Exercise Sheet 4

Assignment: November 28, 2012
Delivery: None, Discussion on December 4, 2012

1 Linear time construction of a Schnyder realizer.

Let $G$ be a maximal planar graph with $n$ vertices. Can a Schnyder labeling and a Schnyder realizer be constructed in time $O(n)$.

Hint: Find a Connection between a canonical ordering and the ordering in which the edge contraction for the construction of a Schnyder labeling is executed.

2 Property of a Schnyder realizer.

Let $G$ be a maximal planar graph with vertices $a$, $b$, $c$ on the outer face. Let $T_a$, $T_b$, $T_c$ be the red, the blue and the green trees of a Schnyder realizer, with sinks at vertices $a$, $b$, $c$, respectively. Let $v$ be an internal vertex of $G$ and denote by $P_a(v)$, $P_b(v)$, $P_c(v)$ the paths connecting $v$ with $a$, $b$, $c$ in $T_a$, $T_b$, $T_c$, respectively. Show that paths $P_a(v)$, $P_b(v)$ and $P_c(v)$ do not have common vertices, except for $v$.

3 Induced path in a Schnyder realizer.

A path of a graph $G$ is called induced if the vertices of this path are connected only by the edges of the path, i.e. path on vertices $v_1, \ldots, v_k$ is induced if for any $1 \leq i, j \leq n$ such that $|i-j| > 1$, edge $(v_i, v_j)$ does not belong to $G$. Let $G$ be a maximal planar graph and let $T_a, T_b, T_c$ be a Schnyder realizer of $G$. Assume that the edges of $T_a, T_b, T_c$ are colored red, blue and green, respectively. Show that a directed monochromatic path in $T_a, T_b, T_c$ is an induced path of $G$.

4 $st$-Ordering and $st$-Graphs

Let $G = (V, E)$ be a biconnected planar graph and let $f : V \rightarrow \mathbb{N}$ be the function giving an $st$-ordering of the vertices of $G$. In the following an undirected edge between vertices $u$ and $v$ is denoted by $\{u, v\}$, and an edge directed from $u$ to $v$ is denoted by $(u, v)$. Let $\overrightarrow{G} = (V, \overrightarrow{E})$ be a directed graph, where $\overrightarrow{E} = \{(u, v) | \{u, v\} \in E \land f(u) < f(v)\}$. I.e. $\overrightarrow{G}$ is just an orientation of $G$, where each edge $\{u, v\}$ is assigned a direction from $u$ to $v$ if $f(u) < f(v)$ or a direction from $v$ to $u$, otherwise. Prove that $\overrightarrow{G}$ is an $st$-digraph.

Hint: To achieve that prove that:

(a) $\overrightarrow{G}$ contains a single source vertex and a single sink vertex. A source (sink) of a directed graph is a vertex without incoming (outgoing) edges.

(b) $\overrightarrow{G}$ is acyclic, i.e. it does not contain any directed cycle.
5 Property of st-Ordering

Let $G = (V, E)$ be a biconnected planar graph with a given embedding and let $v_1, \ldots, v_n$ be an st-ordering of $G$ such that $v_1, v_n$ belong to the outer face of $G$. Let $G_i$ denote the plane subgraph of $G$ induced by the vertices $v_1, \ldots, v_i$. Prove that $v_{i+1}$ belongs to the outer face of $G_i$.

6 Ear decomposition.

Let $G = (V, E)$ such that for each edge $\{s, t\} \in E$, $G$ has an open ear decomposition that starts with $\{s, t\}$. Show that $G$ is 2-connected.