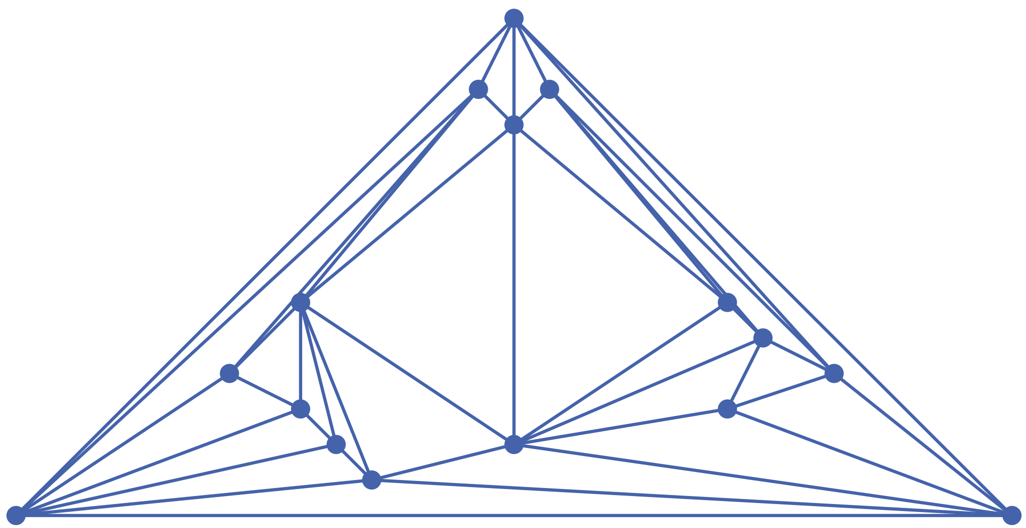
Algorithms for graph visualization

Layouts for planar graphs. Shift method.

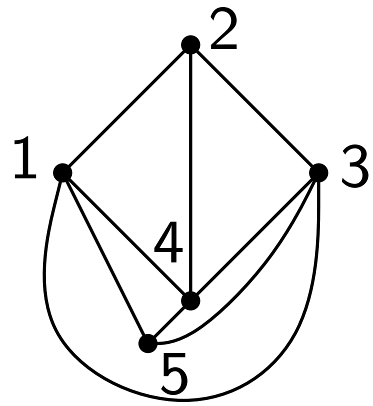
WINTER SEMESTER 2012/2013

Tamara Mchedlidze – MARTIN NÖLLENBURG – IGNAZ RUTTER



History

- Straight line drawing of a planar graph



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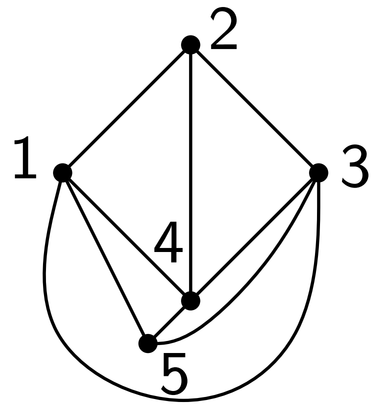

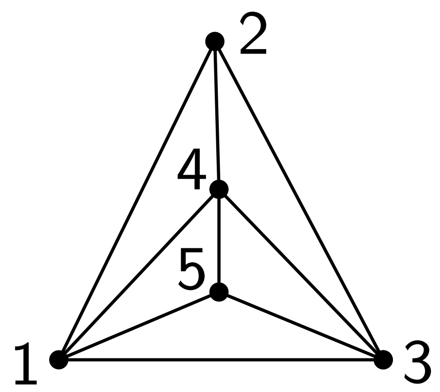

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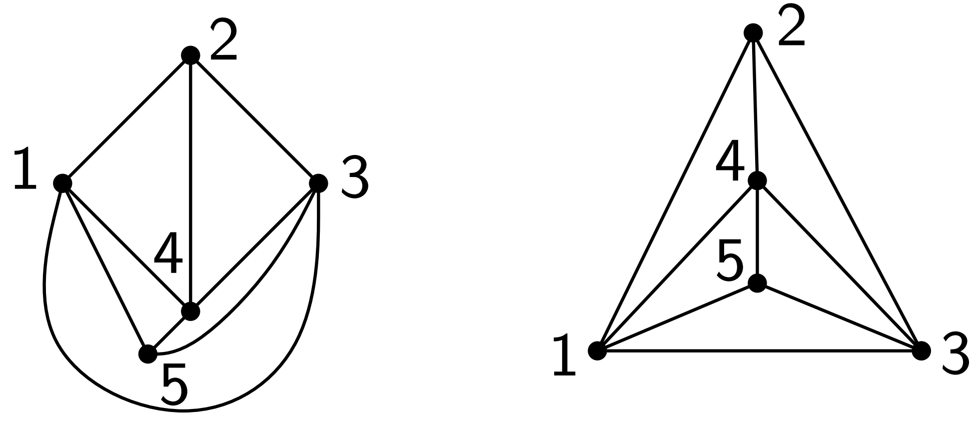

Theorem [Wagner ’36, Fary ’48, Stein ’51]

Every planar graph has a planar straight-line drawing.

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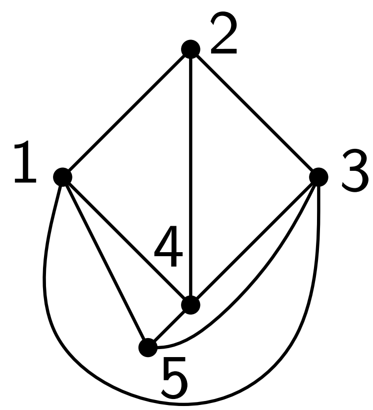

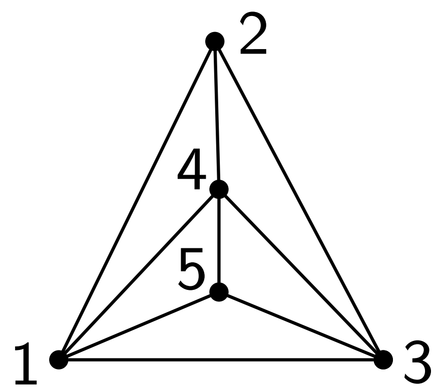

Theorem [Wagner '36, Fary '48, Stein '51]

Every planar graph has a planar straight-line drawing.

- These algorithms produce drawings with area **not bounded** by any polynomial on n.

Next

This lecture:

Theorem [De Fraysseix, Pach, Pollack '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(2n-4) \times (n-2)$.

Next lecture:

Theorem [Schnyder '90]

Every n-vertex planar graph has a planar straight-line drawing of a size $(n-2) \times (n-2)$.

Canonical Ordering

Definition: Canonical Ordering

Let $G = (V, E)$ be a triangulated planar embedded graph of $n \geq 3$ vertices. An ordering $\pi = (v_1, v_2, \ldots, v_n)$ is called a canonical ordering, if the following conditions hold for each k, $3 \leq k \leq n$.

- (C1) Vertices $\{v_1, \ldots v_k\}$ induce a 2-connected internally triangulated graph, call it G_k

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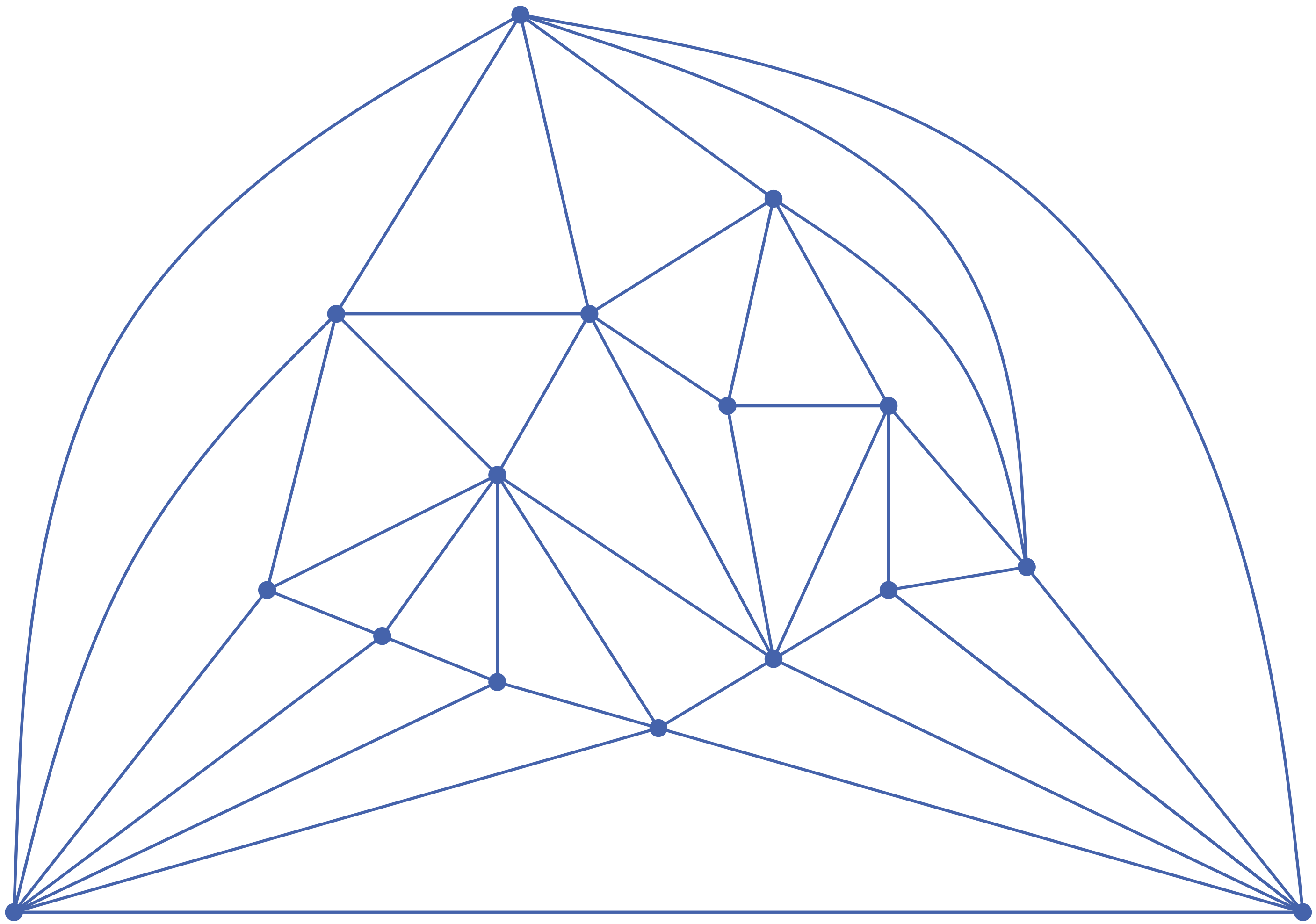
- (C2) Edge (v_1, v_2) belongs to the outer face of G_k
- (C3) If $k < n$ then vertex v_{k+1} lies in the outer face of G_k, and all neighbors of v_{k+1} in G_k appear on the boundary of G_k consecutively.

Example of Canonical Ordering



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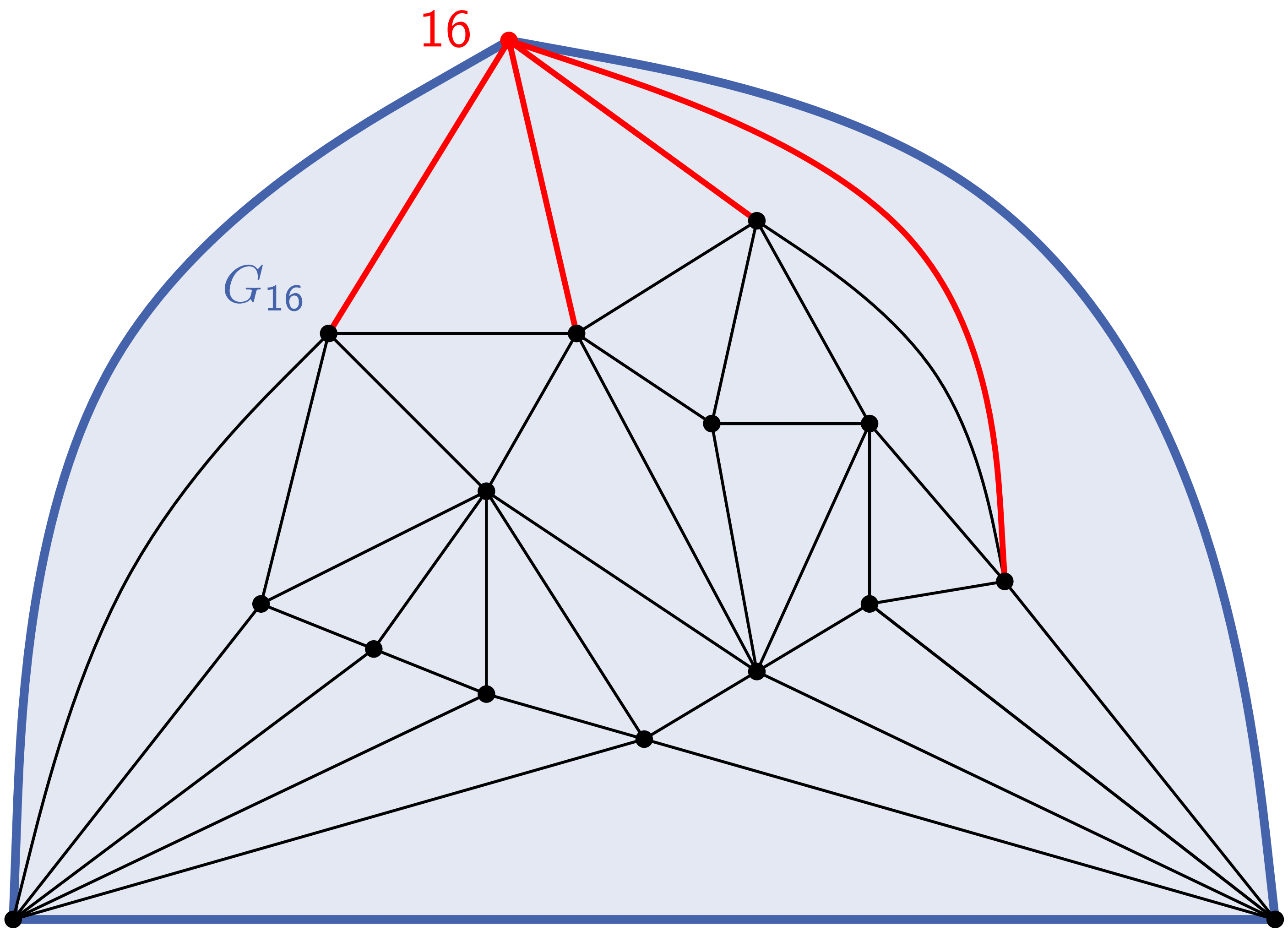

