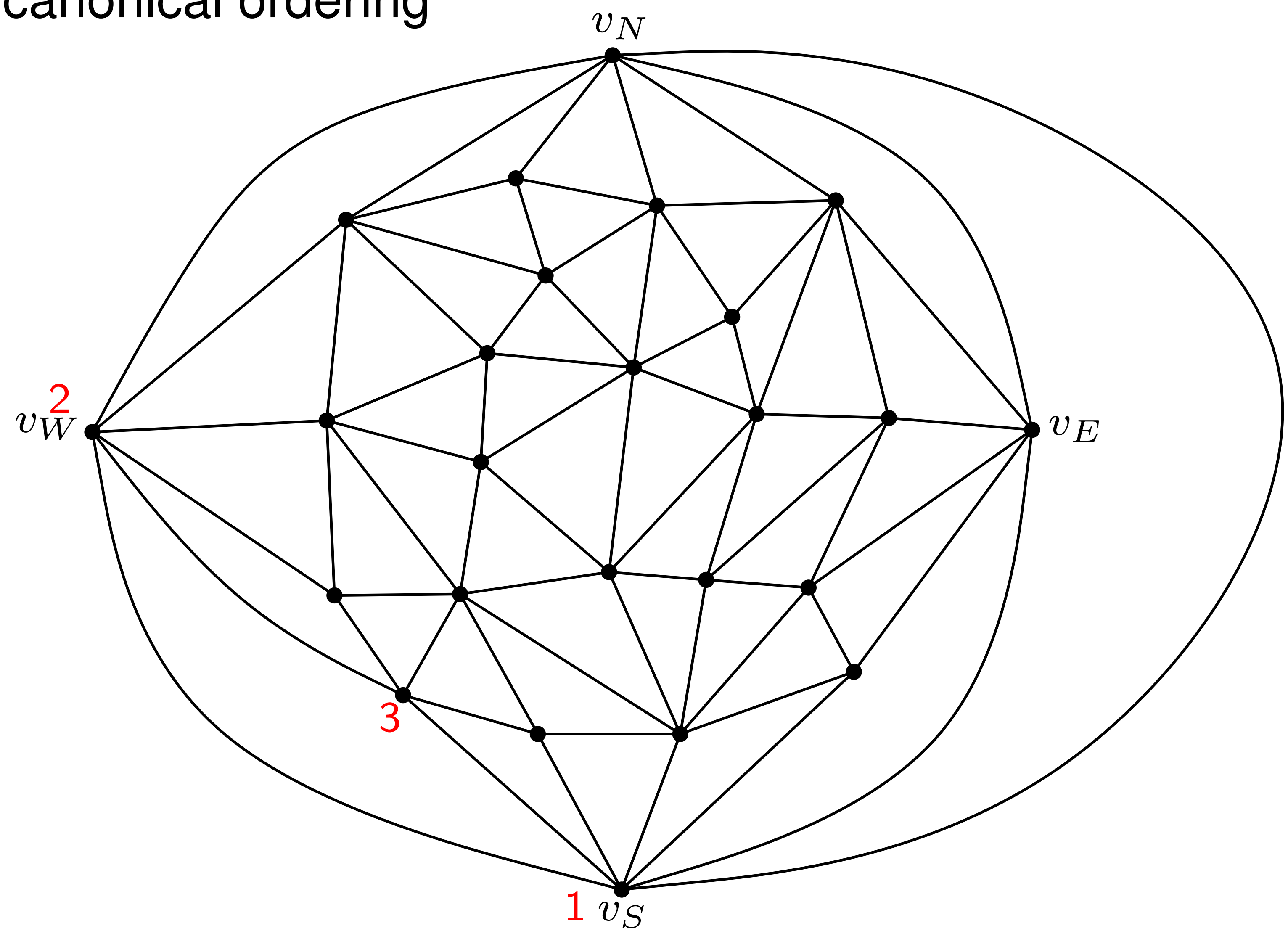
Rectangular Dual

Refined canonical ordering



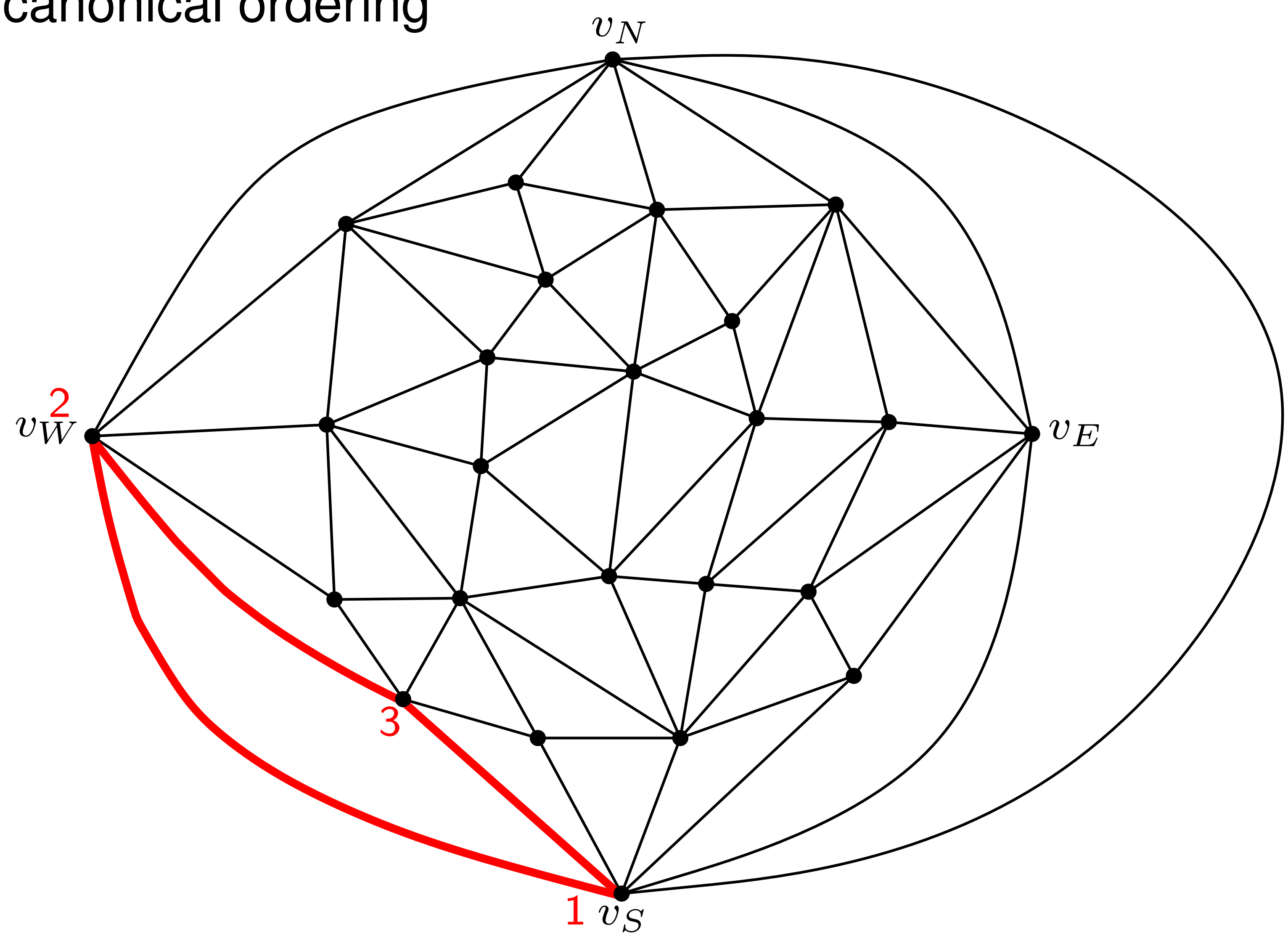
Rectangular Dual

Refined canonical ordering



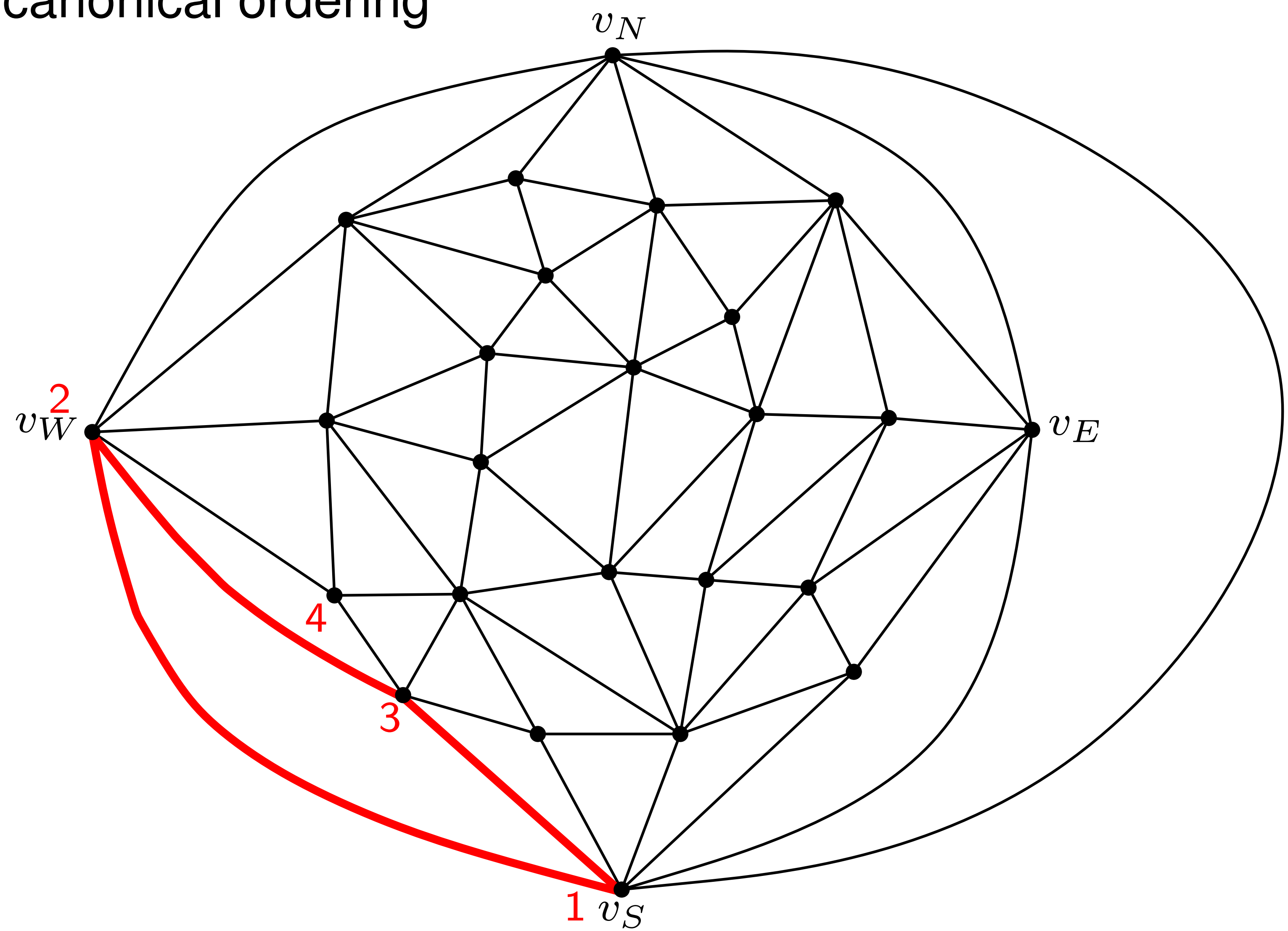
Rectangular Dual

Refined canonical ordering



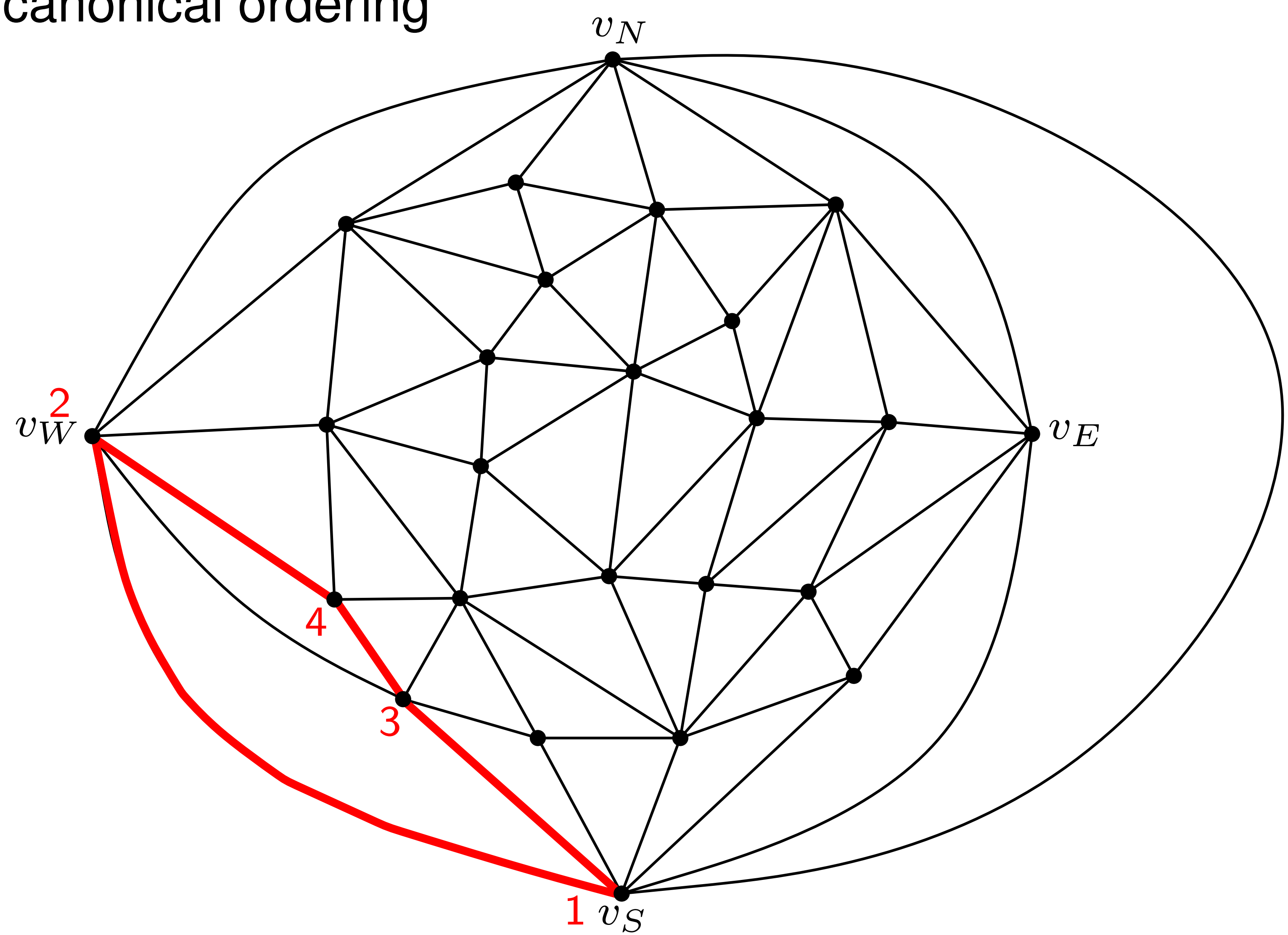
Rectangular Dual

Refined canonical ordering



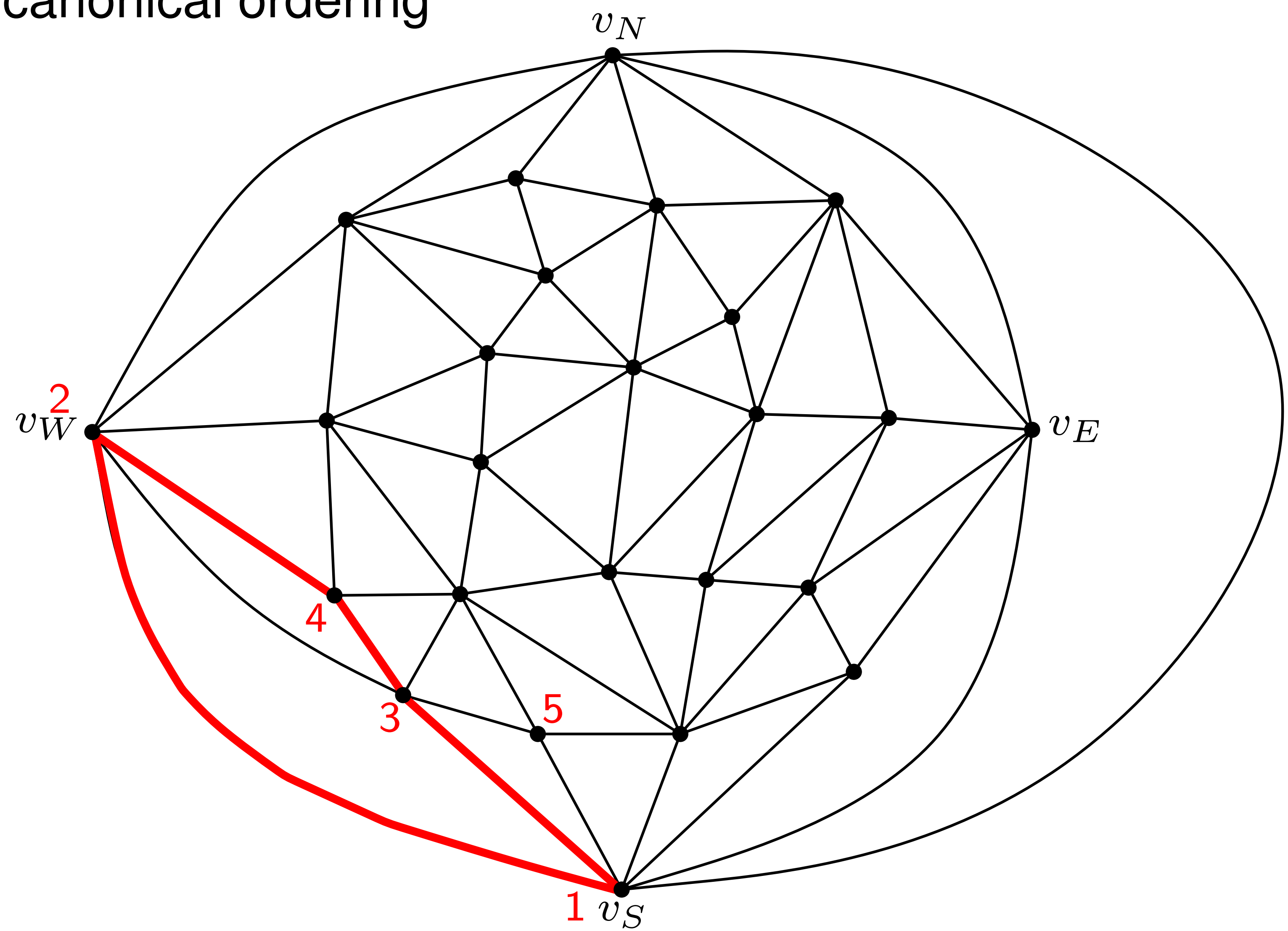
Rectangular Dual

Refined canonical ordering



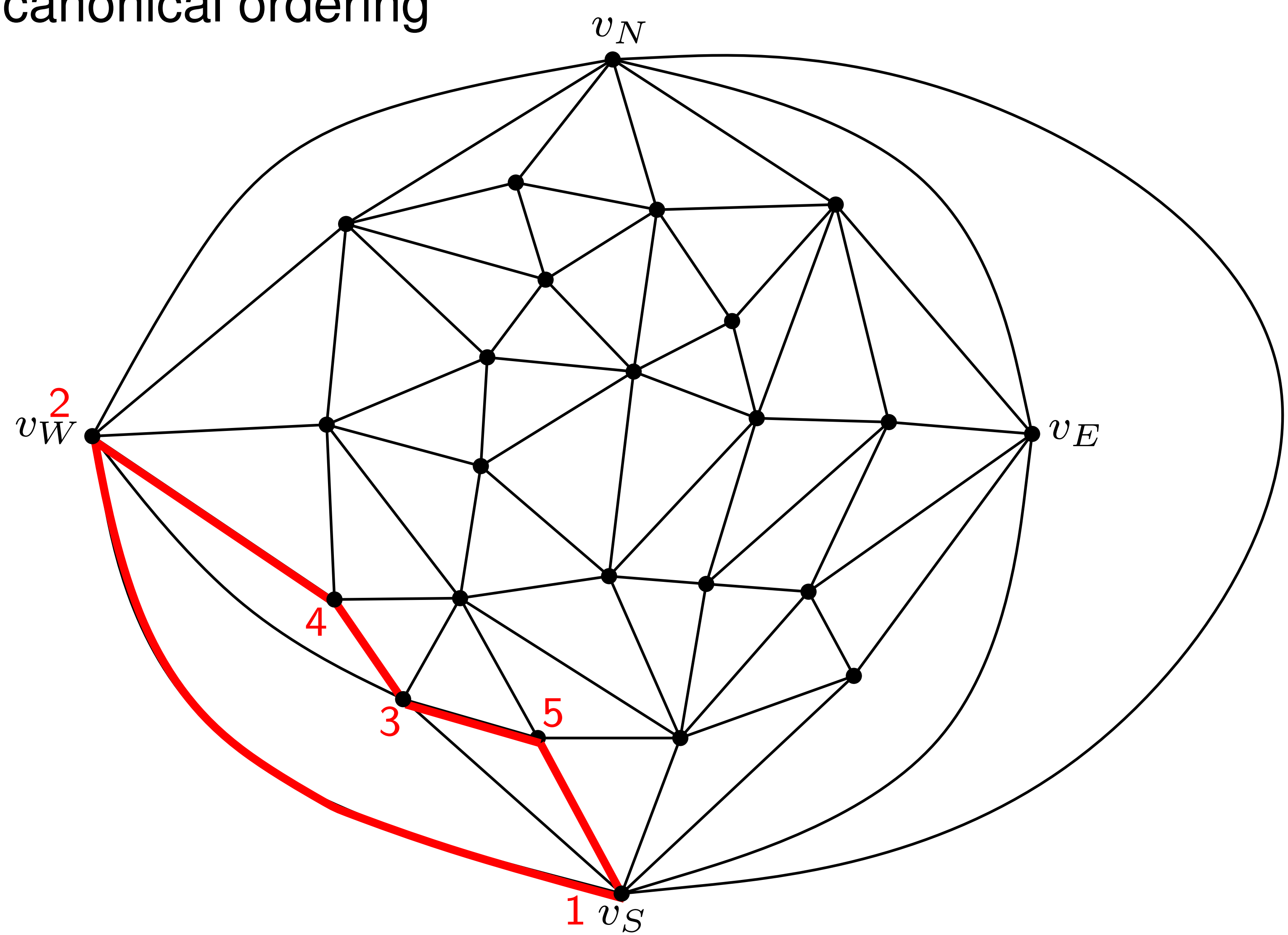
Rectangular Dual

Refined canonical ordering



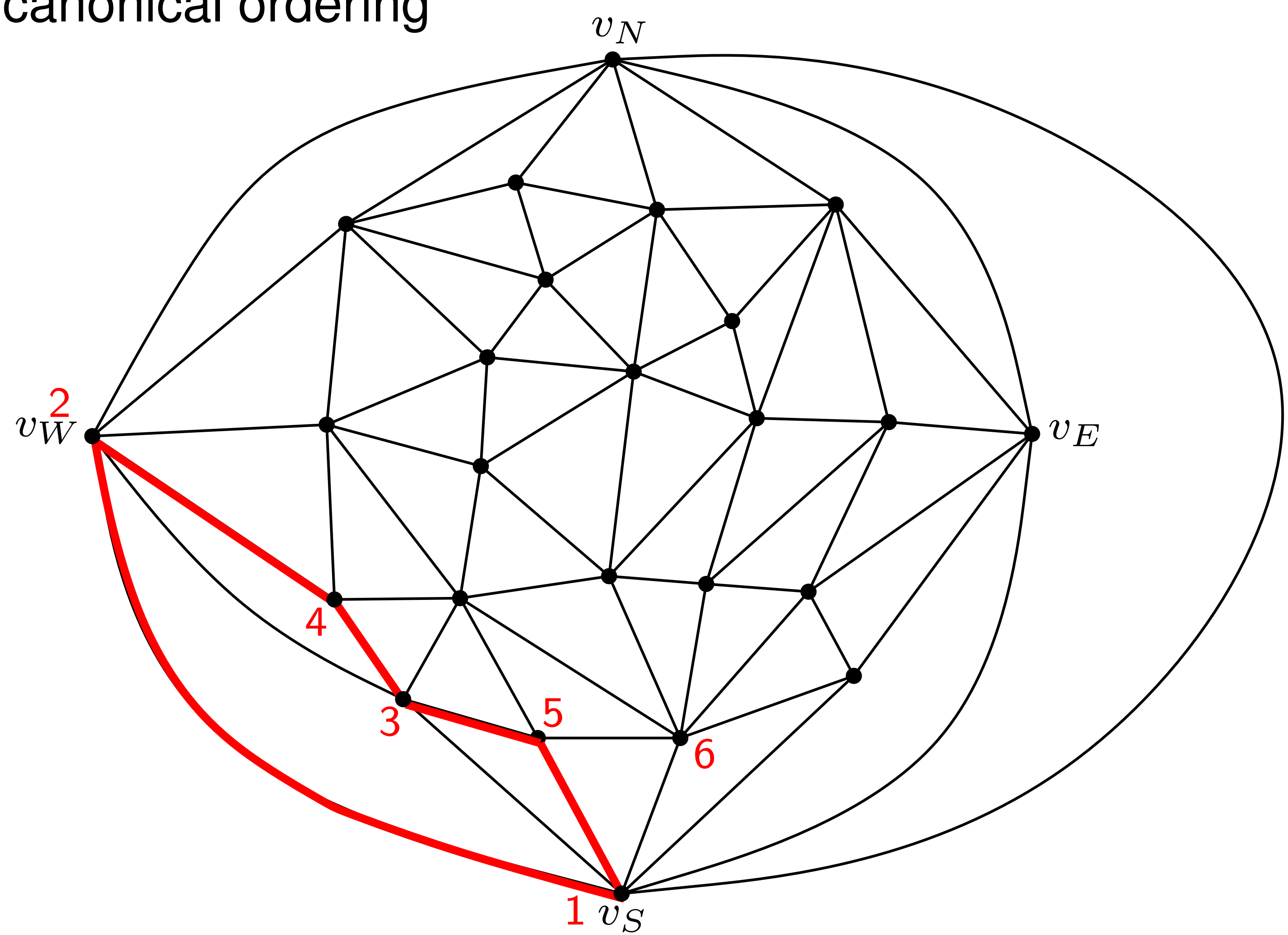
Rectangular Dual

Refined canonical ordering



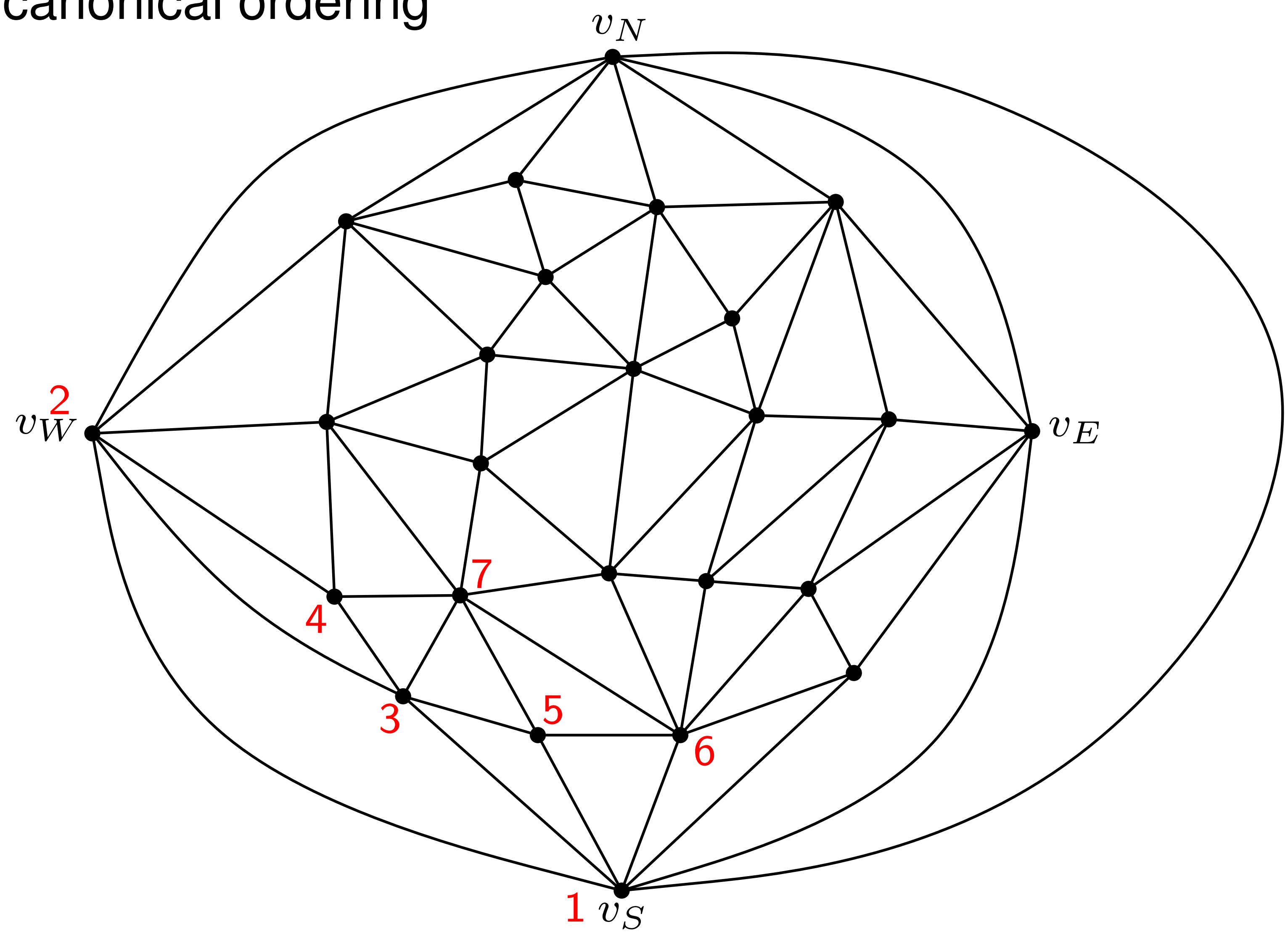
Rectangular Dual

Refined canonical ordering



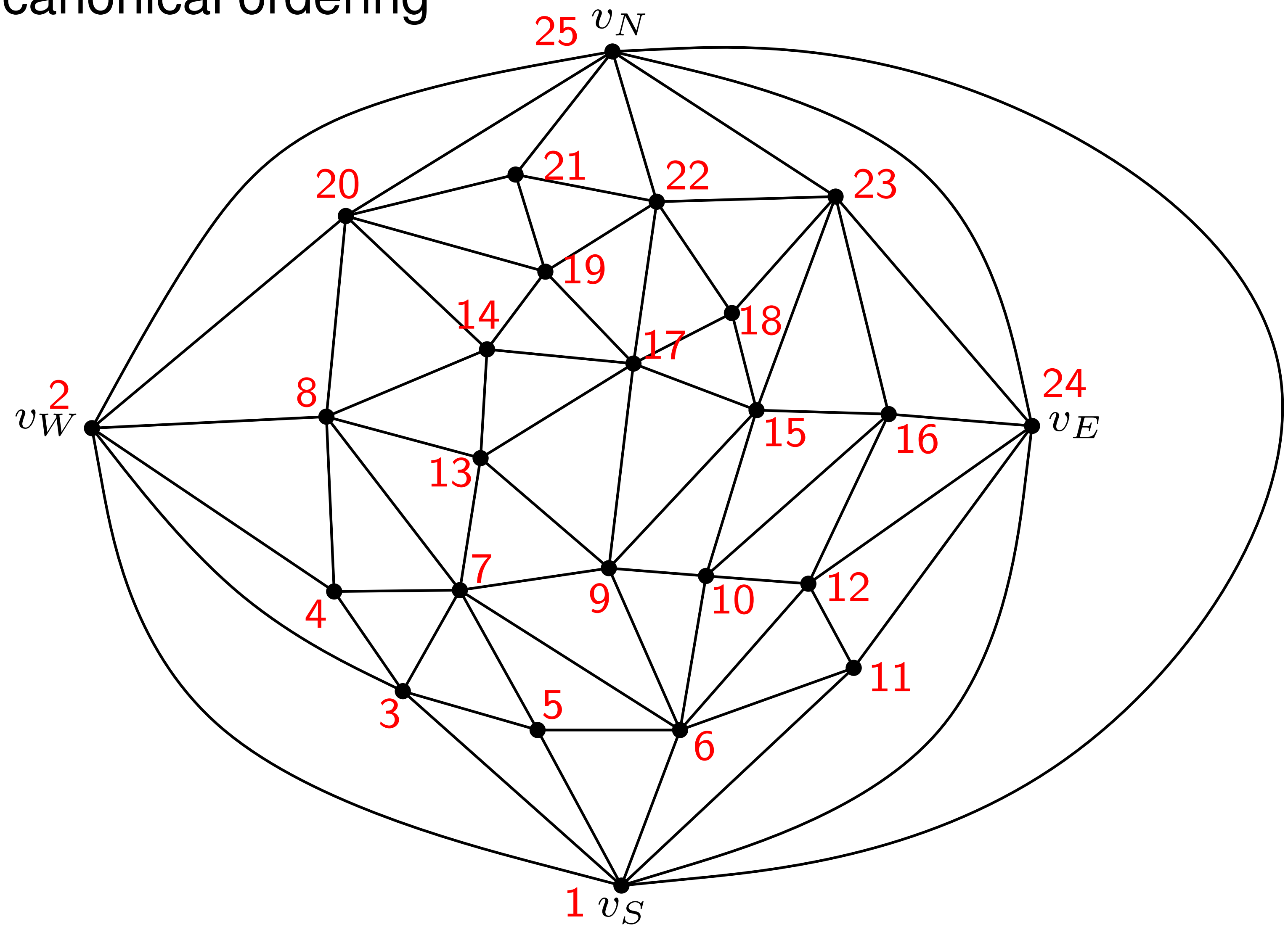
Rectangular Dual

Refined canonical ordering



Rectangular Dual

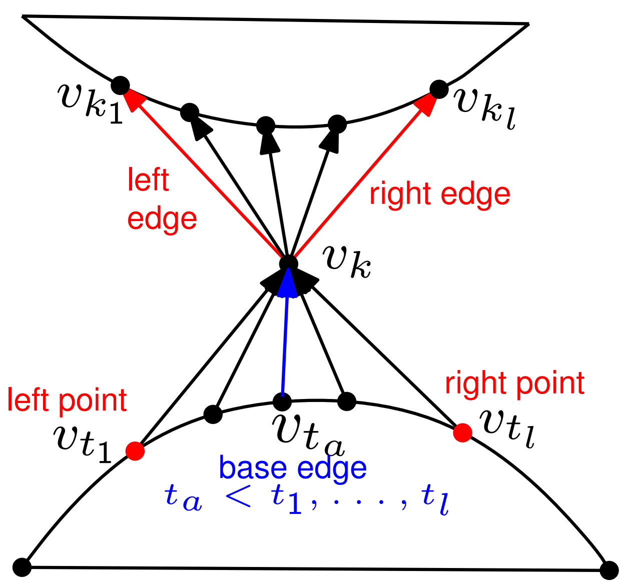
Refined canonical ordering



Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

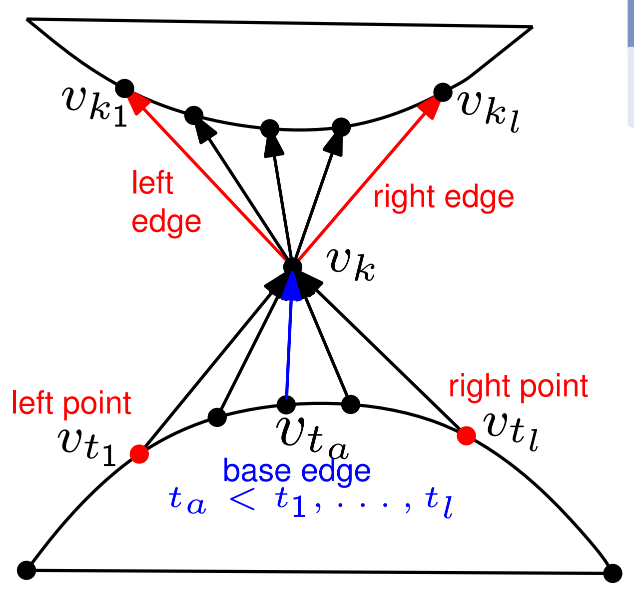
- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



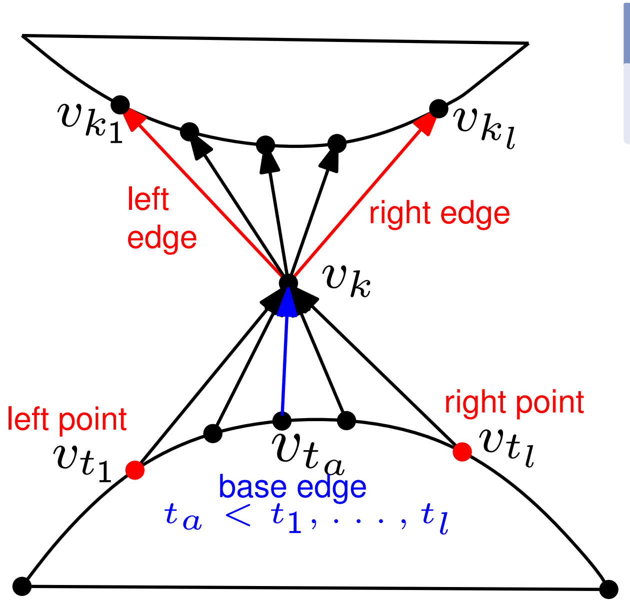
Lemma 1

Left edge or right edge can not be a base edge.

Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



Lemma 1

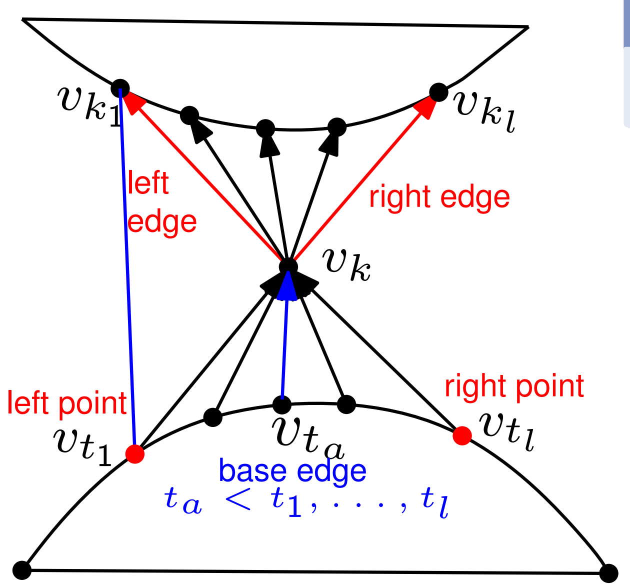
Left edge or right edge can not be a base edge.

Proof: Assume that left edge (v_k, v_{k_1}) is the base edge of v_{k_1}.

Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \dots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \dots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



Lemma 1

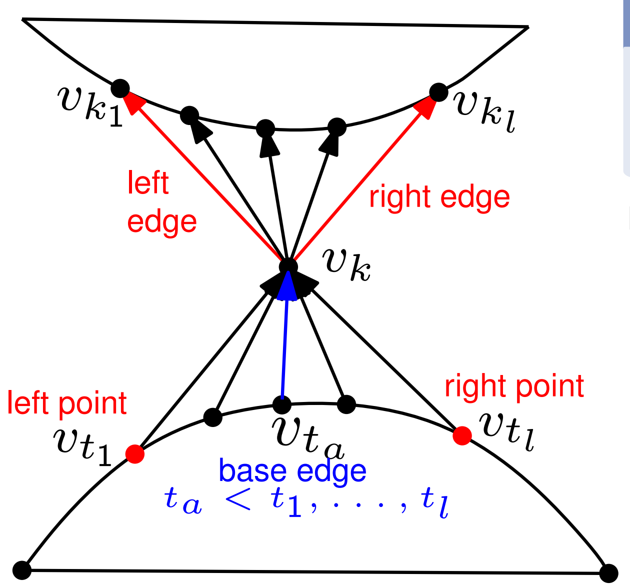
Left edge or right edge can not be a base edge.

Proof: Assume that left edge (v_k, v_{k_1}) is the base edge of v_{k_1}.

Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



