On Finding the Rectangular Duals of Planar Triangular Graphs
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Abstract
We present a new linear time algorithm for finding rectangular duals of planar triangular graphs. The algorithm is conceptually simpler than the previous known algorithm. The coordinates of the rectangular dual constructed by our algorithm are integers and have pure combinatorial meaning. This allows us to discuss the heuristics for minimizing the size of the rectangular duals.

Key words: Algorithm, Planar graph, Rectangular dual.
AMS(MOS) subject classifications: 05C05, 05C38, 68Q25, 68R10.

1. Introduction
Let $R$ be a rectangle. A rectangular subdivision system of $R$ is a partition of $R$ into a set $\Phi = \{R_1, R_2, \ldots, R_n\}$ of non-intersecting smaller rectangles such that no four rectangles in $\Phi$ meet at the same point. A rectangular dual of a graph $G = (V, E)$ is a rectangular subdivision system $\Phi$ and a one-to-one correspondence $f : V \to \Phi$ such that two vertices $u$ and $v$ are adjacent in $G$ if and only if their corresponding rectangles $f(u)$ and $f(v)$ share a common boundary. Figures 1.1 and 1.2 show a graph $G$ and its rectangular dual. If $G$ has a rectangular dual, clearly $G$ must be a planar graph.

The rectangular dual of a graph $G$ finds applications in the floor planning of electronic chips and in architectural design [5, 9]. Each vertex of $G$ represents a circuit module and the edges represent module adjacencies. A rectangular dual provides a placement of the circuit modules that preserves the required adjacencies.

The problem of finding rectangular duals has been studied in [2, 3, 6, 8, 12]. A linear time algorithm for this problem was given in [3]. This algorithm is rather complicated and requires real arithmetic for the coordinates of the rectangular dual. We present a new linear time algorithm for solving this problem. The coordinates of the rectangular dual $R$ constructed by our algorithm are

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integers and have pure combinatorial meaning. This allows us to discuss heuristics for reducing the size of $R$.

The present paper is organized as follows. Sections 2 introduces some definitions and lemmas needed by our algorithm. Section 3 presents the algorithm. Section 4 proves its correctness. In section 5, we discuss the heuristics for reducing the size of the rectangular dual. Section 6 concludes the paper.

2. Regular Edge Labeling of Planar Triangular Graphs

Let $G = (V, E)$ be a planar graph. Consider a fixed plane embedding of $G$. The embedding divides the plane into a number of regions. The unbounded region is called the exterior face. Other regions are called interior faces. The vertices and the edges on the exterior face are called exterior vertices and exterior edges, respectively. A cycle $C$ of $G$ divides the plane into its interior region and exterior region. If $C$ contains at least one vertex in its interior region, $C$ is called a separating cycle of $G$. For each vertex $v$, $N(v)$ denotes the set of neighbors of $v$ and $Star(v)$ denotes the set of edges incident to $v$. Whenever these notations are used, it is understood that the members in each set are listed in counterclockwise order around $v$ in the embedding.

Consider a planar graph $H = (V, E)$. Let $v_0, v_1, v_2, v_3$ be four vertices on the exterior face of $H$ in counterclockwise order. Let $P_i$ ($i = 0, 1, 2, 3$) be the four paths on the exterior face of $H$ consisting of the vertices between $v_i$ and $v_{i+1}$ (where the addition is mod 4). We seek a rectangular dual $R_H$ of $H$ such that the four vertices $v_0, v_1, v_2, v_3$ correspond to the four corner rectangles of $R_H$ and the vertices on $P_0$ ($P_1, P_2, P_3$, respectively) correspond to the rectangles located along the north (west, south, east, respectively) boundary of $R_H$. Necessary and sufficient conditions for testing if $H$ has a rectangular dual were discussed in [2, 3, 6]. These conditions, however, can be easily reduced to the following simpler form.

In order to simplify the problem, we modify $H$ as follows: Add four new vertices $v_N, v_W, v_S, v_E$ and connect $v_N$ ($v_W, v_S, v_E$, respectively) to every vertex on $P_0$ ($P_1, P_2, P_3$, respectively). Then add four new edges ($v_N, v_W$), ($v_W, v_S$), ($v_S, v_E$), ($v_E, v_N$). Let $G$ be the resulting graph. It is easy to see that $H$ has a rectangular dual $R_H$ with $v_0, v_1, v_2, v_3$ corresponding to the four corner rectangles of $R_H$ if and only if $G$ has a rectangular dual $R$ with exactly four rectangles on the boundary of $R$. Without loss of generality, we will only discuss planar graphs with exactly four vertices on its exterior face.

If $G$ has a rectangular dual $R$, then every face of $G$, except the exterior face, must be a triangle (since no four rectangles of $R$ meet at the same point). Moreover, since at least four rectangles are
needed to fully enclose some non-empty area on the plane, any separating cycle of $G$ must have length at least 4. The following theorem states that these two conditions are also sufficient for $G$ to have a rectangular dual.

**Theorem 1** [6]: A planar graph $G = (V, E)$ has a rectangular dual $R$ with four rectangles on the boundary of $R$ if and only if the following two conditions hold: (1) Every interior face of $G$ is a triangle and the exterior face of $G$ is a quadrangle; (2) $G$ has no separating triangles.

A different form of Theorem 1 was given in [2, 3]. A graph satisfying the two conditions of Theorem 1 is called a *proper triangular planar* (PTP) graph. From now on, we only discuss such graphs.

