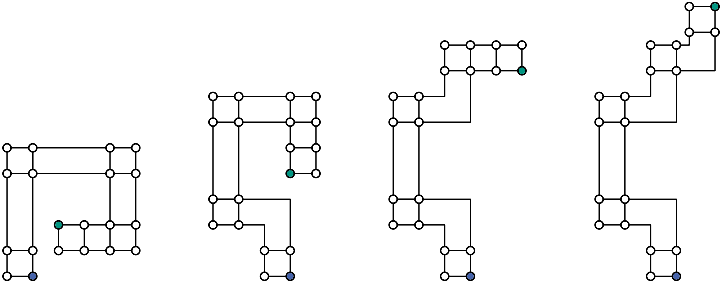
Orthogonal Graph Drawing with Flexibility Constraints

Thomas Bläsius, Marcus Krug, Ignaz Rutter, and Dorothea Wagner | February 4, 2013

INSTITUTE OF THEORETICAL INFORMATICS · ALGORITHMICS I · PROF. DR. DOROTHEA WAGNER



Motivation

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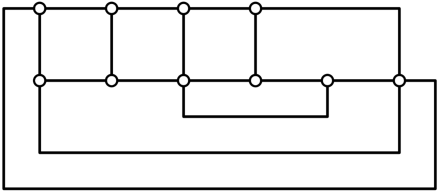
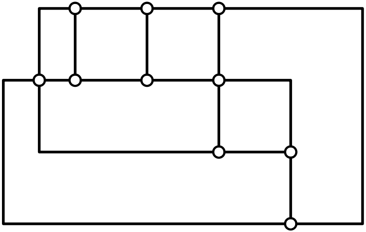
Classical Approach

Minimize total number of bends or maximum number of bends per edge.

Is minimization of bends always the best choice?



9 bends in total; at most 4 bends per edge



7 bends in total; at most 2 bends per edge

What should be done if different edges have different importance?

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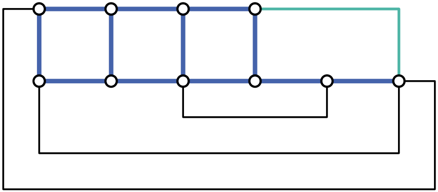

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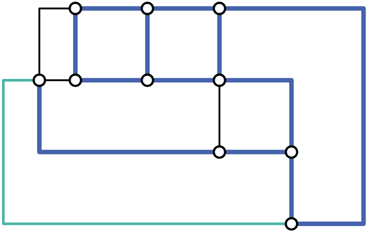

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What should be done if different edges have different importance?

The FLEXDRAW Problem

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FLEXDRAW Problem

Given

- a 4-planar graph $G = (V, E)$
- and a function $\text{flex} : E \longrightarrow \mathbb{N}_0$ assigning a *flexibility* to every edge.

Find

- a planar orthogonal drawing of G
- such that every edge $e \in E$ has at most $\text{flex}(e)$ bends.

Related Work

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Fixed Planar Embedding

- Minimize the total number of bends. [**Tamassia** 1987]
- Modification solves FLEXDRAW with fixed embedding.

Variable planar Embedding

FLEXDRAW generalizes β-embeddability

- 0-embeddability is $\mathcal{NP}$-hard. [**Garg, Tamassia** 2001]
- 2-embeddability can be solved in polynomial time. [**Biedl, Kant** 1994] [**Liu, Morgana, Simeone** 1998]
- 1-embeddability $\subseteq$ this talk

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Our Contribution

Theorem

FLEXDRAW *can be solved in* $O(n^{5/2})$ *time for 4-planar graphs with positive flexibility.*

- 1-embeddability can be solved in polynomial time.
- Means: Characterize all possible shapes of orthogonal drawings.

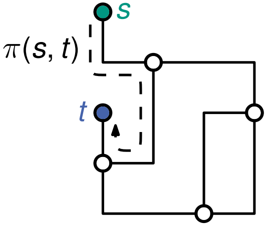
The Shape of a Drawing

The Path $\pi(s, t)$

Path from s to t on the outer face in counter-clockwise direction.

