Algorithms for graph visualization

Incremental algorithms. Orthogonal drawing.
Definition: Orthogonal Drawing

A drawing $\Gamma$ of a graph $G = (V, E)$ is called **orthogonal** if its vertices are drawn as points and each edge is represented as a sequence of alternating horizontal and vertical segments.
**Definition**

**Definition: Orthogonal Drawing**

A drawing $\Gamma$ of a graph $G = (V, E)$ is called **orthogonal** if its vertices are drawn as points and each edge is represented as a sequence of alternating horizontal and vertical segments.
Definition: Orthogonal Drawing

A drawing $\Gamma$ of a graph $G = (V, E)$ is called \textit{orthogonal} if its vertices are drawn as points and each edge is represented as a sequence of alternating horizontal and vertical segments.

- Edges lie on the grid, i.e., \textit{bends} lie on grid points.
**Definition: Orthogonal Drawing**

A drawing $\Gamma$ of a graph $G = (V, E)$ is called **orthogonal** if its vertices are drawn as points and each edge is represented as a sequence of alternating horizontal and vertical segments.

- Edges lie on the grid, i.e., **bends** lie on grid points
- Degree of each vertex has to be at most 4
Definition: *st*-ordering

An *st*-ordering of a graph $G = (V, E)$ is an ordering of the vertices \{v_1, v_2, \ldots, v_n\}, such that for each $j$, $2 \leq j \leq n - 1$, vertex $v_j$ has at least one neighbour $v_i$ with $i < j$, and at least one neighbour $v_k$ with $k > j$. 

**Definition: st-ordering**

An *st-ordering* of a graph $G = (V, E)$ is an ordering of the vertices \{\(v_1, v_2, \ldots, v_n\}\), such that for each \(j, 2 \leq j \leq n - 1\), vertex \(v_j\) has at least one neighbour \(v_i\) with \(i < j\), and at least one neighbour \(v_k\) with \(k > j\).

![Example of an st-ordering](image)
**Definition: **st-ordering

An **st-ordering** of a graph \( G = (V, E) \) is an ordering of the vertices \( \{v_1, v_2, \ldots, v_n\} \), such that for each \( j, 2 \leq j \leq n - 1 \), vertex \( v_j \) has at least one neighbour \( v_i \) with \( i < j \), and at least one neighbour \( v_k \) with \( k > j \).

**Theorem [Lempel, Even, Cederbaum, 66]**

Let \( G \) be a biconnected graph \( G \) and let \( s, t \) be vertices of \( G \). \( G \) has an st-ordering such that \( s \) appears as the first and \( t \) as the last vertex in this ordering.

Example of an **st-ordering**
Biedl & Kant Orthogonal Drawing Algorithm
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
Biedl & Kant Orthogonal Drawing Algorithm

first vertex indegree = 1
Biedl & Kant Orthogonal Drawing Algorithm

first vertex indegree = 1
Biedl & Kant Orthogonal Drawing Algorithm

first vertex

indegree = 1

indegree = 2
Biedl & Kant Orthogonal Drawing Algorithm

first vertex

indegree = 1

indegree = 2
Biedl & Kant Orthogonal Drawing Algorithm

first vertex

indegree = 1

indegree = 2

indegree = 3
Biedl & Kant Orthogonal Drawing Algorithm

first vertex
indegree = 1
indegree = 2
indegree = 3
Lemma (Area of Biedl & Kant drawing)

The width is \( m - n + 1 \) and the height at most \( n + 1 \).
Lemma (Area of Biedl & Kant drawing)

The width is $m - n + 1$ and the height at most $n + 1$.

Proof

**Width:** At each step we increase the number of columns by $\text{outdeg}(v_i) - 1$, if $i > 1$ and $\text{outdeg}(v_1)$ for $v_1$. 
Lemma (Area of Biedl & Kant drawing)

The width is $m - n + 1$ and the height at most $n + 1$.

Proof

- **Width**: At each step we increase the number of columns by $\text{outdeg}(v_i) - 1$, if $i > 1$ and $\text{outdeg}(v_1)$ for $v_1$.

- **Height**: Every vertex except for $v_2$ is placed at a new row. Vertex $v_n$ uses one more row if $\text{indeg}(v_n) = 4$. 

Biedl & Kant Orthogonal Drawing Algorithm

Lemma (Area of Biedl & Kant drawing)

The width is \( m - n + 1 \) and the height at most \( n + 1 \).