**Definition 1:** A *regular edge labeling* (REL) of a PTP graph $G = (V, E)$ is a partition of the interior edges of $G$ into two subsets $\{T_1, T_2\}$ of directed edges such that the following hold:

1. For each interior vertex $v$, the edges in $\text{Star}(v)$ appear in counterclockwise order around $v$ as follows: a set of edges in $T_1$ leaving $v$; a set of edges in $T_2$ entering $v$; a set of edges in $T_1$ entering $v$; a set of edges in $T_2$ leaving $v$.

2. All interior edges incident to $v_N$ are in $T_1$ and entering $v_N$. All interior edges incident to $v_W$ are in $T_2$ and leaving $v_W$. All interior edges incident to $v_S$ are in $T_1$ and leaving $v_S$. All interior edges incident to $v_E$ are in $T_2$ and entering $v_E$.

From Theorem 1, we can easily prove the following:

**Theorem 2:** Every PTP graph $G = (V, E)$ has a REL.

**Proof:** By Theorem 1, $G$ has a rectangular dual $R$. For each $v \in V$, let $R(u)$ denote the rectangle in $R$ corresponding to $v$. For each interior vertex $v$, label each edge $(v, u) \in \text{Star}(v)$ as follows: If $R(u)$ is above $R(v)$, $e$ is in $T_1$ and directed leaving $v$. If $R(u)$ is below $R(v)$, $e$ is in $T_1$ and directed entering $v$. If $R(u)$ is to the left of $R(v)$, $e$ is in $T_2$ and directed entering $v$. If $R(u)$ is to the right of $R(v)$, $e$ is in $T_2$ and directed leaving $v$. This labeling satisfies the two conditions of Definition 1. $\square$

Although Theorem 2 is proved from Theorem 1, our algorithm goes another way around: We find a REL of $G$ first and construct a rectangular dual of $G$ from the REL. We first prove some properties of the REL.

Let $G = (V, E)$ be a PTP graph and $\{T_1, T_2\}$ be a REL of $G$. Let $G_1$ be the directed subgraph of $G$ induced by the edges in $T_1$ and the four exterior edges directed as $v_S \rightarrow v_W; v_W \rightarrow v_N; v_S \rightarrow v_E; v_E \rightarrow v_N$. Let $E_1$ denote the edge set of $G_1$. ($E_1$ is the union of $T_1$ and the four exterior edges.) Let $G_2$ be the directed subgraph of $G$ induced by the edges in $T_2$ and the four exterior edges directed as $v_E \rightarrow v_S; v_S \rightarrow v_E; v_W \rightarrow v_N; v_N \rightarrow v_E$. Let $E_2$ denote the edge set of $G_2$. ($E_2$ is the union of
$T_2$ and the four exterior edges.) We will call $G_1$ the S-N net and $G_2$ the W-E net of $G$ derived from the REL $\{T_1, T_2\}$.

Figure 1.1 shows a PTP graph $G$. An S-N net $G_1$ and the corresponding W-E net $G_2$ are shown in Figures 1.3 and 1.4. (Ignore the integers in the small boxes in Figures 1.3 and 1.4 for now.)

Figure 1

**Lemma 3:** (1) $G_1$ is acyclic with $v_S$ as the only source and $v_N$ as the only sink.
(2) $G_2$ is acyclic with $v_W$ as the only source and $v_E$ as the only sink.

**Proof:** By way of contradiction. Suppose either $G_1$ or $G_2$ contains a directed cycle. Let $C = \{v_1, \ldots, v_l\}$ be such a cycle such that the total number of vertices that are on $C$ or in the interior of $C$ is minimized. Without loss of generality, suppose $C$ is a cycle in $G_1$ and directed in clockwise direction. (The proof of other cases are similar.)

Case 1: $C$ contains no vertices in its interior. If $l = 3$, then there is no edge in $T_2$ leaving $v_2$. This contradicts the condition (1) of Definition 1. Suppose $l > 3$. Then there is an edge $e = (v_i, v_j) \in E$ contained in the interior of $C$. $e$ cannot be in $T_1$ since otherwise we would have a smaller cycle in $G_1$ which contradicts the choice of $C$. So $e$ must be in $T_2$. But regardless of the direction of $e$ in $T_2$, the condition (1) of Definition 1 is violated either at $v_i$ or at $v_j$.

Case 2: $C$ contains at least one vertex $u_1$ in its interior. Start at $u_1$, we can reach another vertex $u_2$ by using a $T_2$ edge. Similarly from $u_2$ we can reach another vertex $u_3$ by using a $T_2$ edge. Since every vertex $u$ has an incident edge in $T_2$ leaving $u$, this process can be repeated again and again. Since $C$ is the smallest cycle in both $G_1$ and $G_2$, we cannot have a cycle in $G_2$ completely contained in the interior of $C$. Thus we must reach a vertex $v_j \in C$. Then the condition (1) of Definition 1 is violated at $v_j$.

Since we get contradictions in all cases, both $G_1$ and $G_2$ are acyclic. Since every vertex $v$, other than $v_S$ and $v_N$, has indegree and outdegree at least 1 in $G_1$, $v_S$ is the only source and $v_N$ is the only sink of $G_1$. Similarly, $v_W$ is the only source and $v_E$ is the only sink of $G_2$. □

Both $G_1$ and $G_2$ are the so-called s-t planar graphs. (An s-t planar graph is a directed acyclic planar graph with exactly one source $s$ and exactly one sink $t$ on its exterior face.) The properties of these graphs have been studied in [7, 10, 13]. Using these properties, the structure of $G_1$ can be summarized as follows:

