Exercise Sheet 3

Assignment: November 20, 2012
Delivery: None, Discussion on November 27, 2012

1 Properties of *st*-Graphs

Let $D = (V, A)$ be a planar *st*-graph with a given embedding. Prove or disprove:

(a) D is bimodal.
(b) The boundary of each face f consists of two directed paths from start(f) to target(f).
(c) For every vertex $v \in V$ there is a simple directed *st*-path that contains v.

2 Duals of *st*-Graphs

Let D be a planar embedded *st*-graph. For a directed edge $e = (u, v)$, let $\ell(e)$ denote the face left of e, and let $r(e)$ denote the face right of e. Without loss of generality assume that D is embedded such that $r(s, t)$ is the external face. The directed dual graph $D^\star = (V^\star, A^\star)$ of D is defined as follows:

- $V^\star$ is the set of faces of D, where $s^\star = r(s, t)$ and $t^\star = \ell(s, t)$.
- $A^\star = \{(\ell(e), r(e)) \mid e \in A \setminus \{(s, t)\}\} \cup \{(s^\star, t^\star)\}$

(a) Prove that $D^\star$ is a planar *st*-graph.
(b) Prove that for any two faces f and g of D exactly one of the following properties holds:
   i) D contains a directed path from target(f) to start(g)
   ii) D contains a directed path from target(g) to start(f)
   iii) $D^\star$ contains a directed path from f to g
   iv) $D^\star$ contains a directed path from g to f

Hint: Consider a topological numbering $\sigma : V \to \mathbb{N}$ of the nodes of D, such that for every $(u, v) \in A$ it holds that $\sigma(u) < \sigma(v)$.

3 Canonical Ordering

Let G be a plane graph with vertices v_1, v_2, v_n on the outer face. Let P be a simple path in G connecting vertices v_1 and v_2 and not containing v_n. Let G_p be the subgraph of G bounded by path P and edge (v_1, v_2). Prove that there exists a canonical ordering of G such that all the vertices of G' appear as initial subsequnce of this ordering.

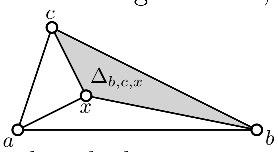
4 Barycentric Coordinates

Let $\Delta_{a,b,c}$ be a triangle on the plane on vertices a, b and c. For each point x laying inside triangle $\Delta_{a,b,c}$ there exists a triple (x_a, x_b, x_c) such that $x_a \cdot a + x_b \cdot b + x_c \cdot c = x$ and $x_a + x_b + x_c = 1$. The triple (x_a, x_b, x_c) is called *barycentric coordinates* of x with respect to $\Delta_{a,b,c}$.

Prove that:

(a) If $A(\Delta)$ denotes the area of the triangle A, then

$$x_a = \frac{A(\Delta_{b,c,x})}{A(\Delta_{a,b,c})},\ x_b = \frac{A(\Delta_{a,c,x})}{A(\Delta_{a,b,c})},\ x_c = \frac{A(\Delta_{a,b,x})}{A(\Delta_{a,b,c})}$$



(b) Equations $x_a = 0$, $x_b = 0$, $x_c = 0$ represent lines through bc, ab and ab, respectively.

(c) Let (x_a, x_b, x_c) be barycentric coordinates of point x in triagnle Δ_{abc}. The set of points $\{(x_a, x'_b, x'_c) : x'_b, x'_c \in \mathbb{R}\}$ represents a line parallel to edge bc passing through point x. Similarly, sets of points $\{(x'_a, x_b, x'_c) : x'_a, x'_c \in \mathbb{R}\}$, $\{(x'_a, x'_b, x_c) : x'_a, x'_b \in \mathbb{R}\}$ represent lines parallel to edges ac, ab, respectively, passing through point x.

5 Baricentric representation

A *Baricentric representation* of a graph G is an injective function $f : v \in V(G) \to (v_a, v_b, v_c) \in \mathbb{R}^3$ satisfying the following two conditions:

(1) $v_a + v_b + v_c = 1$ for all vertices v; and

(2) for each edge (x, y) there is no vertex $z \notin \{x, y\}$, such that $\max\{x_k, y_k\} > z_k$ for each $k \in \{a, b, c\}$.

Let f be a barycentric representation of graph G, and let a, b, c be non collinear points on the plane. Prove that the function $g : v \in G(V) \to v_a a + v_b b + v_c c \in \mathbb{R}^2$ yelds a planar straight line drawing of G inside the triangle $\Delta_{a,b,c}$.