Proof

- **Width:** At each step we increase the number of columns by 
  \( \text{outdeg}(v_i) - 1 \), if \( i > 1 \) and \( \text{outdeg}(v_1) \) for \( v_1 \).

- **Height:** Every vertex except for \( v_2 \) is placed at a new row. Vertex \( v_n \) uses one more row if \( \text{indeg}(v_n) = 4 \).

Lemma (Number of bends in Biedl & Kant drawing)

There are at most \( 2m - 2n + 4 \) bends.
Lemma (Area of Biedl & Kant drawing)

The width is $m - n + 1$ and the height at most $n + 1$.

Proof

- **Width:** At each step we increase the number of columns by $\text{outdeg}(v_i) - 1$, if $i > 1$ and $\text{outdeg}(v_1)$ for $v_1$.

- **Height:** Every vertex except for $v_2$ is placed at a new row. Vertex $v_n$ uses one more row if $\text{indeg}(v_n) = 4$.

Lemma (Number of bends in Biedl & Kant drawing)

There are at most $2m - 2n + 4$ bends.

Proof

- Each vertex $v_i, i \neq 1, n$, introduces $\text{indeg}(v_i) - 1$ and $\text{outdeg}(v_i) - 1$ new bends.
<table>
<thead>
<tr>
<th>Lemma (Number of bends per edge in Biedl &amp; Kant drawing)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All edges but one bent at most twice. The exceptional edge bents at most three times.</td>
</tr>
</tbody>
</table>
Lemma (Number of bends per edge in Biedl & Kant drawing)

All edges but one bent at most twice. The exceptional edge bends at most three times.

Proof

Let \((v_i, v_j), i < j, i, j \neq 1, n.\) Then \(\text{outdeg}(v_i), \text{indeg}(v_j) \leq 3.\) i.e \((v_i, v_j)\) gets at most one bend after placement of \(v_i\) and at most one before placement of \(v_j.\) Edges outgoing from \(v_1\) can me made 2-bend by using the column below \(v_1\) for the edge \((v_1, v_2).\)
Biedl & Kant Orthogonal Drawing Algorithm

Lemma (Number of bends per edge in Biedl & Kant drawing)
All edges but one bent at most twice. The exceptional edge bends at most three times.

Proof
Let \((v_i, v_j), i < j, i, j \neq 1, n\). Then \(\text{outdeg}(v_i), \text{indeg}(v_j) \leq 3\). I.e \((v_i, v_j)\) gets at most one bend after placement of \(v_i\) and at most one before placement of \(v_j\). Edges outgoing from \(v_1\) can be made 2-bend by using the column below \(v_1\) for the edge \((v_1, v_2)\).

Lemma (planarity)
For planar embedded graphs, with \(v_1\) and \(v_n\) on the outer face, the algorithm produces a planar drawing.
**Lemma (Number of bends per edge in Biedl & Kant drawing)**

All edges but one bent at most twice. The exceptional edge bends at most three times.

**Proof**

- Let \((v_i, v_j), i < j, i, j \neq 1, n\). Then \(\text{outdeg}(v_i), \text{indeg}(v_j) \leq 3\). I.e \((v_i, v_j)\) gets at most one bend after placement of \(v_i\) and at most one before placement of \(v_j\). Edges outgoing from \(v_1\) can me made 2-bend by using the column below \(v_1\) for the edge \((v_1, v_2)\).

**Lemma (planarity)**

For planar embedded graphs, with \(v_1\) and \(v_n\) on the outer face, the algorithm produces a planar drawing.

**Proof**

- Consider a planar embedding of \(G\). Let \(v_1, \ldots, v_n\) be an \(st\)-ordering of \(G\). Let \(G_i\) be the graph induced by \(v_1, \ldots, v_i\). We will prove later that if \(G\) is planar, vertex \(v_{i+1}\) lies on the outer face of \(G_i\).
Lemma (planarity)

For planar embedded graphs, with \( v_1 \) and \( v_n \) on the outer face, the algorithm produces a planar drawing.

Proof (Continuation)

Let \( E_i \) be the edges outgoing from the vertices of \( G_i \) in the order they appear in the embedded \( G \).
Lemma (planarity)
For planar embedded graphs, with $v_1$ and $v_n$ on the outer face, the algorithm produces a planar drawing.

Proof (Continuation)
- Let $E_i$ be the edges outgoing from the vertices of $G_i$ in the order they appear in the embedded $G$.
- By induction we can show that edges of $E_i$ appear in the same order in the orthogonal drawing of $G_i$. 
Lemma (planarity)

For planar embedded graphs, with $v_1$ and $v_n$ on the outer face, the algorithm produces a planar drawing.