(a) For each vertex $v$ other than $v_S$ and $v_N$, the edges entering $v$ appear consecutively around $v$ in $G_1$. The edges leaving $v$ appear consecutively around $v$ in $G_1$. Let $e_1$ and $e_2$ be the left-most and the right-most edges in $G_1$ entering $v$. Let $e_3$ and $e_4$ be the left-most and the right-most edges in
Let $G = (V, E)$ be a PTP graph and $\{T_1, T_2\}$ be a REL of $G$. Consider the S-N net $G_1$ derived from $\{T_1, T_2\}$. For each edge $e \in E_1$, let $left(e)$ ($right(e)$, respectively) denote the face of $G_1$ on the left (right, respectively) of $e$. Define the dual graph, denoted by $G_1^*$, of $G_1$ as follows. The node set of $G_1^*$ is the set of the interior faces of $G_1$ plus the two exterior faces $f_W$ and $f_E$. For each edge $e \in E_1$, there is a corresponding arc $e^*$ in $G_1^*$ directed from the face $left(e)$ to the face $right(e)$. Since $G_1$ is an s-t graph, $G_1^*$ is also an s-t graph [11]. Namely $G_1^*$ is a directed acyclic planar graph with $f_W$ as the only source and $f_E$ as the only sink.

Similarly, define the dual graph $G_2^*$ of $G_2$ as follows. For each edge $e \in E_2$, let $above(e)$ ($below(e)$, respectively) denote the face of $G_2$ on the left (right, respectively) of $e$. The nodes of $G_2^*$ are the interior faces of $G_2$ plus the two exterior faces $f_S$ and $f_N$. For each edge $e \in E_2$, there is a directed arc $e^*$ in $G_2^*$ from the face $below(e)$ to the face $above(e)$. $G_2^*$ is a directed acyclic planar graph with $f_S$ as the only source and $f_N$ as the only sink.

**Definition 2:** A consistent numbering of order $k_1$ of $G_1^*$ is a surjective mapping $F_1$ from the node set of $G_1^*$ to the set of integers $\{0, 1, \ldots, k_1\}$ such that: (1) $F_1(f_W) = 0$ and $F_1(f_E) = k_1$; and (2) if there is an arc from the node $f$ to the node $g$ in $G_1^*$ then $F_1(f) < F_1(g)$. 

3. Algorithm

Let $G = (V, E)$ be a PTP graph and $\{T_1, T_2\}$ be a REL of $G$. Consider the S-N net $G_1$ derived from $\{T_1, T_2\}$. For each edge $e \in E_1$, let $left(e)$ ($right(e)$, respectively) denote the face of $G_1$ on the left (right, respectively) of $e$. Define the dual graph, denoted by $G_1^*$, of $G_1$ as follows. The node set of $G_1^*$ is the set of the interior faces of $G_1$ plus the two exterior faces $f_W$ and $f_E$. For each edge $e \in E_1$, there is a corresponding arc $e^*$ in $G_1^*$ directed from the face $left(e)$ to the face $right(e)$. Since $G_1$ is an s-t graph, $G_1^*$ is also an s-t graph [11]. Namely $G_1^*$ is a directed acyclic planar graph with $f_W$ as the only source and $f_E$ as the only sink.

Similarly, define the dual graph $G_2^*$ of $G_2$ as follows. For each edge $e \in E_2$, let $above(e)$ ($below(e)$, respectively) denote the face of $G_2$ on the left (right, respectively) of $e$. The nodes of $G_2^*$ are the interior faces of $G_2$ plus the two exterior faces $f_S$ and $f_N$. For each edge $e \in E_2$, there is a directed arc $e^*$ in $G_2^*$ from the face $below(e)$ to the face $above(e)$. $G_2^*$ is a directed acyclic planar graph with $f_S$ as the only source and $f_N$ as the only sink.

**Definition 2:** A consistent numbering of order $k_1$ of $G_1^*$ is a surjective mapping $F_1$ from the node set of $G_1^*$ to the set of integers $\{0, 1, \ldots, k_1\}$ such that: (1) $F_1(f_W) = 0$ and $F_1(f_E) = k_1$; and (2) if there is an arc from the node $f$ to the node $g$ in $G_1^*$ then $F_1(f) < F_1(g)$.
For an example, a topological ordering \([1, 4]\) of \(G^*_1\) is a consistent numbering. As another example, if we define \(F_1(f)\) to be the length of the longest path in \(G^*_1\) from \(f_W\) to \(f\) (with \(F_1(f_W) = 0\)), \(F_1\) is also a consistent numbering. Define the length of \(G^*_1\) to be the length of the longest path from \(f_W\) to \(f_E\) in \(G^*_1\). Note that if the length of \(G^*_1\) is \(k\), then any consistent numbering of \(G\) has order at least \(k\) by Definition 2. The consistent numbering of \(G^*_2\) can be defined similarly. We now can present our algorithm as follows.

**Algorithm DUAL:**

Input: A PTP graph \(G = (V, E)\).

1. Find a REL \(\{T_1, T_2\}\) of \(G\).

2a. Construct the S-N net \(G_1\) derived from \(\{T_1, T_2\}\) and its dual graph \(G^*_1\).

2b. Compute a consistent numbering \(F_1\) of \(G^*_1\). Let \(k_1 = F_1(f_E)\).

2c. For each vertex \(v \in V\) other than \(v_S\) and \(v_N\), let \(f_1 = \text{left}(v)\) and \(f_2 = \text{right}(v)\) in \(G_1\). Let \(x_1(v) = F_1(f_1)\) and \(x_2(v) = F_1(f_2)\). Define \(x_1(v_N) = x_1(v_S) = 1\) and \(x_2(v_N) = x_2(v_S) = k_1 - 1\).

3a. Construct the W-E net \(G_2\) derived from \(\{T_1, T_2\}\) and its dual graph \(G^*_2\).