Lemma 2

An edge is either a left edge, a right edge or a base edge.

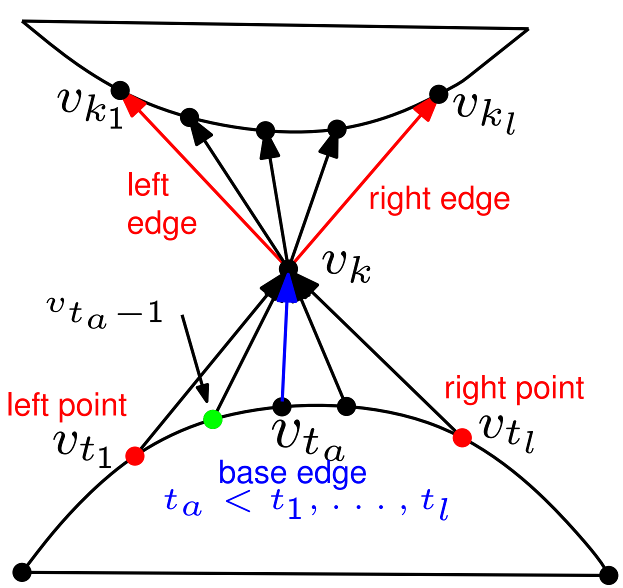
Proof:

- The exclusive “or” follows from Lemma 1.
- Let (v_{t_a}, v_k) be base edge of v_k.
- v_{t_a} is right point of $v_{t_{a-1}}$, $v_{t_{a-1}}$ is right point of $v_{t_{a-2}}$, generally $v_{t_{i+1}}$ is right point of v_{t_i}, $1 \leq i < a - 1$
- Edges (v_{t_i}, v_k), $1 \leq i < a - 1$, are right edges;
- Similarly we prove that edges (v_{t_i}, v_k), $a+1 \leq i < l$, are left edges;

Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j, for $i < j$;
- Base edge of v_k is (v_l, v_k), where $l < k$ is minimal.
- v_k has incoming edges from $v_{t_1}, \ldots, v_{t_l}$, we say that v_{t_1} is left point of v_k and v_{t_l} is right point of v_k.
- If $v_{k_1}, \ldots, v_{k_l}$ are higher numbered neighbors of v_k, we call (v_k, v_{k_1}) left edge and (v_k, v_{k_l}) right edge.



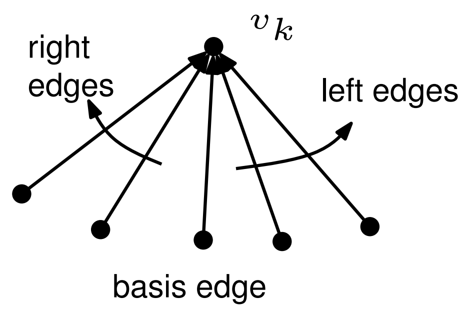
Lemma 2

An edge is either a left edge, a right edge or a base edge.

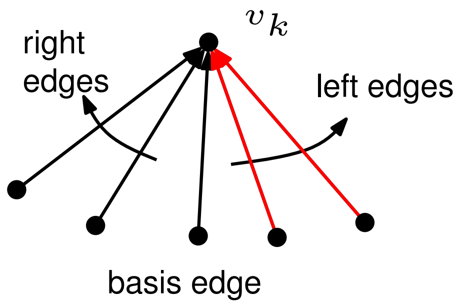
Proof:

- The exclusive “or” follows from Lemma 1.
- Let (v_{t_a}, v_k) be base edge of v_k.
- v_{t_a} is right point of $v_{t_{a-1}}$, $v_{t_{a-1}}$ is right point of $v_{t_{a-2}}$, generally $v_{t_{i+1}}$ is right point of v_{t_i}, $1 \leq i < a-1$
- Edges (v_{t_i}, v_k), $1 \leq i < a-1$, are right edges;
- Similarly we prove that edges (v_{t_i}, v_k), $a+1 \leq i < l$, are left edges;

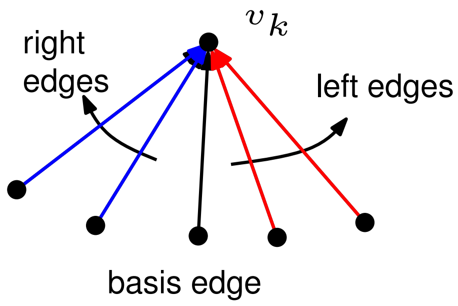
Rectangular Dual



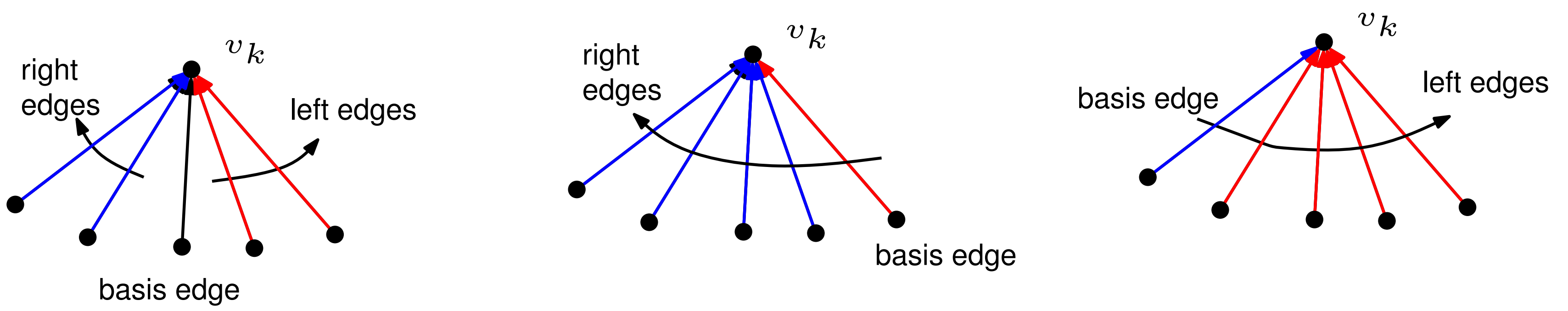
Rectangular Dual



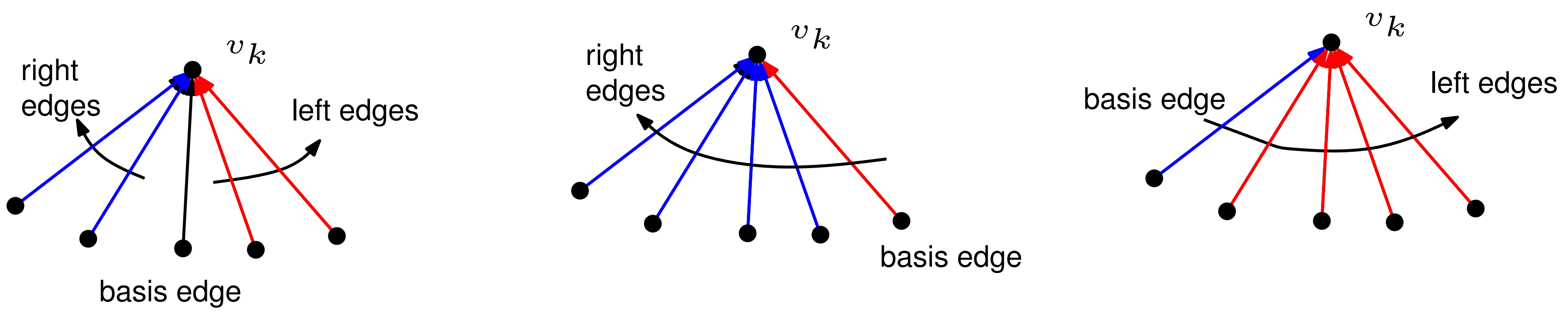
Rectangular Dual



Rectangular Dual

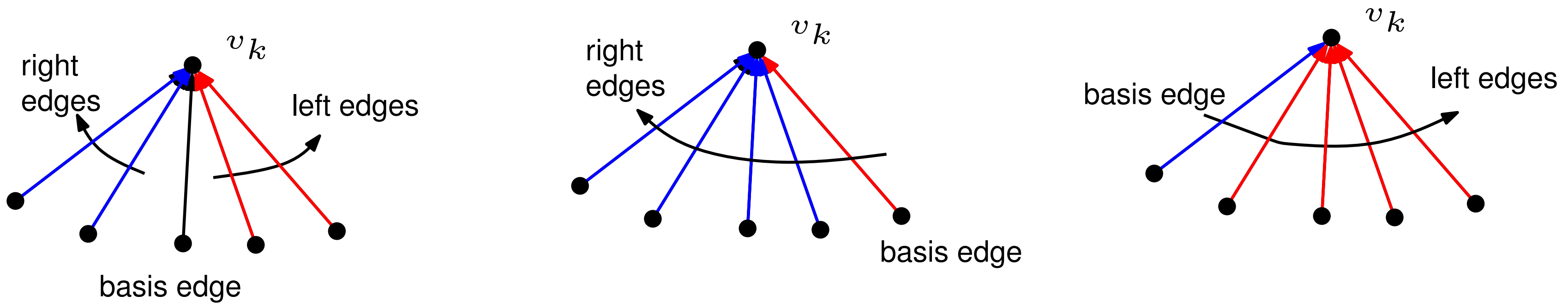


Rectangular Dual



We call T_b blue edges and T_r red edges.