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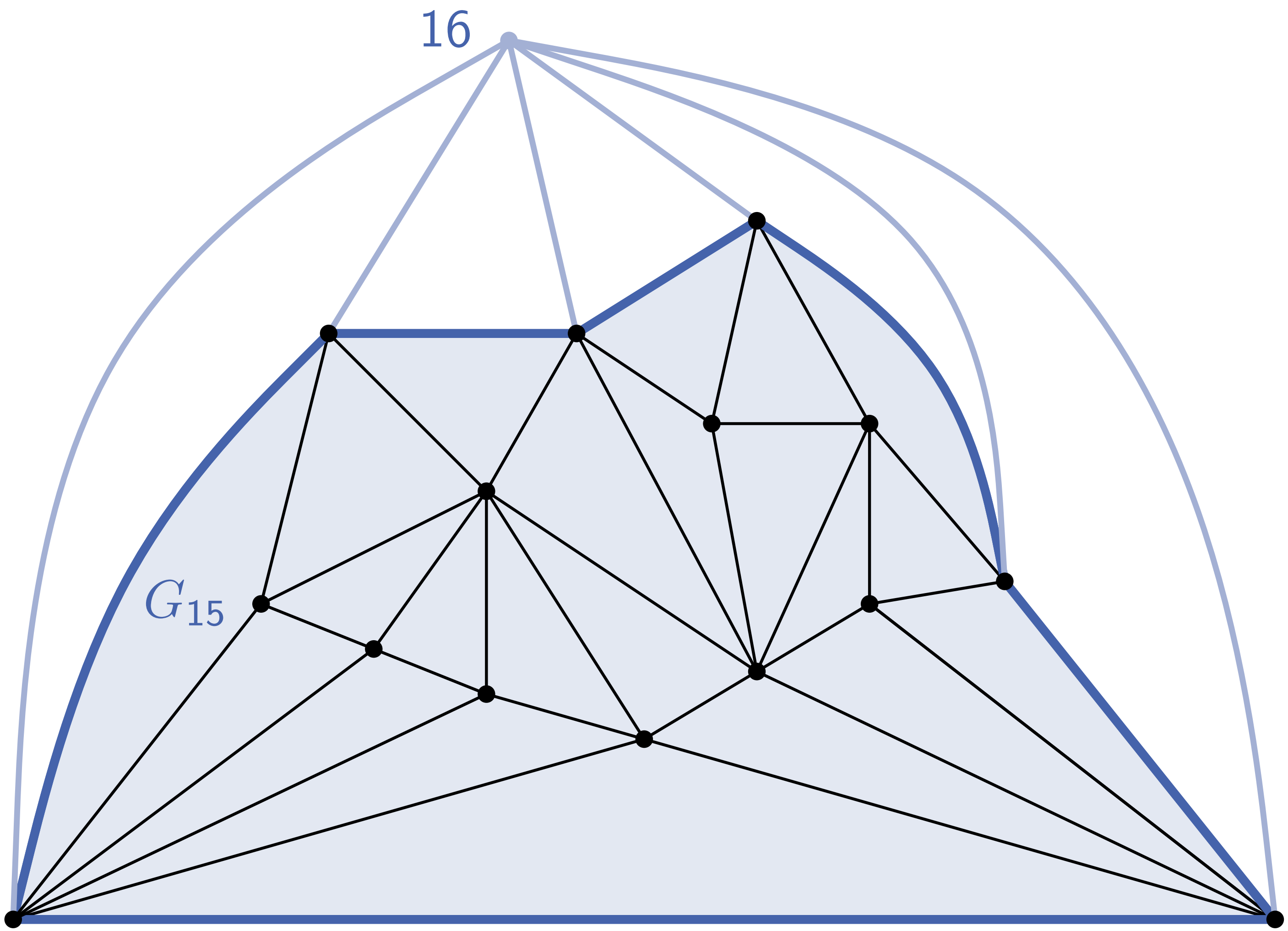

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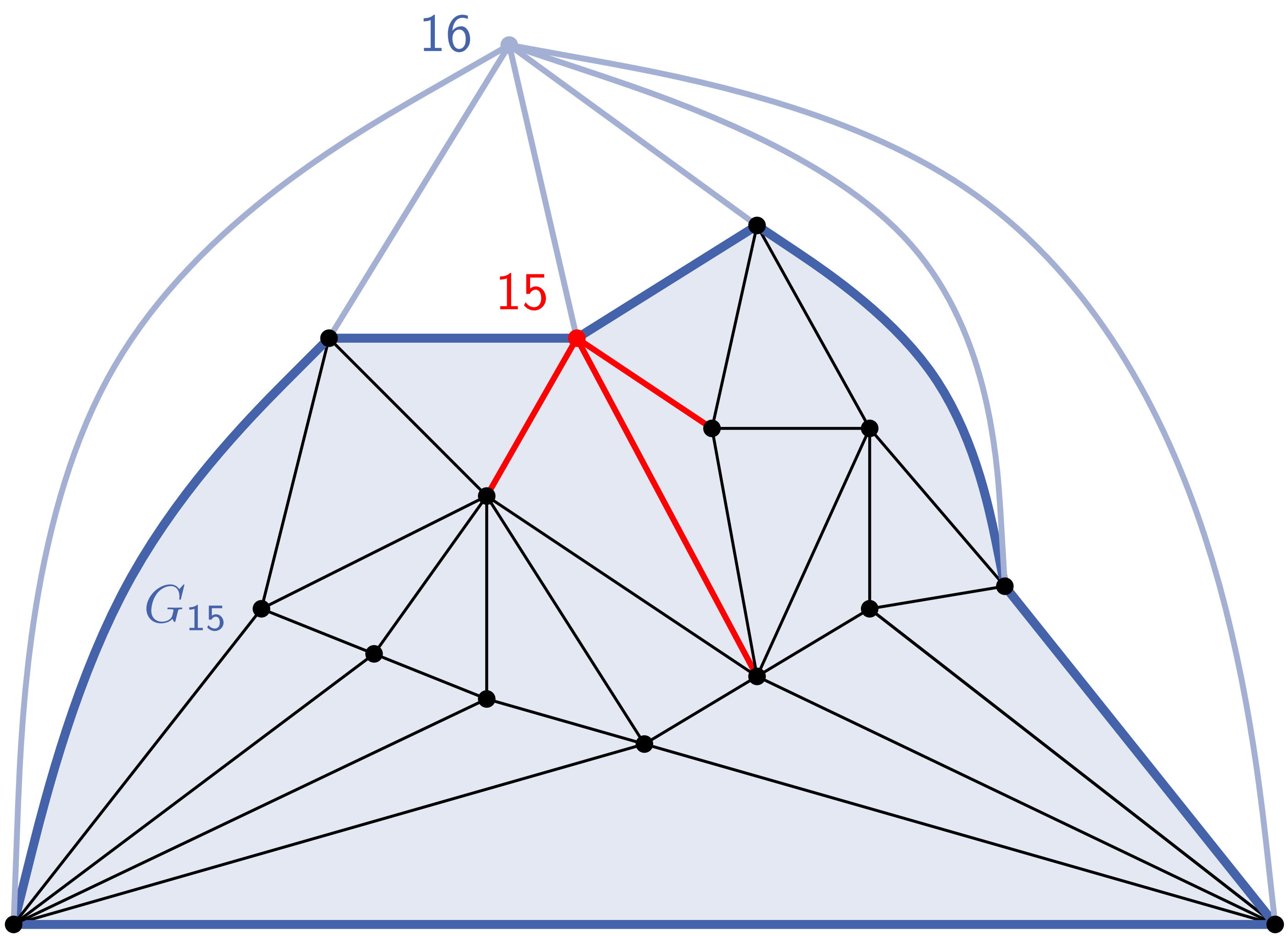

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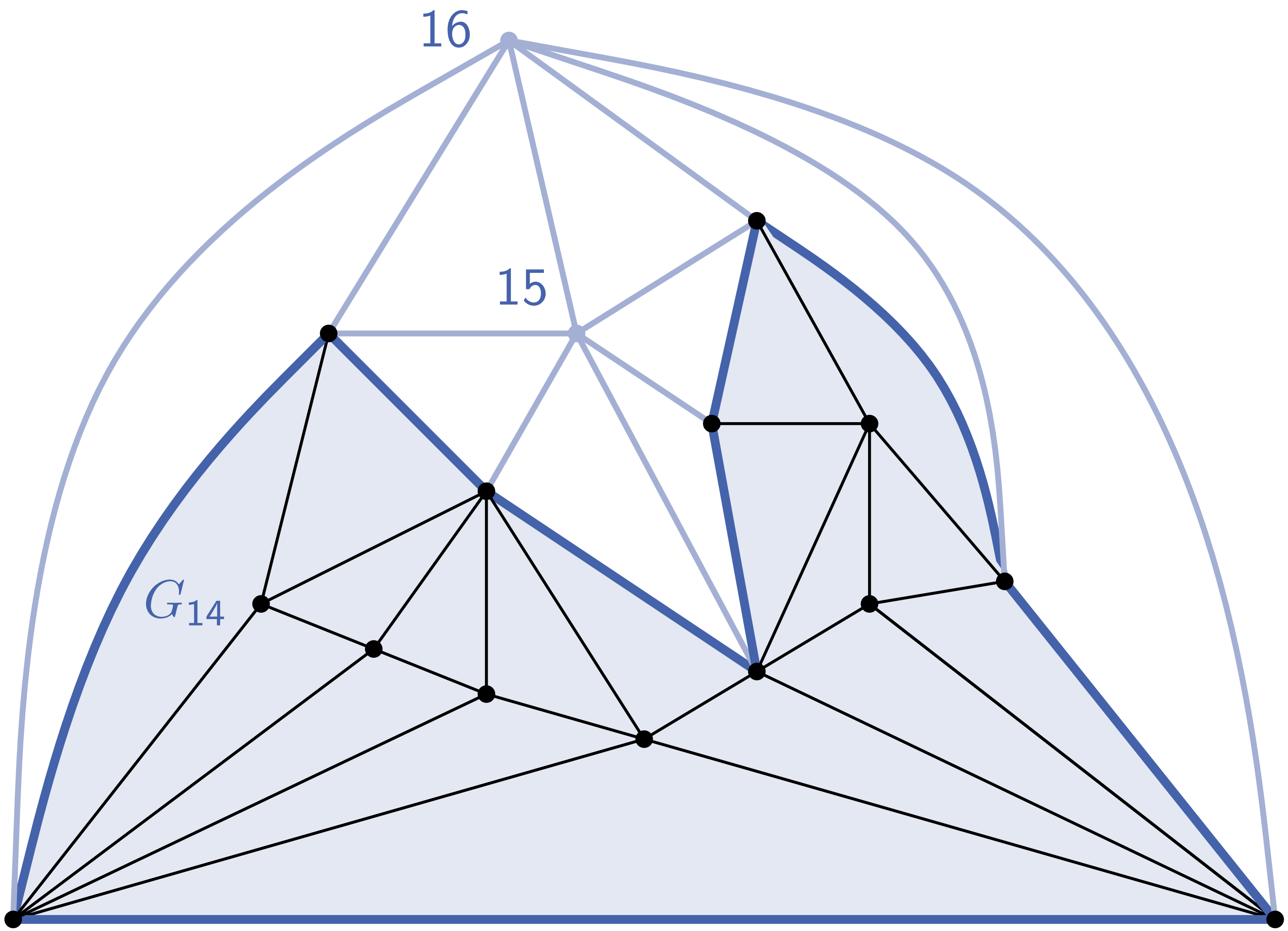

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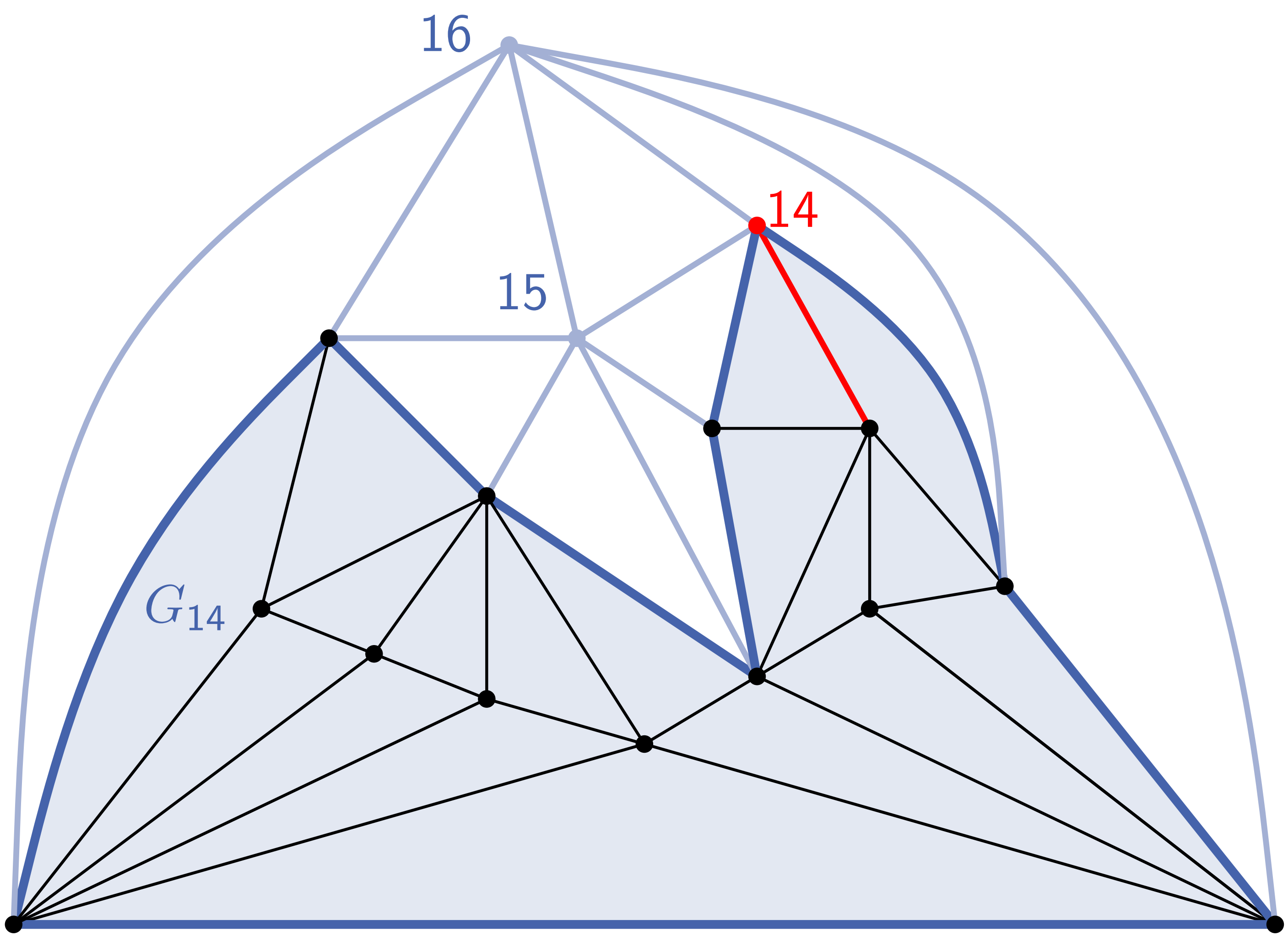