3b. Compute a consistent numbering \(F_2\) of \(G^*_2\). Let \(k_2 = F_2(f_N)\).

3c. For each vertex \(v \in V\), let \(g_1 = \text{below}(v)\) and \(g_2 = \text{above}(v)\) in \(G_2\). Let \(y_1(v) = F_2(g_1)\) and \(y_2(v) = F_2(g_2)\).

4. For each vertex \(v \in V\), assign \(v\) a rectangle \(R(v)\) bounded by two vertical lines with \(x\)-coordinates \(x_1(v), x_2(v)\) and two horizontal lines with \(y\)-coordinates \(y_1(v), y_2(v)\).

End.

In section 4, we will prove the algorithm DUAL correctly computes a \(k_1 \times k_2\) rectangular dual of \(G\). For an example, the rectangular dual shown in Figure 1.2 is constructed from the information indicated in Figures 1.3 and 1.4. In this example, \(F_1(f)\) is the length of the longest path from \(f_W\) to \(f\) in \(G^*_1\). \(F_2(g)\) is the length of the longest path from \(f_S\) to \(g\) in \(G^*_2\). In Figure 1.3, the integers in the small boxes are the \(F_1\)-numbers of the faces of \(G_1\). In Figure 1.4, the numbers in the small boxes are the \(F_2\)-numbers of the faces of \(G_2\).

To implement the algorithm DUAL, we assume the embedding of \(G\) is given. (If not, it can be computed by using the well known linear time planarity algorithms.) Step 1 can be carried
out by using the $O(n)$ algorithm in [3]. (The algorithm in [3] finds the set $T_1$ which is called the path digraph). For Step (2a), the graph $G_1$ and the dual graph $G_1^*$ can be constructed from the embedding information of $G$. The implementation of Step (2b) depends on the choice of $F_1$. The most natural choice, the length of the longest path from $f_S$ to $f$ in $G_1^*$, can be calculated according to the topological ordering of $G_1^*$ [1, 4]. For Step (2c), the left face and the right face of each vertex can be determined from the embedding information. All these steps take $O(n)$ time. Step (3) can be implemented similarly. Step (4) clearly takes $O(n)$ time. Thus the total running time of the algorithm is $O(n)$.

4. Correctness Proof

Before we prove the correctness of the algorithm DUAL, we need several definitions. Consider an S-N net $G_1$ of $G$. An S-N path is a directed path $P$ in $G_1$ from $v_S$ to $v_N$. Let $P_1$ and $P_2$ be two S-N paths of $G_1$. ($P_1$ and $P_2$ are not necessarily edge disjoint.) We say $P_2$ is to the right of $P_1$ if every edge $e \in P_2$ is either on $P_1$ or to the right of $P_1$.

**Definition 3:** A path system of $G_1$ is a collection $\{P_0, \ldots, P_{l-1}\}$ of S-N paths of $G_1$ such that:

1. The union of the paths $P_i$ ($0 \leq i \leq l - 1$) is the edge set $E_1$ of $G_1$.
2. $P_i$ is to the right of $P_{i-1}$ for $1 \leq i \leq l - 1$.

**Definition 4:** Let $F_1$ be a consistent numbering of $G_1^*$ of order $k_1$. For each $0 \leq i \leq k_1$, define:

1. $FACE_i = \{f \mid f$ is a face of $G_1$ with $F_1(f) = i\}$.
2. $LB_i = \{e \in E_1 \mid e$ is on the left boundary of a face $f \in FACE_i\}$.
3. $RB_i = \{e \in E_1 \mid e$ is on the right boundary of a face $f \in FACE_i\}$.
4. Define the standard path system $\{P_0, \ldots, P_{k_1-1}\}$ of $G_1$ as follows:

$$P_0 = RB_0; \quad P_i = P_{i-1} - LB_i \cup RB_i \quad \text{for } 1 \leq i \leq k_1 - 1.$$ 

Note that: $FACE_0 = \{f_W\}$, $LB_0 = \emptyset$, $RB_0 = \{(v_S,v_W),(v_W,v_N)\}$. $FACE_{k_1} = \{f_E\}$, $LB_{k_1} = \{(v_S,v_E),(v_E,v_N)\}$, $RB_{k_1} = \emptyset$.

We make the following observations. Consider any edge $e \in E_1$. Let $g_1 = \text{left}(e)$, $g_2 = \text{right}(e)$, $p = F_1(g_1)$, $q = F_1(g_2)$. Since $e$ is on the right boundary of $g_1$ and on the left boundary of $g_2$, $e \in RB_p$ and $e \in LB_q$. Since $e$’s corresponding arc $e^*$ is directed from $g_1$ to $g_2$ in $G_1^*$, we have $p < q$. So $LB_i \cap RB_i = \emptyset$ for all $0 \leq i \leq k_1$. Since each $e \in E_1$ is in exactly one $RB_i$ ($0 \leq i \leq k_1 - 1$), $E_1$ is the disjoint union of the sets $RB_i$ ($0 \leq i \leq k_1 - 1$). Similarly $E_1$ is the disjoint union of the sets $LB_i$ ($1 \leq i \leq k_1$).

**Lemma 4:** Let $F_1$ be a consistent numbering of $G_1^*$ of order $k_1$. Then
(a) The standard path system $\{P_0, P_1, \ldots, P_{k_1-1}\}$ in Definition 4 is a path system of $G_1$. 

(b) For each vertex \( v \in V \), let \( f_1 = \text{left}(v) \) and \( f_2 = \text{right}(v) \) in \( G_1 \). Define \( x_1(v) = F_1(f_1) \) and 
\( x_2(v) = F_1(f_2) \). Then \( v \) is on the path \( P_i \) if and only if \( x_1(v) \leq i \leq x_2(v) - 1 \).