Proof (Continuation)

- Let $E_i$ be the edges outgoing from the vertices of $G_i$ in the order they appear in the embedded $G$.
- By induction we can show that edges of $E_i$ appear in the same order in the orthogonal drawing of $G_i$.
- Since $v_{i+1}$ is on the outer face of $G_i$, it can be placed without creating any crossing.
Lemma (planarity)

For planar embedded graphs, with $v_1$ and $v_n$ on the outer face, the algorithm produces a planar drawing.

Proof (Continuation)

- Let $E_i$ be the edges outgoing from the vertices of $G_i$ in the order they appear in the embedded $G$.
- By induction we can show that edges of $E_i$ appear in the same order in the orthogonal drawing of $G_i$.
- Since $v_{i+1}$ is on the outer face of $G_i$, it can be placed without creating any crossing.
Lemma (planarity)
For planar embedded graphs, with $v_1$ and $v_n$ on the outer face, the algorithm produces a planar drawing.

Proof (Continuation)
- Let $E_i$ be the edges outgoing from the vertices of $G_i$ in the order they appear in the embedded $G$.
- By induction we can show that edges of $E_i$ appear in the same order in the orthogonal drawing of $G_i$.
- Since $v_{i+1}$ is on the outer face of $G_i$, it can be placed without creating any crossing.
### Theorem (Biedl & Kant 98)

A biconnected graph $G$ with vertex-degree at most 4 admits an orthogonal drawing such that:

- Area is $(m - n + 1) \times n + 1$
- Each edge (except maybe for one) has at most 2 bends
- The exceptional edge has at most 3 bends
- The total number of bends is at most $2m - 2n + 4$
- If $G$ is plane, the orthogonal drawing is planar
- Finally, provided an $st$-ordering such a drawing can be constructed in $O(n)$ time.
Theorem (Biedl & Kant 98)

A biconnected graph $G$ with vertex-degree at most 4 admits an orthogonal drawing such that:
- Area is $(m - n + 1) \times n + 1$
- Each edge (except maybe for one) has at most 2 bends
- The exceptional edge has at most 3 bends
- The total number of bends is at most $2m - 2n + 4$
- If $G$ is plane, the orthogonal drawing is planar
- Finally, provided an $st$-ordering such a drawing can be constructed in $O(n)$ time.

For the construction we have used an $st$-ordering of $G$!
Definition: st-digraph

Let $G$ be a directed graph. A vertex $s$ (resp. $t$) is called source (resp. sink) of $G$ if it has only outgoing (resp. incoming) edges. A directed acyclic graph with one source and one sink is called st-digraph.
**Definition: st-digraph**

Let $G$ be a directed graph. A vertex $s$ (resp. $t$) is called **source** (resp. **sink**) of $G$ if it has only outgoing (resp. incomming) edges. A directed acyclic graph with one source and one sink is called **st-digraph**.

**Definition: topological ordering**

A **topological ordering** of a directed graph $G$ (with $n$ vertices) is an assignment of numbers $\{1, \ldots, n\}$ to the vertices of $G$, such that for every edge $(u, v)$, $\text{number}(v) > \text{number}(u)$.
**Definition: st-digraph**

Let $G$ be a directed graph. A vertex $s$ (resp. $t$) is called **source** (resp. **sink**) of $G$ if it has only outgoing (resp. incoming) edges. A directed acyclic graph with one source and one sink is called **st-digraph**.

**Definition: topological ordering**

A **topological ordering** of a directed graph $G$ (with $n$ vertices) is an assignment of numbers $\{1, \ldots, n\}$ to the vertices of $G$, such that for every edge $(u, v)$, $\text{number}(v) > \text{number}(u)$. 

![Diagram](image_url)
**Definition: st-digraph**

Let $G$ be a directed graph. A vertex $s$ (resp. $t$) is called **source** (resp. **sink**) of $G$ if it has only outgoing (resp. incoming) edges. A directed acyclic graph with one source and one sink is called **st-digraph**.

**Definition: topological ordering**

A **topological ordering** of a directed graph $G$ (with $n$ vertices) is an assignment of numbers $\{1, \ldots, n\}$ to the vertices of $G$, such that for every edge $(u, v)$, $\text{number}(v) > \text{number}(u)$.
st-digraph, topological ordering

**Definition: st-digraph**

Let $G$ be a directed graph. A vertex $s$ (resp. $t$) is called **source** (resp. **sink**) of $G$ if it has only outgoing (resp. incoming) edges. A directed acyclic graph with one source and one sink is called **st-digraph**.