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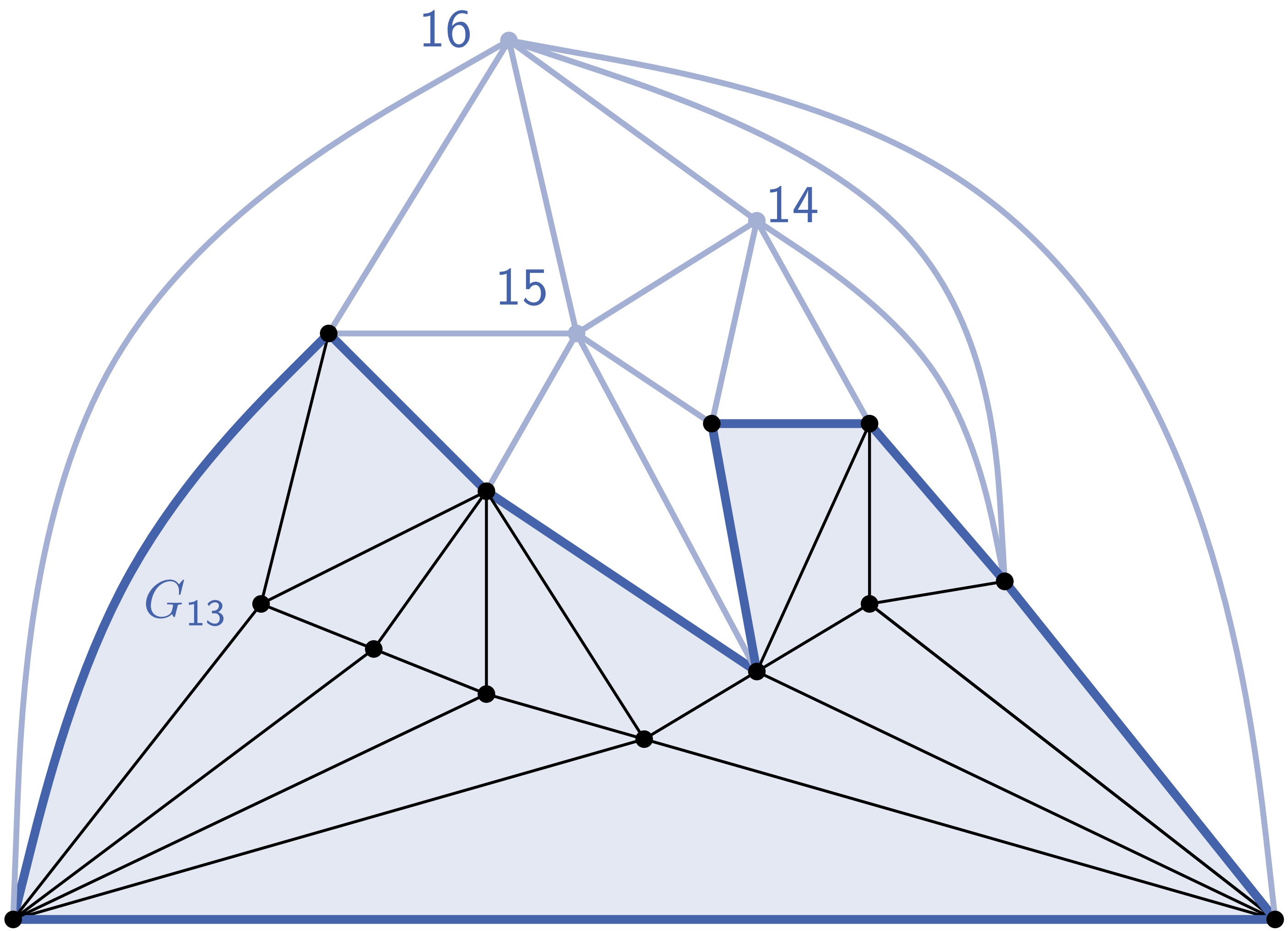

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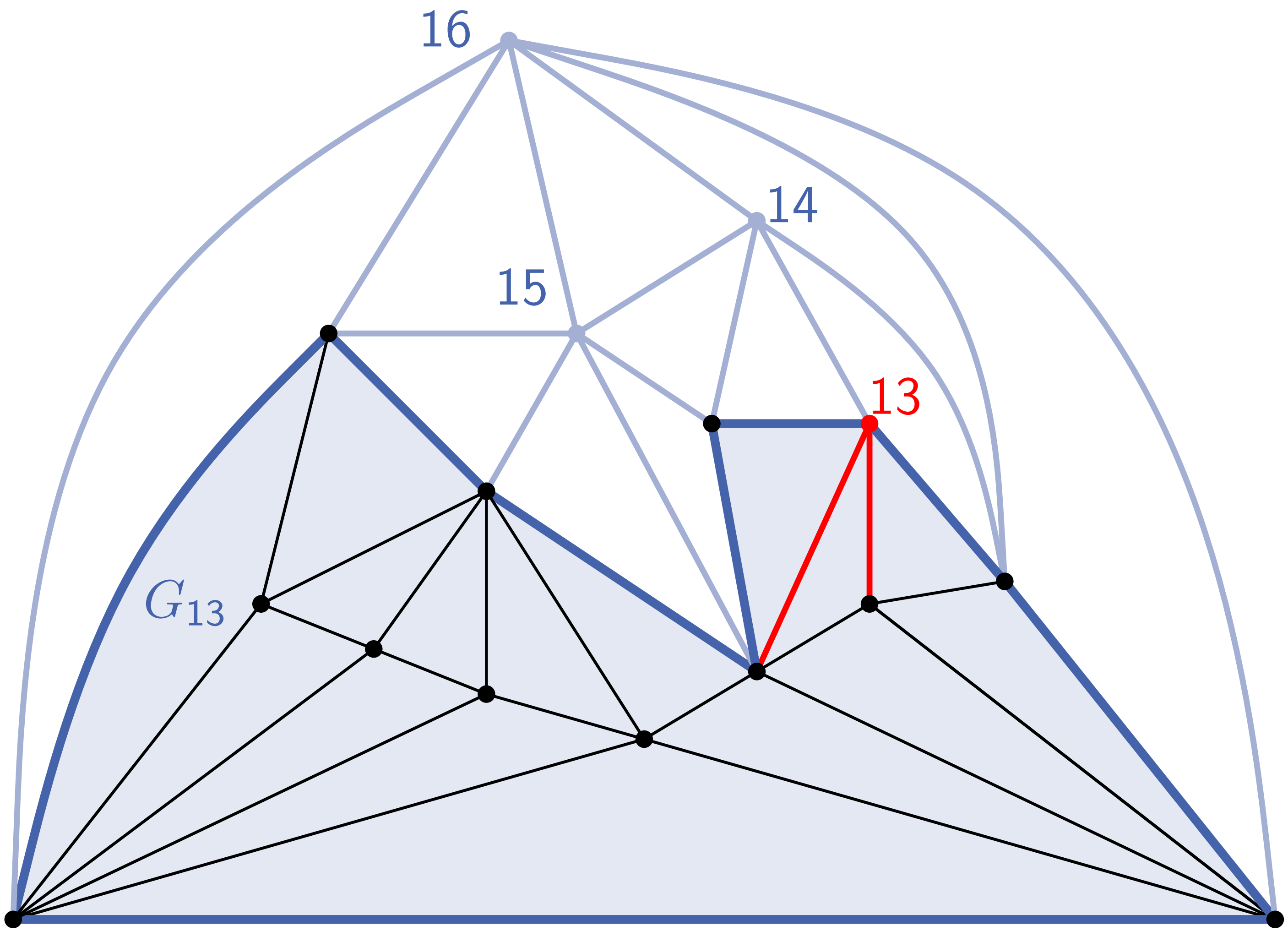

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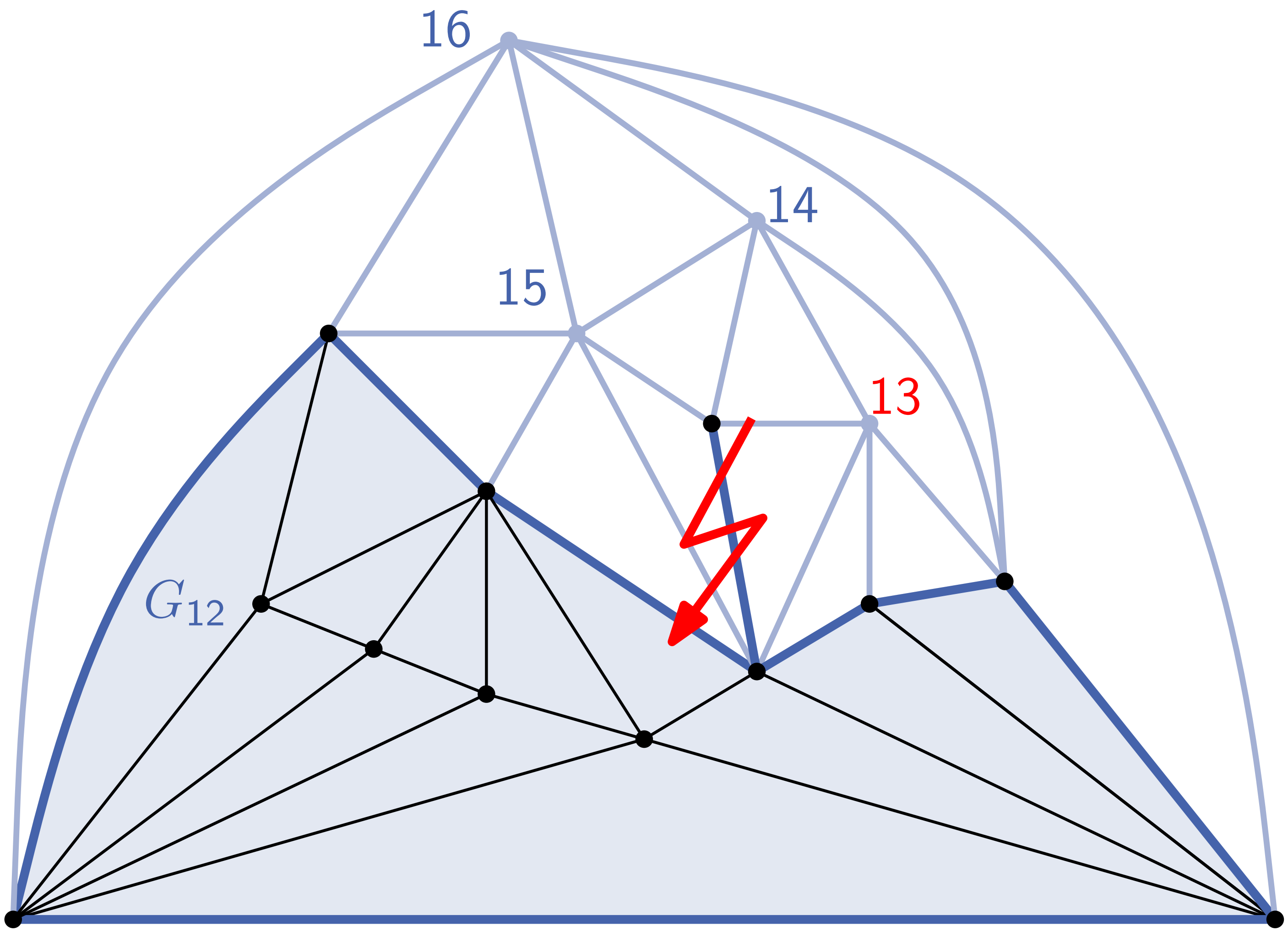

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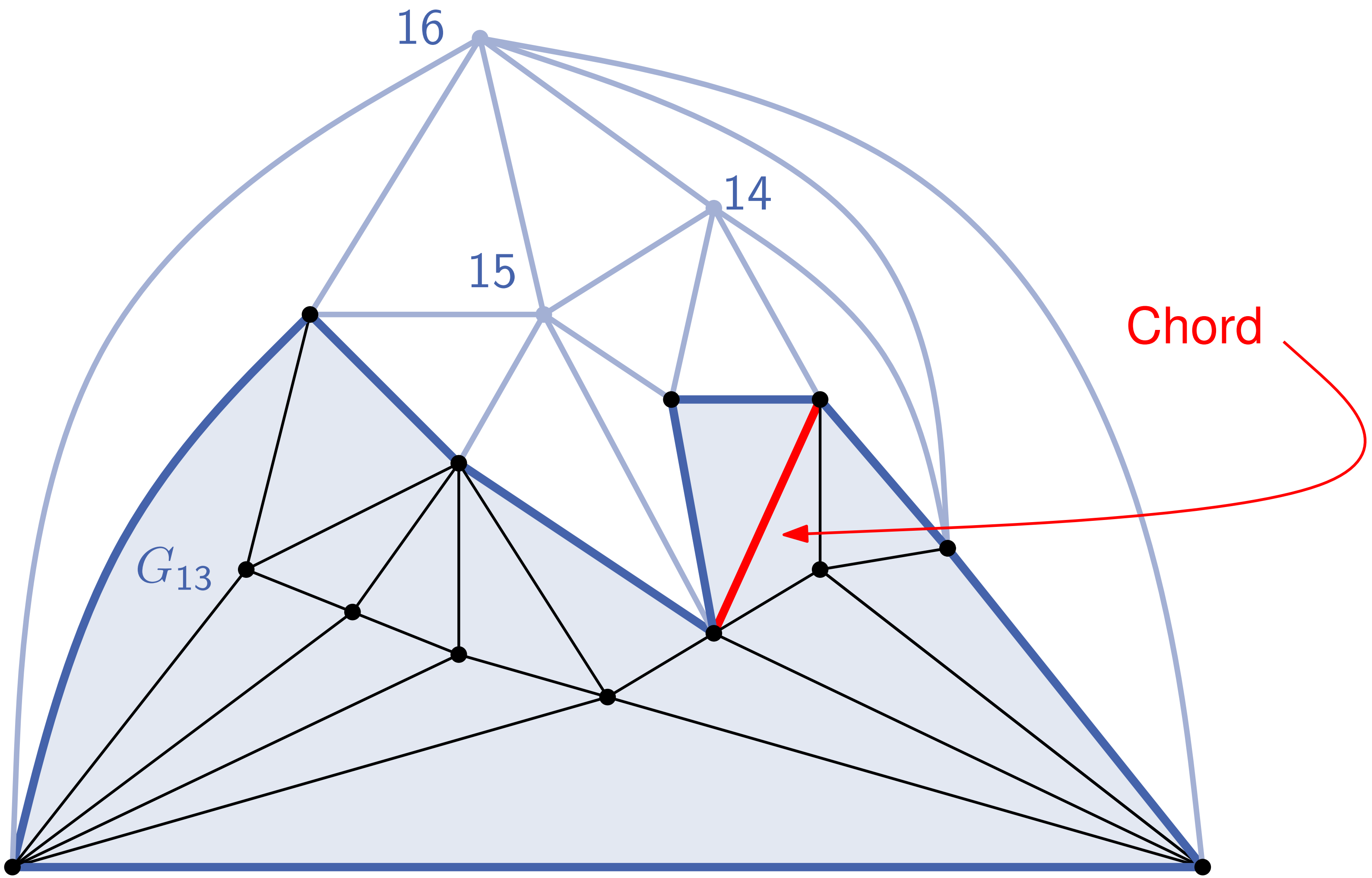