**Proof:** (a) We prove, by induction, the following hold for each \( i \) (\( 0 \leq i \leq k_1 - 1 \)): (1) \( P_i \) is an S-N path of \( G_1 \); and (2) \( LB_{i+1} \subseteq P_i \).

Base step \( i = 0 \): (1) \( P_0 = \{(x_S, x_W), (x_W, x_N)\} \) is an S-N path of \( G_1 \).

(2) Let \( e \) be an edge in \( LB_1 \). Then \( e \) is on the left boundary of a face \( f \in FACE_1 \). Let \( e^* \) be the arc in \( G^*_1 \) corresponding to \( e \). Since \( F_1(f) = 1 \), \( e^* \) must be directed from \( f_E \) to \( f \) in \( G^*_1 \). This implies \( e \in RB_0 = P_0 \). Since this is true for all \( e \in LB_1 \), we have \( LB_1 \subseteq P_0 \).

Induction step: Assume the claims (1) and (2) are true for \( i - 1 \), we show they are true for \( i \).

(1) By induction hypothesis, \( P_{i-1} \) is an S-N path. Suppose \( FACE_i = \{h_1, \ldots, h_l\} \) for some \( l \).

Let \( A_j \) and \( B_j \) be the left and the right boundary of \( h_j \) respectively (\( 1 \leq j \leq l \)). Since \( P_{i-1} \) is true for \( P_{i-1} \), the paths \( A_j \) (\( 1 \leq j \leq l \)) are sub-paths of \( P_{i-1} \). Since \( A_j \) and \( B_j \) (\( 1 \leq j \leq l \)) start at the same vertex and end at the same vertex and \( P_i \) is obtained from \( P_{i-1} \) by replacing each \( A_j \) with \( B_j \), \( P_i \) is an S-N path of \( G_1 \).

(2) Consider any edge \( e \in LB_{i+1} \). Let \( g_1 = \text{left}(e) \) and \( g_2 = \text{right}(e) \). Since \( e \in LB_{i+1} \), \( F_1(g_2) = i + 1 \). Suppose \( F_1(g_1) = q \) for some \( q \). Then \( e \in RB_1 \). Since \( e^* \) is directed from \( g_1 \) to \( g_2 \) in \( G^*_1 \), \( q < i + 1 \). By definition, \( e \) is added into \( P_i \) and deleted when \( P_{i+1} \) is constructed. So \( e \) is in \( P_i \) for all \( q \leq r \leq i \). In particular \( e \in P_i \). Thus \( LB_{i+1} \subseteq P_i \). This completes the induction.

Each \( e \in E_1 \) is in some \( RB_i \) (\( 0 \leq i \leq k_1 - 1 \)) and hence in \( P_i \). Therefore \( E_1 \) is the union of \( P_i \)'s (\( i = 0, \ldots, k_1 - 1 \)). From the definition of \( P_i \), it is easy to see \( P_i \) is to the right of \( P_{i-1} \) for all \( 1 \leq i \leq k_1 - 1 \). Thus \( \{P_0, \ldots, P_{k_1-1}\} \) is a path system of \( G_1 \).

(b) Since \( v \) is on the right boundary of \( f_1 \), it is added into the path \( P_{x_1(v)} \). Since \( v \) is on the left boundary of \( f_2 \), it is removed when the path \( P_{x_2(v)} \) is constructed. Hence \( v \) is on the paths \( P_i \) for exactly those indices \( i \) with \( x_1(v) \leq i \leq x_2(v) - 1 \). \( \square \)

All above discussion can be repeated on the W-E net \( G_2 \) and its dual graph \( G^*_2 \). Let \( F_2 \) be a consistent numbering of \( G^*_2 \) of order \( k_2 \). We can construct the standard path system \( \{Q_0, \ldots, Q_{k_2 - 1}\} \) of \( G_2 \) from \( F_2 \) similar to Definition 4. For each vertex \( v \) of \( G \), let \( g_1 = \text{below}(v) \) and \( g_2 = \text{above}(v) \) in \( G_2 \). Define \( y_1(v) = F_2(g_1) \) and \( y_2(v) = F_2(g_2) \). Similar to Lemma 4, it can be shown that \( v \) is on the path \( Q_j \) if and only if \( y_1(v) \leq j \leq y_2(v) - 1 \).

**Lemma 5:** Let \( G_1 \) and \( G_2 \) be the S-N net and the W-E net derived from a REL \( \{T_1, T_2\} \) of \( G \). Let \( F_1 \) and \( F_2 \) be two consistent numberings of \( G^*_1 \) and \( G^*_2 \), respectively. Let \( u \) and \( v \) be two vertices of \( G \).
(1) If \((u, v) \in T_2\) and is directed from \(u\) to \(v\) in \(G_2\), then \(x_2(u) = x_1(v)\).

(2) If there is a directed path from \(u\) to \(v\) in \(G_2\) with length at least 2, then \(x_2(u) < x_1(v)\).

(3) If \((u, v) \in T_1\) and is directed from \(u\) to \(v\) in \(G_1\), then \(y_2(u) = y_1(v)\).

(4) If there is a directed path from \(u\) to \(v\) in \(G_1\) with length at least 2, then \(y_2(u) < y_1(v)\).