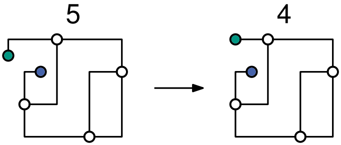
Rotation along $\pi(s, t)$

$\text{rot}(\pi(s, t)) = \#\{\text{bends to the right}\} - \#\{\text{bends to the left}\}$



$\text{rot}(\pi(s, t)) = 2$

Graphs behave like Edges



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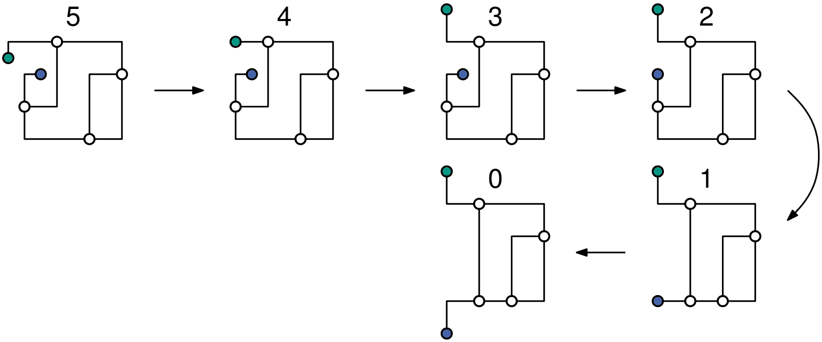

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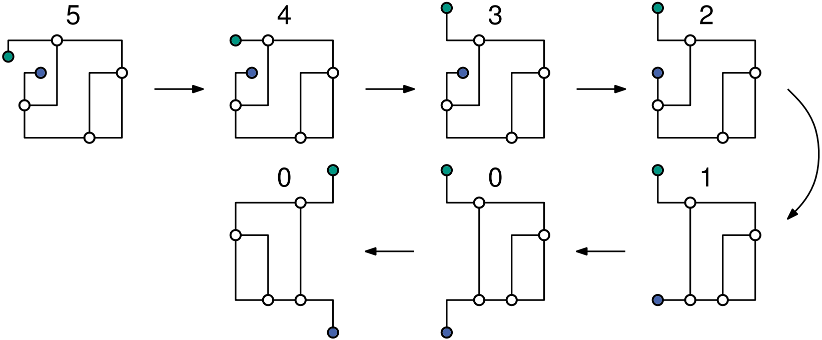

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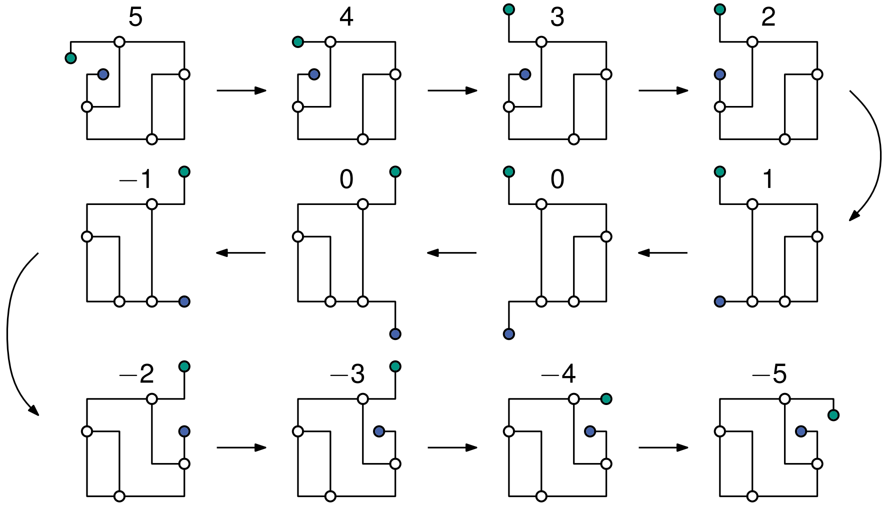
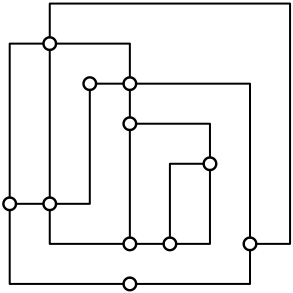


Replacing Subgraphs

Check all Embeddings of a Graph for valid Drawings

- If we know the possible shapes of a subgraph,
- we can replace it by an edge.



⇒ Number of embeddings is reduced.