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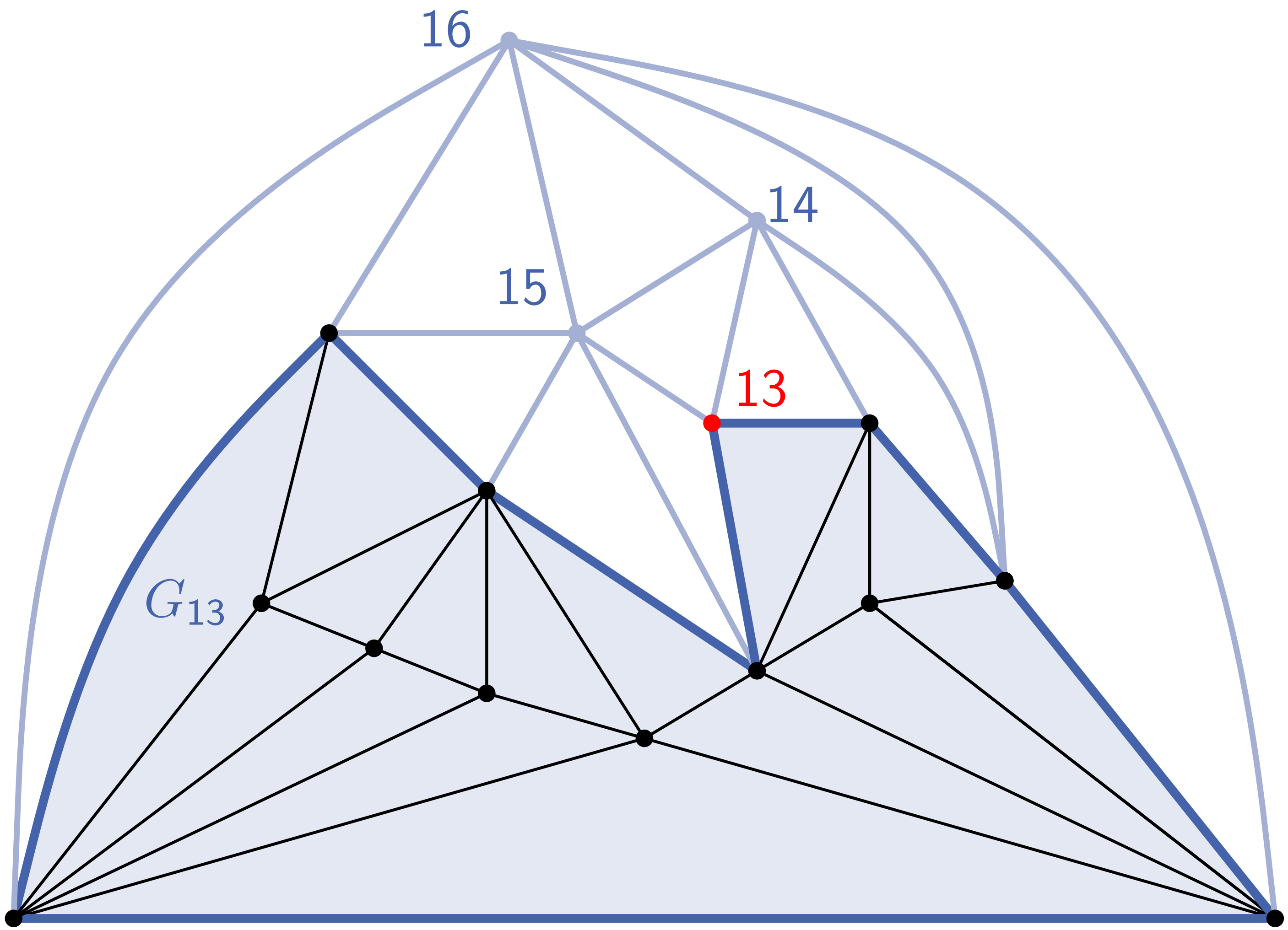

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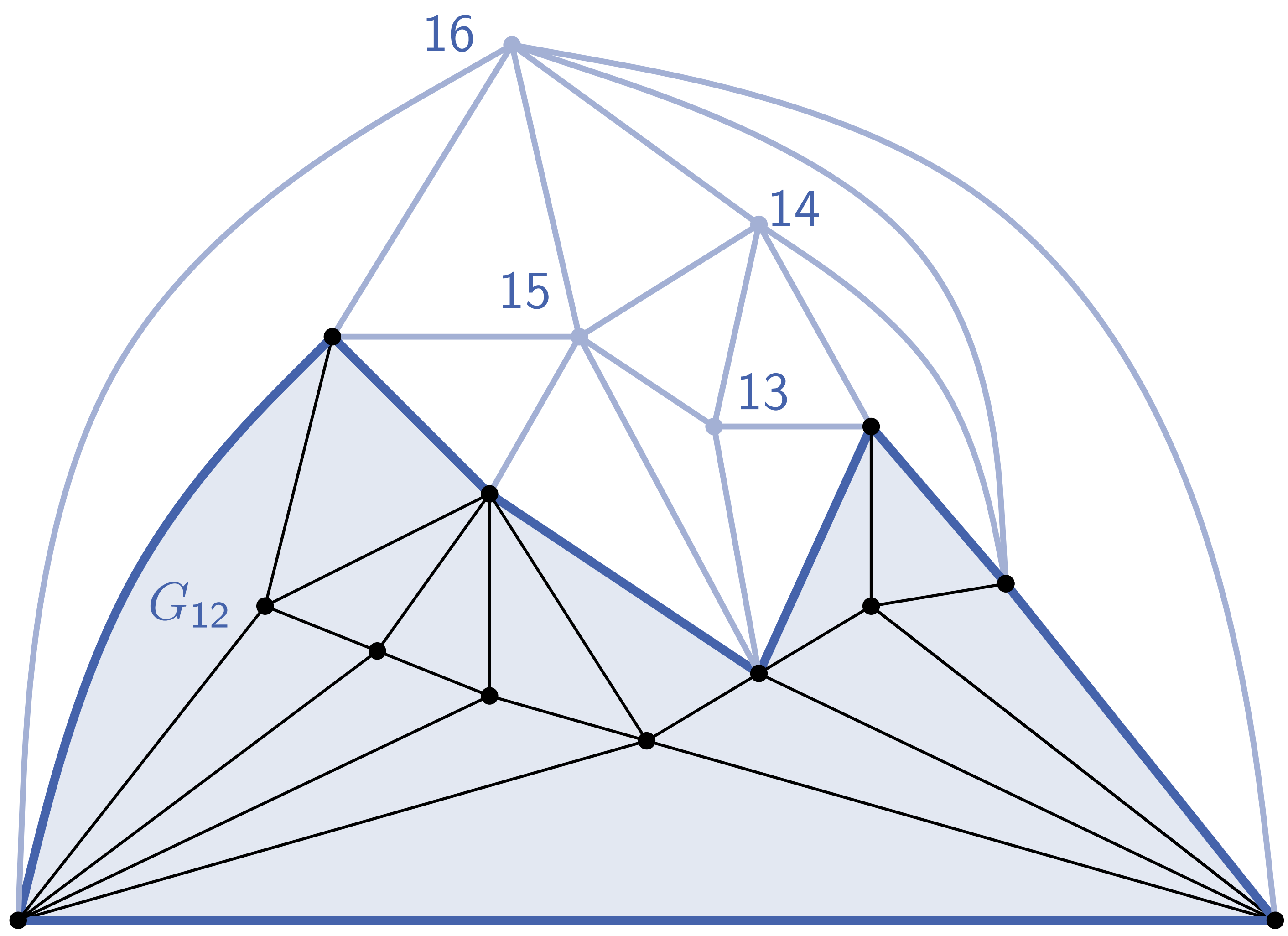

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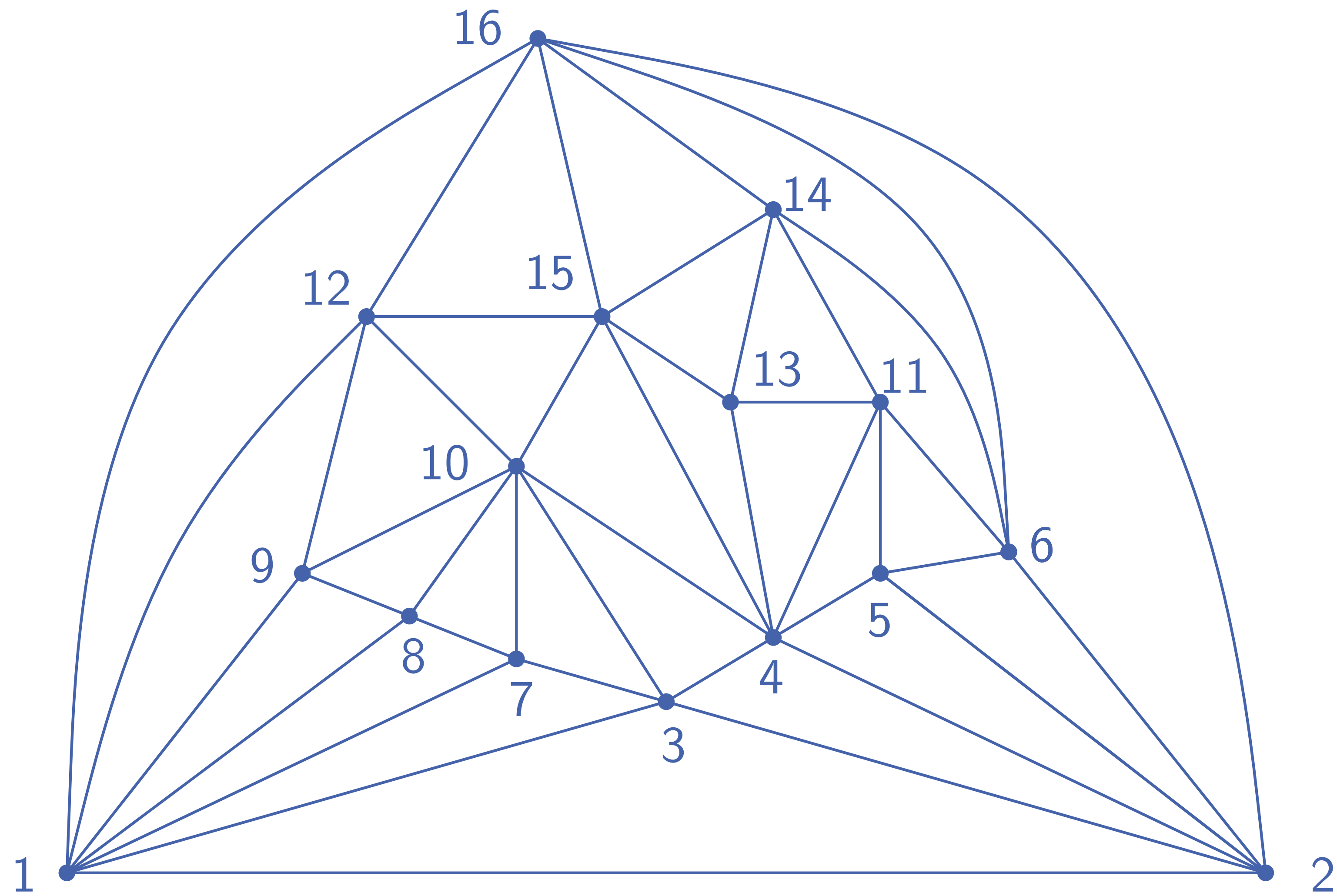

Canonical Ordering Existence

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Every triangulated plane graph has a canonical ordering.

- Let $G_n = G$, and let v_1, v_2, v_n be the vertices of the outer face of G_n. Conditions C1-C3 hold.

Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

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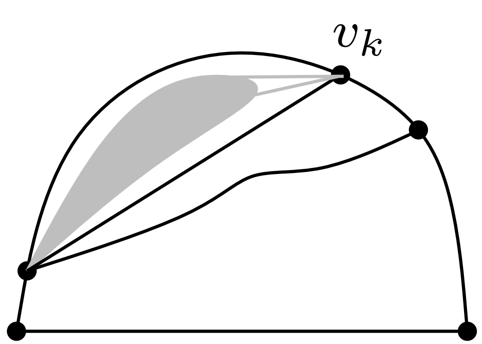

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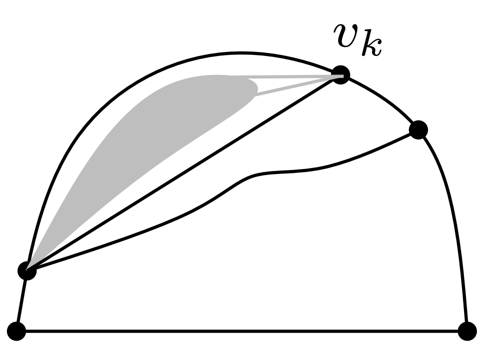

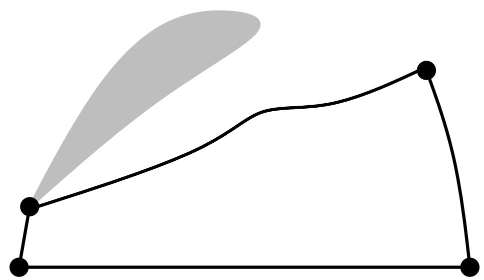

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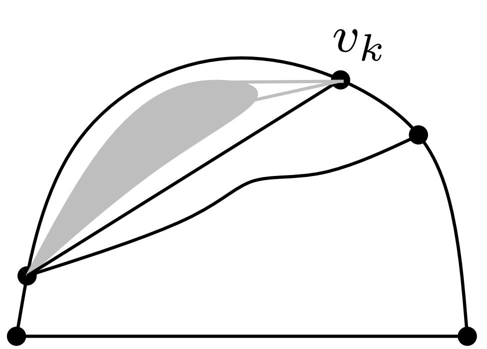

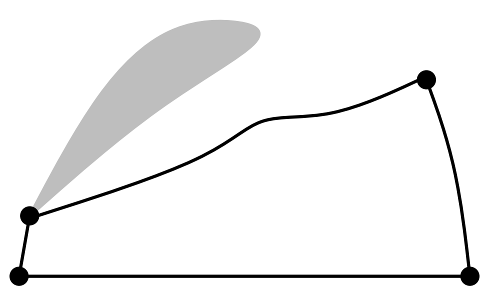

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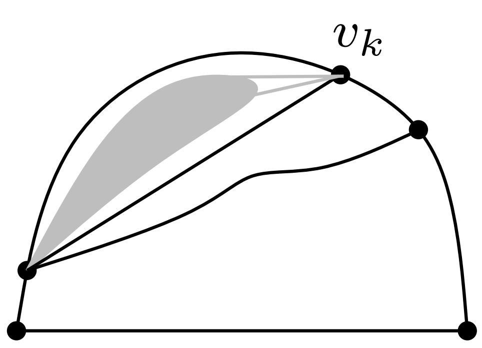

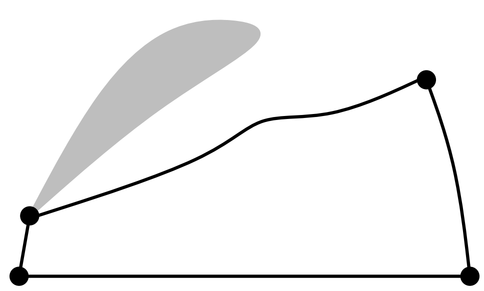

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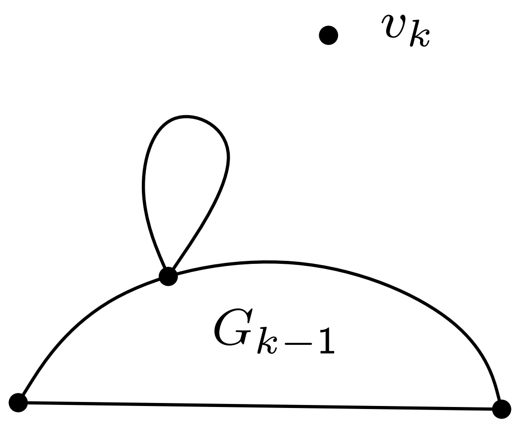
Is it sufficient?