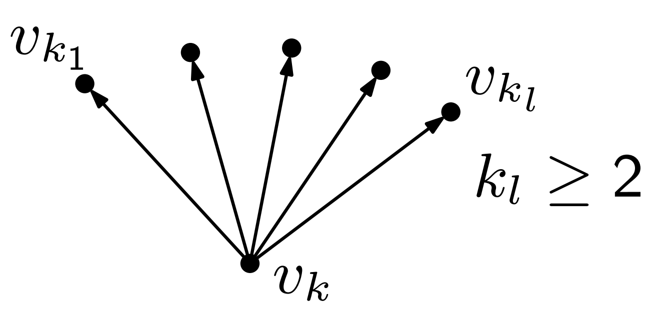
Rectangular Dual



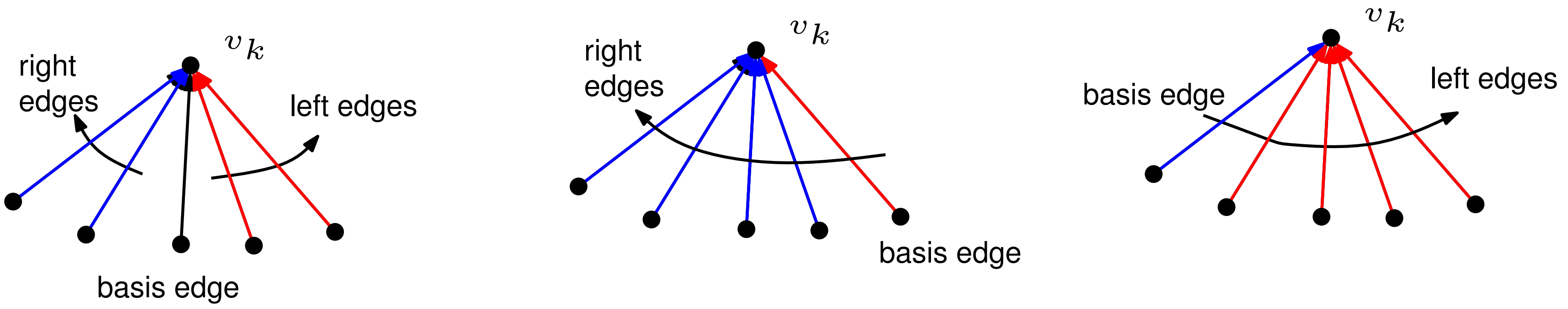
We call T_b blue edges and T_r red edges.

Lemma 3
$\{T_r, T_b\}$ is a regular edge labeling.

Proof:



Rectangular Dual

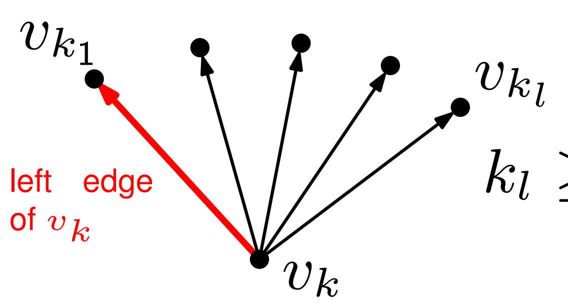


We call T_b blue edges and T_r red edges.

Lemma 3

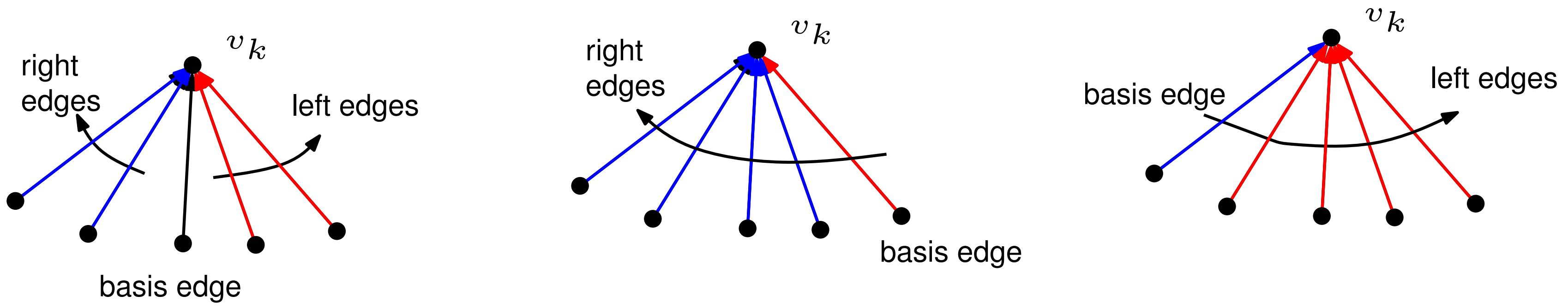
$\{T_r, T_b\}$ is a regular edge labeling.

Proof:



$k_l \geq 2$

Rectangular Dual

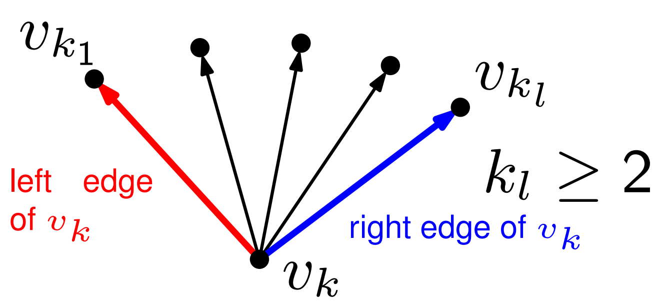


We call T_b blue edges and T_r red edges.

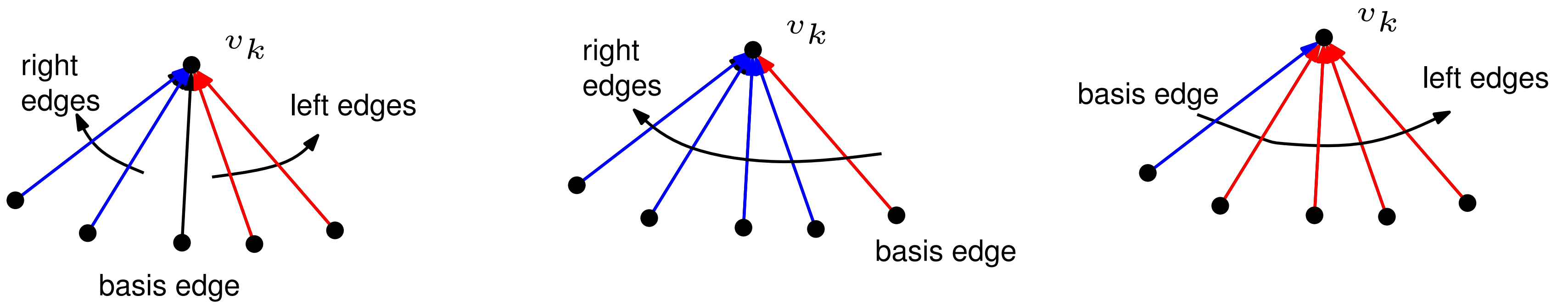
Lemma 3

$\{T_r, T_b\}$ is a regular edge labeling.

Proof:



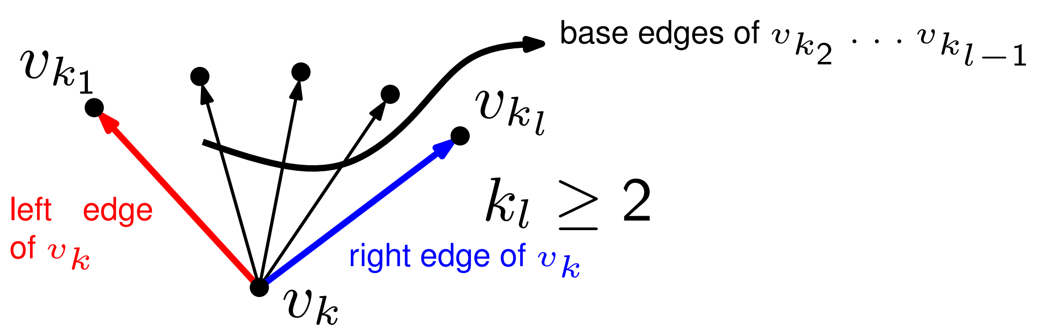
Rectangular Dual



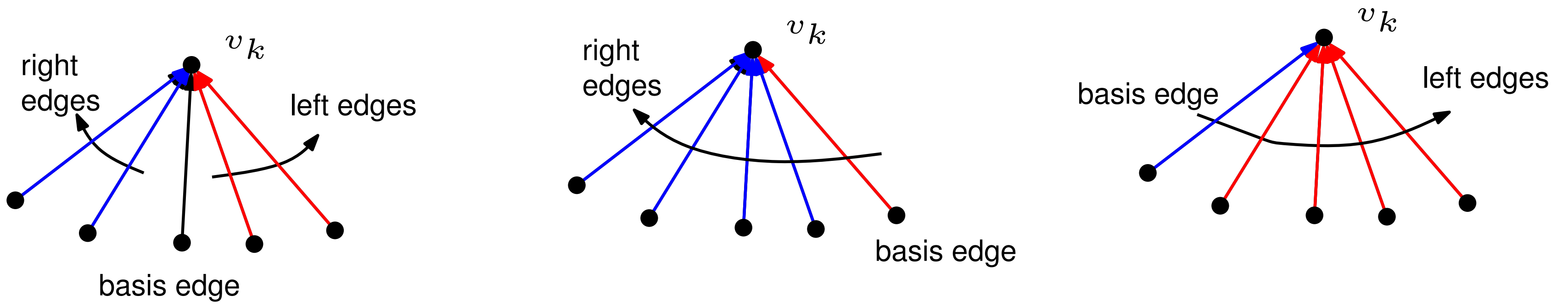
We call T_b blue edges and T_r red edges.

Lemma 3
$\{T_r, T_b\}$ is a regular edge labeling.

Proof:



Rectangular Dual

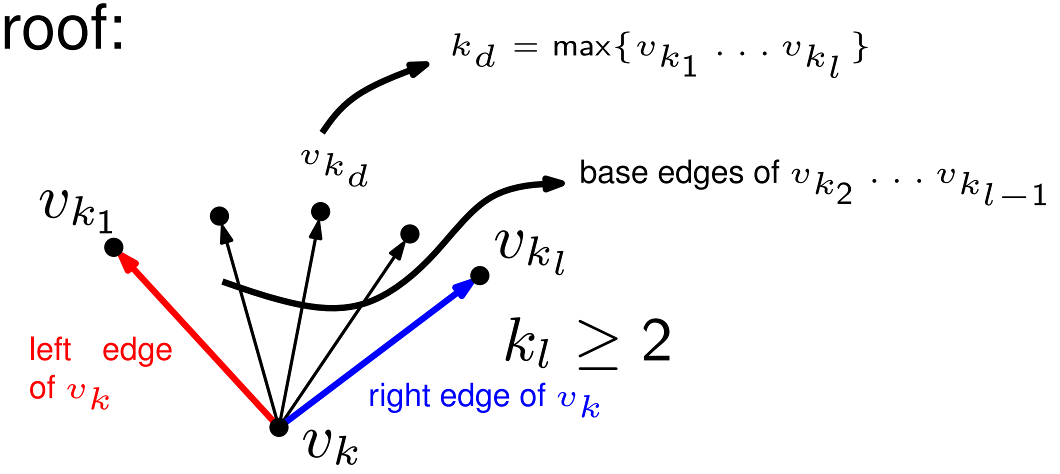


We call T_b blue edges and T_r red edges.

Lemma 3

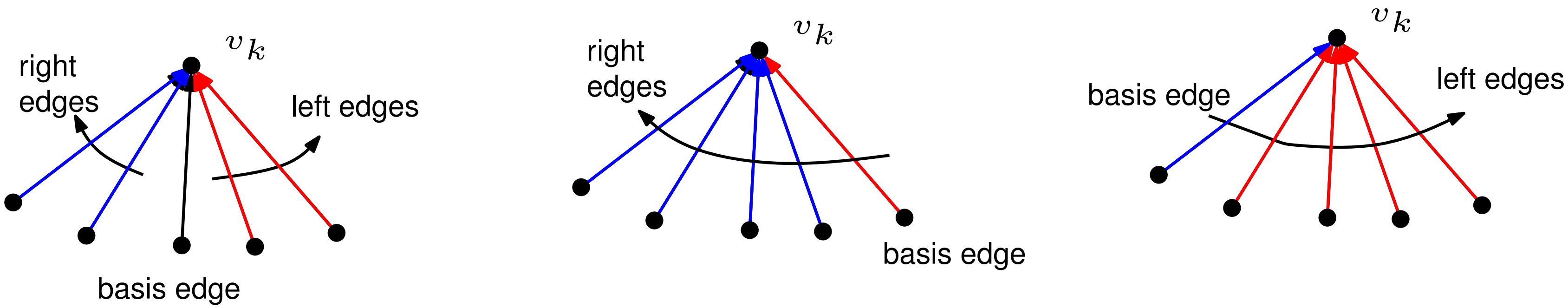
$\{T_r, T_b\}$ is a regular edge labeling.

Proof:



Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

Rectangular Dual



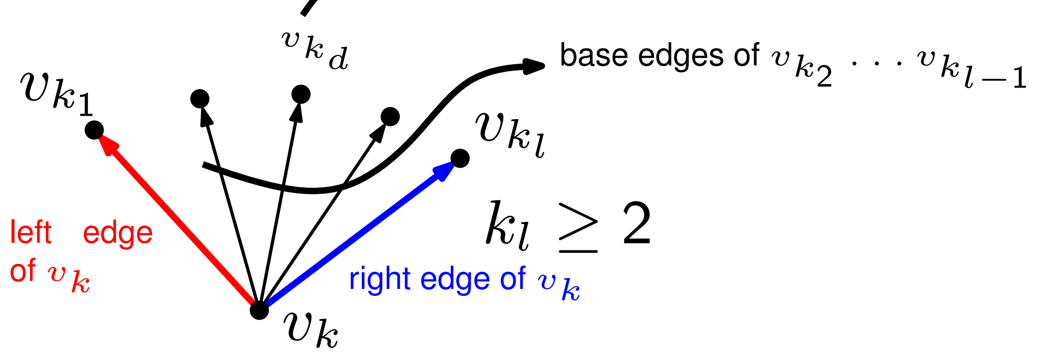
We call T_b blue edges and T_r red edges.

Lemma 3

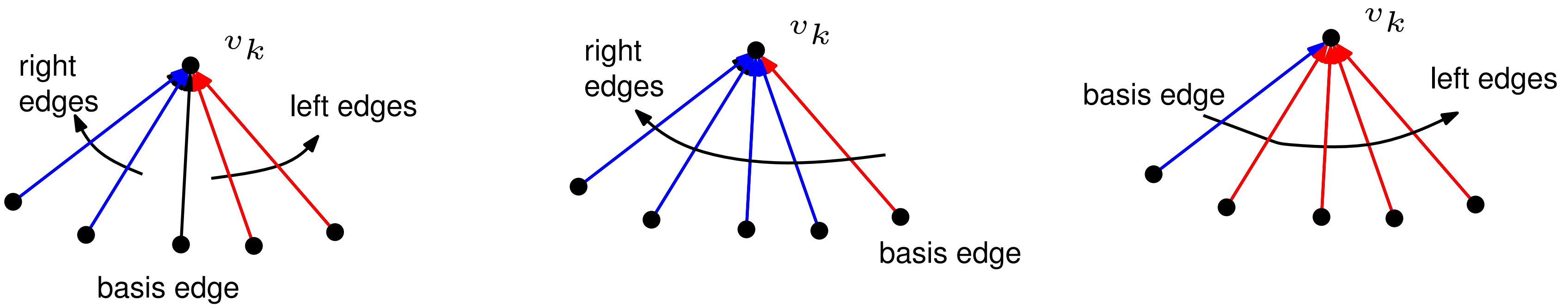
$\{T_r, T_b\}$ is a regular edge labeling.

Proof:

$k_d = \max\{v_{k_1} \dots v_{k_l}\}$ ⇒ $k_1 < k_2 < \dots < k_d$ and $k_d > k_{d+1} > \dots > k_l$



Rectangular Dual



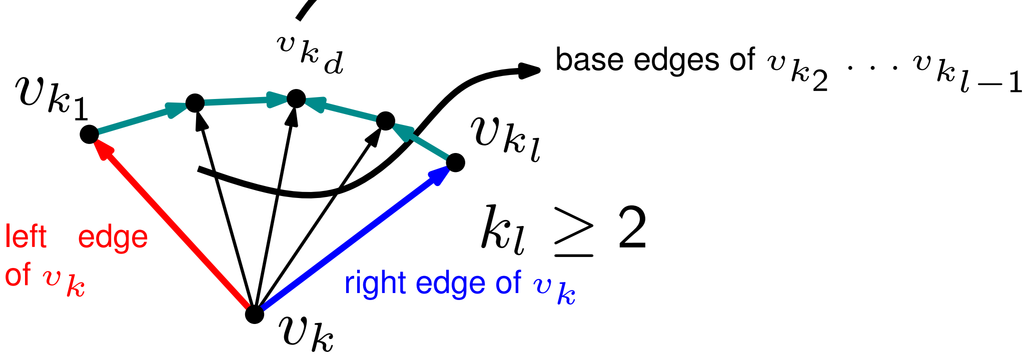
We call T_b blue edges and T_r red edges.

Lemma 3

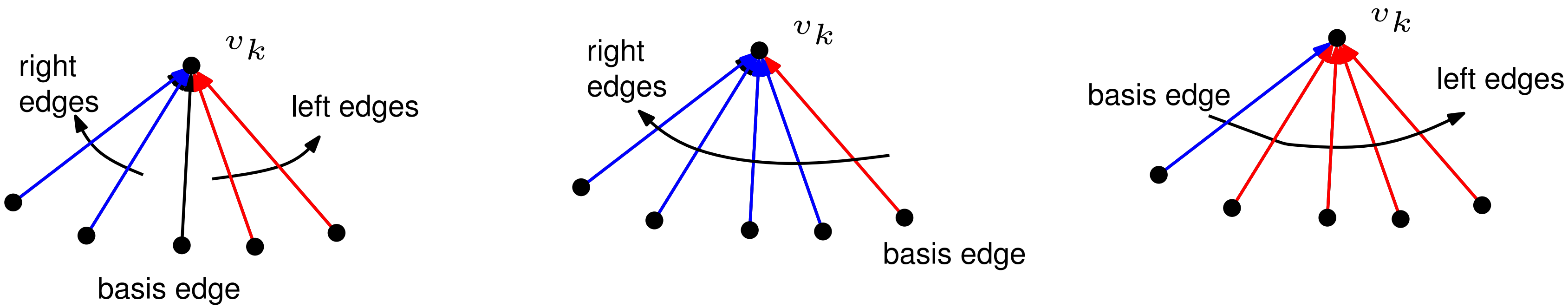
$\{T_r, T_b\}$ is a regular edge labeling.

Proof:

$k_d = \max\{v_{k_1} \dots v_{k_l}\}$ ⇒ $k_1 < k_2 < \dots < k_d$ and $k_d > k_{d+1} > \dots > k_l$



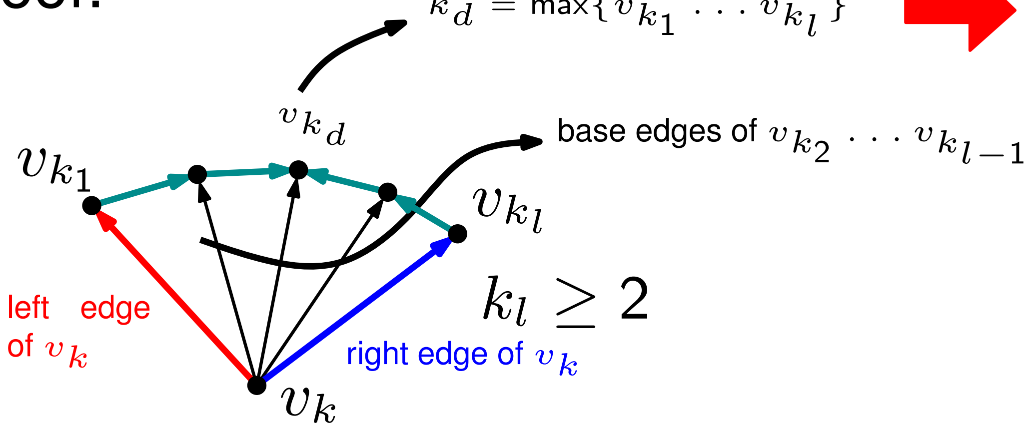
Rectangular Dual



We call T_b blue edges and T_r red edges.

Lemma 3
$\{T_r, T_b\}$ is a regular edge labeling.

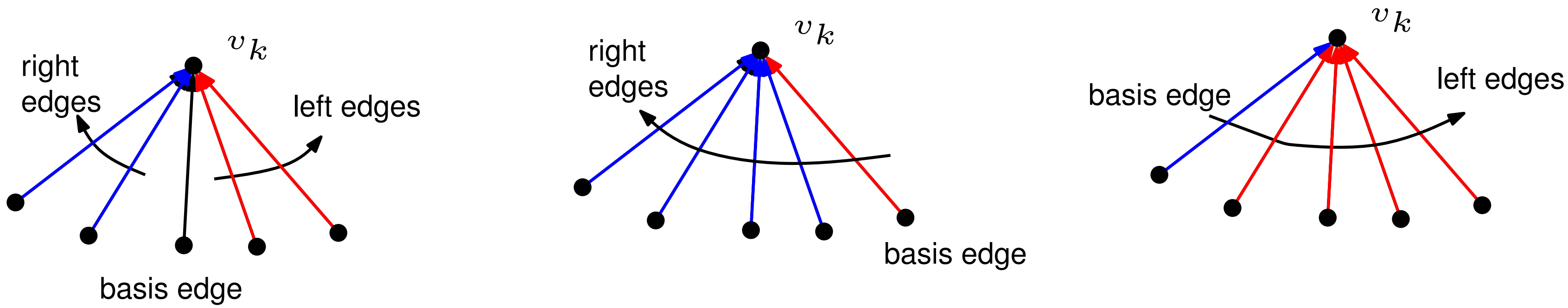
Proof:



$k_1 < k_2 < \dots < k_d$ and
$k_d > k_{d+1} > \dots > k_l$

(v_k, v_{k_i}), $2 \leq i \leq d-1$ are red
(v_k, v_{k_i}), $d+1 \leq i \leq l-1$ are blue
edge (v_k, v_{k_d}) is either red or blue

Rectangular Dual



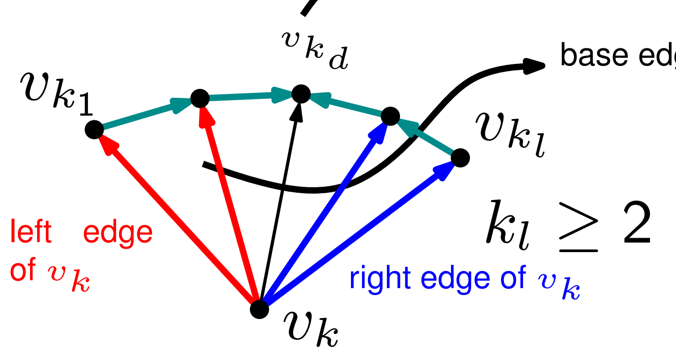
We call T_b blue edges and T_r red edges.

Lemma 3

$\{T_r, T_b\}$ is a regular edge labeling.