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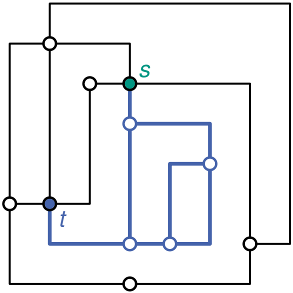

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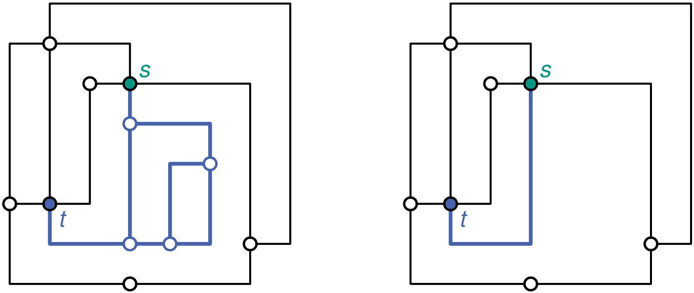

$\Rightarrow$ Number of embeddings is reduced.

Outline

1. Fixed Embedding
   - Unwinding a given Drawing
   - Characterization of all Possible Shapes
2. Variable Embedding
   - Characterization of all Possible Shapes
   - Replacing Subgraphs by simple Gadgets
   - Considering all Planar Embeddings – SPQR-Tree

Reducing the Rotation

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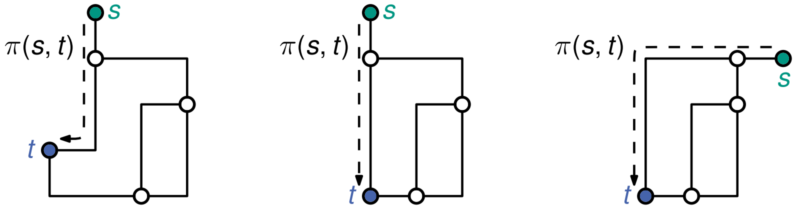
Lemma

Given a graph with positive flexibility and a drawing with $\mathrm{rot}(\pi(s,t)) \geq 0$*.*
We can reduce the rotation by 1.



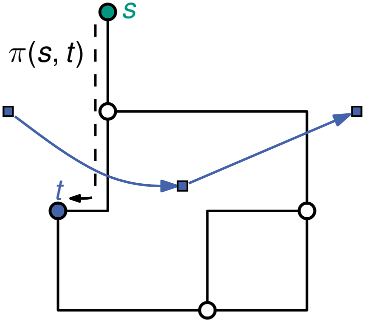
$\Rightarrow$ The possible values form an interval.

$\Rightarrow$ There are no “rigid” graphs.

Reducing the Rotation – Proof

The Flex Graph

- Bend along a path crossing $\pi(s, t)$
- Split the outer face into f_ℓ and f_r. Consider the dual graph.
- Remove edges harming flexibility constraints.



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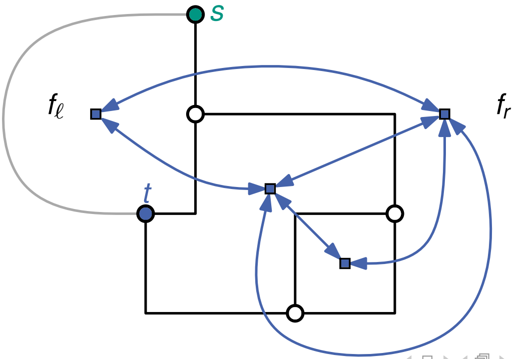

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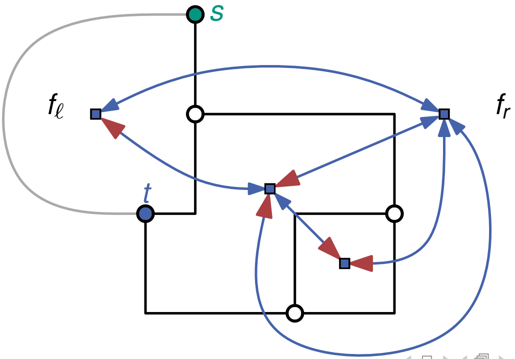