Canonical Ordering Existence

Statement If v_k is not adjacent to a chord then removal of v_k leaves the graph biconnected.



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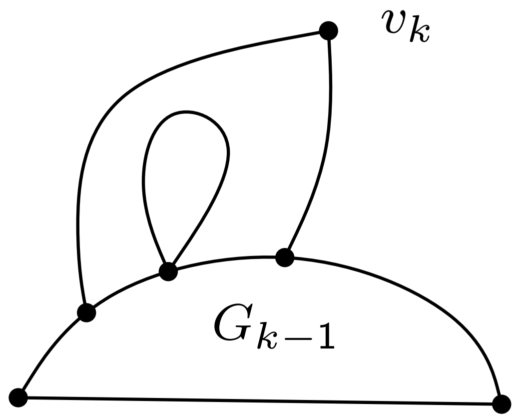

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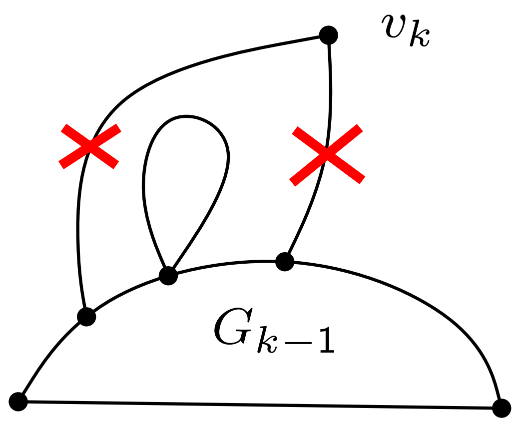

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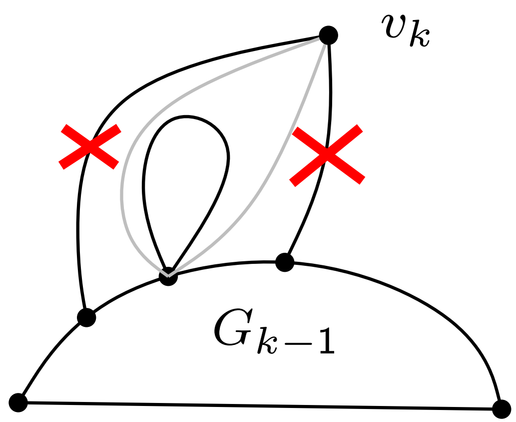

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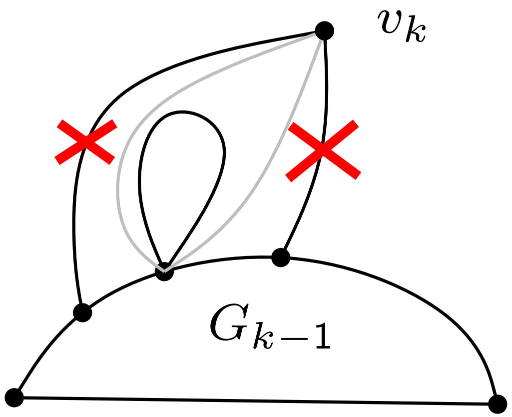

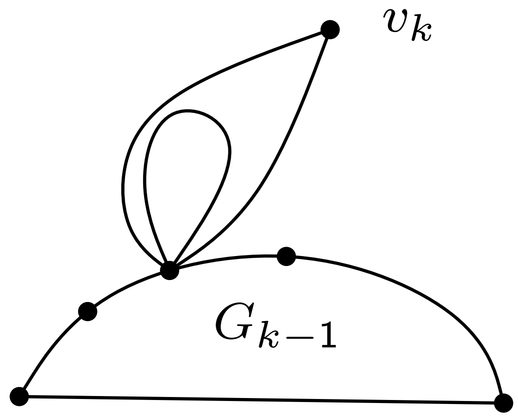

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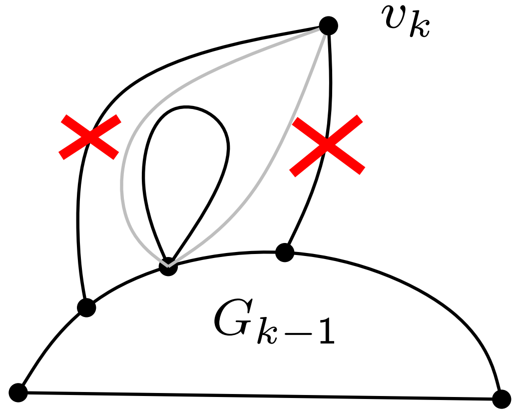

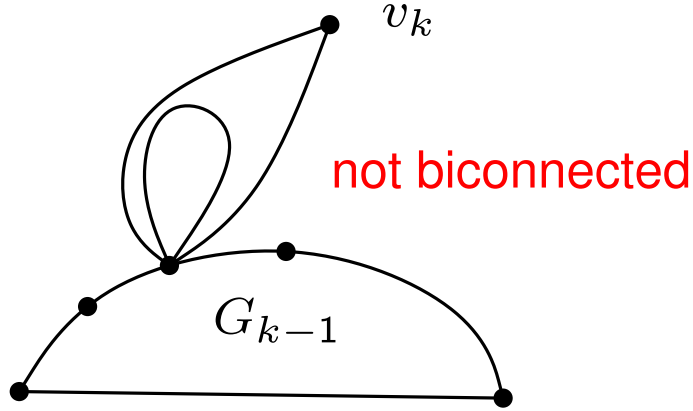

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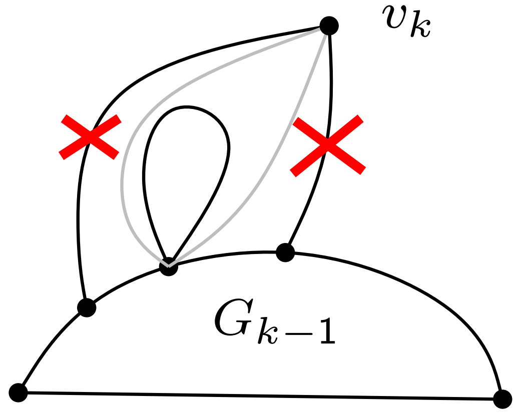

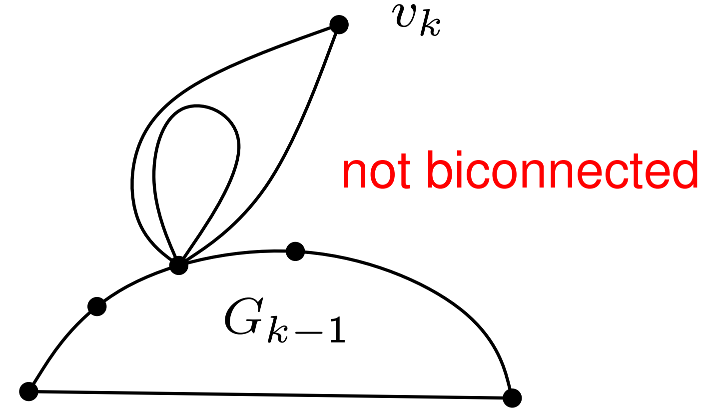

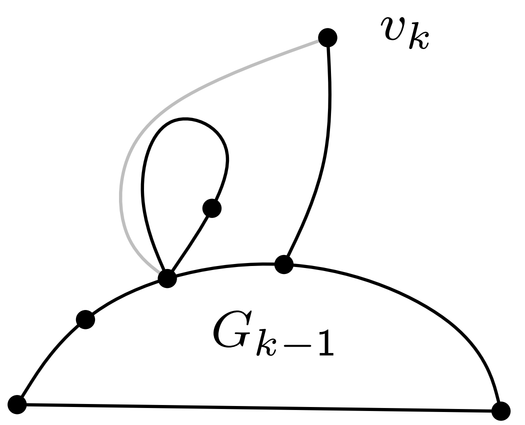

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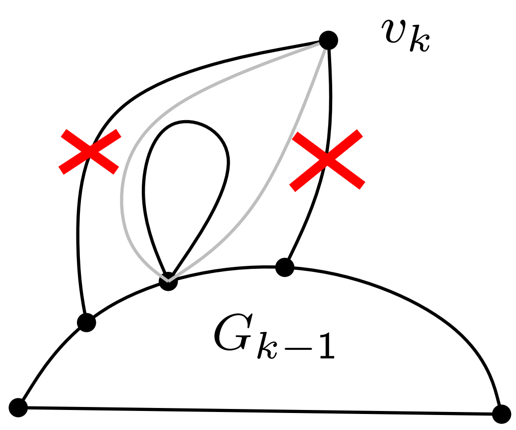

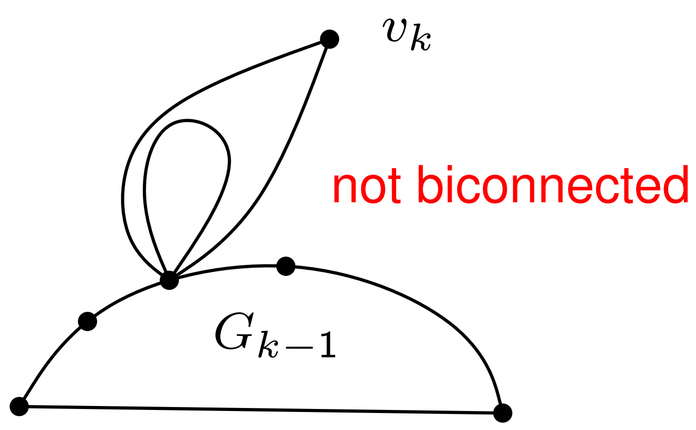

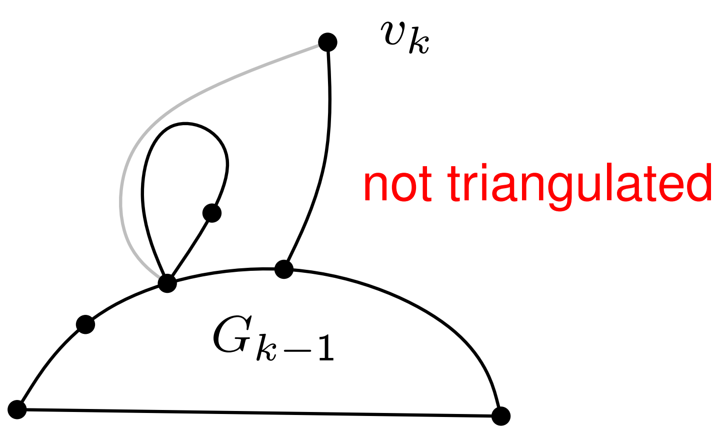

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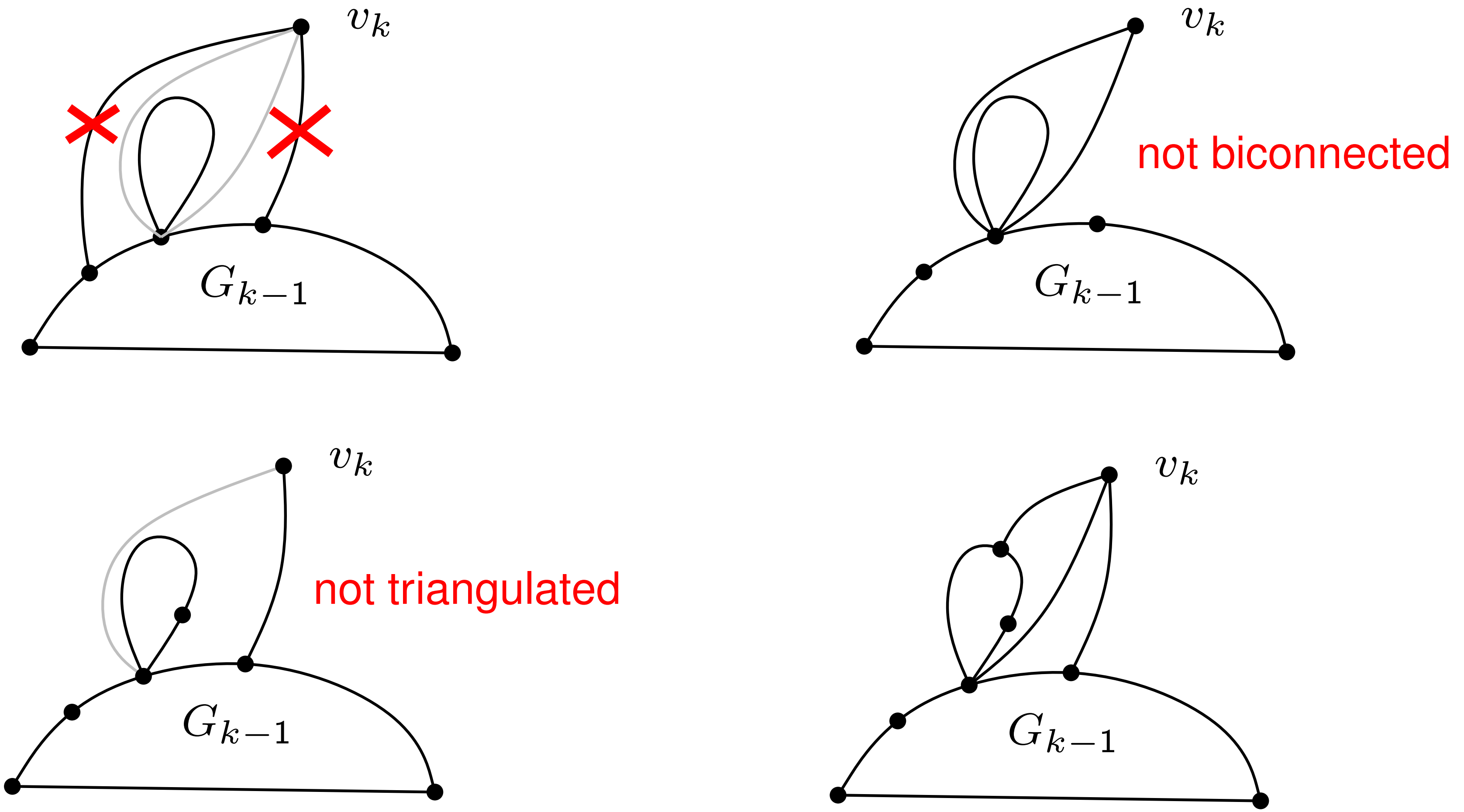