Proof:

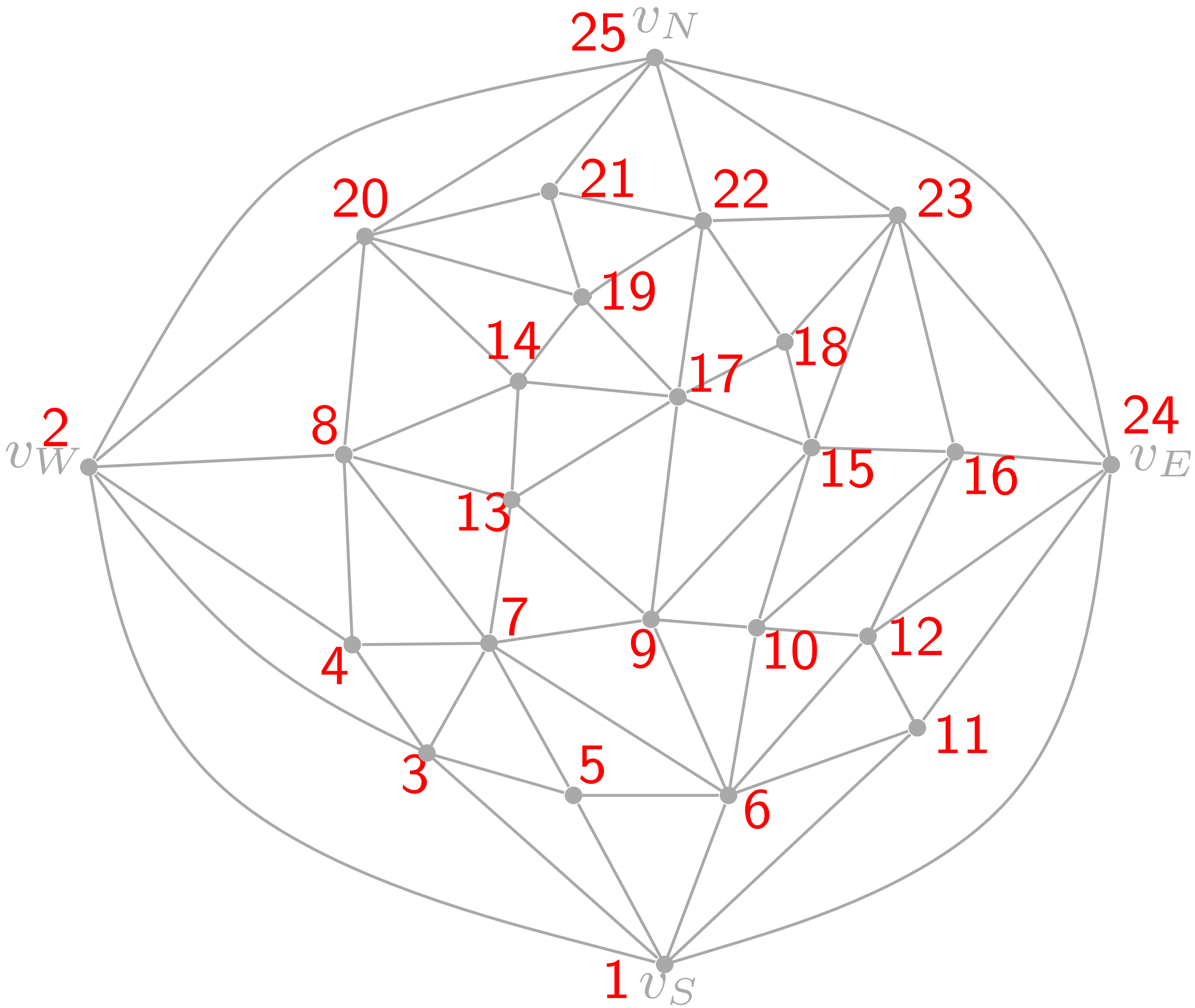
$k_d = \max\{v_{k_1} \dots v_{k_l}\}$



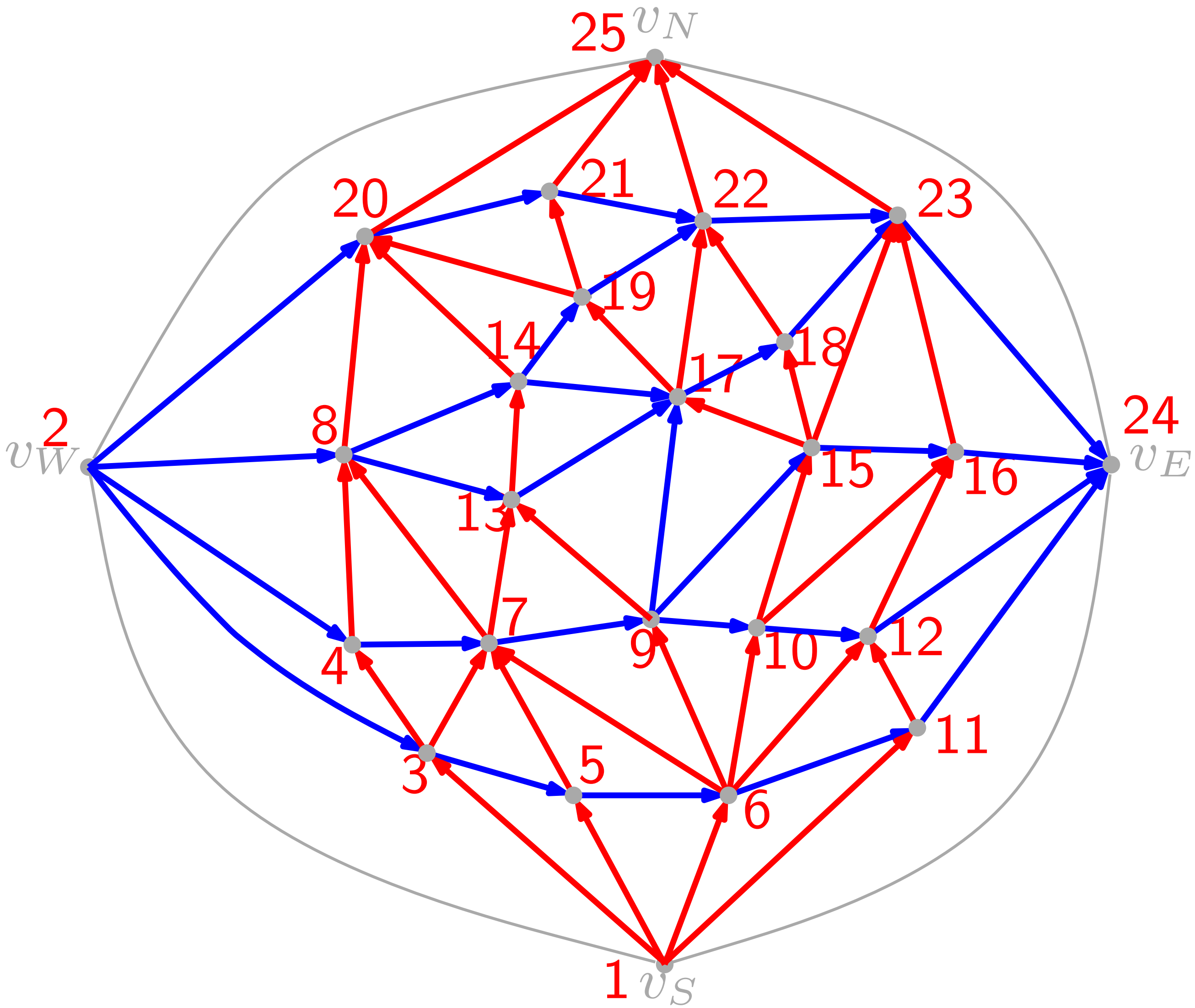
$k_1 < k_2 < \dots < k_d$ and
$k_d > k_{d+1} > \dots > k_l$

(v_k, v_{k_i}), $2 \leq i \leq d-1$ are red
(v_k, v_{k_i}), $d+1 \leq i \leq l-1$ are blue
edge (v_k, v_{k_d}) is either red or blue