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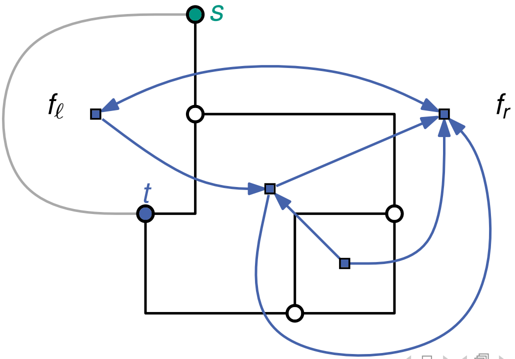
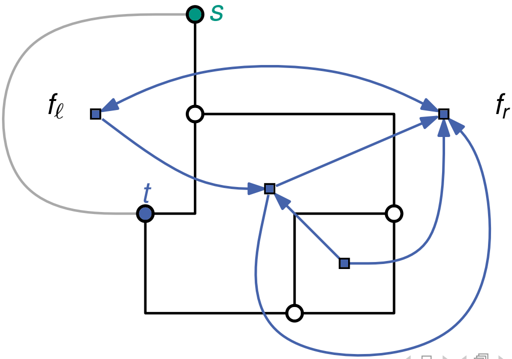


Reducing the Rotation – Proof

Lemma

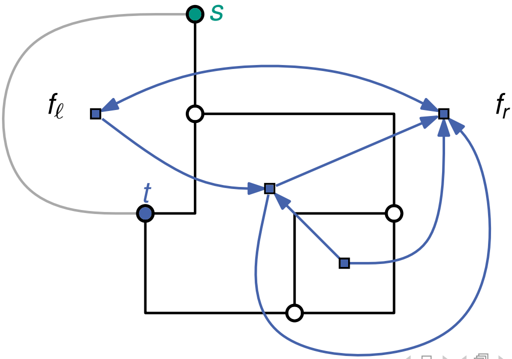
Positive flexibility ⇒ such a path exists.



Reducing the Rotation – Proof

The Flex Graph contains a Path from f_ℓ to f_r

- Fact: positive flexibility and $\mathrm{rot}(\pi(s,t)) \geq 0 \Rightarrow$ edge (f_ℓ, f_u) exists
- If $f_u = f_r$ we are done.
- Else we remove the corresponding edge in G and continue with the remaining graph.



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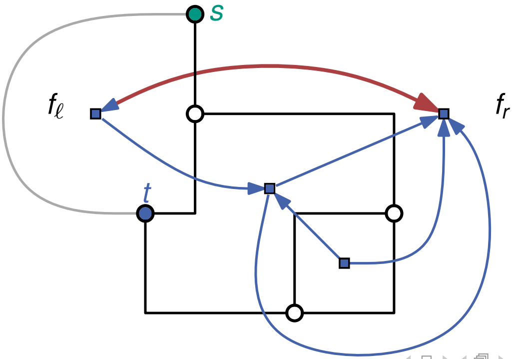

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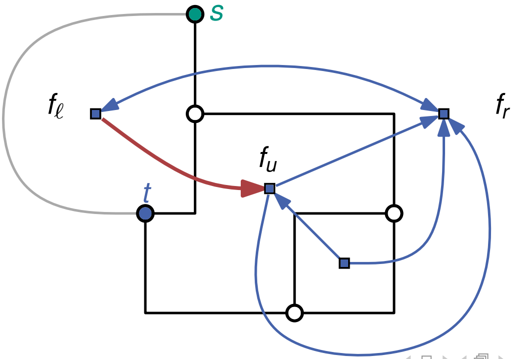

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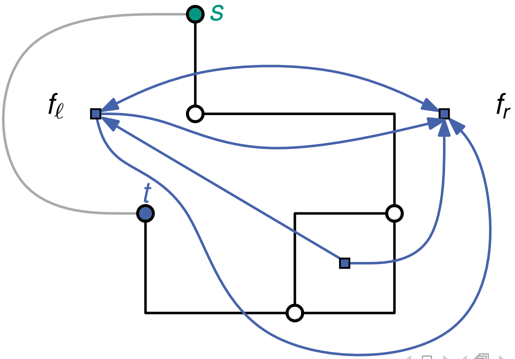

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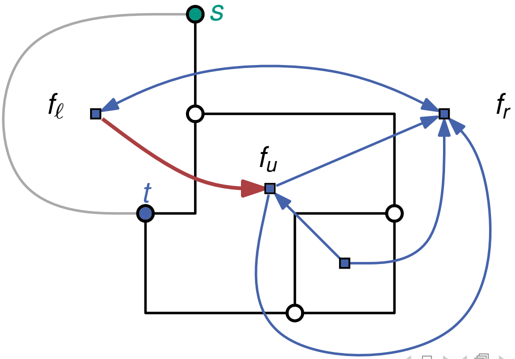

Shape of a Graph

For the set Ω of all valid drawings with respect to the embedding $\mathcal{E}$ define

$$\text{maxrot}_{\mathcal{E}}(G) = \max_{\mathcal{R} \in \Omega} \text{rot}_{\mathcal{R}}(\pi(s,t))$$

Theorem

For $\rho \in [-1, \text{maxrot}_{\mathcal{E}}(G)]$ a valid drawing with $\text{rot}(\pi(s,t)) = \rho$ exists.