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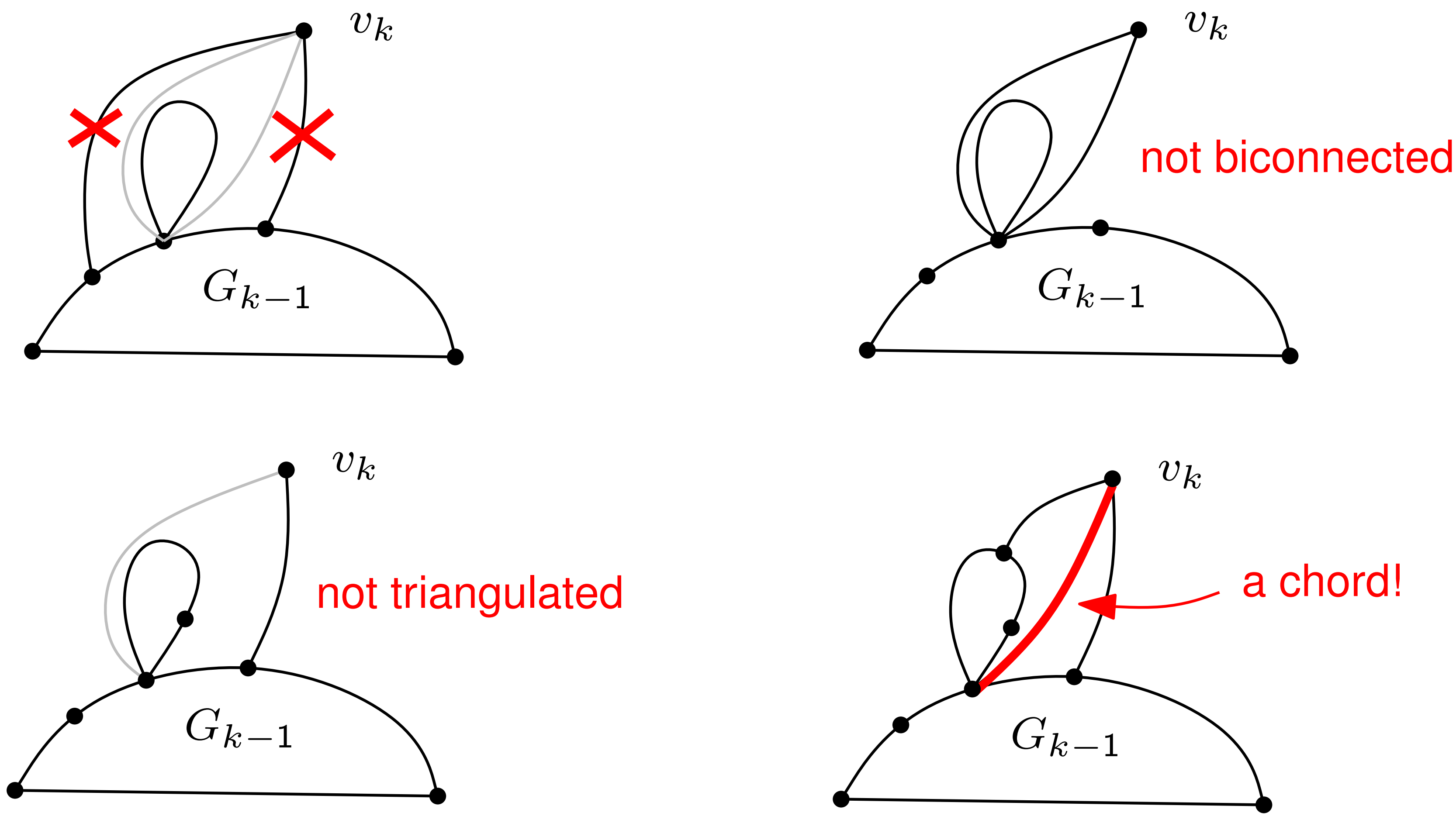
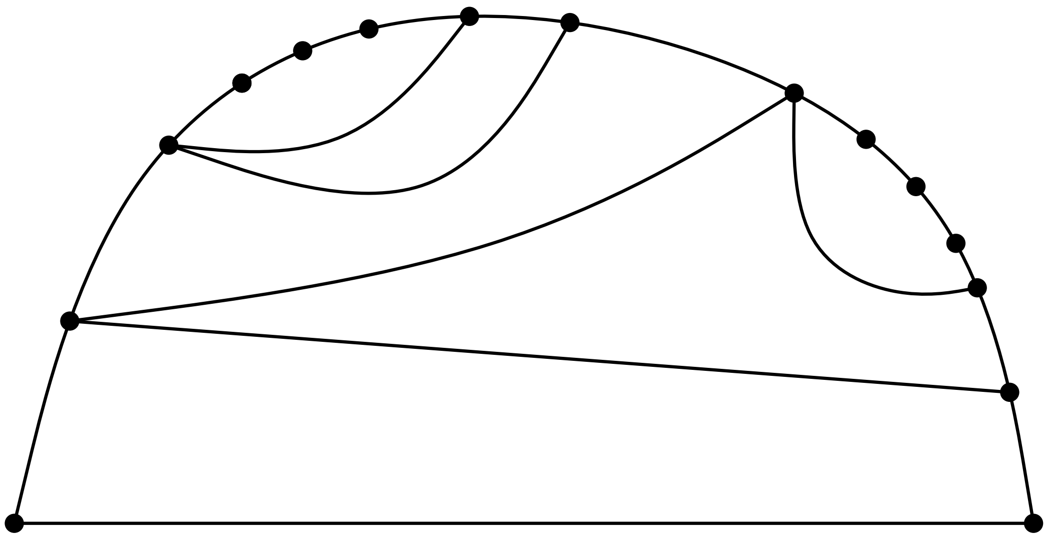


Canonical Ordering Existence

- Why a vertex not adjacent to a chord exists?



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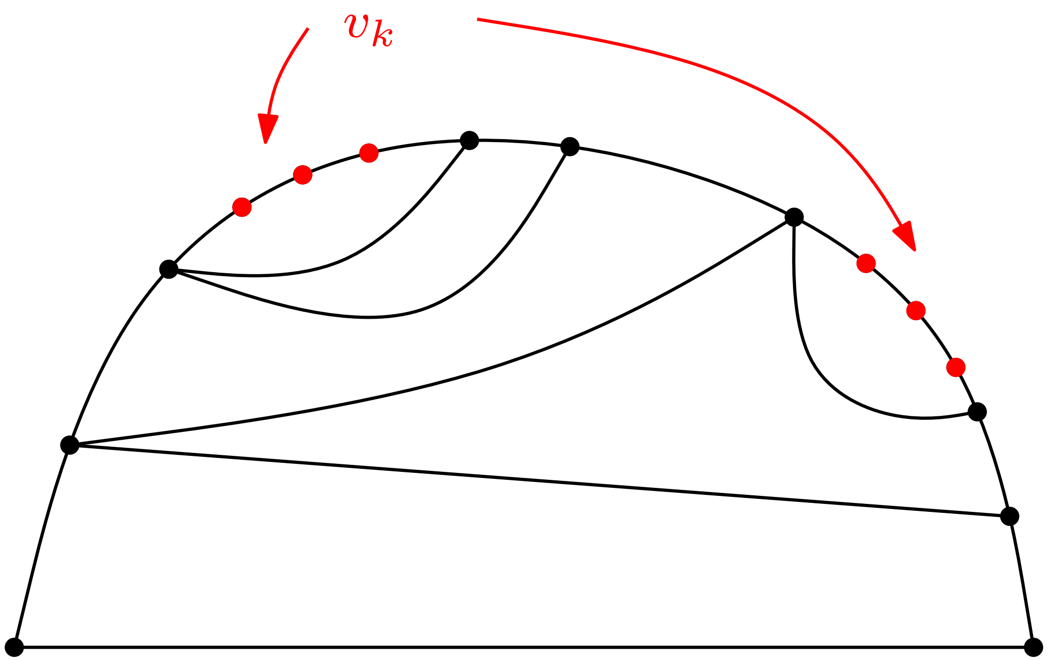

Computing Canonical Ordering

KIT Karlsruhe Institute of Technology

Algorithm CO

forall the $v \in V$ **do**
  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false;

out(v_1), out(v_2), out(v_n) $\leftarrow$ true;
for $k = n$ **to** 3 **do**
  choose $v \neq v_1, v_2$ such that mark(v) = false, out(v) = true, chords(v) = 0;
  $v_k \leftarrow v$; mark(v) $\leftarrow$ true;
  // Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{k-1}; and let $w_p, \ldots, w_q$ be the unmarked neighbors v_k;
  out(w_i) $\leftarrow$ true for all $p < i < q$;
  update number of chords for w_i and its neighbors;

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  update number of chords for w_i and its neighbors;

- chord(v) - number of chords adjacent to v
- mark(v) = true iff vertex v was numbered
- out(v)=true iff v is the outer vertex of current plane graph

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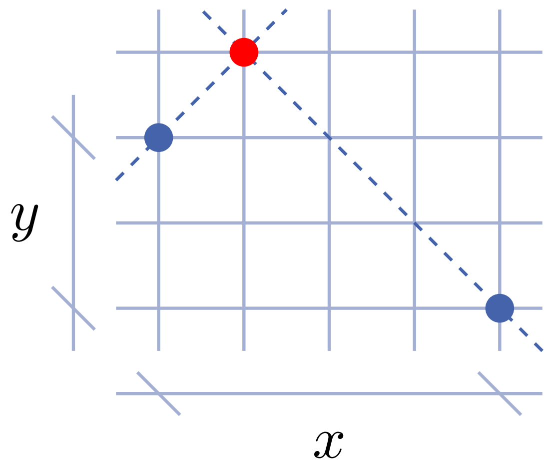
Lemma

Algorithm CO computes a canonical ordering of a graph in $O(n)$ time.

De Fraysseix Pach Pollack (Shift) Algorithm



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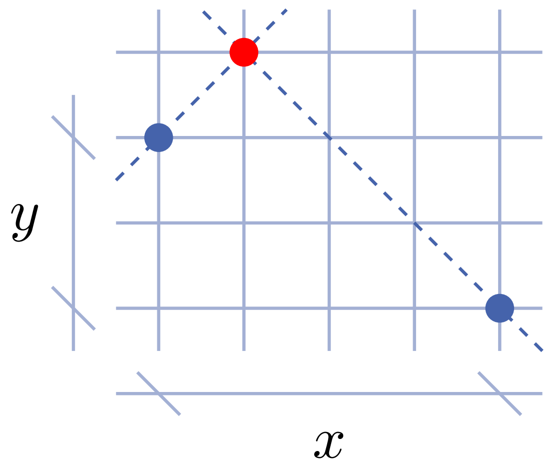
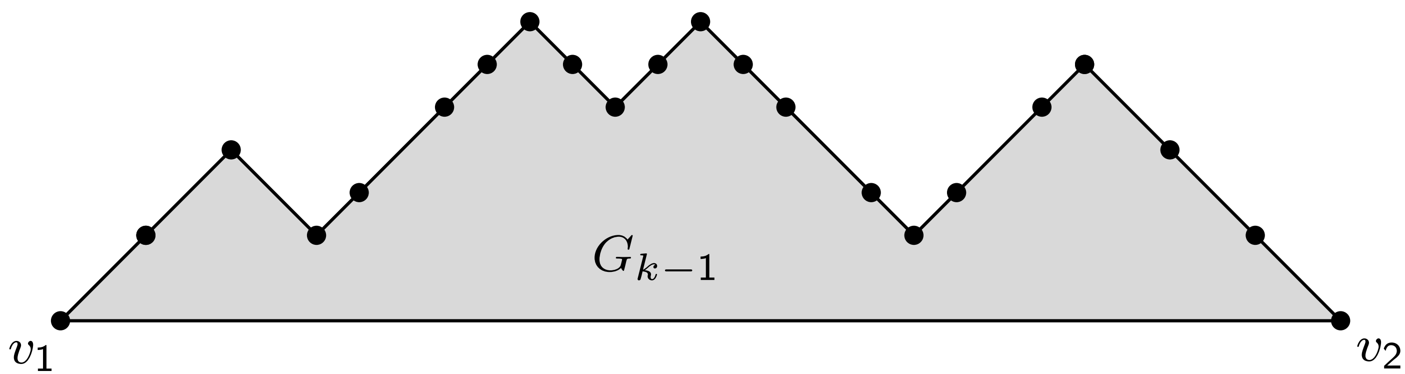


Algorithm constraints: G_{k-1} is drawn such that

- v_1 is on $(0, 0)$, v_2 is on $(2k-6, 0)$
- Boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn x-monotone
- Each edge of the boundary of G_{k-1} (minus edge (v_1, v_2)) is drawn with slopes ± 1



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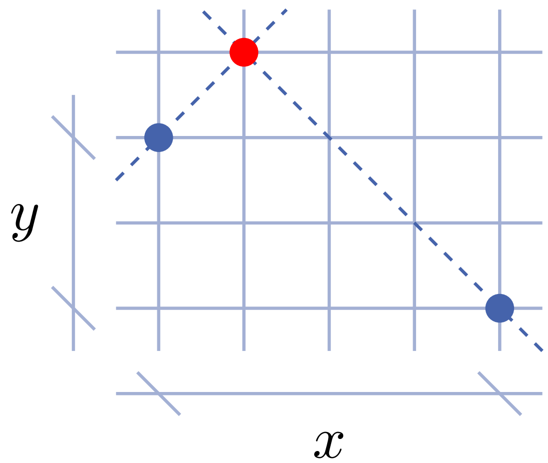