**Definition: topological ordering**

A **topological ordering** of a directed graph $G$ (with $n$ vertices) is an assignment of numbers $\{1, \ldots, n\}$ to the vertices of $G$, such that for every edge $(u, v)$, $\text{number}(v) > \text{number}(u)$.

How to construct a topological ordering?
Construction of an $st$-ordering:

$G$ is undirected biconnected graph
Construction of an \textit{st}-ordering:

$G$ is undirected \textit{biconnected} graph

Orient edges of $G$
Construction of an \textit{st}-ordering:

\begin{itemize}
  \item \textit{G} is undirected biconnected graph
  \item Orient edges of \textit{G}
  \item \textit{G'} is an \textit{st}-digraph
\end{itemize}
Construction of an $st$-ordering:

$G$ is an undirected, biconnected graph

$G'$ is an $st$-digraph

Let $v_1, \ldots, v_n$ be a topological ordering of $G'$

Orient edges of $G$
Construction of an \(st\)-ordering:

- \(G\) is undirected biconnected graph
- Orient edges of \(G\)
- \(G'\) is an \(st\)-digraph
- Let \(v_1, \ldots, v_n\) be a topological ordering of \(G'\)

Since \(G'\) is an \(st\)-digraph, for \(v_i\) (\(i \neq 1, n\)) \(\exists (v_j, v_i)\) and \((v_i, v_k)\). By the property of topological ordering \(j < i\) and \(i < k\).
Construction of an $st$-ordering:

$G$ is undirected biconnected graph

Orient edges of $G$

$G'$ is an $st$-digraph

Let $v_1, \ldots, v_n$ be a topological ordering of $G'$

Since $G'$ is an $st$-digraph, for $v_i$ $(i \neq 1, n)$ there exist $(v_j, v_i)$ and $(v_i, v_k)$. By the property of topological ordering $j < i$ and $i < k$.

$v_1, \ldots, v_n$ is an $st$-ordering of $G$
**Construction of an st-ordering:**

- **G** is an undirected biconnected graph
- Orient edges of **G**
- **G′** is an st-digraph
- Let **v_1, ..., v_n** be a topological ordering of **G′**

Since **G′** is an st-digraph, for **v_i** (i ≠ 1, n) ∃ (v_j, v_i) and (v_i, v_k). By the property of topological ordering j < i and i < k.

**v_1, ..., v_n** is an st-ordering of **G**

**EXAMPLE**
Construction of an \textit{st}-ordering:

\textit{G} is undirected biconnected graph \hspace{2cm} \text{HOW?} \hspace{2cm} \textit{G}' is an \textit{st}-digraph \hspace{2cm} Let \(v_1, \ldots, v_n\) be a topological ordering of \textit{G}'

Since \textit{G}' is an \textit{st}-digraph, for \(v_i\) \((i \neq 1, n) \exists (v_j, v_i)\) and \((v_i, v_k)\). By the property of topological ordering \(j < i\) and \(i < k\).

\(v_1, \ldots, v_n\) is an \textit{st}-ordering of \textit{G}

\textbf{EXAMPLE}
Definition: Ear decomposition

An ear decomposition \( D = (P_0, \ldots, P_r) \) of an undirected graph \( G = (V, E) \) is a partition of \( E \) into an ordered collection of edge disjoint paths \( P_0, \ldots, P_r \), such that:

- \( P_0 \) is an edge
- \( P_0 \cup P_1 \) is a simple cycle
- both end-vertices of \( P_i \) belong to \( P_0 \cup \cdots \cup P_{i-1} \)
- no internal vertex of \( P_i \) belong to \( P_0 \cup \cdots \cup P_{i-1} \)

An ear decomposition of open if \( P_0, \ldots, P_r \) are simple paths.
Lemma (Ear decomposition)
Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. $G$ has an open ear decomposition $(P_0, \ldots, P_r)$, where $P_0 = (s, t)$. 
Lemma (Ear decomposition)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. $G$ has an open ear decomposition $(P_0, \ldots, P_r)$, where $P_0 = (s, t)$.

Proof

- Let $P_0 = (s, t)$ and $P_1$ be path between $s$ and $t$, it exists since $G$ is biconnected.
Lemma (Ear decomposition)
Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. $G$ has an open ear decomposition $(P_0, \ldots, P_r)$, where $P_0 = (s, t)$.