In $O(n^{3/2})$ time we can

- check if G admits a valid orthogonal drawing with respect to $\mathcal{E}$.
- compute $\text{maxrot}_{\mathcal{E}}(G)$.

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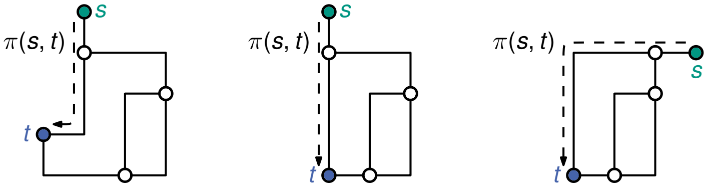
Summary – Fixed Planar Embedding

No “rigid” drawings: the rotation can always be reduced.



Theorem

For $\rho \in [-1, \mathrm{maxrot}_{\mathcal{E}}(G)]$ *a valid drawing with* $\mathrm{rot}(\pi(s, t)) = \rho$ *exists.*

Now we drop the fixed planar embedding.

Shape of a Graph

For the set Ψ of all embeddings with s and t on the outer face define

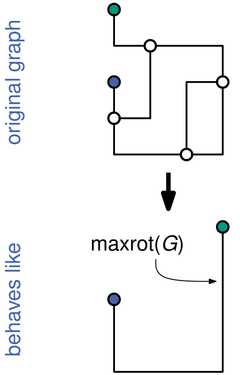
$$\operatorname{maxrot}(G) = \max_{\mathcal{E} \in \Psi} \operatorname{maxrot}_{\mathcal{E}}(G)$$

Theorem

If $\deg(s) = \deg(t) = 1$ *then* G *admits a valid drawing with* $\operatorname{rot}(\pi(s,t)) = \rho$ *if and only if* $\rho \in [-\operatorname{maxrot}(G), \operatorname{maxrot}(G)]$.

Replacing Subgraphs by Gadgets

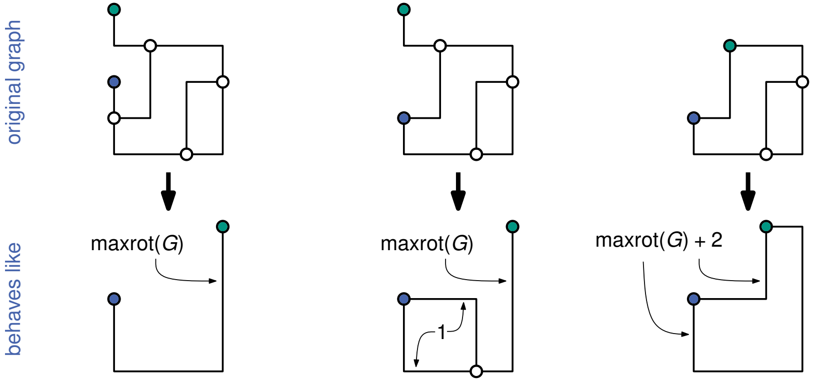
$\deg(s) = 1, \deg(t) = 1$



$[-\operatorname{maxrot}(G), \operatorname{maxrot}(G)]$

Replacing Subgraphs by Gadgets

$\deg(s) = 1, \deg(t) = 1$ | $\deg(s) = 1, \deg(t) = 2$ | $\deg(s) = 2, \deg(t) = 2$



$[-\,\mathrm{maxrot}(G), \mathrm{maxrot}(G)]$ | $[-\,\mathrm{maxrot}(G) - 1, \mathrm{maxrot}(G)]$ | $[-\,\mathrm{maxrot}(G) - 2, \mathrm{maxrot}(G)]$