**Proof:** We only prove (1) and (2). The proof of (3) and (4) is similar.

1. Suppose \((u, v) \in T_2\) and is directed from \(u\) to \(v\). Let \(e_1\) (or \(e_2\)) be the rightmost outgoing (incoming, respectively) edge of \(u\) in \(G_1\). Let \(e_3\) (or \(e_4\)) be the leftmost outgoing (incoming, respectively) edge of \(v\) in \(G_1\). Let \(f\) be the face of \(G_1\) with \(e_1, e_2, e_3, \) and \(e_4\) on its boundary. Then \(f = right(u) = left(v)\) and \(x_2(u) = x_1(v) = F_1(f)\).

2. Let \(u = u_0, u_1, \ldots, u_p = v\) \((p \geq 2)\) be a directed path in \(G_2\) from \(u\) to \(v\). By (1), \(x_2(u-l) = x_1(u_l)\) for all \(1 \leq l \leq p\). Since \(x_1(u_l) < x_2(u_l)\) for all \(0 \leq l \leq p\) and \(p \geq 2\), we have \(x_2(u) = x_2(u_0) < x_1(u_p) = x_1(v)\). \(\square\)

From above two lemmas, we can prove the following:

**Theorem 6:** The algorithm DUAL correctly constructs a rectangular dual of \(G\) in \(O(n)\) time.

**Proof:** We have shown the algorithm can be implemented in linear time. We next prove the correctness of the algorithm. Let \(\{P_0, \ldots, P_{k_1-1}\}\) be the standard path system of \(G_1\) derived from \(F_1\) and let \(\{Q_0, \ldots, Q_{k_2-1}\}\) be the standard path system of \(G_2\) derived from \(F_2\). In the rectangular dual \(R\) constructed by the algorithm DUAL, each S-N path \(P_i\) \((0 \leq i \leq k_1 - 1)\) corresponds to a vertical strip bounded by the two vertical lines with \(x\)-coordinates \(i\) and \(i + 1\). Each W-E path \(Q_j\) \((0 \leq j \leq k_2 - 1)\) corresponds to a horizontal strip bounded by the two horizontal lines with \(y\)-coordinates \(j\) and \(j + 1\). Let \(R(v)\) be the rectangle with coordinates \(x_1(v), x_2(v), y_1(v), y_2(v)\). To show the set \(\{R(v) \mid v \in V\}\) forms a rectangular dual of \(G\), we need to prove the following claims.

1. We show that each unit square \(R_{ij}\) \((0 \leq i \leq k_1 - 1 \text{ and } 0 \leq j \leq k_2 - 1)\) with \(x\)-coordinates \(i, i + 1\) and \(y\)-coordinates \(j, j + 1\) is occupied by a rectangle \(R(v)\) for a unique \(v \in V\). Consider the S-N path \(P_i\) and the W-E path \(Q_j\). Except the four special cases (a) \(i = 0, j = 0\) (b) \(i = k_1 - 1, j = 0\) (c) \(i = 0, j = k_2 - 1\) (d) \(i = k_1 - 1, j = k_2 - 1\), \(P_i\) and \(Q_j\) intersect at a unique vertex \(v \in V\). By Lemma 4 (b), \(v\) is the unique vertex satisfying all of the following inequalities: \(x_1(v) \leq i, i + 1 \leq x_2(v), y_1(v) \leq j, j + 1 \leq y_2(v)\). Hence \(R(v)\) is the unique rectangle occupying \(R_{ij}\). For the four special cases, this claim is not true. (For example, both \(v_S\) and \(v_W\) belong to the intersection of \(P_0\) and \(Q_0\).) The four special cases correspond to the four corner unit squares of \(R\). However, the special definition \(x_1(v_S) = x_1(v_N) = 1\) and \(x_2(v_S) = x_2(v_N) = k_1 - 1\) at the Step (2b) of the algorithm DUAL ensures that each of the four unit corner squares of \(R\) is occupied by one of \(R(v_W), R(v_E)\).
(2) We show if $e = (u, v)$ is an edge in $G$, then the corresponding rectangles $R(u)$ and $R(v)$ share a common boundary. If $e$ is an exterior edge, this is ensured by the definition of $R(v_S), R(v_E), R(v_N)$. So assume $e$ is an interior edge. Suppose $e \in T_1$ and is directed from $u$ to $v$. (Other cases are similar.) Let $P_i$ be an S-N path containing $e$. By Lemma 4 (b), $x_1(u) \leq i \leq x_2(u) - 1$ and $x_1(v) \leq i \leq x_2(v) - 1$. By Lemma 5 (3), $y_2(u) = y_1(v) = j$ for some $j$. Thus $R(u)$ and $R(v)$ have the line segment connecting two points $(i, j)$ and $(i + 1, j)$ as their common boundary.

(3) We show if two rectangles $R(u)$ and $R(v)$ share a common boundary, then $(u, v)$ is an edge in $G$. Assume the common boundary of $R(u)$ and $R(v)$ contains a horizontal line segment $I$ connecting two points $(i, j)$ and $(i + 1, j)$. (Other cases are similar.) Since $x_1(u) \leq i$, $i + 1 \leq x_2(u)$ and $x_1(v) \leq i$, $i + 1 \leq x_2(v)$, both $u$ and $v$ are on the S-N path $P_i$. We need to show $(u, v)$ is an edge on $P_i$. If not, there exists a directed path from $u$ to $v$ in $G_1$ of length at least 2. By Lemma 5 (4), we have $y_2(u) < y_1(v)$. This contradicts the assumption that $R(u)$ and $R(v)$ share $I$ as their common boundary.