Proof
- Let $P_0 = (s, t)$ and $P_1$ be path between $s$ and $t$, it exists since $G$ is biconnected.
- Induction hypothesis: $P_0, \ldots, P_i$ are ears.
Lemma (Ear decomposition)

Let \( G = (V, E) \) be a biconnected graph \( G \) and let \((s, t) \in E\). \( G \) has an open ear decomposition \((P_0, \ldots, P_r)\), where \( P_0 = (s, t) \).

Proof

- Let \( P_0 = (s, t) \) and \( P_1 \) be path between \( s \) and \( t \), it exists since \( G \) is biconnected.
- Induction hypothesis: \( P_0, \ldots, P_i \) are ears.
- Let \((u, v)\) be an edge in \( G \) such that \( u \in P_0 \cup \cdots \cup P_i \) and \( v \not\in P_0 \cup \cdots \cup P_i \). Let \((u, u')\), such that \( u' \in P_0 \cup \cdots \cup P_i \). Let \( P \) be a path between \( v \) and \( u' \), not passing through \( u \). \( P \) exists since \( G \) is biconnected.
**st-ordering**

---

**Lemma (Ear decomposition)**

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. $G$ has an open ear decomposition $(P_0, \ldots, P_r)$, where $P_0 = (s, t)$.

**Proof**

- Let $P_0 = (s, t)$ and $P_1$ be path between $s$ and $t$, it exists since $G$ is biconnected.

- Induction hypothesis: $P_0, \ldots, P_i$ are ears.

- Let $(u, v)$ be an edge in $G$ such that $u \in P_0 \cup \cdots \cup P_i$ and $v \notin P_0 \cup \cdots \cup P_i$. Let $(u, u')$, such that $u' \in P_0 \cup \cdots \cup P_i$. Let $P$ be a path between $v$ and $u'$, not passing through $u$. $P$ exists since $G$ is biconnected.
Lemma (Ear decomposition)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. $G$ has an open ear decomposition $(P_0, \ldots, P_r)$, where $P_0 = (s, t)$.

Proof

- Let $P_0 = (s, t)$ and $P_1$ be path between $s$ and $t$, it exists since $G$ is biconnected.

- Induction hypothesis: $P_0, \ldots, P_i$ are ears.

- Let $(u, v)$ be an edge in $G$ such that $u \in P_0 \cup \cdots \cup P_i$ and $v \notin P_0 \cup \cdots \cup P_i$. Let $(u, u')$, such that $u' \in P_0 \cup \cdots \cup P_i$. Let $P$ be a path between $v$ and $u'$, not passing through $u$. $P$ exists since $G$ is biconnected.

- Let $w$ be the first vertex of $P$ that is contained in $P_0 \cup \cdots \cup P_i$. Set $P_{i+1} = (u, v) \cup P(v - \cdots - w)$. 

\[ P_0 \cup \cdots \cup P_i \]

\[ P \]

\[ u' \]

\[ v \]

\[ w \]
Lemma (Ear decomposition)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. $G$ has an open ear decomposition $(P_0, \ldots, P_r)$, where $P_0 = (s, t)$.

Proof

- Let $P_0 = (s, t)$ and $P_1$ be path between $s$ and $t$, it exists since $G$ is biconnected.
- Induction hypothesis: $P_0, \ldots, P_i$ are ears.
- Let $(u, v)$ be an edge in $G$ such that $u \in P_0 \cup \cdots \cup P_i$ and $v \not\in P_0 \cup \cdots \cup P_i$. Let $(u, u')$, such that $u' \in P_0 \cup \cdots \cup P_i$. Let $P$ be a path between $v$ and $u'$, not passing through $u$. $P$ exists since $G$ is biconnected.
- Let $w$ be the first vertex of $P$ that is contained in $P_0 \cup \cdots \cup P_i$. Set $P_{i+1} = (u, v) \cup P(v \cdots \cdots w)$.
Lemma ($st$-orientation)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$. 
Lemma ($st$-orientation)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$.

Proof

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$. 
Lemma \((st\text{-}orientation)\)

Let \(G = (V, E)\) be a biconnected graph \(G\) and let \((s, t) \in E\). There is an orientation \(G'\) of \(G\) which represents an \(st\)-digraph. \(G'\) is called \(st\)-orientation of \(G\).

Proof

- Let \(D = (P_0, \ldots, P_r)\) be an ear decomposition of \(G = (V, E)\). Notice that \(G = P_0 \cup \cdots \cup P_r\).