Rectangular Dual

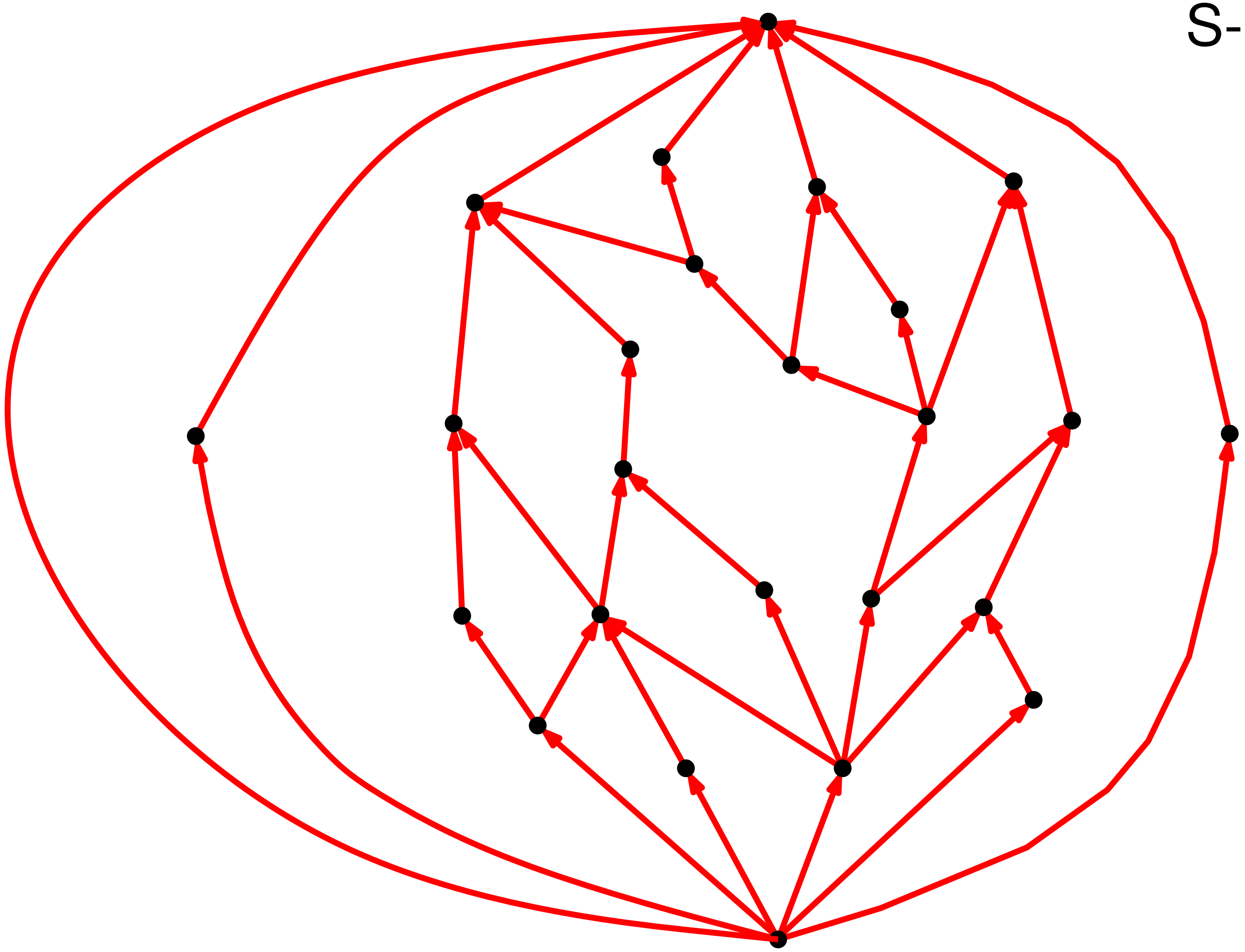


Rectangular Dual



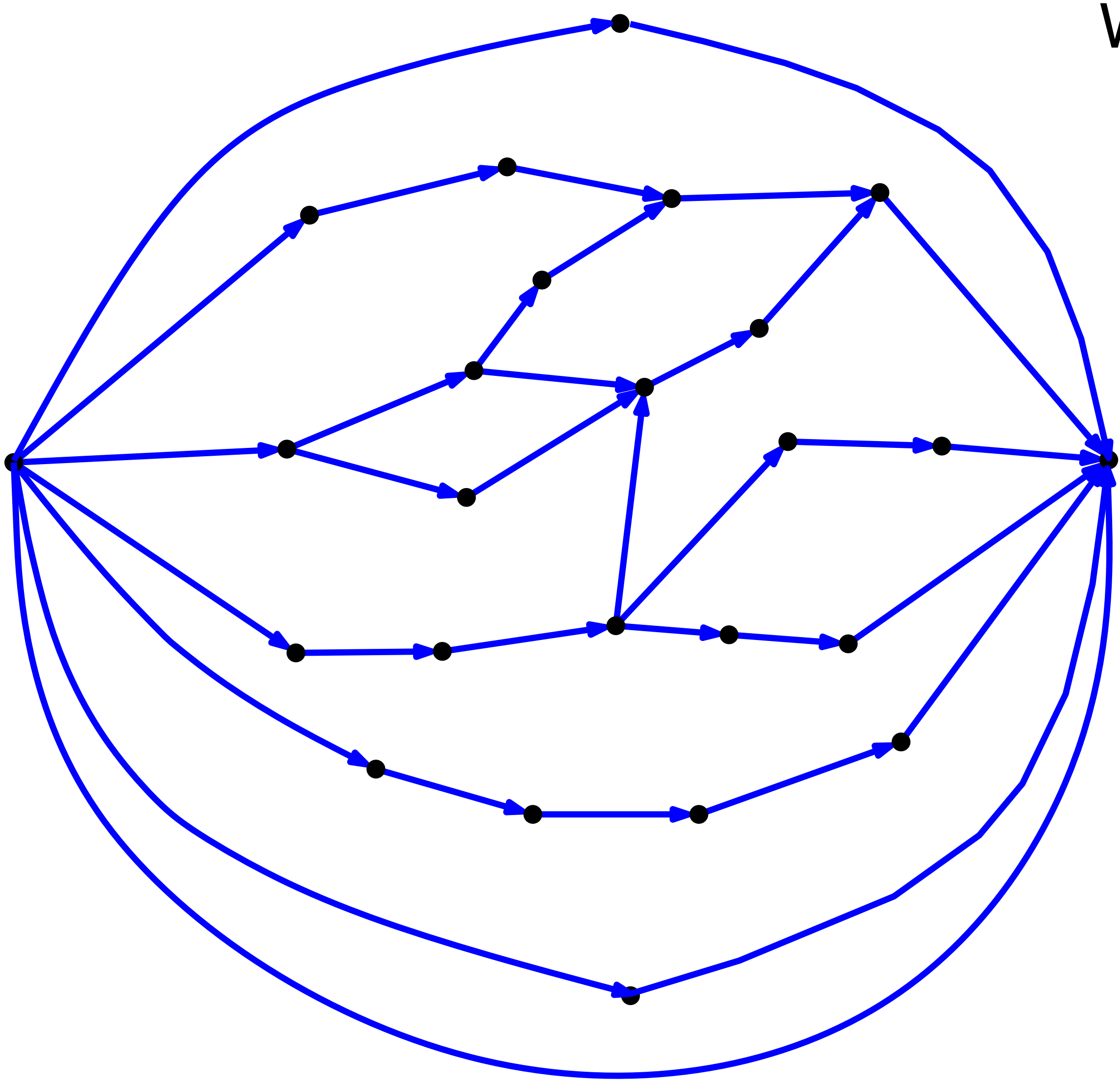
Rectangular Dual

S-N net G_{S-N}



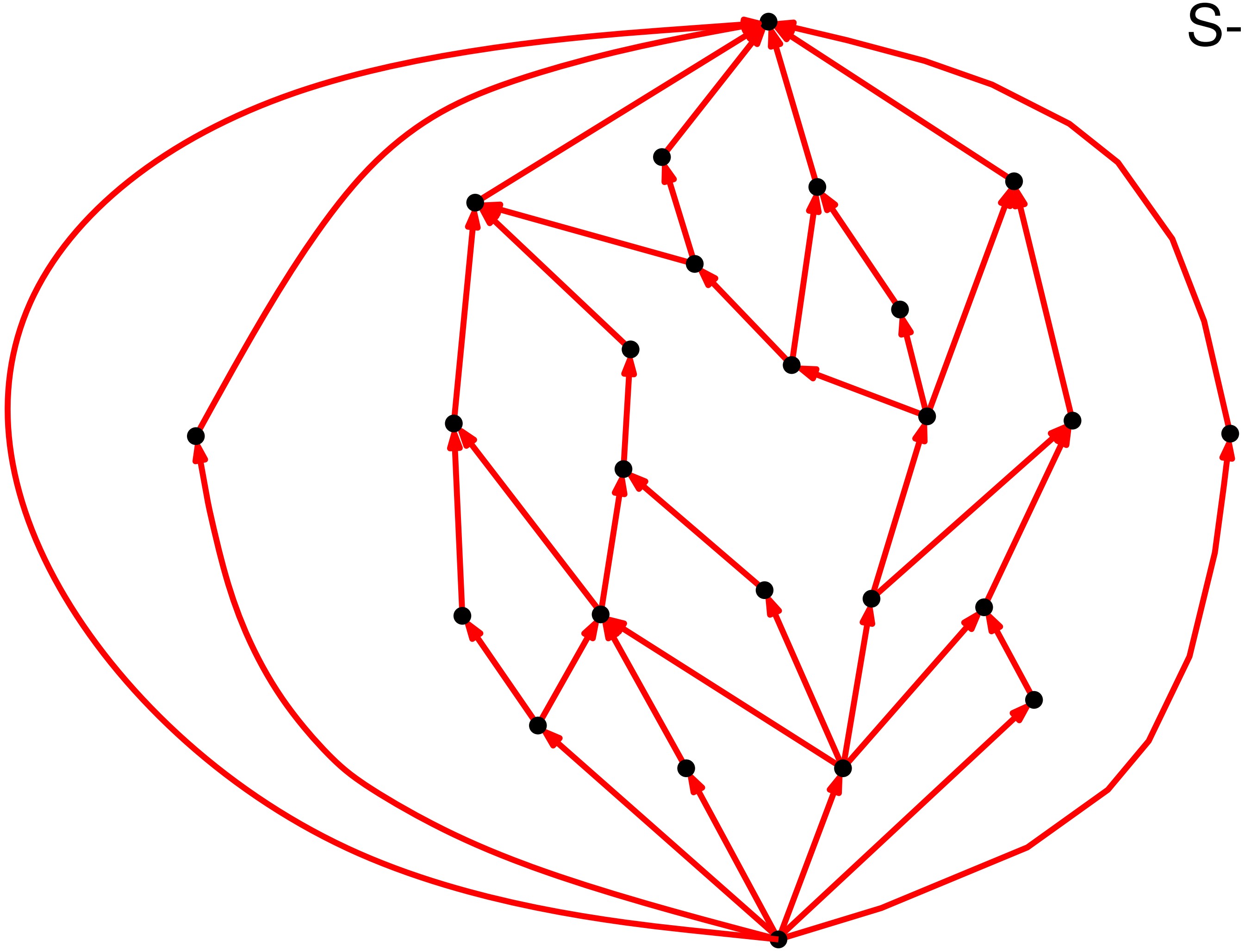
Rectangular Dual

W-E net G_{W-E}



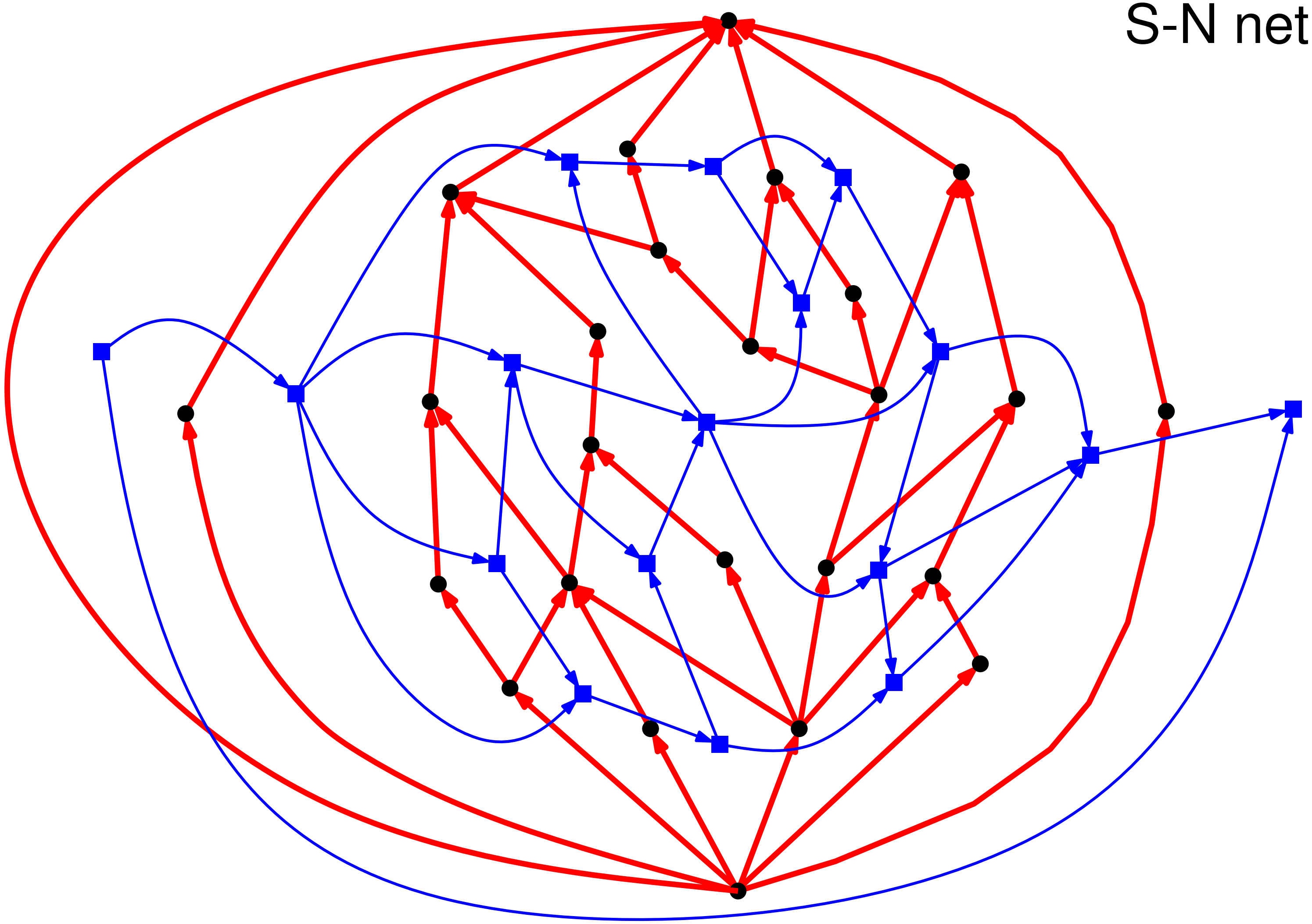
Rectangular Dual

S-N net G_{S-N}

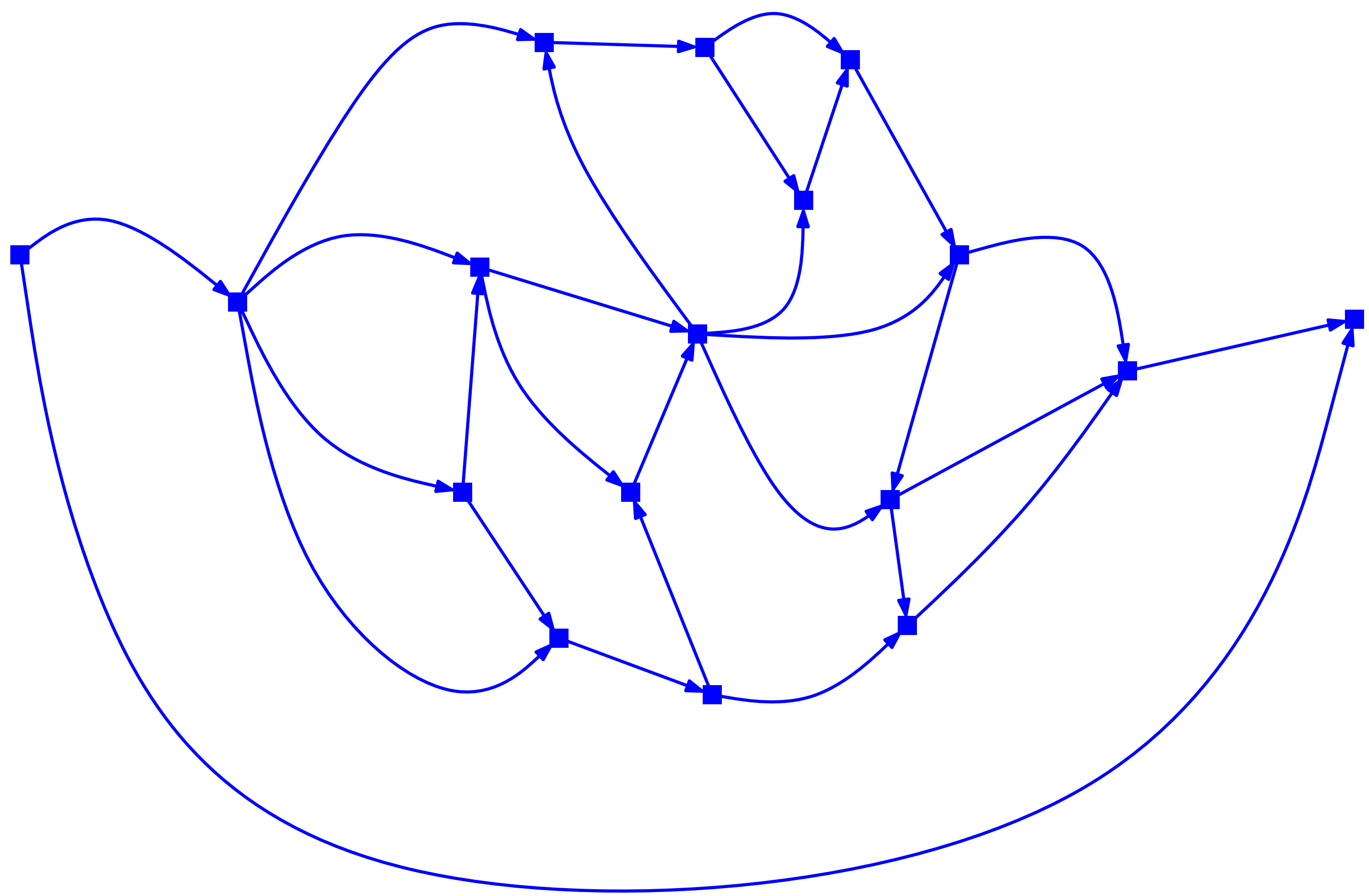


Rectangular Dual

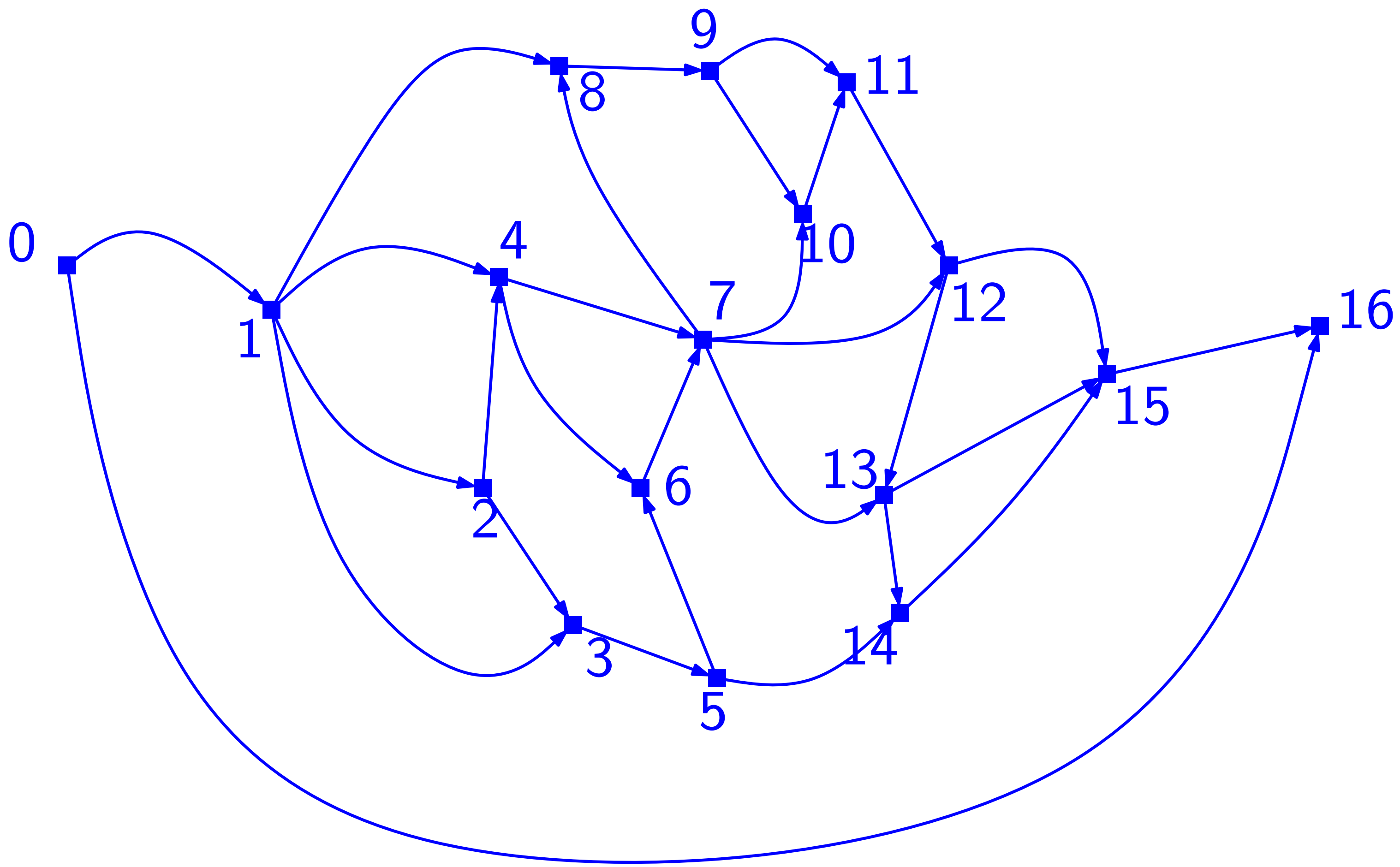
S-N net G_{S-N}



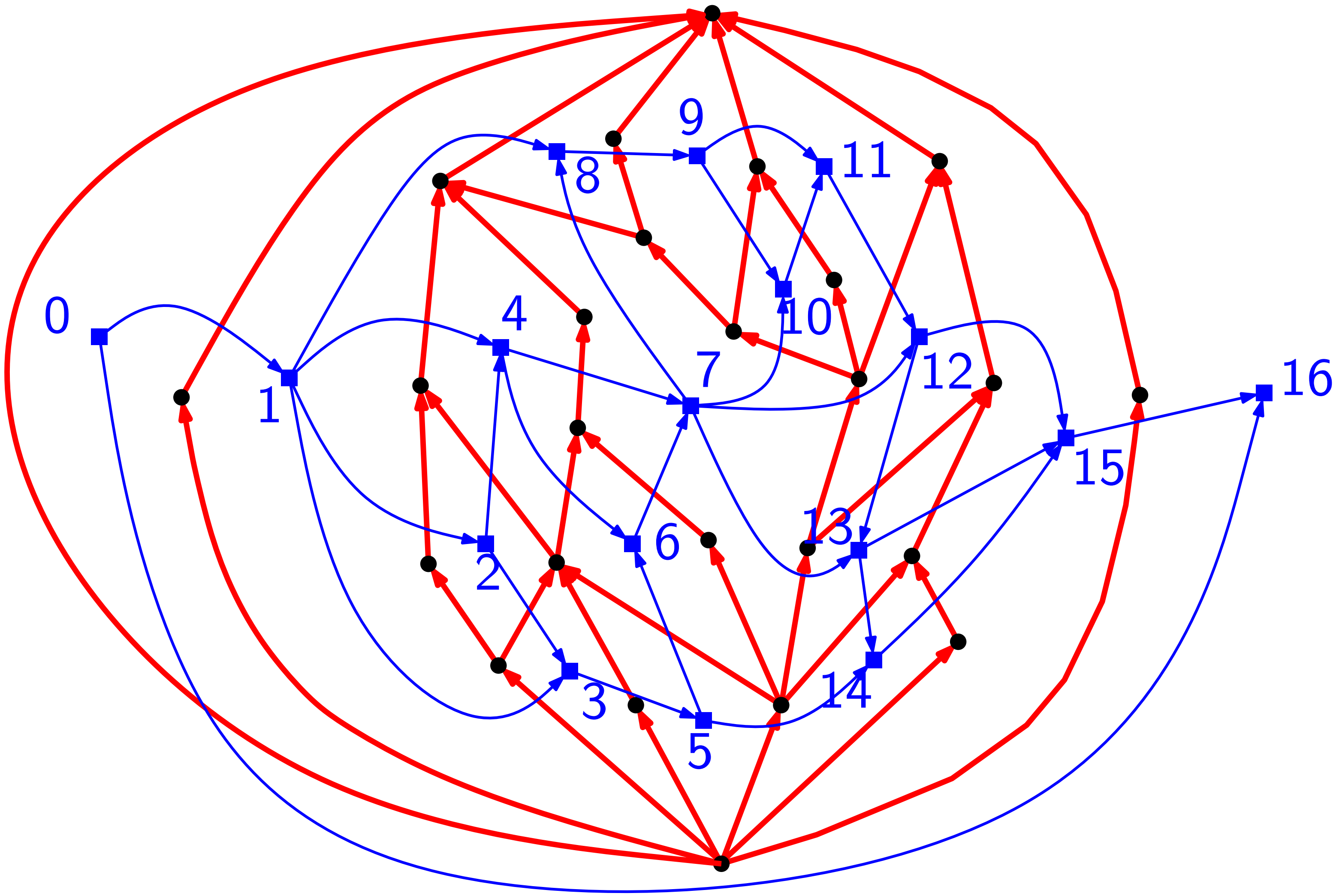
Rectangular Dual



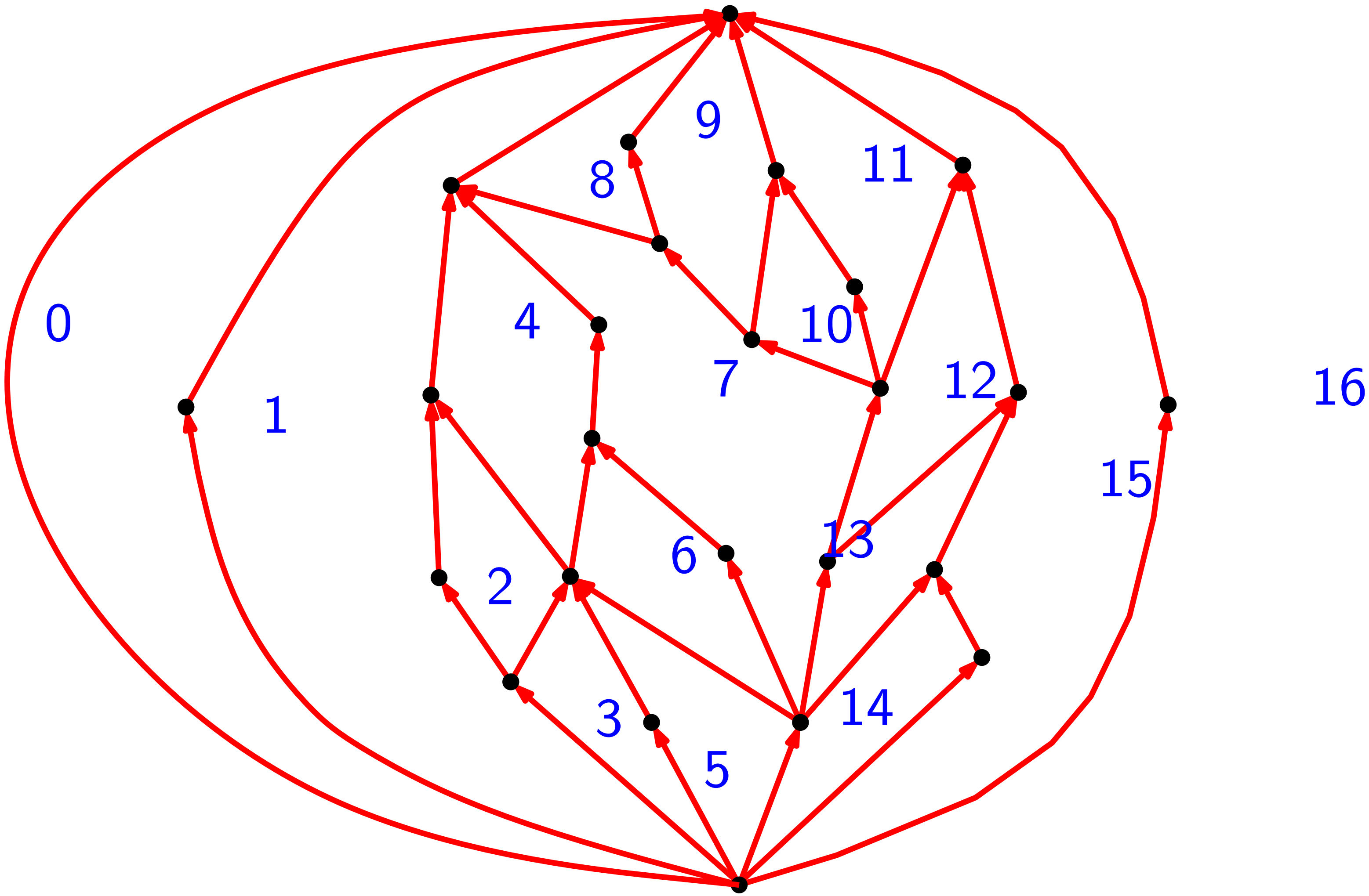
Rectangular Dual



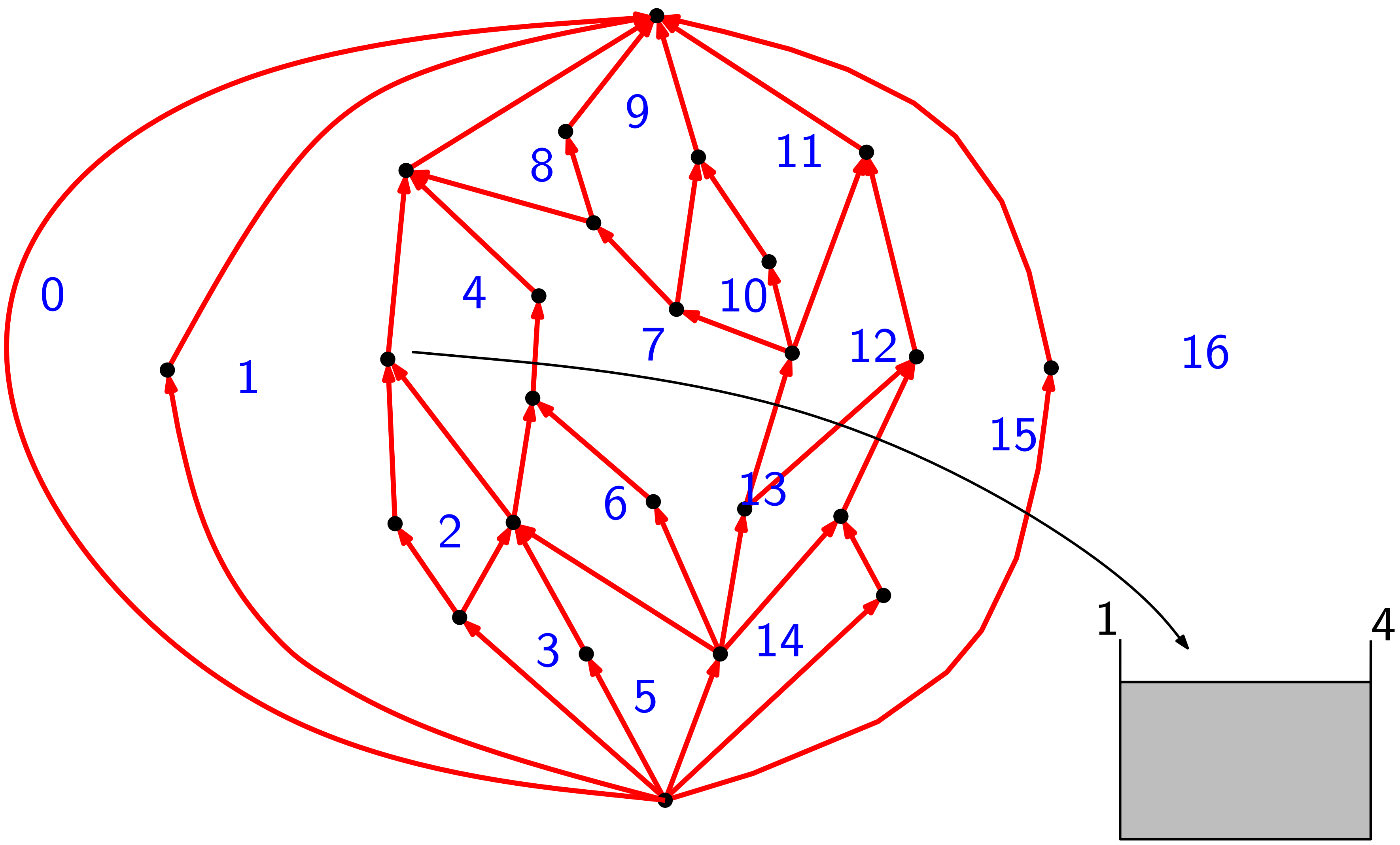
Rectangular Dual



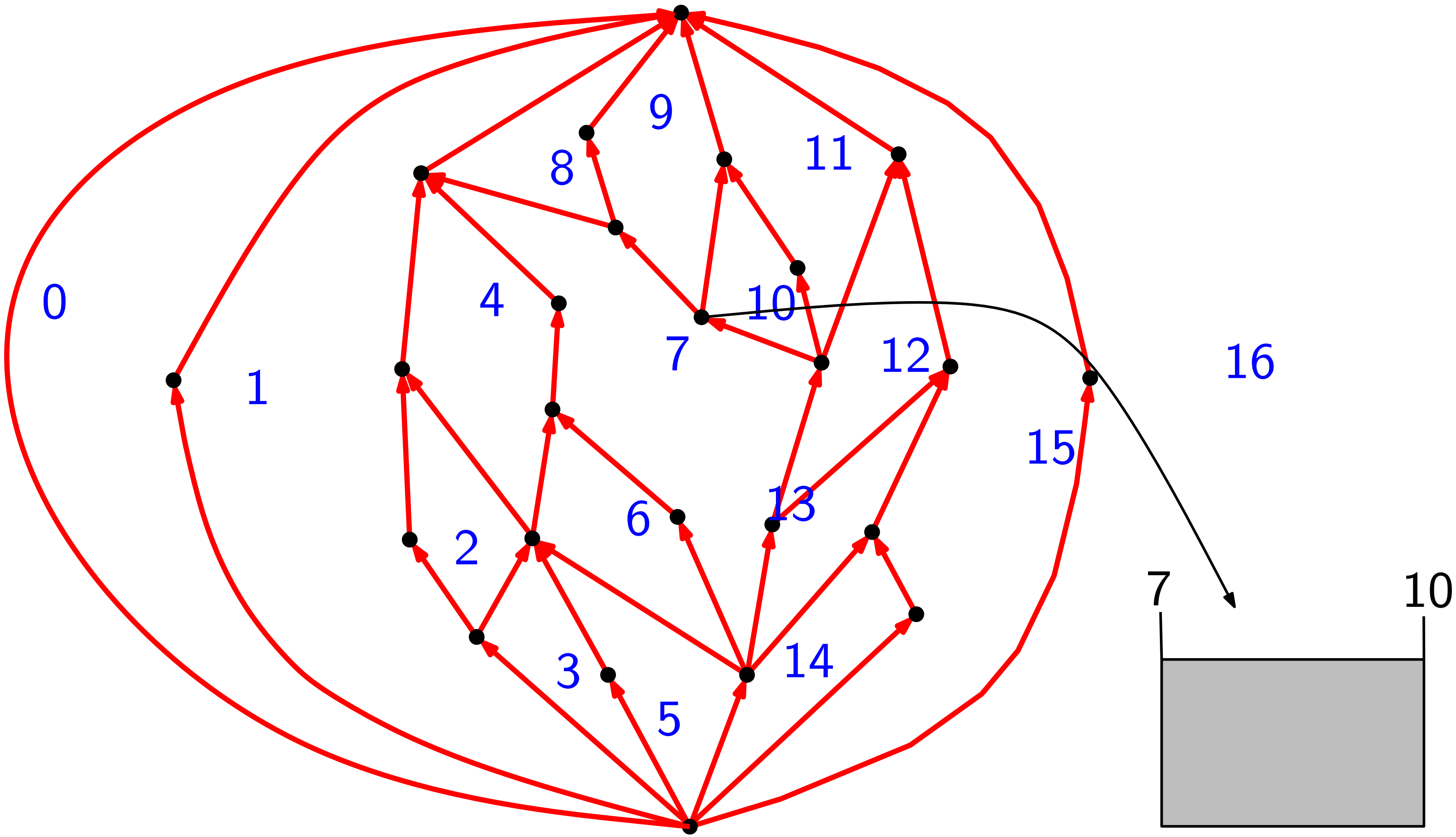
Rectangular Dual



Rectangular Dual



Rectangular Dual



Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a S-N net G_{S-N} of G (consists of T_r plus outer edges)
- Construct the dual $G^\star_{S-N}$ of G_{S-N} and compute a topological ordering f_{sn} of $G^\star_{S-N}$
- For each vertex $v \in V$, let f and g be the face on the left and face on the right of v. Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a net of G (consists of plus outer edges)
- Construct the dual o and compute a topological ordering of
- For each vertex $v \in V$, let f and g be the face and face v. Set $= f_{sn}(f)$ and $= f_{sn}(g)$.
- Define and

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a W-E net G_{W-E} of G (consists of T_b plus outer edges)
- Construct the dual o and compute a topological ordering of
- For each vertex $v \in V$, let f and g be the face and face v. Set $= f_{sn}(f)$ and $= f_{sn}(g)$.
- Define and

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a $\boxed{\text{W-E}}$ net $\boxed{G_{W-E}}$ of G (consists of $\boxed{T_b}$ plus outer edges)
- Construct the dual $\boxed{G^{\star}_{W-E}}$ o $\boxed{G_{W-E}}$ and compute a topological ordering $\boxed{f_{we}}$ of $\boxed{G^{\star}_{W-E}}$
- For each vertex $v \in V$, let f and g be the face $\boxed{}$ and face $\boxed{}$ v. Set $\boxed{}$ $= f_{sn}(f)$ and $\boxed{}$ $= f_{sn}(g)$.
- Define $\boxed{}$ and $\boxed{}$

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a $\boxed{\text{W-E}}$ net $\boxed{G_{W-E}}$ of G (consists of $\boxed{T_b}$ plus outer edges)
- Construct the dual $\boxed{G^\star_{W-E}}$ o $\boxed{G_{W-E}}$ and compute a topological ordering $\boxed{f_{we}}$ of $\boxed{G^\star_{W-E}}$
- For each vertex $v \in V$, let f and g be the face $\boxed{\text{below}}$ and face $\boxed{\text{above}}$ v. Set $\boxed{y_1(v)} = f_{sn}(f)$ and $\boxed{y_2(v)} = f_{sn}(g)$.
- Define $\boxed{}$ and $\boxed{}$

Rectangular Dual

Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

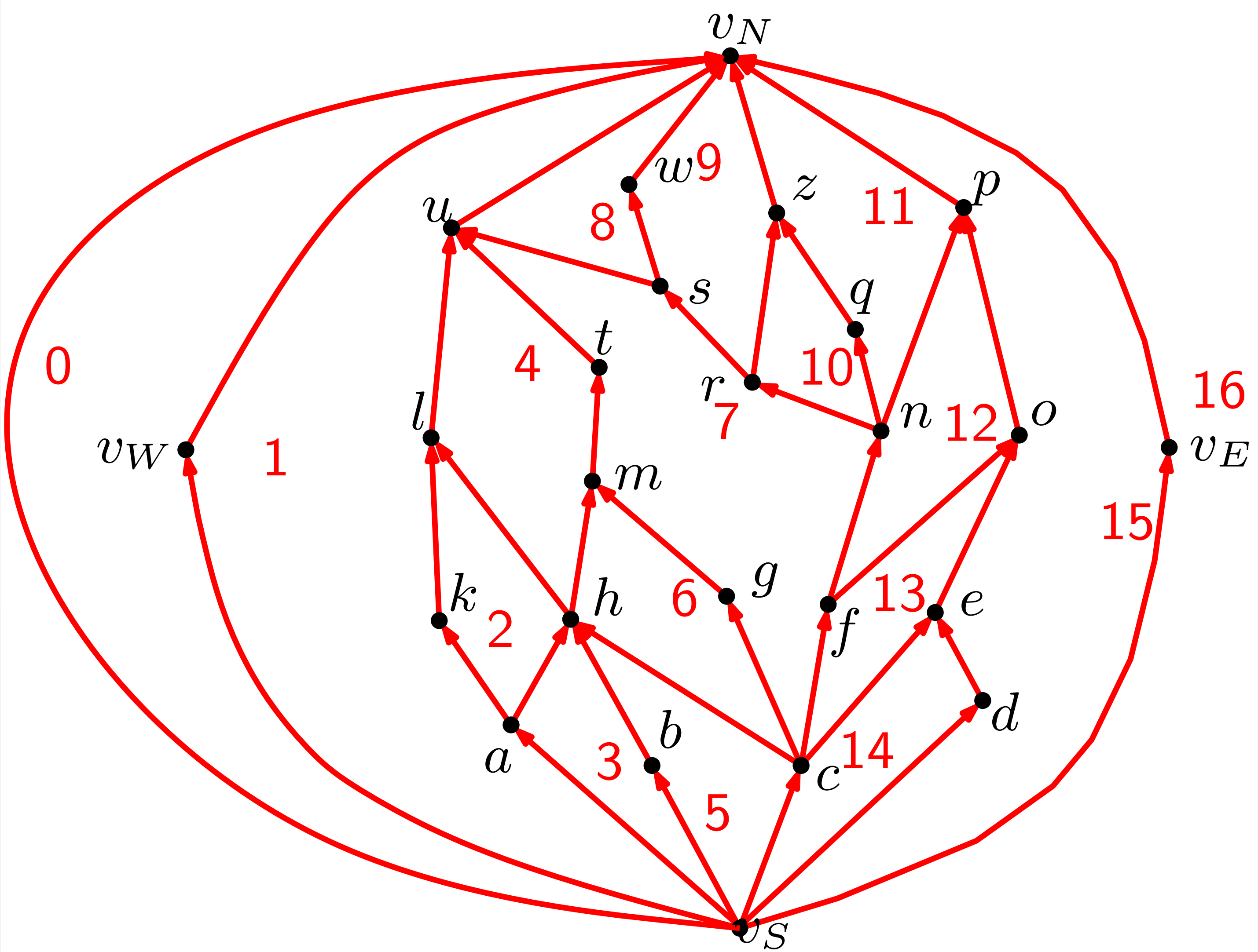
- Find a REL T_r, T_b of G;
- Construct a W-E net G_{W-E} of G (consists of T_b plus outer edges)
- Construct the dual $G^{\star}_{W-E}$ of G_{W-E} and compute a topological ordering f_{we} of $G^{\star}_{W-E}$
- For each vertex $v \in V$, let f and g be the face below and face above v. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$.
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$