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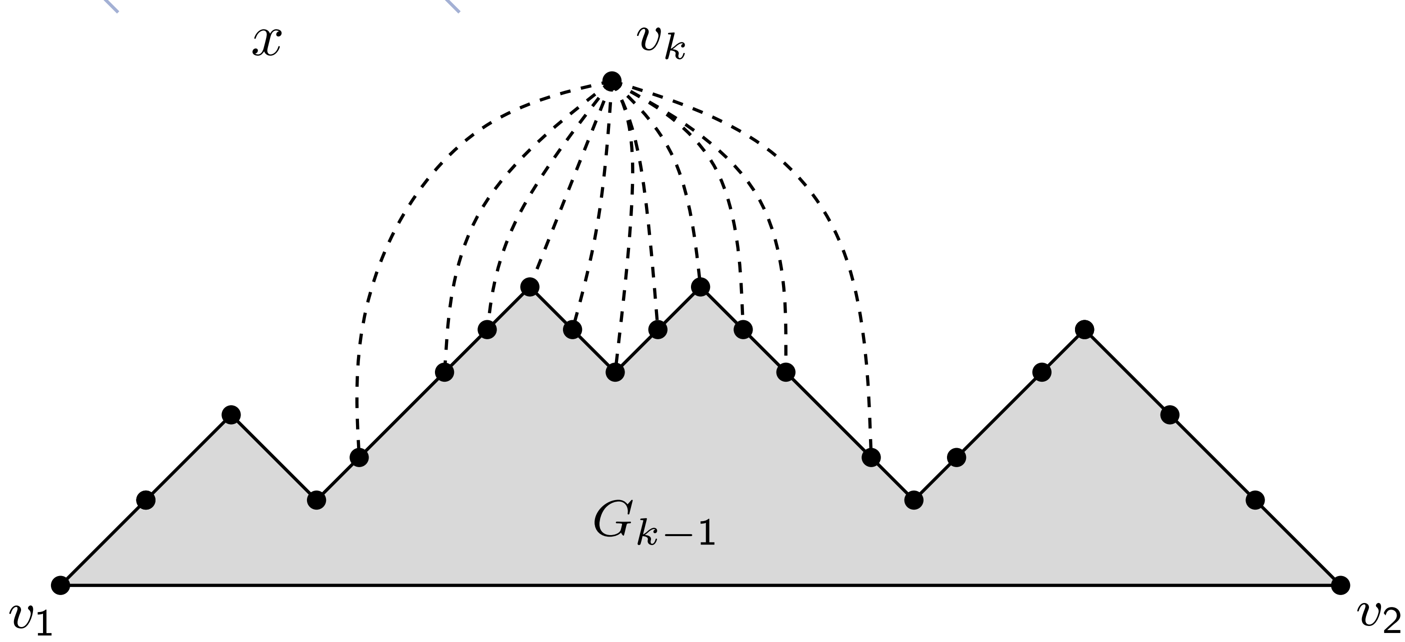
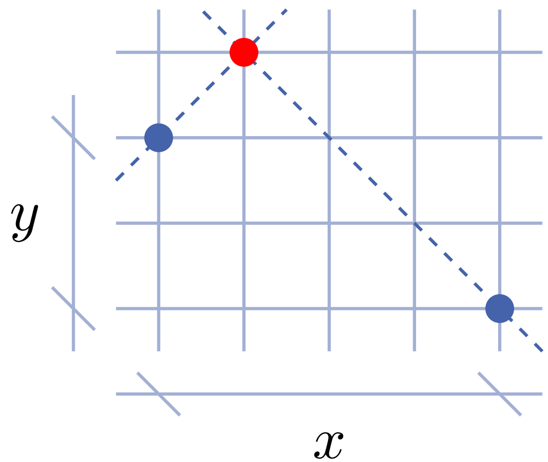


De Fraysseix Pach Pollack (Shift) Algorithm



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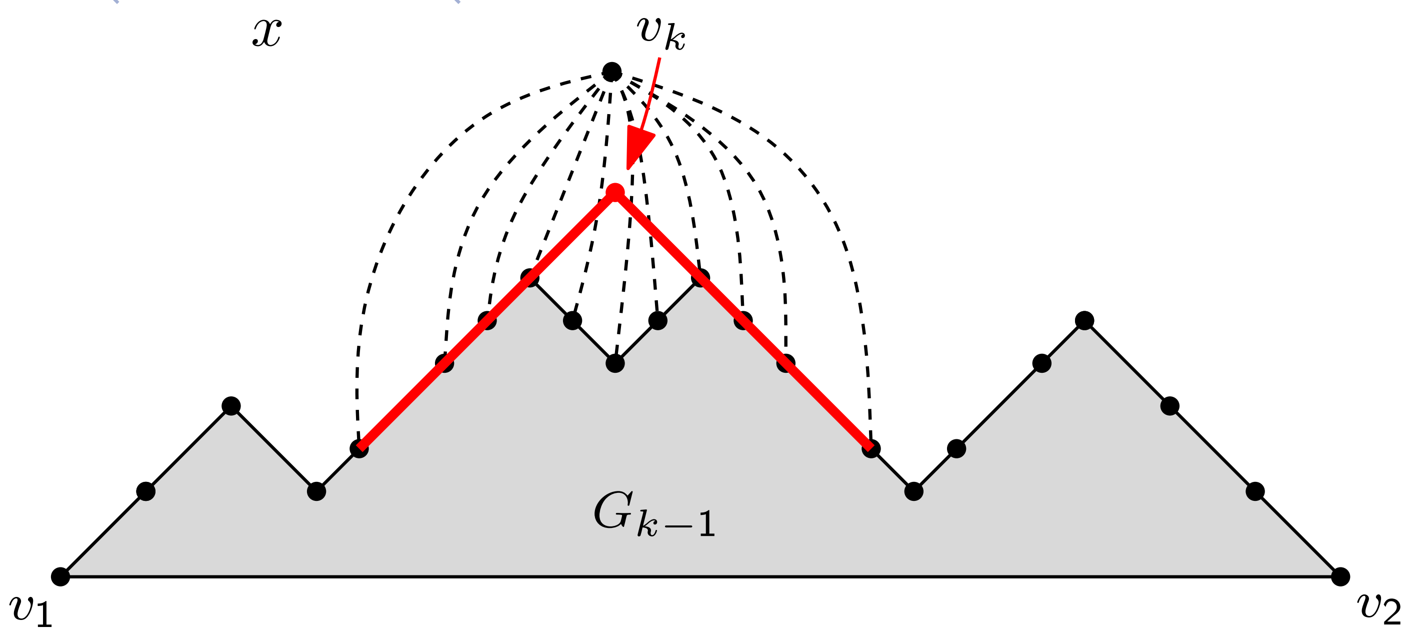

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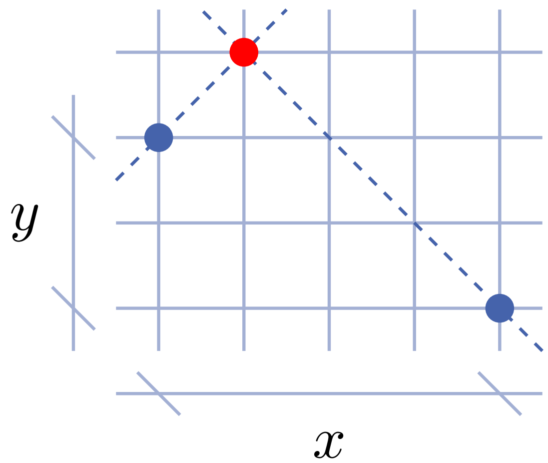

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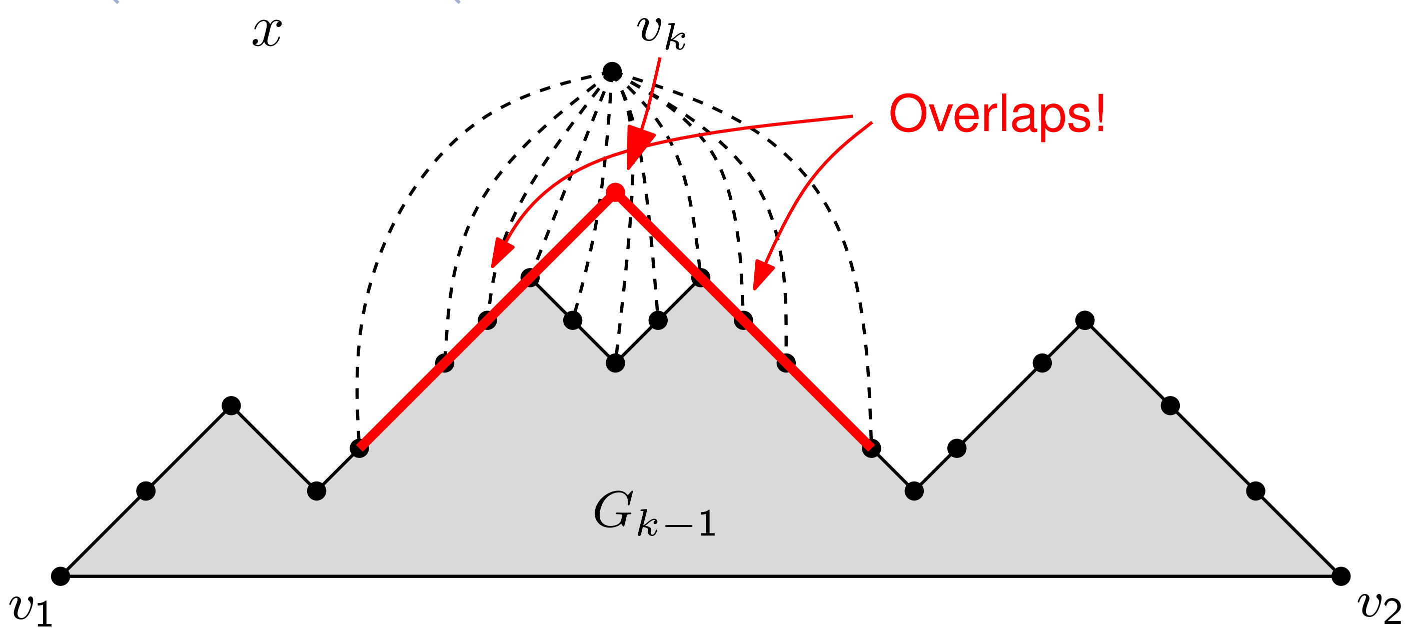

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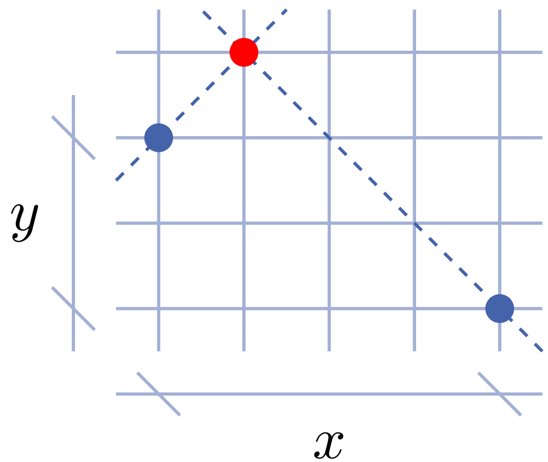

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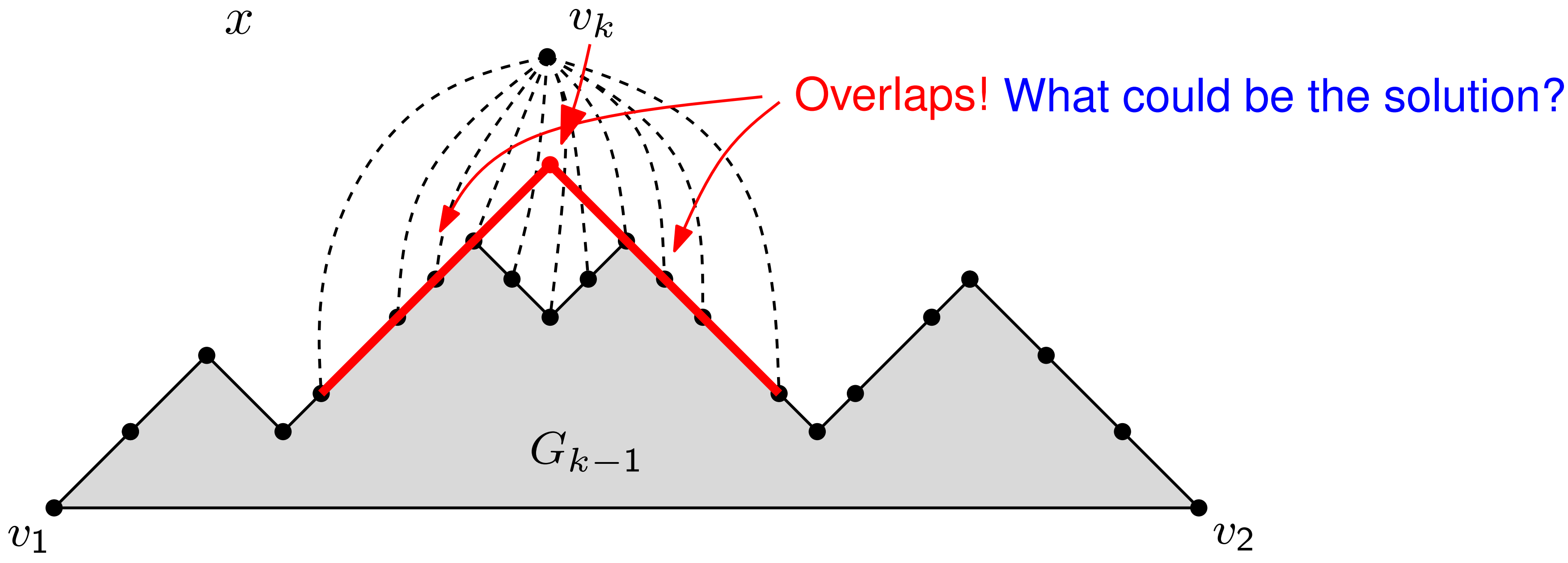

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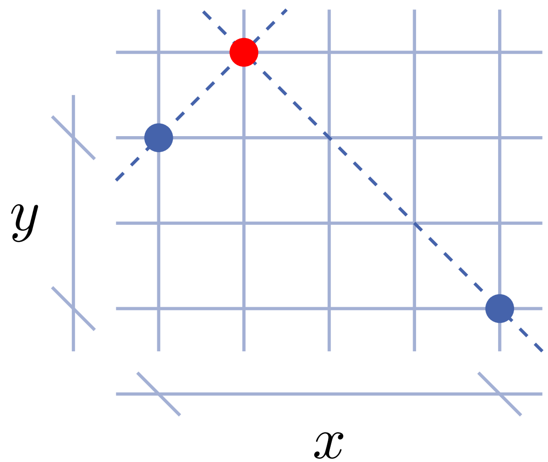