- Let \(G_i = P_0 \cup \cdots \cup P_i\). We prove that \(G_i\) has an \(st\)-orientation by induction on \(i\).
**Lemma (st-orientation)**

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$.

**Proof**

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an $st$-orientation by induction on $i$. 

![Diagram](https://via.placeholder.com/150)
Lemma (st-orientation)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an st-digraph. $G'$ is called st-orientation of $G$.

**Proof**

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an st-orientation by induction on $i$. 

![Diagram of st-orientation](image)
**Lemma (st-orientation)**

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$.

**Proof**

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.
- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an $st$-orientation by induction on $i$. 
**Lemma (st-orientation)**

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$.

**Proof**

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an $st$-orientation by induction on $i$. 

![Diagram of a biconnected graph with an ear decomposition and an st-orientation](image-url)
Lemma (\textit{st}-orientation)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an \textit{st}-digraph. $G'$ is called \textit{st}-orientation of $G$.

Proof

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an \textit{st}-orientation by induction on $i$.

- Distinguish two cases based on whether $u$ and $v$ are connected by a directed path or not.
Lemma (st-orientation)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$.

Proof

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an $st$-orientation by induction on $i$.

  - Distinguish two cases based on whether $u$ and $v$ are connected by a directed path or not.
**st-ordering**

---

**Lemma (st-orientation)**

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an st-digraph. $G'$ is called st-orientation of $G$.

**Proof**

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an st-orientation by induction on $i$.

- Distinguish two cases based on whether $u$ and $v$ are connected by a directed path or not.
Lemma (st-orientation)

Let $G = (V, E)$ be a biconnected graph $G$ and let $(s, t) \in E$. There is an orientation $G'$ of $G$ which represents an $st$-digraph. $G'$ is called $st$-orientation of $G$.

Proof

- Let $D = (P_0, \ldots, P_r)$ be an ear decomposition of $G = (V, E)$. Notice that $G = P_0 \cup \cdots \cup P_r$.

- Let $G_i = P_0 \cup \cdots \cup P_i$. We prove that $G_i$ has an $st$-orientation by induction on $i$.

- Distinguish two cases based on whether $u$ and $v$ are connected by a directed path or not.
Construction of an $st$-ordering:

$G$ is undirected biconnected graph

**HOW?**

Orient edges of $G$

$G'$ is an $st$-digraph

Let $v_1, \ldots, v_n$ be a topological ordering of $G'$

Since $G'$ is an $st$-digraph, for $v_i$ ($i \neq 1, n$) $\exists (v_j, v_i)$ and $(v_i, v_k)$. By the property of topological ordering $j < i$ and $i < k$.

$v_1, \ldots, v_n$ is an $st$-ordering of $G$
Construction of an $st$-ordering:

$G$ is an undirected biconnected graph

HOW?

Orient edges of $G$

$G'$ is an $st$-digraph

Let $v_1, \ldots, v_n$ be a topological ordering of $G'$

Since $G'$ is an $st$-digraph, for $v_i$

$(i \neq 1, n) \exists (v_j, v_i)$

and $(v_i, v_k)$. By the property of topological ordering $j < i$ and $i < k$.

$v_1, \ldots, v_n$ is an $st$-ordering of $G$.

Ear decomposition of $G$
Construction of an \( st \)-ordering:

\( G \) is undirected biconnected graph

\( G' \) is an \( st \)-digraph

Let \( v_1, \ldots, v_n \) be a topological ordering of \( G' \)

Since \( G' \) is an \( st \)-digraph, for \( v_i \) 

\((i \neq 1, n) \exists (v_j, v_i) \) 

and \((v_i, v_k)\). By the property of topological ordering \( j < i \) and 

\( i < k \).

\( v_1, \ldots, v_n \) is an \( st \)-ordering of \( G \)
Construction of an \(st\)-ordering:

\(G\) is undirected biconnected graph

**HOW?**

Orient edges of \(G\)

\(G'\) is an \(st\)-digraph

Let \(v_1, \ldots, v_n\) be a topological ordering of \(G'\)

Since \(G'\) is an \(st\)-digraph, for \(v_i\) \((i \neq 1, n)\) \(\exists (v_j, v_i)\) and \((v_i, v_k)\). By the property of topological ordering \(j < i\) and \(i < k\).

\(v_1, \ldots, v_n\) is an \(st\)-ordering of \(G\)

Ear decomposition of \(G\)

Orient ears
**Direct construction of $st$-ordering from ear decomposition**
Direct construction of $st$-ordering from ear decomposition

- We construct it incrementally, considering $G_i = P_0 \cup \cdots \cup P_i$, $i = 0, \ldots, r$. 