Rectangular Dual

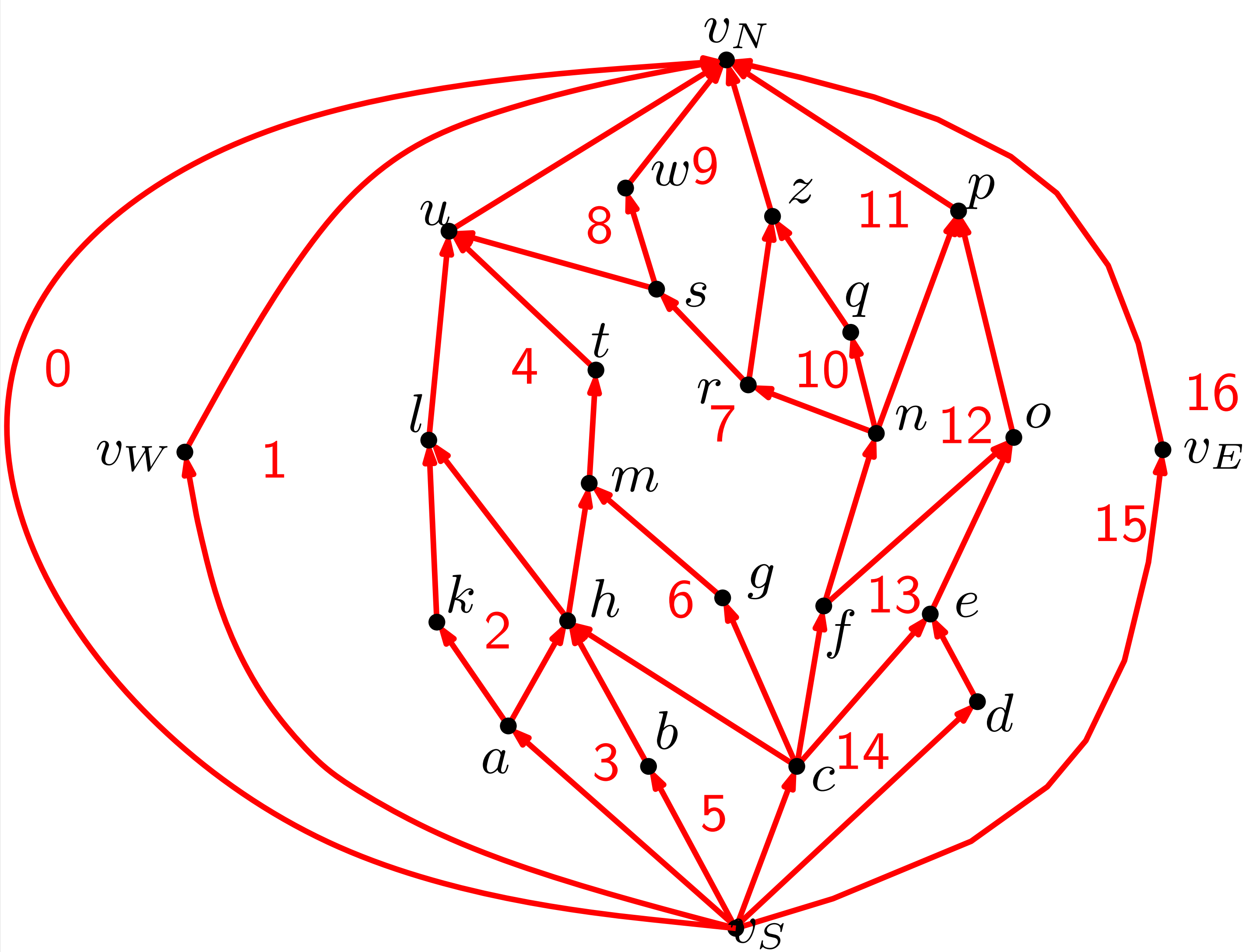
Algorithm Rectangular dual
Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G;
- Construct a W-E net G_{W-E} of G (consists of T_b plus outer edges)
- Construct the dual $G^\star_{W-E}$ of G_{W-E} and compute a topological ordering f_{we} of $G^\star_{W-E}$
- For each vertex $v \in V$, let f and g be the face below and face above v. Set $y_1(v) = f_{sn}(f)$ and $y_2(v) = f_{sn}(g)$.
- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$
- For each $v \in V$, assign a rectangle $R(v)$ bounded by x-coordinates $x_1(v),\ x_2(v)$ and y-coordinates $y_1(v),\ y_2(v)$.

Rectangular Dual

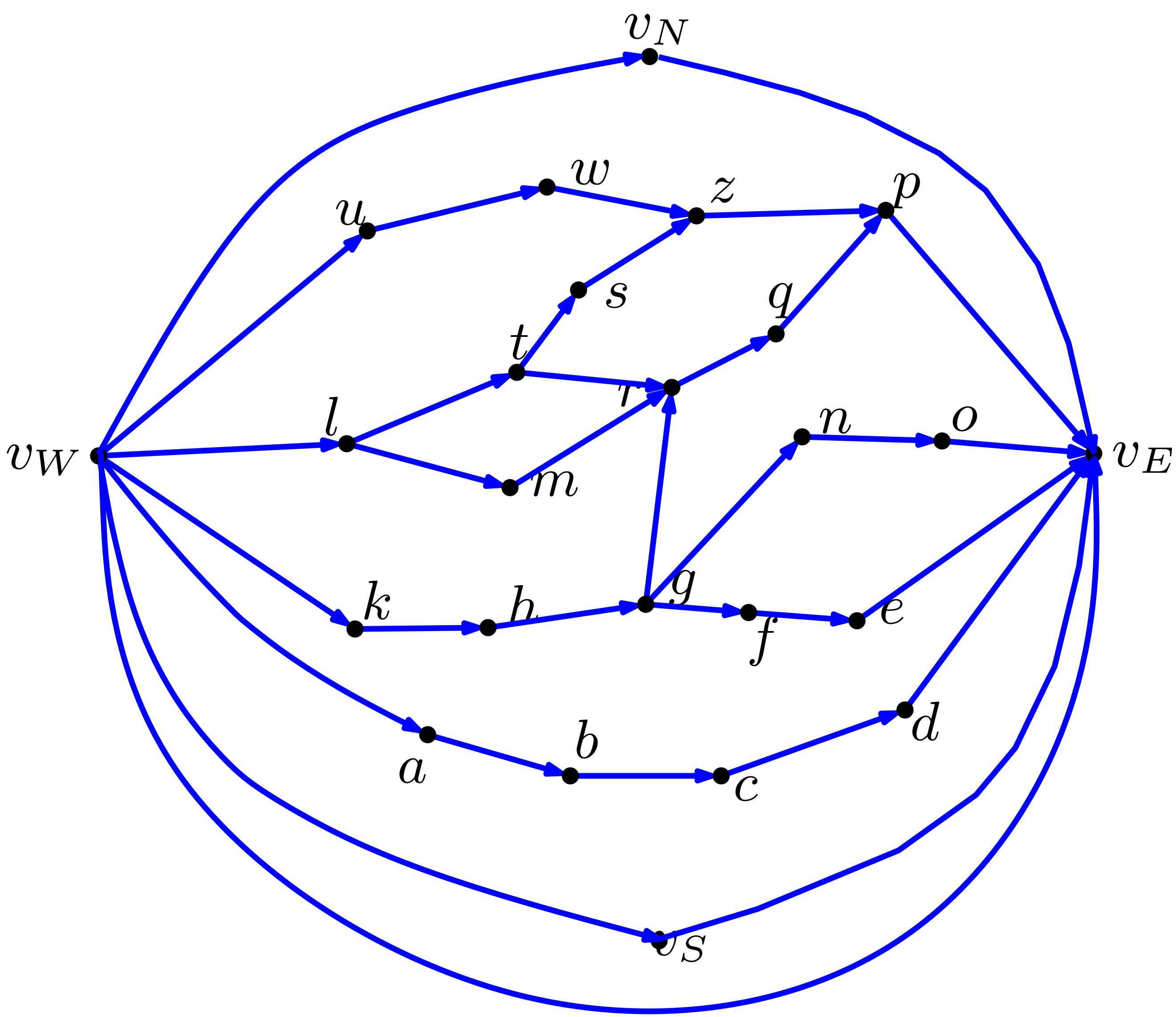


Rectangular Dual



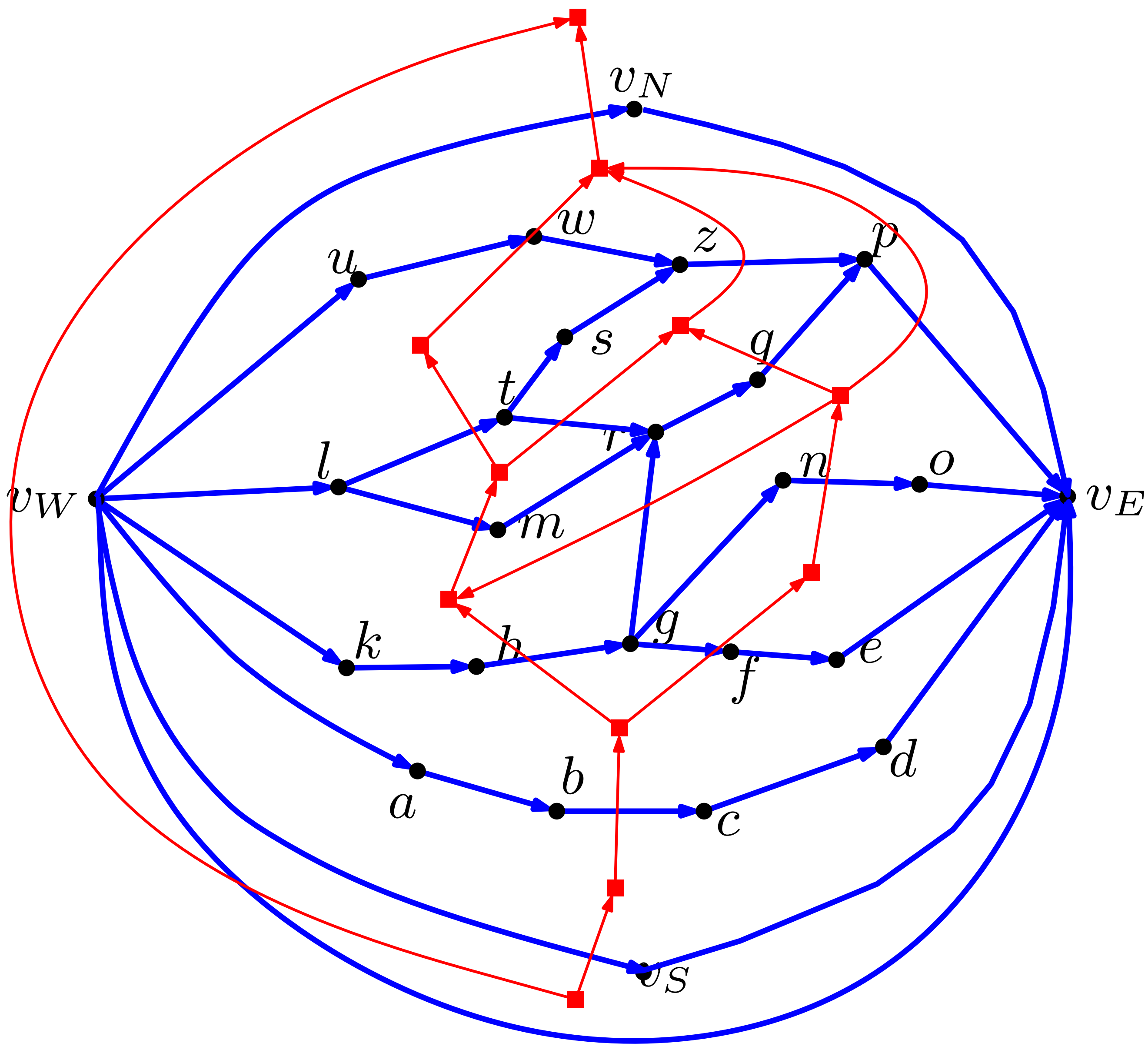
$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



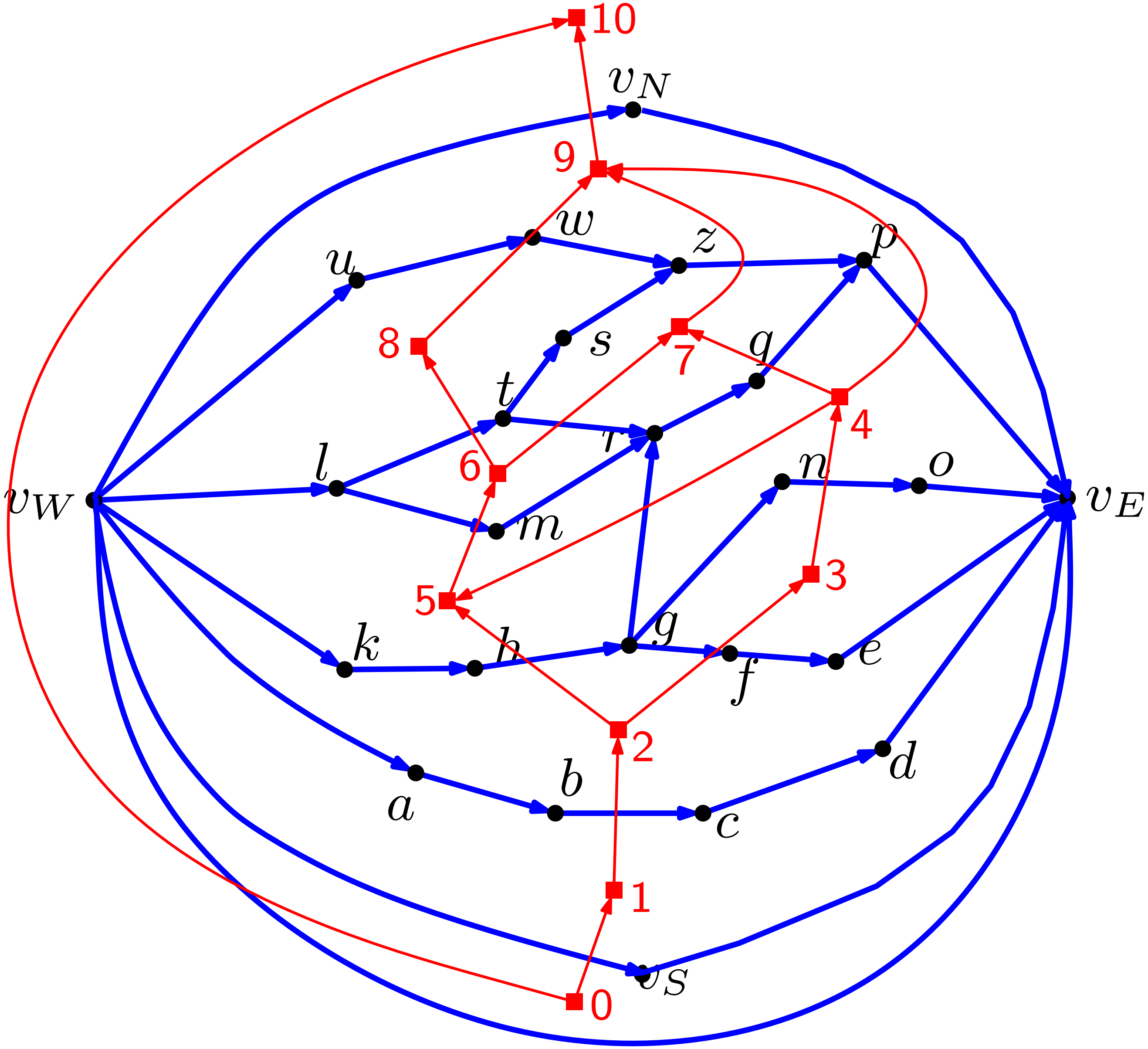
$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



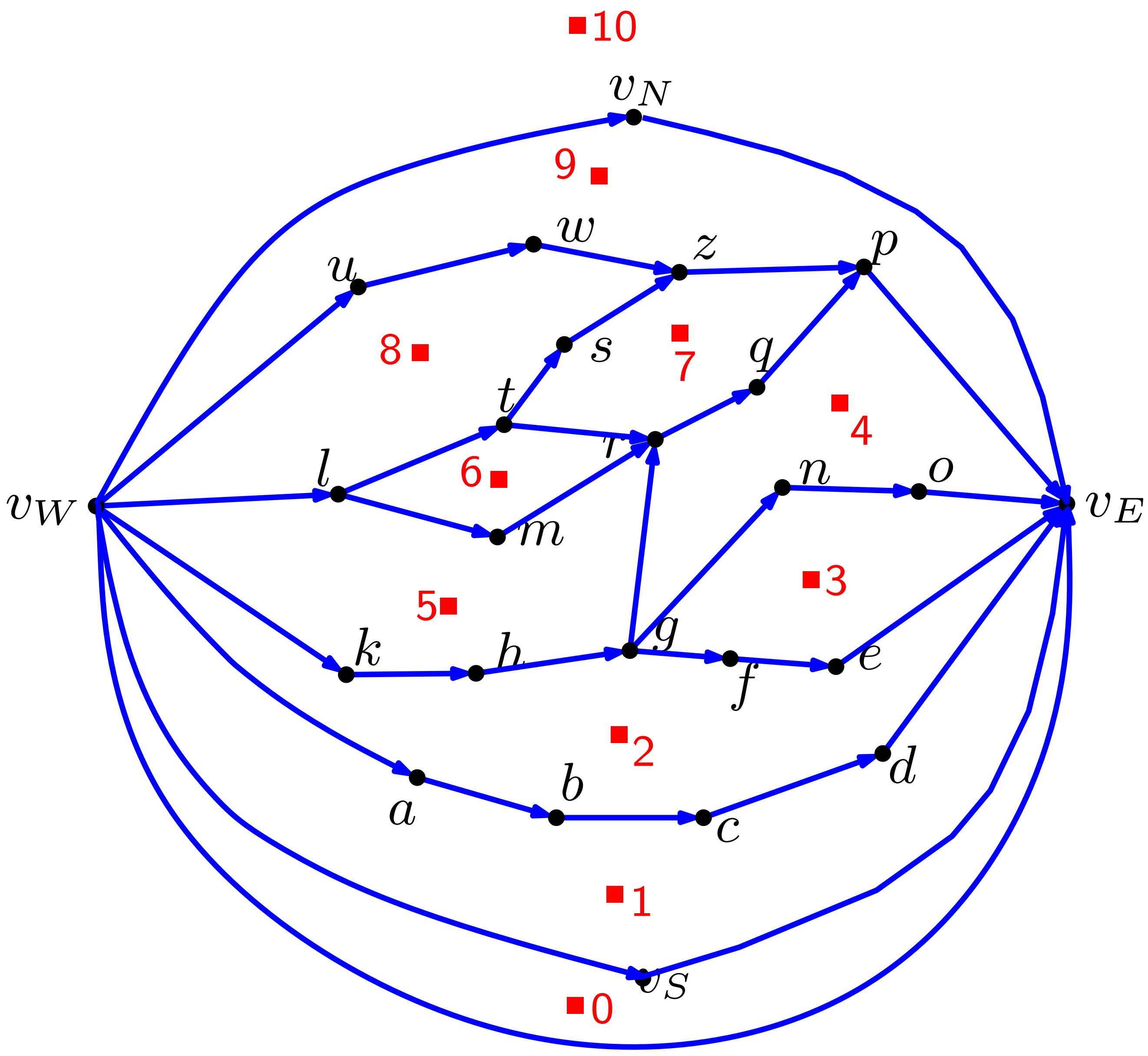
$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



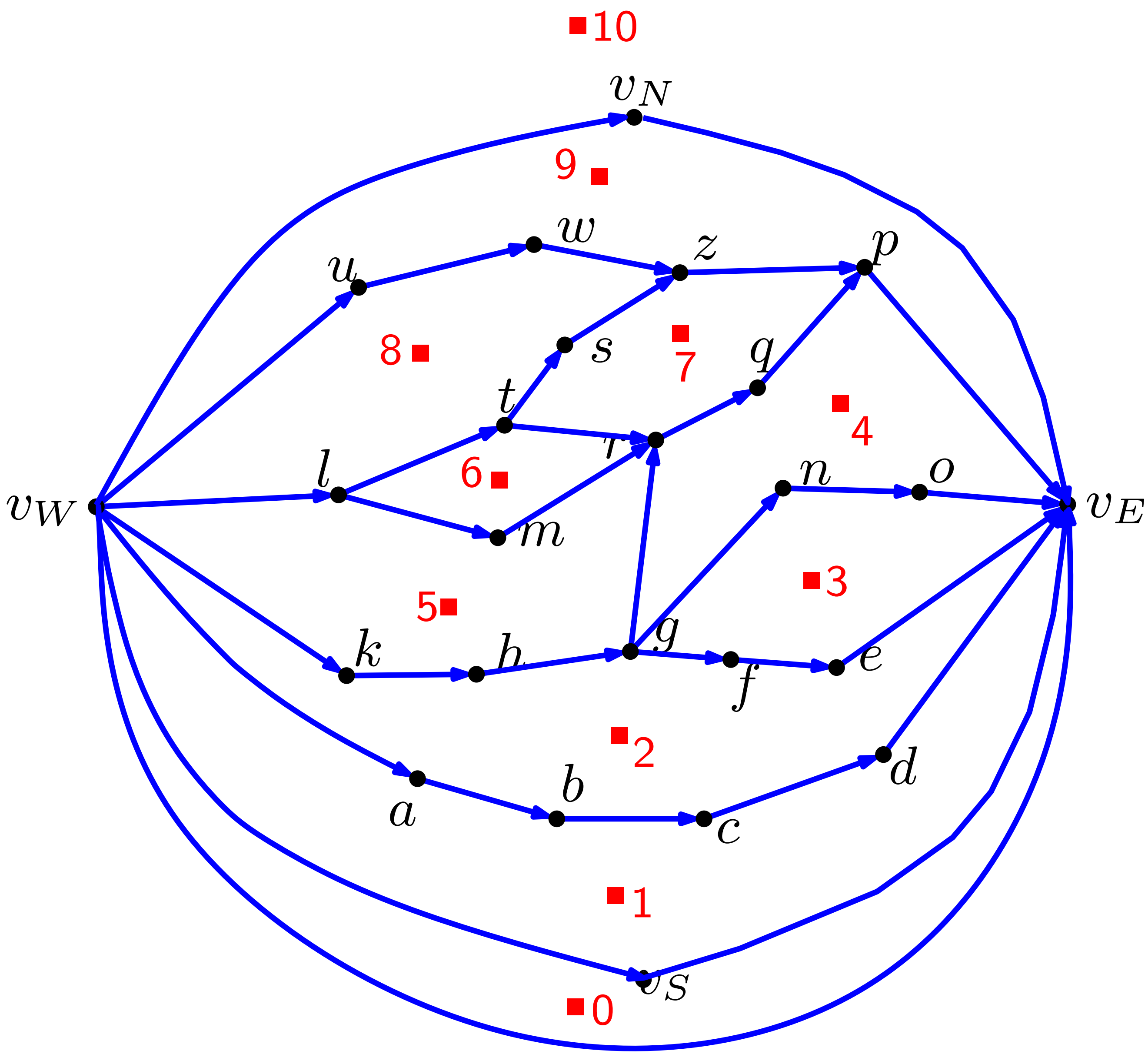
$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