Thus $e = (u, v)$ is an edge of $G$ if and only if $R(u)$ and $R(v)$ share a common boundary. Hence $\{R(v) \mid v \in V\}$ form a rectangular dual of $G$. □

5. Heuristics for Reducing the Size of the Rectangular Dual

The rectangular dual produced by the algorithm DUAL has size $k_1 \times k_2$, where $k_i (i = 1, 2)$ is the order of the consistent numbering of $F_i$ of $G^*_1$. As mentioned earlier, if $k_1$ is the length of the longest path from $f_W$ to $f_E$ in $G^*_1$ and $k_2$ is the length of the longest path from $f_S$ to $f_N$ in $G^*_2$, then any consistent numbering of $G^*_1$ has order at least $k_1$ and any consistent numbering of $G^*_2$ has order at least $k_2$. Thus the size of the smallest rectangular dual that can be produced from a given REL $\{T_1, T_2\}$ is exactly $k_1 \times k_2$. So in order to reduce the size of the rectangular dual of $G$, we must try to find a good REL. In this section, we present two such heuristics: (1) Delete certain edges from $G_1$ to obtain another S-N net so that the corresponding rectangular dual $R'$ has smaller width (at the cost of possible increase in the height of $R'$). (2) Add certain edges into $G_1$ to obtain another S-N net so that the corresponding rectangular dual $R''$ has the same width and possibly smaller height.

5.1 Reducing the width of the rectangular dual

Let $G^*_1$ be the dual of $G_1$ and $k_1$ be the length of $G^*_1$. In order to reduce the width of the rectangular dual $R$, we must reduce $k_1$.

Let $e = (u, v)$ be an interior edge of $G_1$ directed from $u$ to $v$. We say $e$ is redundant if $u$ has at least two outgoing edges and $v$ has at least two incoming edges in $G_1$. It is easy to show that
the directed graph $G_1 - \{e\}$, obtained from $G_1$ by deleting a redundant edge $e$, is still an S-N net of $G$. Let $G^*$ denote the dual graph of $G_1 - \{e\}$. $G^*$ can be obtained from $G_1^*$ by removing $e$'s corresponding arc $e^*$ and merging the two faces of $G_1$ with $e$ on their boundary into a single face.

Our strategy for reducing the width of $R$ is to keep deleting the redundant edges from $G_1$ until no redundant edges remain. We say the resulting S-N net is unreduceable.

The removal of some redundant edges does reduce the length of $G_1^*$, while the removal of other redundant edges does not. So the order in which the redundant edges are removed is important. Let $e$ be an redundant edge of $G_1$ and let $e^* = (f_1, f_2)$ be its corresponding arc in $G_1^*$. Suppose $p_1 = F_1(f_1)$ and $p_2 = F_1(f_2)$. We say $e$ is critical if $e^*$ is the only arc in $G_1^*$ from a face with $F_1$-number $p_1$ to a face with $F_1$-number $p_2$. It is easy to show that the removal of a critical redundant edge $e$ reduces the length of $G_1^*$ by 1. Thus when deleting redundant edges from $G_1$, we always delete critical edges first. The following is a heuristic algorithm for this strategy. It is to be inserted into the algorithm DUAL following the step (1).

Algorithm Reduce Width:

Input: An S-N net $G_1$ of a PTP graph $G = (V, E)$.

Repeat:

If there is a critical redundant edge $e \in G_1$, then $G_1 \leftarrow G_1 - \{e\}$.

Else find an arbitrary redundant edge $e \in G_1$ and $G_1 \leftarrow G_1 - \{e\}$.

Until $G_1$ is unreducible.

This algorithm can be easily implemented in $O(n^2)$ time. Figure 2 shows an example. When performing this algorithm on the S-N net shown in Figure 1.3, two critical redundant edges $(l, i)$ and $(n, h)$ are deleted. At this point, the edges $(j, i)$ and $(p, j)$ are redundant, but none is critical. So we arbitrarily delete the edge $(j, i)$. After this is done, the edge $(p, j)$ becomes critical and is deleted. The resulting unreducible S-N net and the corresponding rectangular dual $R'$ is shown in Figure 2. The width of $R'$ is reduced from 10 to 7. The height of $R'$ is increased from 11 to 12.

Figure 2

5.2 Reducing the height of the rectangular dual

Given an S-N net $G_1$, it is possible to add certain edges into $G_1$ without increasing the length of the dual graph $G_1^*$. By doing so, we hope to reduce the length of the corresponding dual graph $G_2^*$, and thus reduce the height of the rectangular dual $R$.

Two edges incident to a vertex $v$ are adjacent if they are consecutive around $v$ in the embedding. An interior edge $e = (u, v) \in G_2$ is called an l-candidate edge if it satisfies the following two conditions:
(a) There are at least two edges in $G_2$ incident to $u$ and at least two edges in $G_2$ incident to $v$.

(b.1) $e$ is adjacent to the rightmost outgoing edge of $u$ in $G_1$ and is adjacent to the leftmost incoming edge of $v$ in $G_1$.

Similarly, $e$ is called a $r$-candidate edge if it satisfies (a) and the following condition:

(b.r) $e$ is adjacent to the rightmost incoming edge of $u$ in $G_1$ and is adjacent to the leftmost outgoing edge of $v$ in $G_1$.

If we add an l-candidate edge $e = (u, v)$ into $G_1$ and direct it from $u$ to $v$ (see Figure 3.1, where the shaded lines denote the edges in $G_2$ and the solid lines denote the edges in $G_1$); or if we add a r-candidate edge $e = (u, v)$ into $G_1$ and direct it from $v$ to $u$ (Figure 3.2), it can be shown that the resulting graph $G'_1$ is an S-N net of $G$.