We construct it incrementally, considering $G_i = P_0 \cup \cdots \cup P_i$, $i = 0, \ldots, r$. 

- For $G_1$, let $P_1 = \{u_1, \ldots, u_p\}$, here $u_1 = s$ and $u_p = t$. The sequence $L = \{u_1, \ldots, u_p\}$ is an $st$-ordering of $G_1$.

Assume that $L$ contains an $st$-ordering of $G_i$ and let ear $P_{i+1} = \{v_1, \ldots, v_q\}$. We insert vertices $v_1, \ldots, v_q$ to $L$ after vertex $v_1$.

- Why this is an $st$-ordering?

Let $G'_i$ be an $st$-orientation of $G_i$ as constructed in the previous proof. $L$ is a topological ordering of $G'_i$ and therefore an $st$-ordering of $G_i$ (other argument?)
Direct construction of \textit{st}-ordering from ear decomposition

- We construct it incrementally, considering $G_i = P_0 \cup \cdots \cup P_i$, $i = 0, \ldots, r$.

- For $G_1$, let $P_1 = \{u_1, \ldots, u_p\}$, here $u_1 = s$ and $u_p = t$. The sequence $L = \{u_1, \ldots, u_p\}$ is an \textit{st}-ordering of $G_1$. 
Direct construction of \( st \)-ordering from ear decomposition

- We construct it incrementally, considering \( G_i = P_0 \cup \cdots \cup P_i, i = 0, \ldots, r \).

- For \( G_1 \), let \( P_1 = \{u_1, \ldots, u_p\} \), here \( u_1 = s \) and \( u_p = t \). The sequence \( L = \{u_1, \ldots, u_p\} \) is an \( st \)-ordering of \( G_1 \).

- Assume that \( L \) contains an \( st \)-ordering of \( G_i \) and let ear \( P_{i+1} = \{v_1, \ldots, v_q\} \). We insert vertices \( v_1, \ldots, v_q \) to \( L \) after vertex \( v_1 \).
Direct construction of \( st \)-ordering from ear decomposition

- We construct it incrementally, considering \( G_i = P_0 \cup \cdots \cup P_i, \ i = 0, \ldots, r \).

- For \( G_1 \), let \( P_1 = \{u_1, \ldots, u_p\} \), here \( u_1 = s \) and \( u_p = t \). The sequence \( L = \{u_1, \ldots, u_p\} \) is an \( st \)-ordering of \( G_1 \).

- Assume that \( L \) contains an \( st \)-ordering of \( G_i \) and let ear \( P_{i+1} = \{v_1, \ldots, v_q\} \). We insert vertices \( v_1, \ldots, v_q \) to \( L \) after vertex \( v_1 \).
Direct construction of \textit{st}-ordering from ear decomposition

- We construct it incrementally, considering $G_i = P_0 \cup \cdots \cup P_i$, $i = 0, \ldots, r$.

- For $G_1$, let $P_1 = \{u_1, \ldots, u_p\}$, here $u_1 = s$ and $u_p = t$. The sequence $L = \{u_1, \ldots, u_p\}$ is an \textit{st}-ordering of $G_1$.

- Assume that $L$ contains an \textit{st}-ordering of $G_i$ and let ear $P_{i+1} = \{v_1, \ldots, v_q\}$. We insert vertices $v_1, \ldots, v_q$ to $L$ after vertex $v_1$.

- \textbf{Why this is an \textit{st}-ordering?} Let $G'_{i+1}$ be an \textit{st}-orientation of $G_i$ as constructed in the previous proof. $L$ is a topological ordering of $G'_{i+1}$ and therefore an \textit{st}-ordering of $G_i$ (other argument?)
Algorithm: \textit{st}-ordering (example)
(Implementation details - Based on DFS)
Algorithm: $st$-ordering (example)
(Implementation details - Based on DFS)
Algorithm: \textit{st}-ordering (example)
(Implementation details - Based on DFS)
Algorithm: $st$-ordering (example)
(Implementation details - Based on DFS)
Algorithm: \(st\)-ordering (example)
(Implementation details - Based on DFS)
Algorithm: \textit{st}-ordering (example)

(Implementation details - Based on DFS)

\textbf{Algorithm: \textit{st}-ordering (example)}

\begin{itemize}
\item Start at \textit{s}
\item Visit \textit{b, f, g, t}
\end{itemize}
Algorithm: \textit{st-ordering} (example)