Rectangular Dual



$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

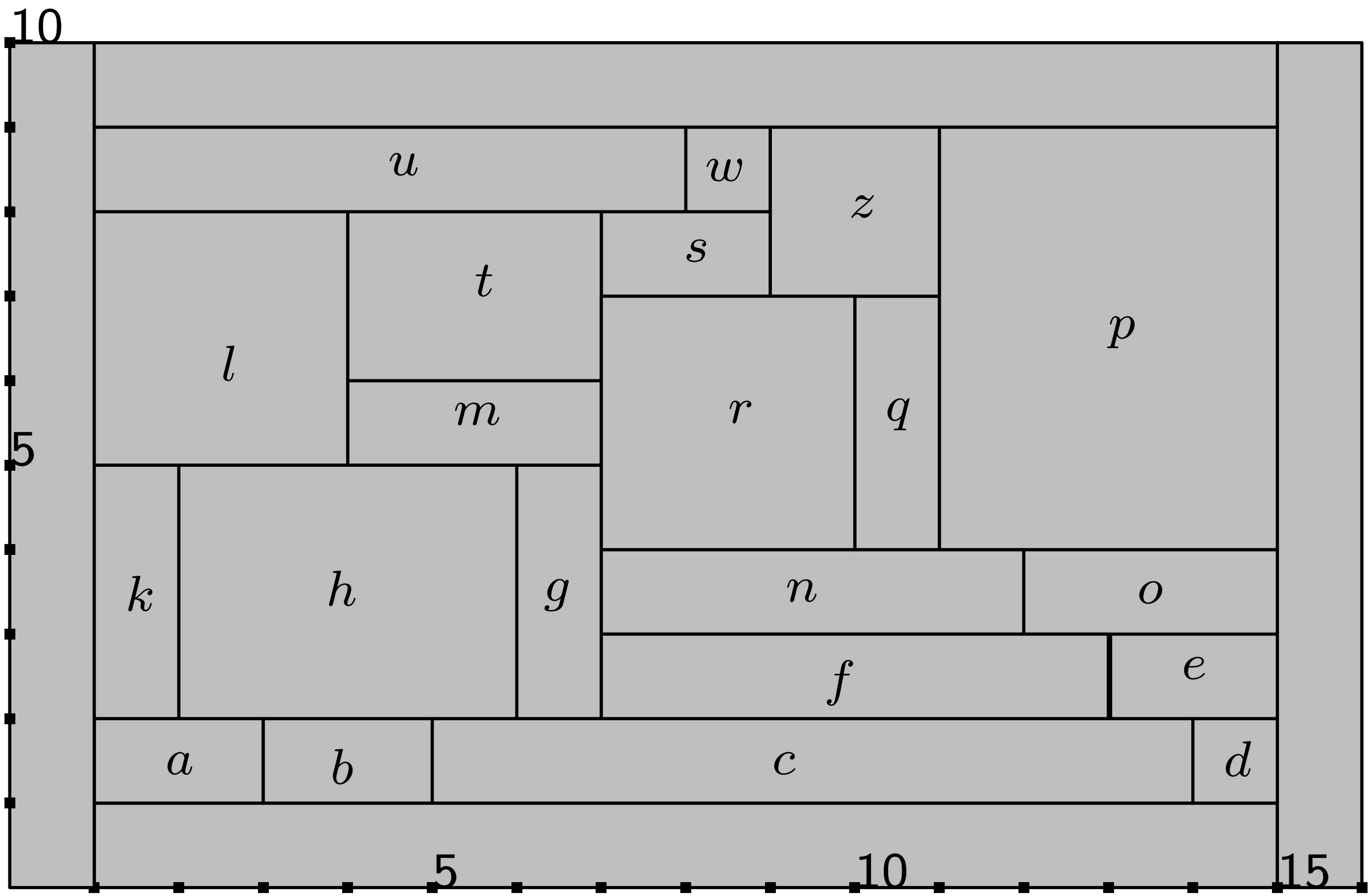
$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual

$x_1(v_N) = 1,\ x_2(v_N) = 15$
$x_1(v_S) = 1,\ x_2(v_S) = 15$
$x_1(v_W) = 0, x_2(v_W) = 1$
$x_1(v_E) = 15,\ x_2(v_E) = 16$
$x_1(a) = 1,\ x_2(a) = 3$
$x_1(b) = 3,\ x_2(b) = 5$
$x_1(c) = 5,\ x_2(c) = 14$
$x_1(d) = 14,\ x_2(d) = 15$
$x_1(e) = 13,\ x_2(e) = 15$

$y_1(v_W) = 0, y_2(v_W) = 10$
$y_1(v_E) = 0,\ y_2(v_E) = 10$
$y_1(v_N) = 9,\ y_2(v_N) = 10$
$y_1(v_S) = 0,\ y_2(v_S) = 1$
$y_1(a) = 1,\ y_2(a) = 2$
$y_1(b) = 1,\ y_2(b) = 2$
$y_1(c) = 1,\ y_2(c) = 2$
$y_1(d) = 1,\ y_2(d) = 2$
$y_1(e) = 2,\ y_2(e) = 3$

Rectangular Dual



$$x_1(v_N) = 1,\ x_2(v_N) = 15$$
$$x_1(v_S) = 1,\ x_2(v_S) = 15$$
$$x_1(v_W) = 0, x_2(v_W) = 1$$
$$x_1(v_E) = 15,\ x_2(v_E) = 16$$
$$x_1(a) = 1,\ x_2(a) = 3$$
$$x_1(b) = 3,\ x_2(b) = 5$$
$$x_1(c) = 5,\ x_2(c) = 14$$
$$x_1(d) = 14,\ x_2(d) = 15$$
$$x_1(e) = 13,\ x_2(e) = 15$$

$$y_1(v_W) = 0, y_2(v_W) = 10$$
$$y_1(v_E) = 0,\ y_2(v_E) = 10$$
$$y_1(v_N) = 9,\ y_2(v_N) = 10$$
$$y_1(v_S) = 0,\ y_2(v_S) = 1$$
$$y_1(a) = 1,\ y_2(a) = 2$$
$$y_1(b) = 1,\ y_2(b) = 2$$
$$y_1(c) = 1,\ y_2(c) = 2$$
$$y_1(d) = 1,\ y_2(d) = 2$$
$$y_1(e) = 2,\ y_2(e) = 3$$

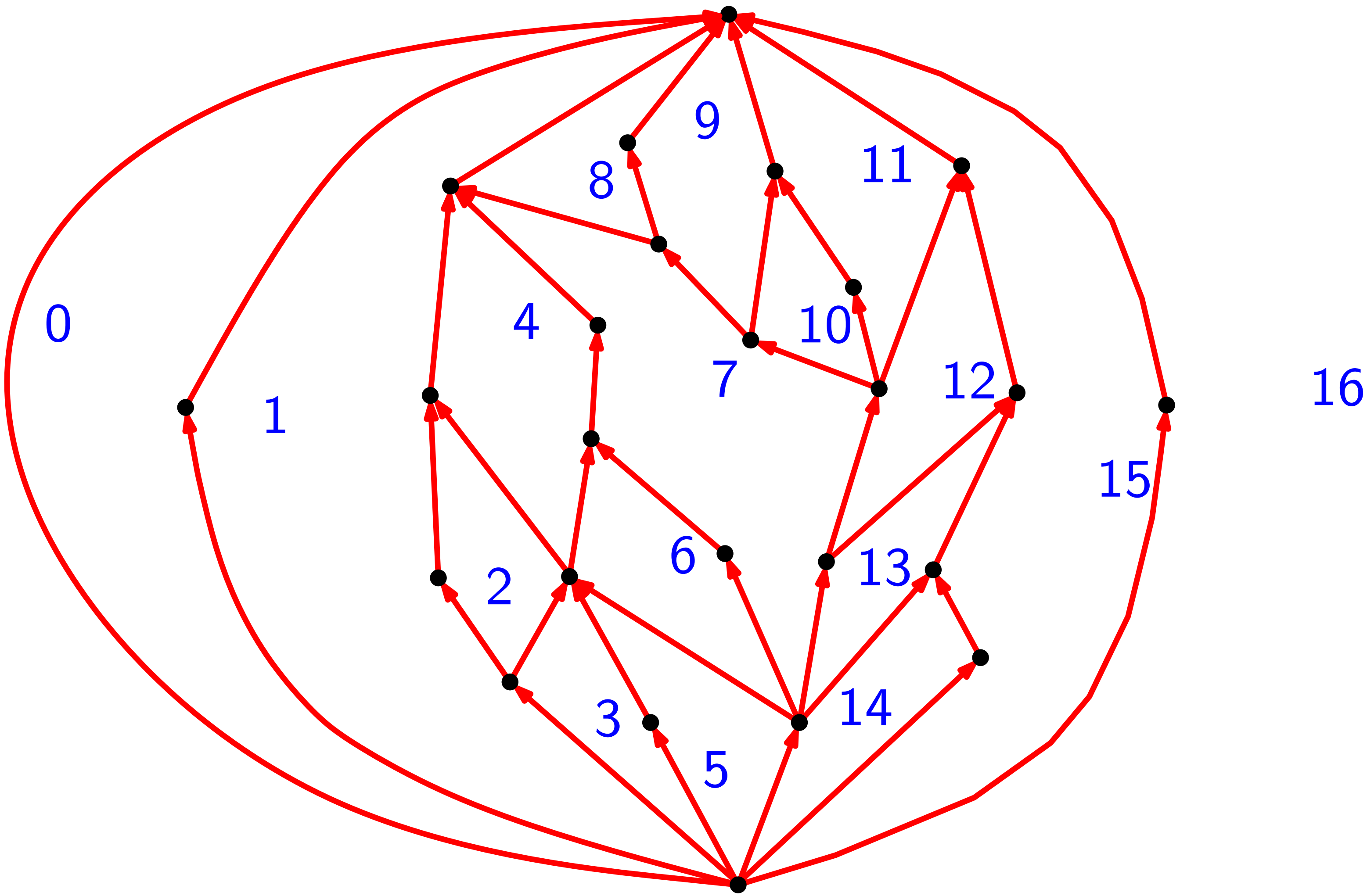
Rectangular Dual

In the following we prove that presented algorithm constructs a rectangular dual of G.

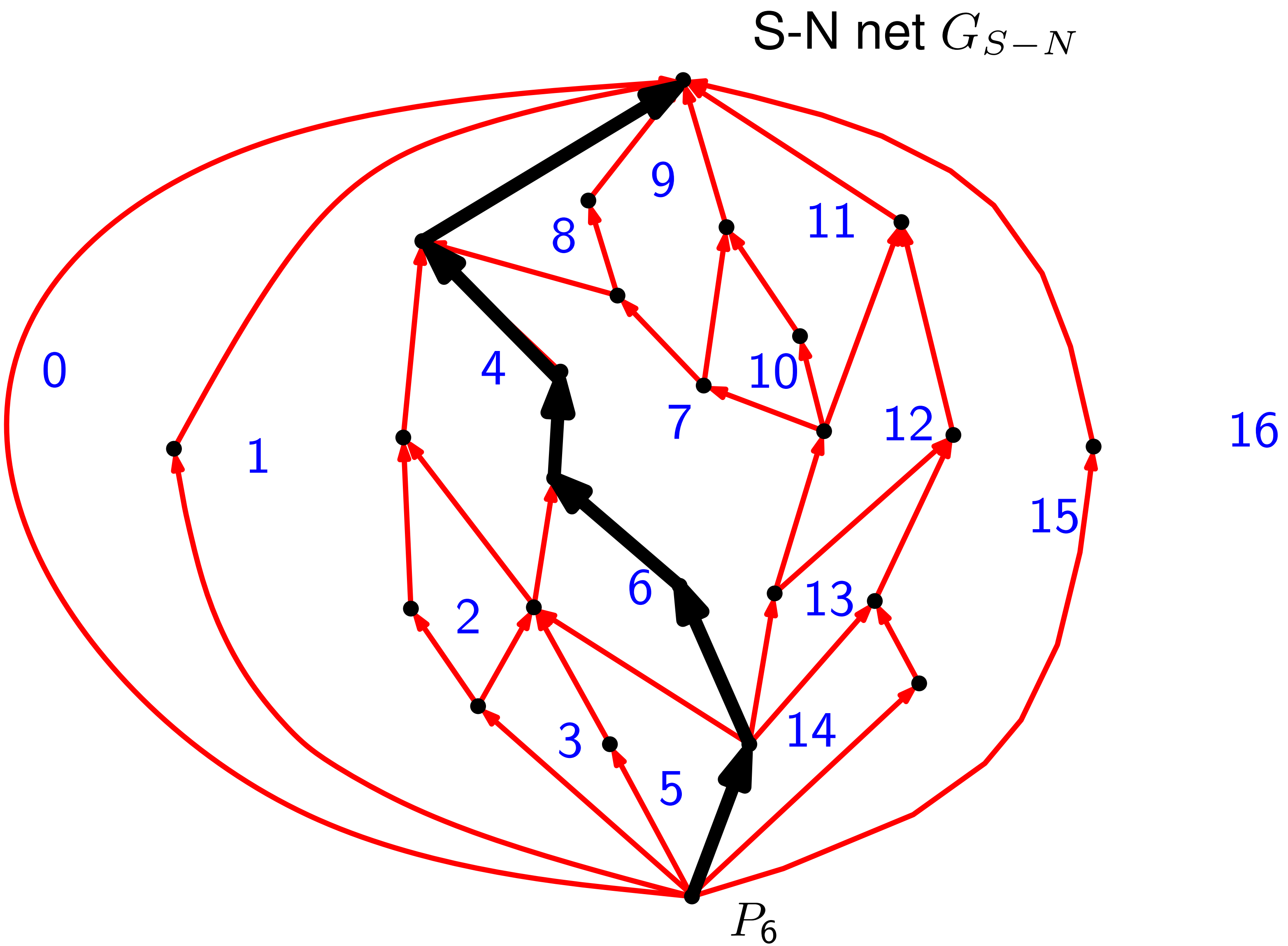
- Let G be a PTP graph and let G_{S-N} (resp. G_{W-E}) be its S-N net, let $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) be the dual of G_{S-N} (resp. G_{W-E})
- Let $f_1, \ldots, f_k$ be the faces of $G^{\star}_{S-N}$ (resp. $G^{\star}_{W-E}$) , enumerated according to st-numbering f_{sn} (resp. f_{we})
- Let G^{i}_{S-N} (resp. G^{i}_{W-E}) denote the subgraph of G that is induced by vertices and edges of $f_1, \ldots, f_i$
- We denote P_i (reps. Q_i) the right (resp. top) boundary of G^{i}_{S-N} (resp. G^{i}_{W-E}) .

Rectangular Dual

S-N net G_{S-N}

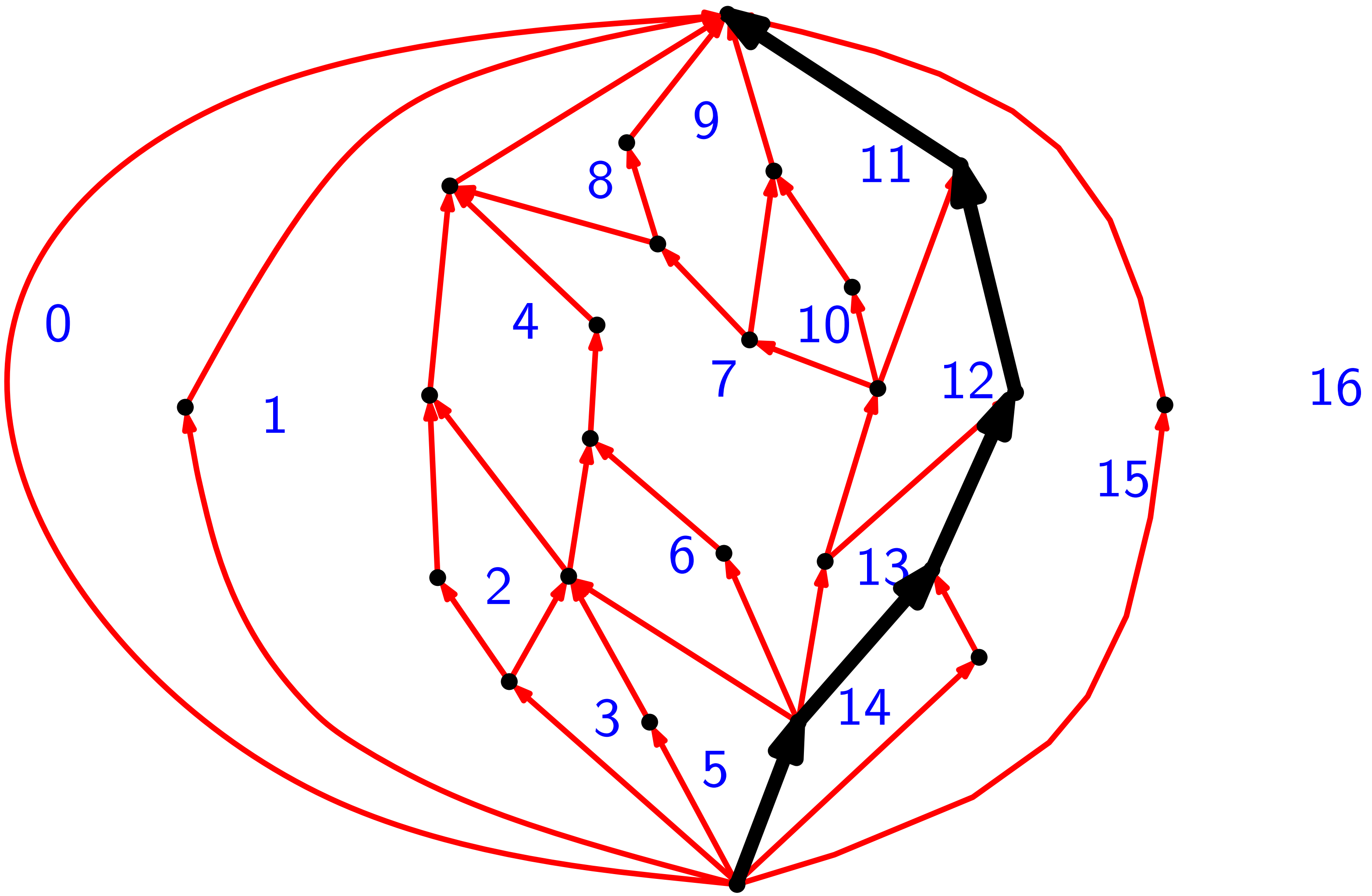


Rectangular Dual

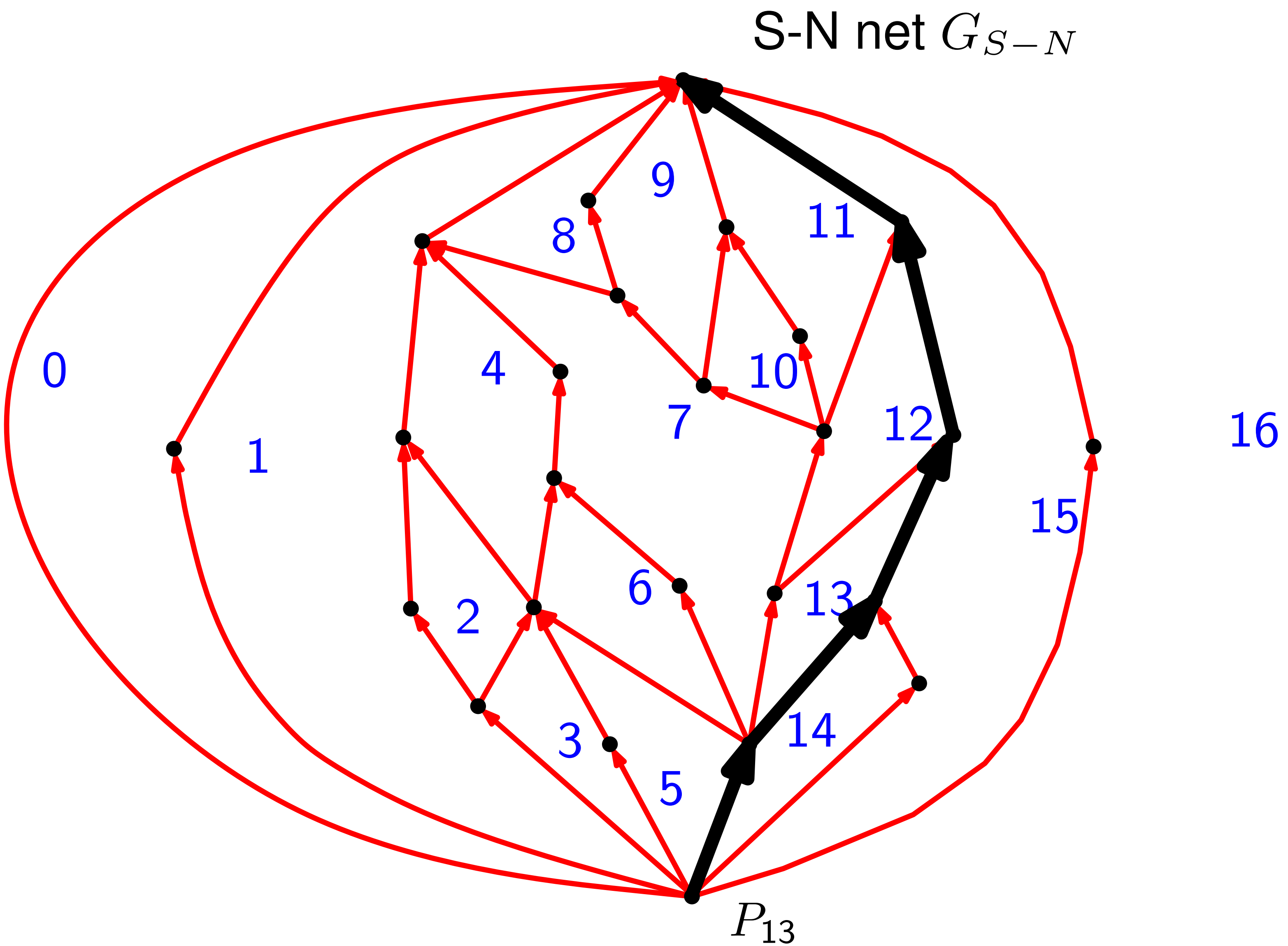


Rectangular Dual

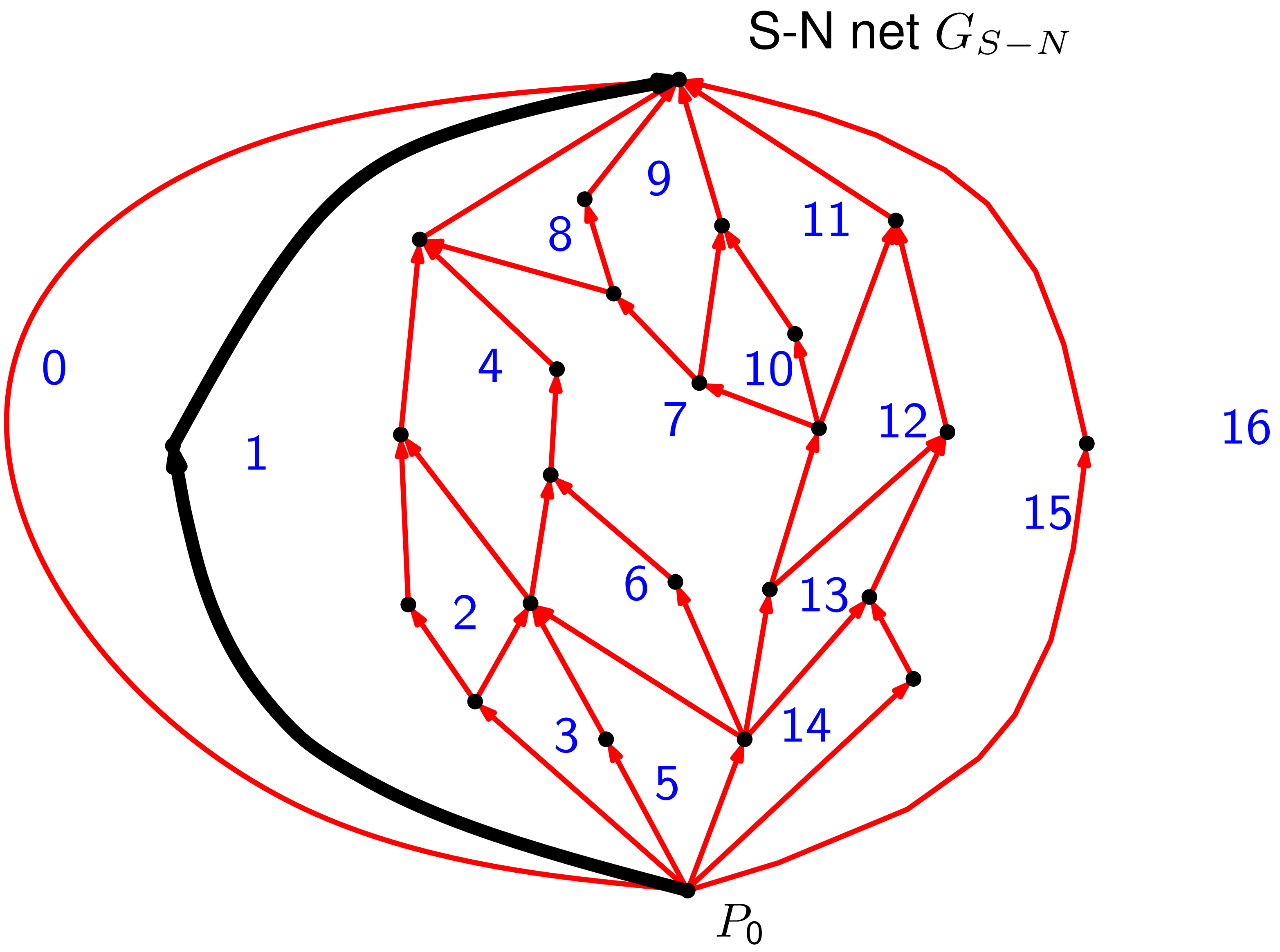
S-N net G_{S-N}



Rectangular Dual



Rectangular Dual



Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

Lemma 4

Let $v \in V$, f and g are the left and the right face of v. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Proof...

Rectangular Dual

- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) **cross at exactly one vertex**.

Lemma 4

Let $v \in V$, f and g are the left and the right face of v. Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Proof...

Lemma 5

Let $v \in V$, f and g are the fases below and above v in G_{W-E}. Let $y_1(v) = f_{we}(f)$ and $y_2(v) = f_{we}(g)$. Vertex v belongs to path Q_j if and only if $y_1(v) \leq j \leq y_2(v) - 1$.