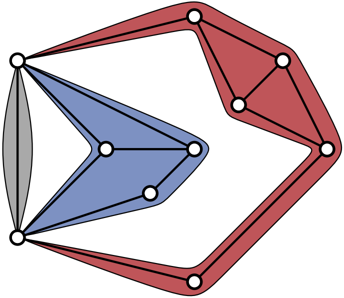
Computing the Maximum Rotation

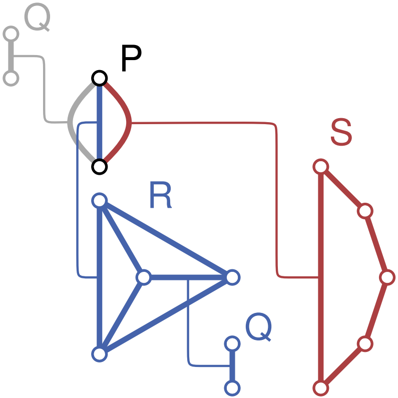
Ingredients for Computing the Maximum Rotation

- Represent all planar embeddings with *s* and *t* on the outer face with the SPQR-tree. [**Di Battista, Tamassia** 1996]
- Traverse the SPQR-tree bottom up and compute the maximum rotation for each node.
- Replace visited nodes by gadgets to reduce the number of embeddings.

The SPQR-Tree

- Decomposition of a biconnected graph in triconnected components.
- Representation of all planar embeddings.





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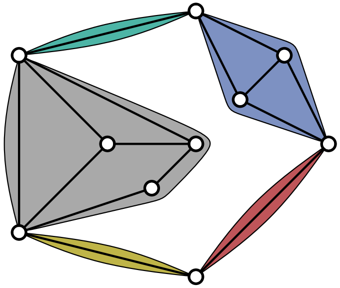

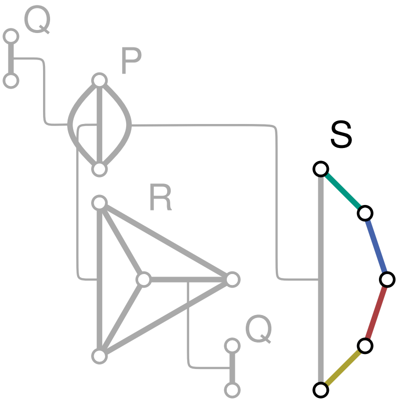

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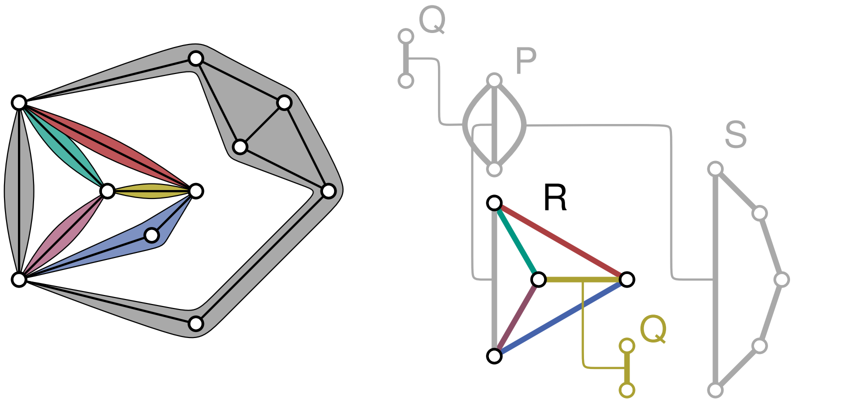

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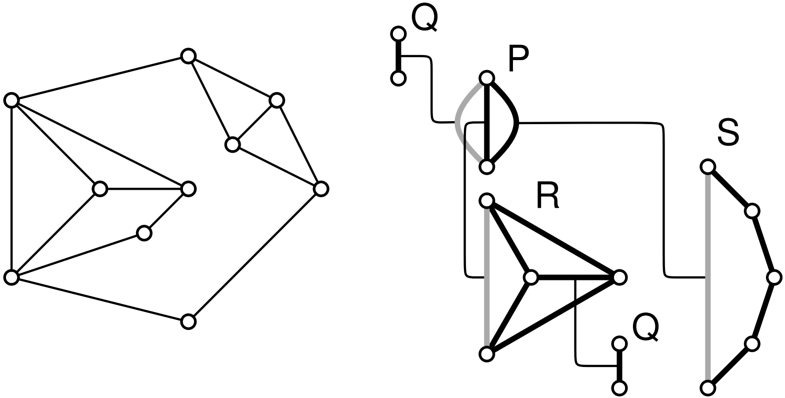
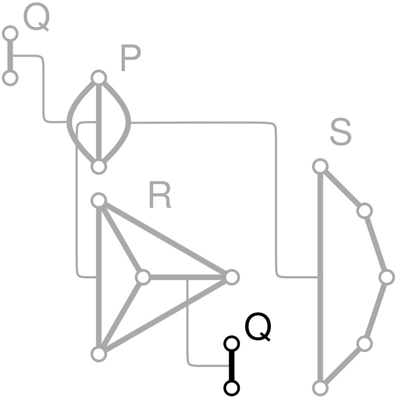