(Implementation details - Based on DFS)
Algorithm: $st$-ordering (example)
(Implementation details - Based on DFS)
Algorithm: \textit{st}-ordering (example)  
(Implementation details - Based on DFS)

\begin{align*}
 s, b, f, g, h, t
\end{align*}
Algorithm: $st$-ordering (example)
(Implementation details - Based on DFS)

$s, b, f, g, h, t$
Algorithm: \textit{st}-ordering (example)

(Implementation details - Based on DFS)

\begin{itemize}
  \item \textit{st}-ordering (example)
  \item (Implementation details - Based on DFS)
\end{itemize}
Algorithm: \textit{st}-ordering \textit{(example)}
(Implementation details - Based on DFS)
st-ordering

Algorithm: st-ordering (example)
(Implementation details - Based on DFS)

Algorithm:
st-ordering (example)
(Implementation details - Based on DFS)
Algorithm: st-ordering (example)
(Implementation details - Based on DFS)

\[ s, e, b, a, f, g, h, t \]
Algorithm: $st$-ordering (example)

(Implementation details - Based on DFS)

$s, e, b, a, f, g, h, t$
Algorithm: \( st \)-ordering (example)
(Implementation details - Based on DFS)

\[ s, e, b, a, f, g, h, t \]
Algorithm: \textit{st}-ordering (example)

(Implementation details - Based on DFS)

\textit{s, e, b, a, f, c, d, g, h, t}
Algorithm \textit{st}-ordering

\textbf{Data:} Undirected biconnected graph $G = (V, E)$, edge \{s, t\} $\in$ E

\textbf{Result:} List $L$ of nodes representing an \textit{st}-ordering of $G$

\begin{algorithmic}
    \begin{verbatim}
    dfs(vertex v) begin
        \State $i \leftarrow i + 1$; $DFS[v] \leftarrow i$
        \While{there exists non-enumerated $e = \{v, w\}$} \Do
            \State $DFS[e] \leftarrow DFS[v];$
            \If{$w$ not enumerated} \Then
                \State $CHILDEDGE[v] \leftarrow e; PARENT[w] \leftarrow v;$
                \State $dfs(w);$ 
            \Else \EndIf
            \State \{w, x\} $\leftarrow$ $CHILDEDGE[w]; D[\{w, x\}] \leftarrow D[\{w, x\}] \cup \{e\}$;
            \If{$x \in L$} \Then
                \State \text{process_ears}(w \rightarrow x);
            \EndIf
        \EndWhile
    \end{verbatim}
    \begin{verbatim}
    begin
        initialize $L$ as \{s, t\};
        $DFS[s] \leftarrow 1$; $i \leftarrow 1$; $DFS[\{s, t\}] \leftarrow 1$; $CHILDEDGE[s] \leftarrow \{s, t\}$;
        $dfs(t);$ 
    \end{verbatim}
\end{algorithmic}
Function `process_ears`:

```plaintext
process_ears(tree edge \( w \rightarrow x \)) begin
  foreach \( v \leftarrow w \in D[w \rightarrow x] \) do
    \( u \leftarrow v; \)
    while \( u \notin L \) do \( u \leftarrow PARENT[u]; \)
    \( P \leftarrow (u \ast \rightarrow v \leftarrow w); \)
    if \( w \rightarrow x \) is oriented from \( w \) to \( x \) (resp. from \( x \) to \( w \)) then
      orient \( P \) from \( w \) to \( u \) (resp. from \( u \) to \( w \));
      paste the inner nodes of \( P \) to \( L \)
      before (resp. after) \( u \);
    foreach tree edge \( w' \rightarrow x' \) of \( P \) do
      \( \text{process_ears}(w' \rightarrow x'); \)
    \( D[\{w, x\}] \leftarrow \emptyset; \)
```
Theorem

The described algorithm produces an $st$-ordering of a given biconnected graph $G = (V, E)$ in $O(E)$ time.
The described algorithm produces an \( st \)-ordering of a given biconnected graph \( G = (V, E) \) in \( O(E) \) time.

**Lemma (Necessary for planarity of orthogonal drawing of planar graphs)**

Let \( G \) be a plane graph and edge \((s, t)\) on the boundary of \( G \). Let \( s = v_1, v_2, \ldots, v_n = t \) be an \( st \)-ordering of \( G \). If \( G_i \) is the graph induced by the vertices \( v_1, \ldots, v_i \) then vertex \( v_{i+1} \) lies on the outer face of \( G_i \).

(Next exercise sheet)