Proof (identical)

Rectangular Dual

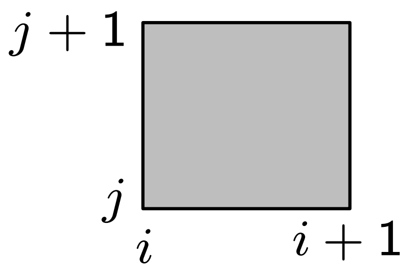
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:

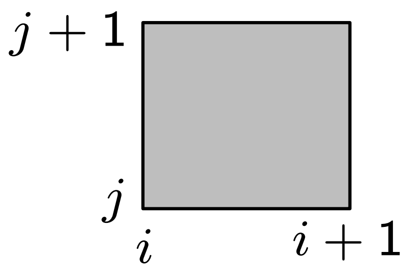


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there **exists a vertex** over this box: $u \in P_i \cap Q_j$

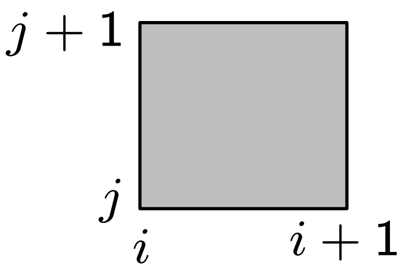


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box

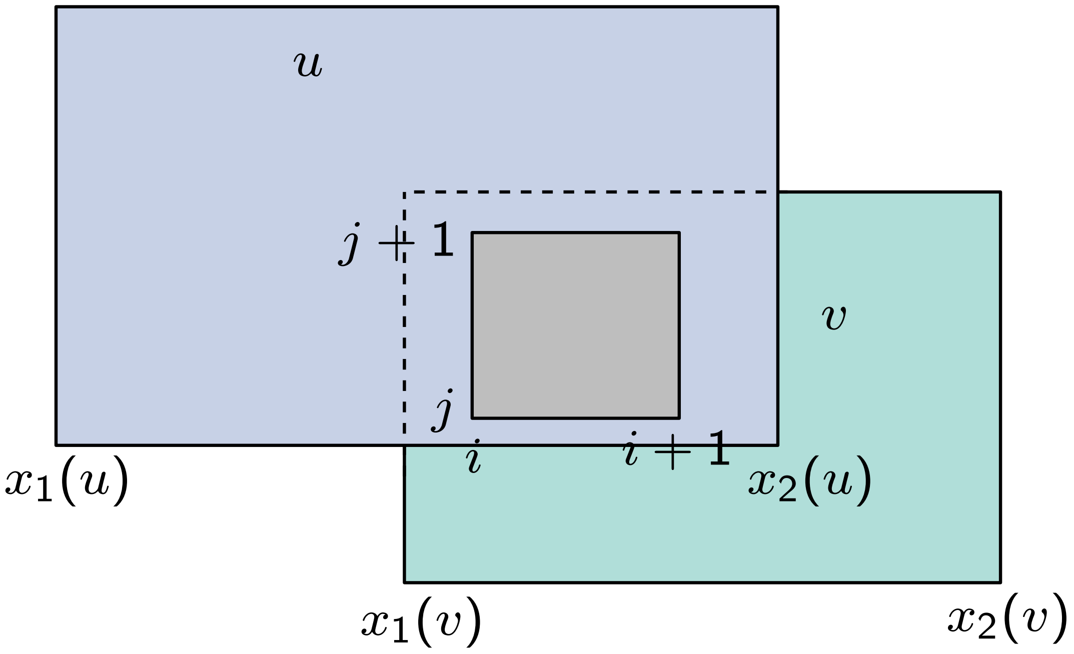


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box

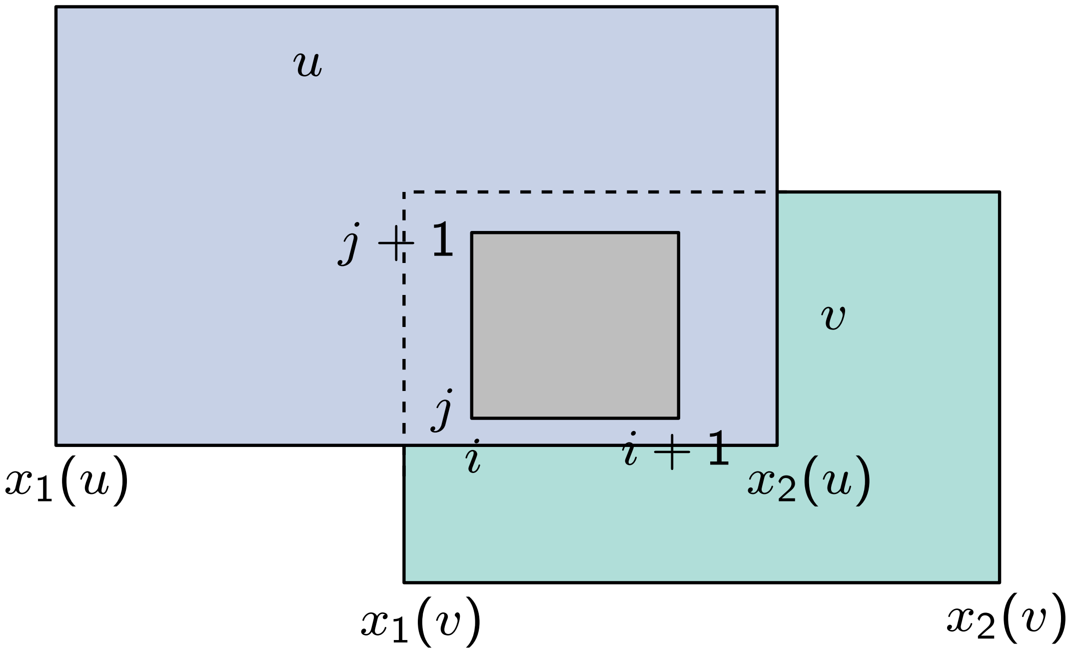


Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



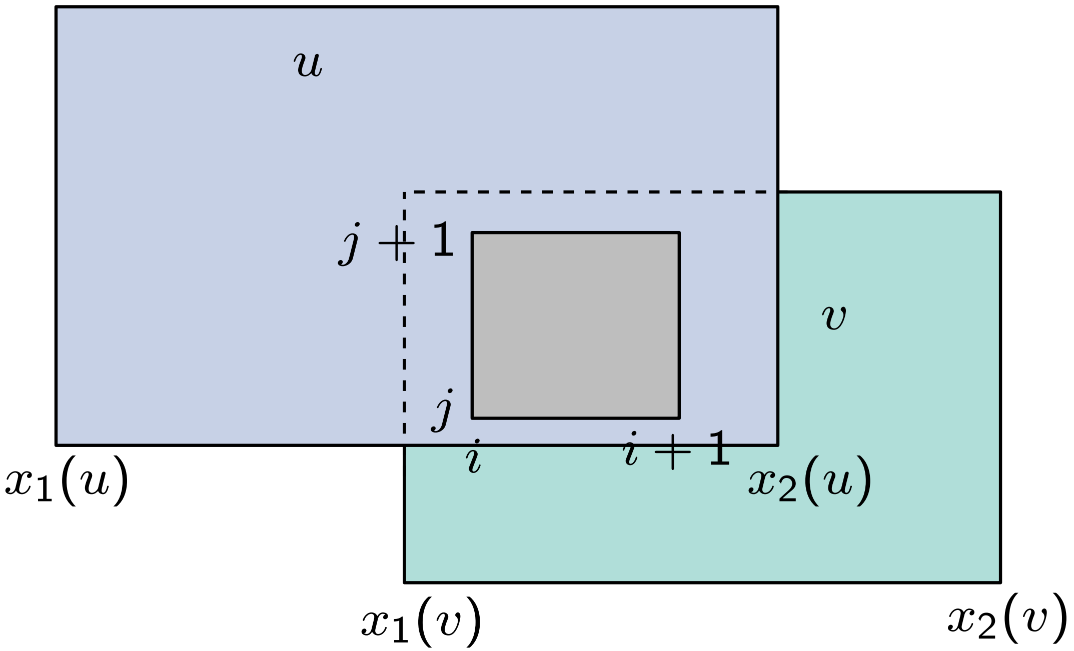
$x_1(u) \leq i$ and $i{+}1 \leq x_2(u)$

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

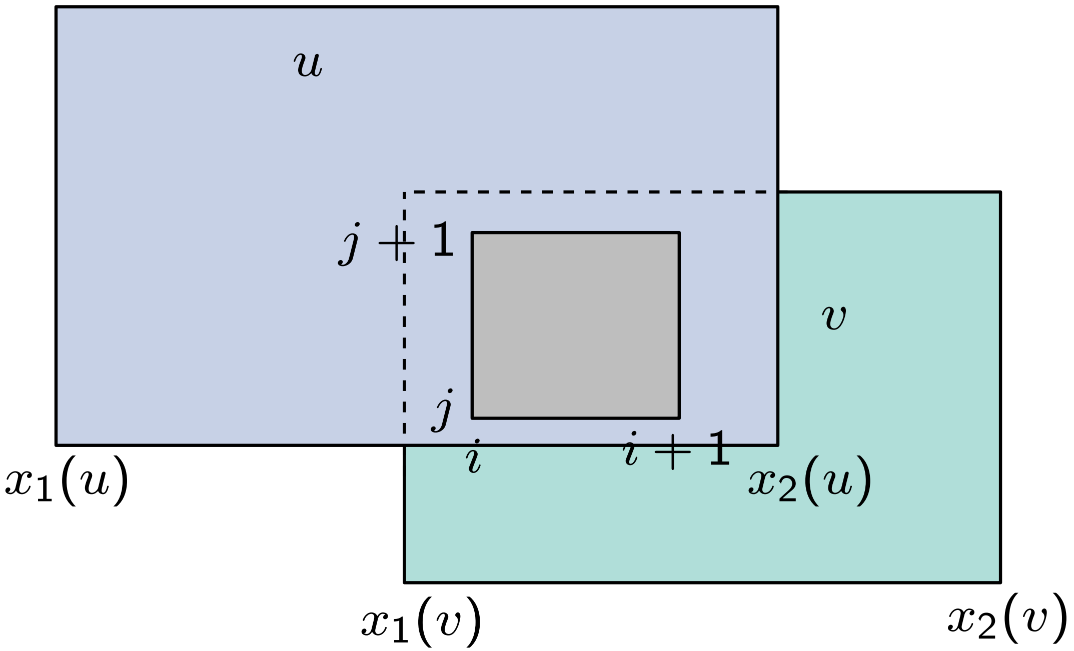
u belongs to P_i

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

u belongs to P_i

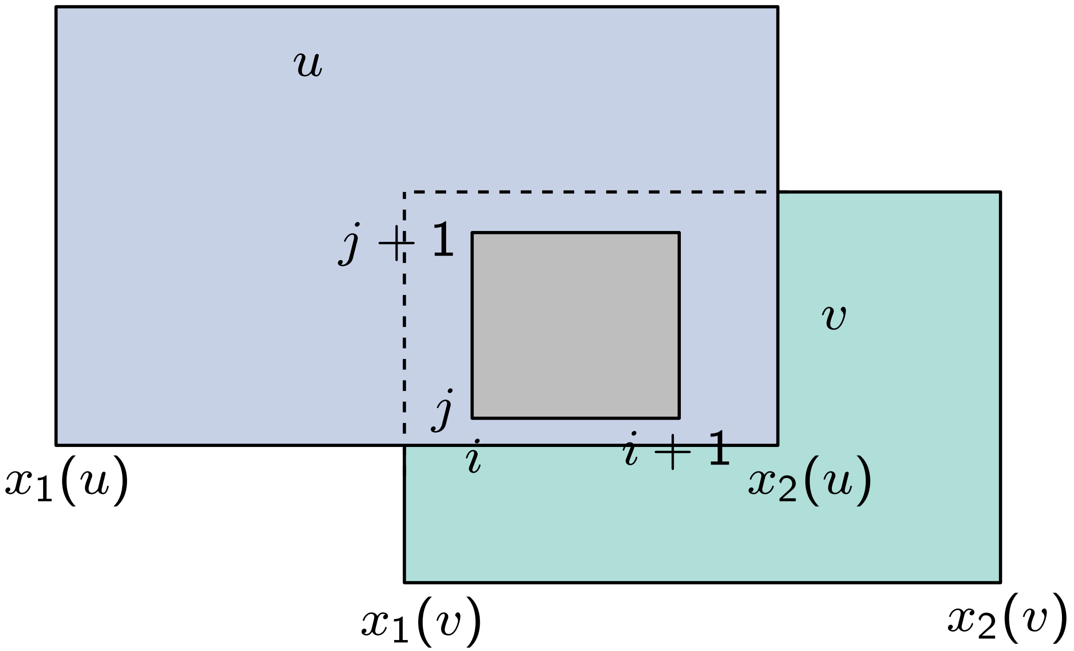
Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i{+}1 \leq x_2(u)$

(Lemma 4)

u belongs to P_i

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

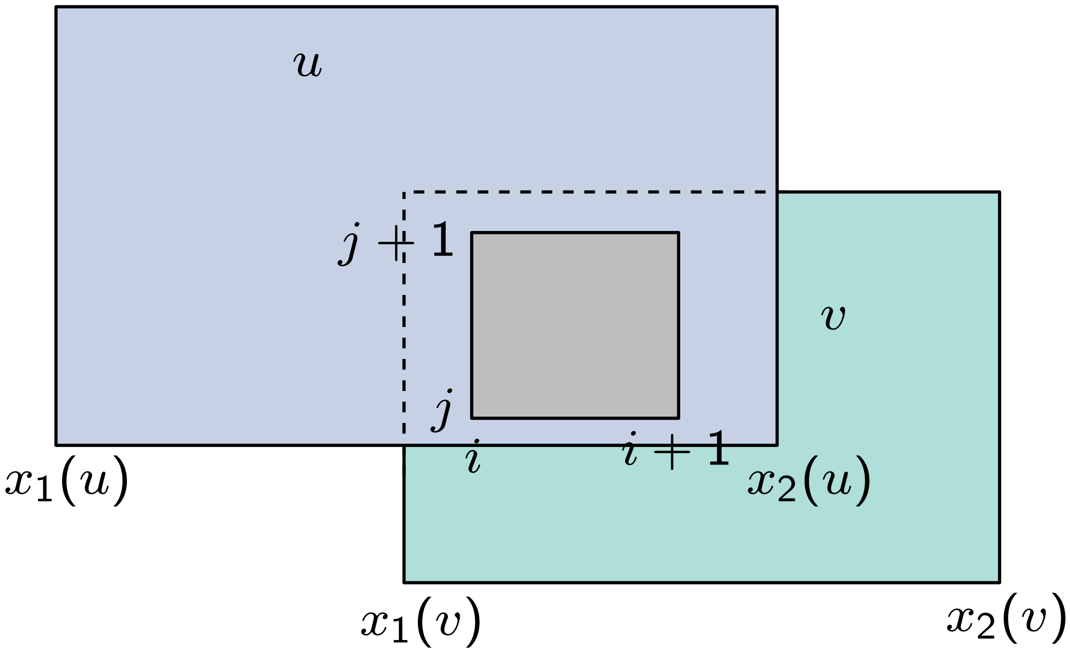
Paths P_i and Q_j intersect at two vertices u and v.

Rectangular Dual

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof: Show that there is **at most one vertex** over this box



$x_1(u) \leq i$ and $i+1 \leq x_2(u)$

(Lemma 4)

u belongs to P_i

Similarly: $v \in P_i$, $u \in Q_j$, $v \in Q_j$.

Paths P_i and Q_j intersect at two vertices u and v.

Which is a contradiction to the property of paths P_i, Q_j except for the cases when: (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$ (corner boxes).

Rectangular Dual

Lemma 7

Let G_{S-N} and G_{W-E}. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from u to v in G_{W-E} containing at least two edges, then $x_2(u) < x_1(v)$
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from u to v in G_{S-N} containing at least two edges, then $y_2(u) < y_1(v)$

Proof...

Rectangular Dual

Lemma 7

Let G_{S-N} and G_{W-E}. The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from u to v in G_{W-E} containing at least two edges, then $x_2(u) < x_1(v)$
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from u to v in G_{S-N} containing at least two edges, then $y_2(u) < y_1(v)$

Proof...

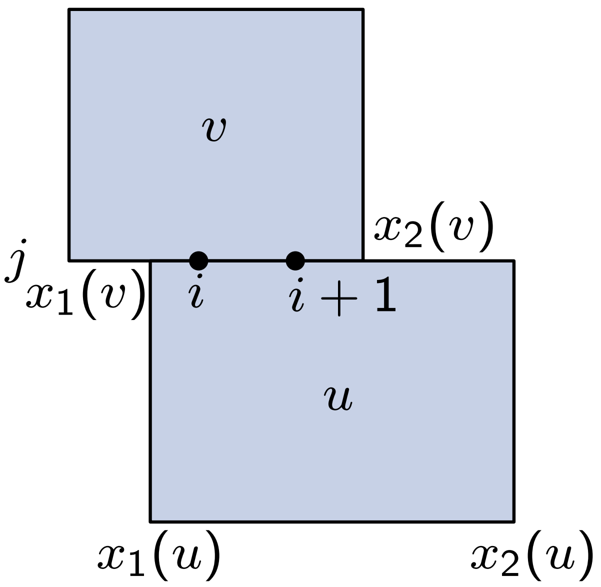
Lemma 8

The assignment provided by the algorithm has the following property: rectangles assigned to vertices u and v have a common segment if and only if there exists edge (u, v) in the graph.

Proof:

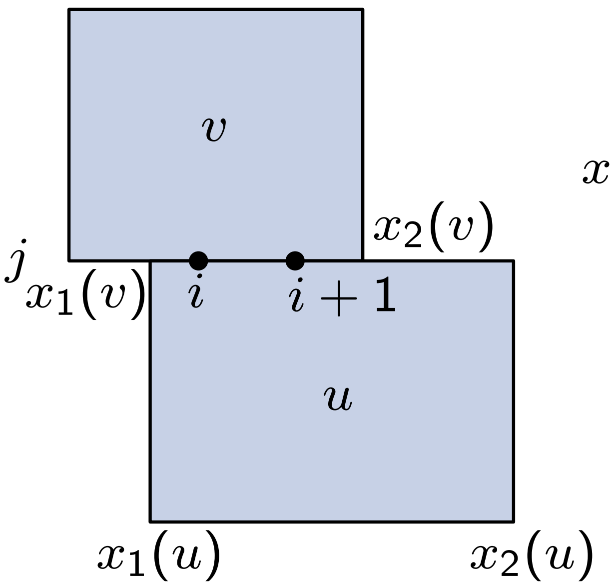
Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



Rectangular Dual

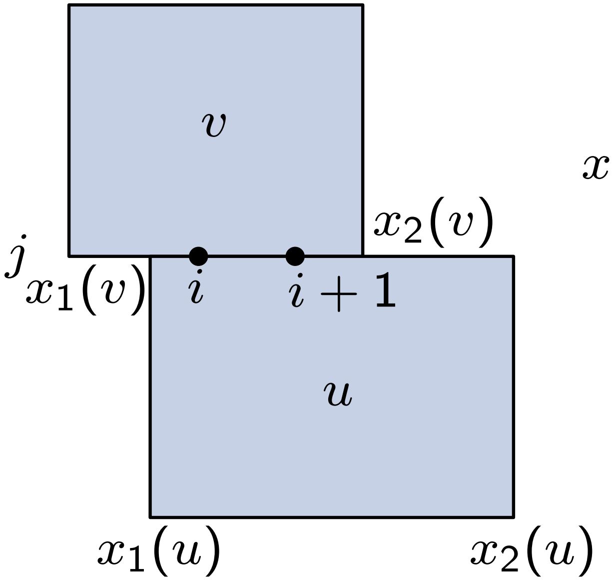
- Assume $R(u)$ and $R(v)$ have a common boundary.



$x_1(v) \leq i,\, i+1 \leq x_2(v)$ and $x_1(u) \leq i,\, i+1 \leq x_2(u)$

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$\Downarrow$

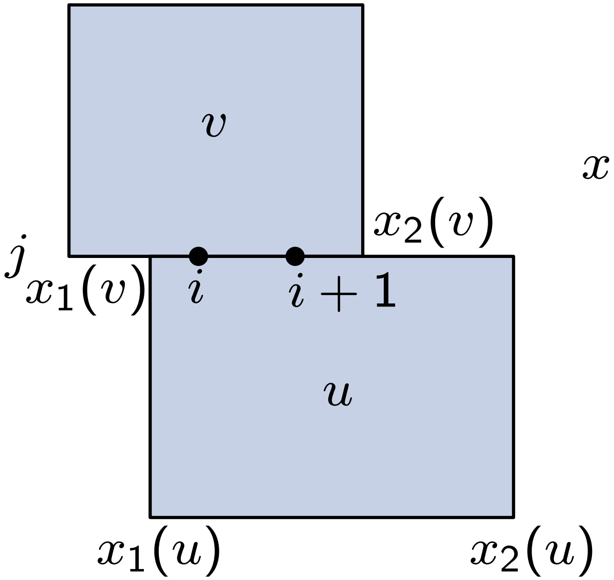
$x_1(v) \leq i$, $i+1 \leq x_2(v)$ and $x_1(u) \leq i$, $i+1 \leq x_2(u)$

$\Downarrow$ (Lemma 4)

u, v belong to P_i

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$x_1(v) \leq i,\ i+1 \leq x_2(v)$ and $x_1(u) \leq i,\ i+1 \leq x_2(u)$

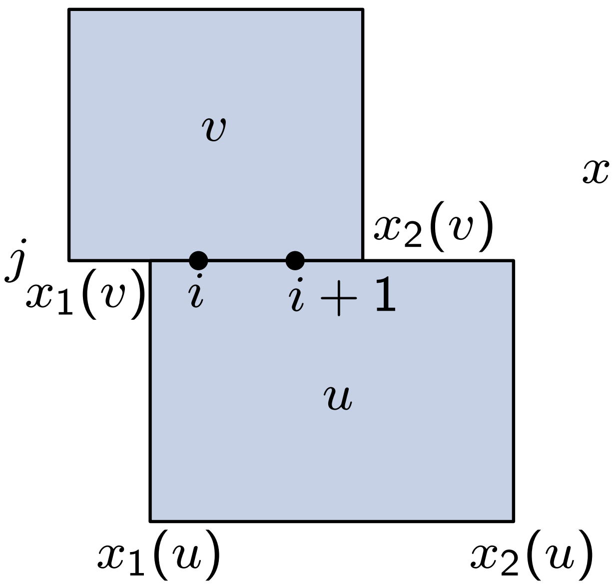
(Lemma 4)

u, v belong to P_i

If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$x_1(v) \leq i$, $i+1 \leq x_2(v)$ and $x_1(u) \leq i$, $i+1 \leq x_2(u)$

(Lemma 4)

u, v belong to P_i

If path between u and v has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

A contradiction to the hypothesis!

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.

$$\Downarrow$$

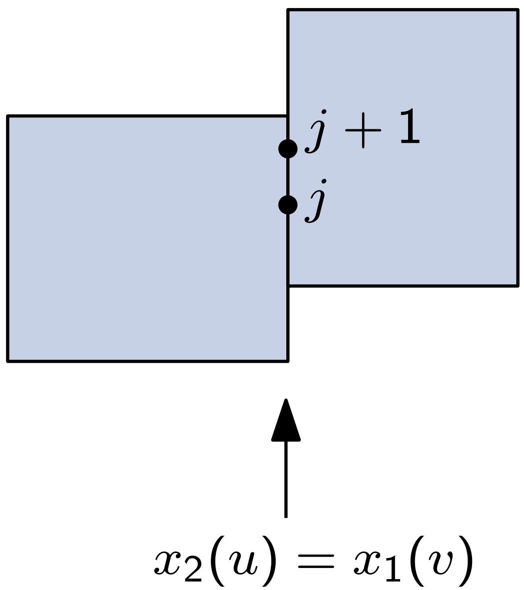
- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j+1 \leq y_2(u)$ and $y_1(v) \leq j$, $j+1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.

$\Downarrow$

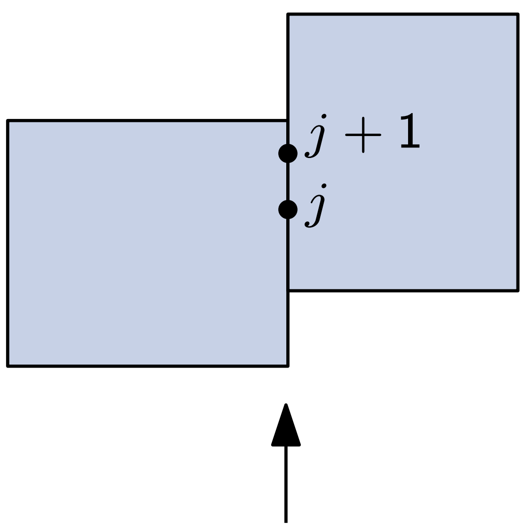
- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$



Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.

- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$



$x_2(u) = x_1(v)$

Lemma 8 is proved!

Rectangular Dual

Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

Rectangular Dual

Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

- Compute a planar embedding of G
- Compute a revised canonical ordering of G
- Traverse the graph and color the edges, construct G_{S-N} and G_{E-W}
- Construct the duals $G^\star_{S-N}$ and $G^\star_{E-W}$ of G_{S-N} and G_{E-W}, respectively
- Compute a topological ordering of $G^\star_{S-N}$ and $G^\star_{E-W}$
- Assing coordinates to the rectangles representing vertices.