Maximum Rotation of a Q-node

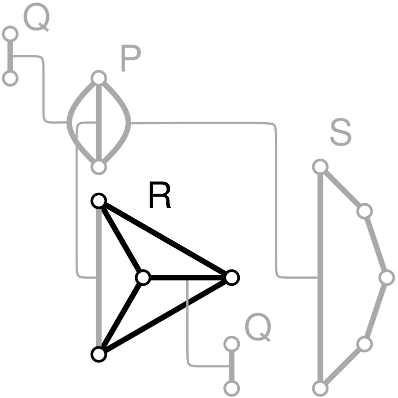
The maximum rotation of a Q-node is simply the flexibility of the corresponding edge.



Maximum Rotation of a R-node

The Skeleton of an R-node is triconnected

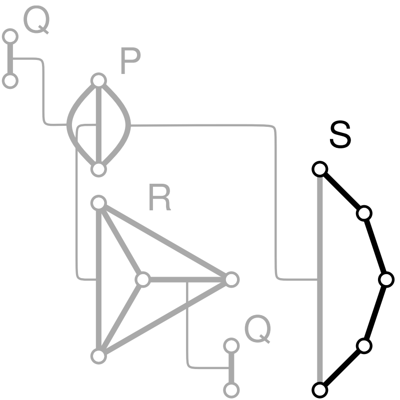
- Replace by gadgets.
- Check both embeddings.



Maximum Rotation of a S-node

An S-node represents a Chain of Graphs

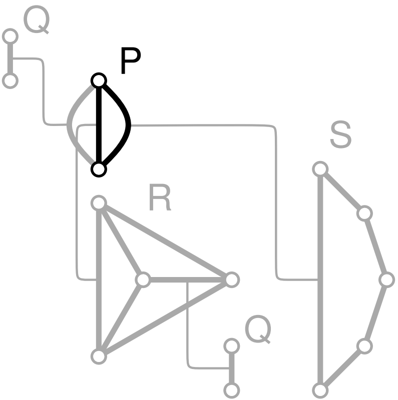
- Maximize the rotation for every graph.
- Put the graphs together with an angle of 90°.
- Only simple calculations.



Maximum Rotation of a P-node

The Skeleton of a P-node has three or four parallel Edges

- Replace by gadgets.
- Check up to six embeddings.



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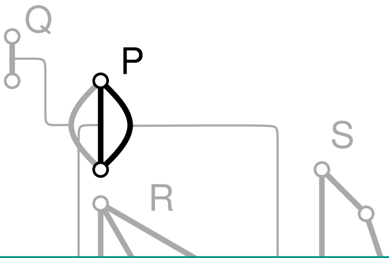

Theorem

For a biconnected graph G with designated vertices s and t we can compute maxrot(G) *in* $O(n^{3/2})$ *time or decide that G has no valid drawing with s and t on the outer face.*

Solving FLEXDRAW

Theorem

FLEXDRAW *can be solved in* $O(n^{5/2})$ *time for 4-planar graphs with* *positive flexibility*.

- Choose every pair of adjacent vertices as s and t.
- Use block-cutvertex tree to solve the general case.

Conclusion

Results

- Solve FLEXDRAW for graphs with positive flexibility.
- Complexity gap between $\mathcal{NP}$-hardness for 0-embeddability and polynomial-time algorithms for 2-embeddability closed.
- Maximum rotation yields characterization of possible shapes.

Open Problems

- Speed up from $O(n^{5/2})$ to $O(n^2)$.
- Can FLEXDRAW be solved efficiently if a subgraph has flexibility 0?
  e.g. matchings, trees, series parallel graphs, graphs with max deg 3
- Some kind of optimization.

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