Figure 3

We next investigate the condition under which a candidate edge can be added into $G_1$ without increasing the length of $G_1^*$. We only discuss the $r$-candidate edges. The condition for the l-candidate edges is similar. Let $e = (u, v)$ be a $r$-candidate edge and $f$ be the face of $G_1$ containing $e$ in its interior. Let $P$ be the left boundary and $Q$ the right boundary of $f$. The vertex $u$ divides $P$ into two paths: $P_1$ ends at $u$ and $P_2$ starts at $u$. Similarly, $v$ divides $Q$ into two paths: $Q_1$ ends at $v$ and $Q_2$ starts at $v$ (Figure 3.3). Let $P_1^*$ and $P_2^*$ be the sets of the arcs in $G_1^*$ corresponding to the edges in $P_1$ and $P_2$, respectively. Let $Q_1^*$ and $Q_2^*$ be the sets of the arcs in $G_1^*$ corresponding to the edges in $Q_1$ and $Q_2$, respectively. An arc $e^* = (g_1, g_2)$ in $G_1^*$ is called a jump arc if $F_1(g_1) < F_1(g_2) - 1$. $e^*$ is called an essential arc if $F_1(g_1) = F_1(g_2) - 1$.

Let $G'_1$ be the graph obtained by adding a $r$-candidate edge $e$ into $G_1$. The dual graph $G_1'^*$ of $G'_1$ can be obtained from $G_1^*$ as follows: The face $f$ is divided into two faces $f_1$ and $f_2$ with $e$ as their common boundary. A new arc $(f_1, f_2)$ from $f_1$ to $f_2$ is introduced. Each arc $(g_1, f) \in P_1^*$ is replaced by a new arc $(g_1, f_1)$. Each arc $(g_2, f) \in P_2^*$ is replaced by a new arc $(g_2, f_2)$. Each arc $(f, h_1) \in Q_1^*$ is replaced by a new arc $(f_1, h_1)$. Each arc $(f, h_2) \in Q_2^*$ is replaced by a new arc $(f_2, h_2)$ (Figure 3.3.)

Let $p$ be the $F_1$-number of the face $f$ in $G_1^*$. Let $p_1$ and $p_2$ be the $F_1$-numbers of the faces $f_1$ and $f_2$ in $G_1'^*$, respectively. If all arcs in $P_1^*$ are jump arcs in $G_1^*$, then $p_1 < p$ and $p_2 = p$. The $F_1$-number of any other face $g \neq f$ in $G_1^*$ remains unchanged in $G_1'^*$. Thus the length of $G_1'^*$ is the same as the length of $G_1^*$. If there is an essential arc in $P_1^*$, then $p_1 = p$ and $p_2 = p + 1$. However, if all arcs in $Q_2^*$ are jump arcs in $G_1^*$, then the $F_1$-number of any other face $g \neq f$ in $G_1^*$ remains unchanged in $G_1'^*$ and the length of $G_1'^*$ is the same as the length of $G_1^*$. If there is an essential arc in $P_1^*$ and an essential arc in $Q_2^*$, then the length of $G_1'^*$ equals the length of $G_1^*$ plus 1.
This observation motivates the following definition: A r-candidate edge is *includable* if either all arcs in $P^*_r$ are jump arcs or all arcs in $Q^*_r$ are jump arcs in $G^*_r$. Similarly, an l-candidate edge is *includable* if either all arcs in $P^*_l$ are jump arcs, or all arcs in $Q^*_l$ are jump arcs in $G^*_l$. As discussed above, if we add an includable r- or l-candidate edge into $G_1$, the resulting directed graph $G'_1$ is an S-N net of $G$, and the length of the corresponding dual graph $G''_1$ remains the same. Our strategy for reducing the height of the rectangular dual $R$ is to keep adding includable candidate edges into $G_1$ until none can be found. We say the resulting S-N net is *saturated*. The following is a heuristic algorithm for this strategy. It is to be inserted into the algorithm DUAL following the step (1).

**Algorithm Reduce Height:**

Input: An S-N net $G_1$ of a PTP graph $G = (V, E)$.

Repeat:

If there is an includable l- or r-candidate edge $e$, then $E_1 ← E_1 ∪ \{e\}$.

Until $G_1$ is saturated.

It is easy to implement this algorithm in $O(n^2)$ time. Figure 4 shows an example. When performing this algorithm on the S-N net shown in Figure 1.3, the includable candidate edges $(s, p)$, $(p, l)$, $(h, f)$, $(f, b)$, and $(q, r)$ are added into $G_1$. (Note that the r-candidate edge $(f, b)$ becomes includable only after the edge $(h, f)$ has been added into $G_1$.) The resulting saturated S-N net and the corresponding rectangular dual $R''$ are shown in Figure 4. The width of $R''$ remains 10. The height of $R''$ is reduced from 11 to 10.

Figure 4

6. Conclusions

A new linear time algorithm for finding a rectangular dual $R$ of a proper triangular planar graph $G$ is presented. The algorithm is conceptually simple and the rectangular dual constructed has integer coordinates. The algorithm is based upon new understanding of the structure of PTP graphs, which is of independent interests. We also presented heuristics for reducing the width and the height of $R$. Several related optimization problems are interesting and deserve further studies. Let $w(R)$ and $h(R)$ denote the width and the height of $R$. How to find a rectangular dual $R$ of $G$ so that $w(R)$ is minimized? $w(R) + h(R)$ is minimized? or $w(R)h(R)$ is minimized?

References

Addison-Wesley, Reading MA, 1974.


(1) A PTP graph G.

(2) A rectangular dual of G.

(3) S-N net $G_1$

(4) W-E net $G_2$

Figure 1. A PTP graph G and a rectangular dual of G.

Figure 2. An unreducible S-N net and the corresponding rectangular dual.
Figure 3. Adding a candidate edge into an S-N net.

Figure 4. A saturated S-N net and the corresponding rectangular dual.