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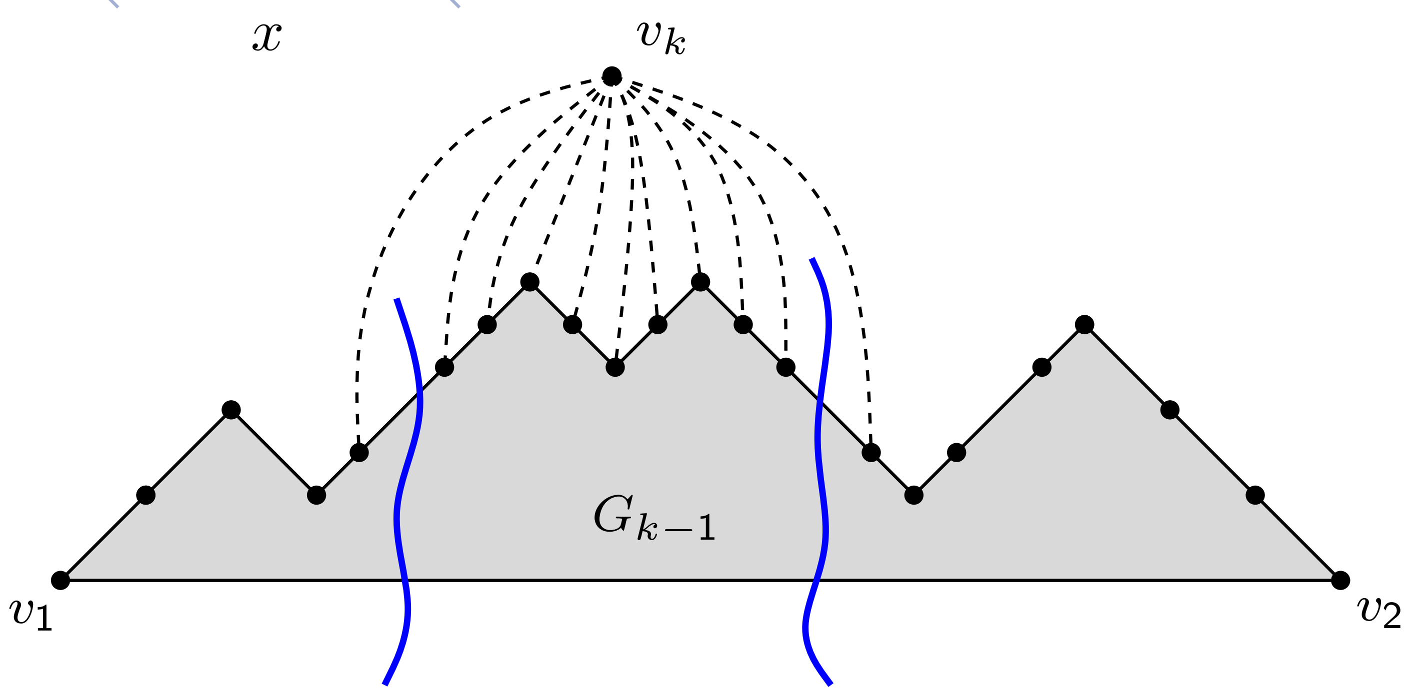

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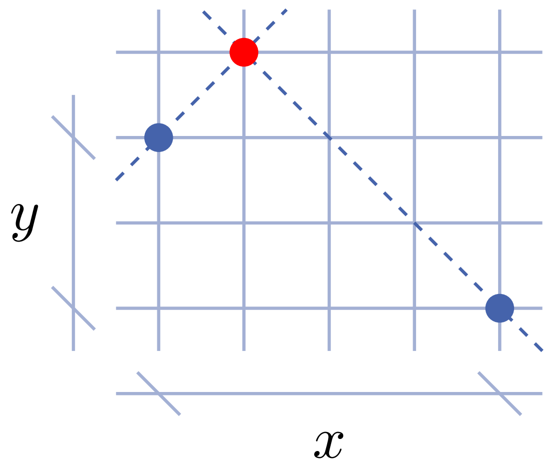

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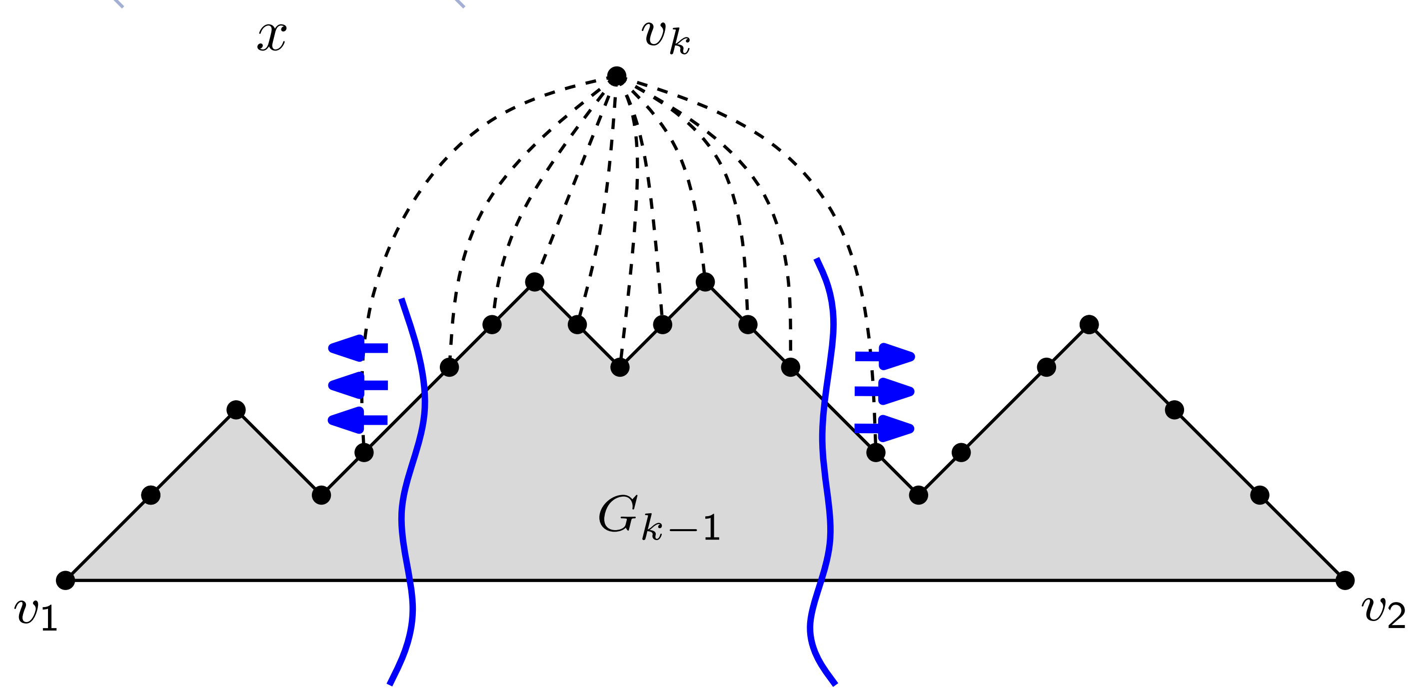

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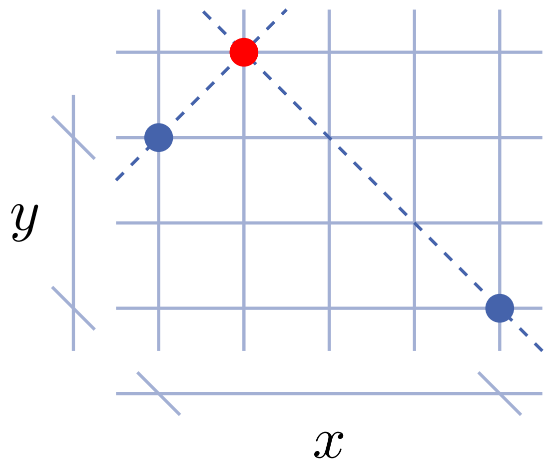

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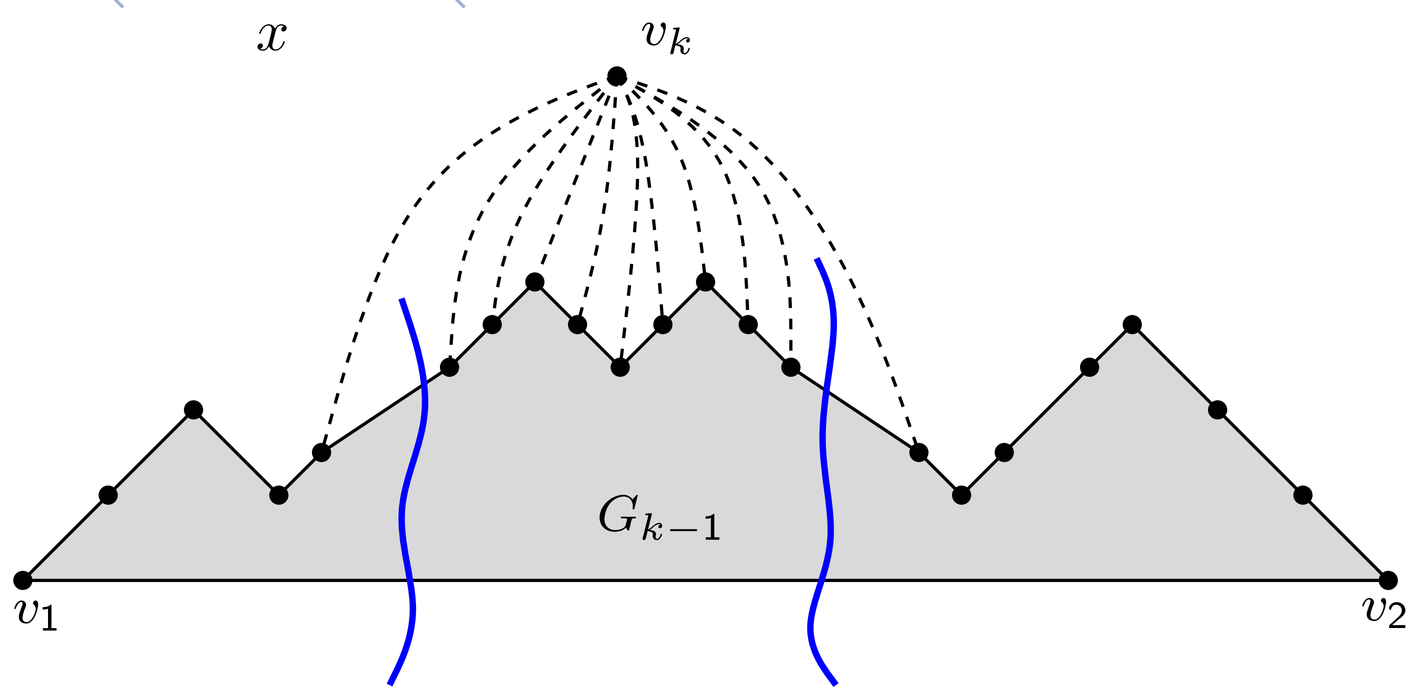

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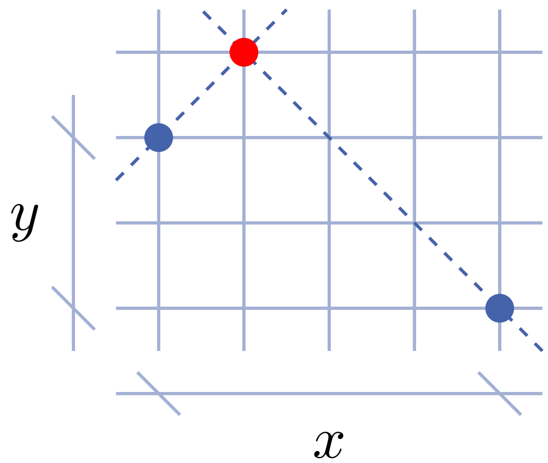

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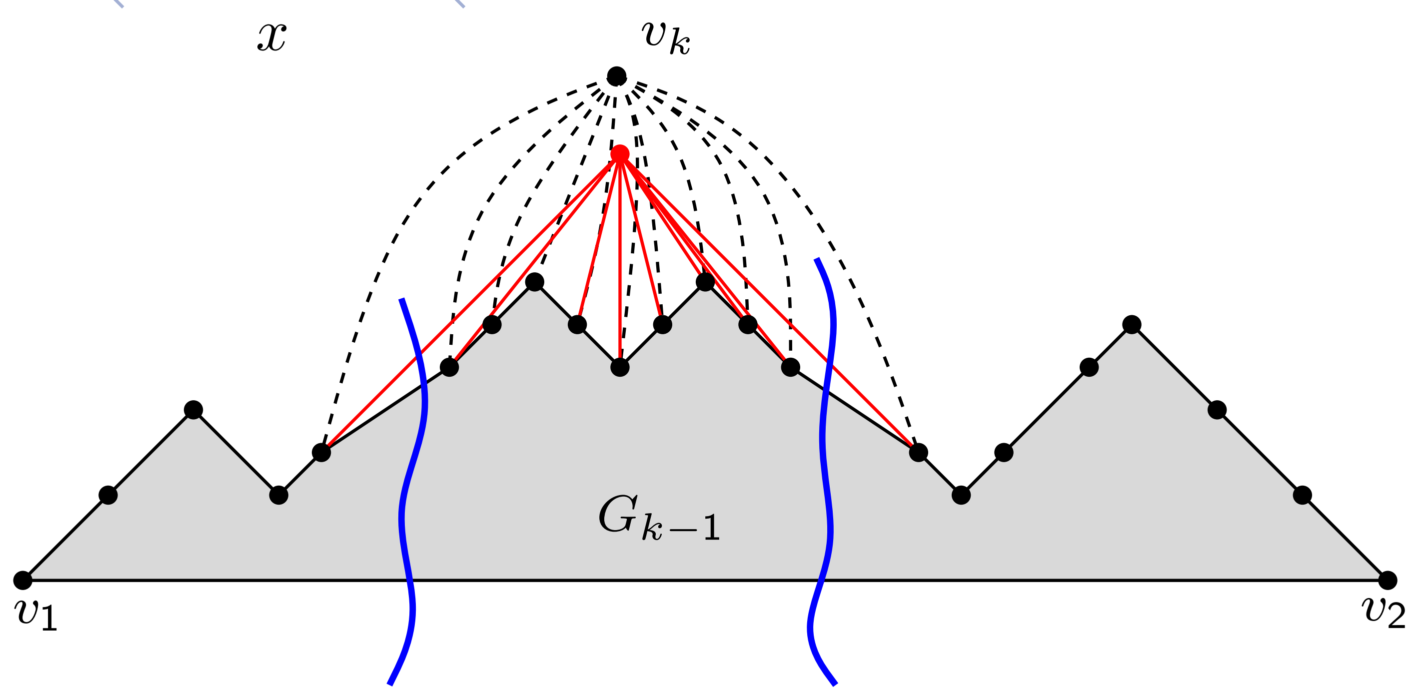

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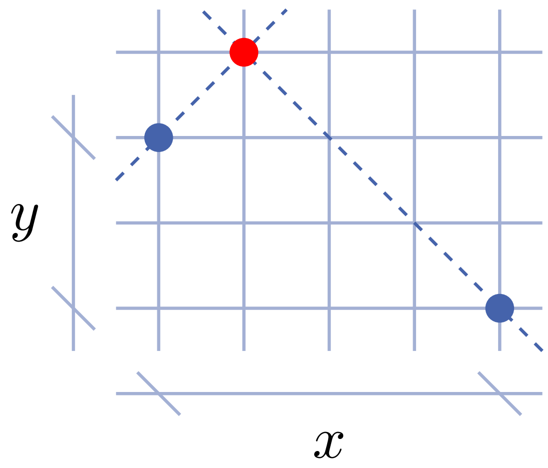

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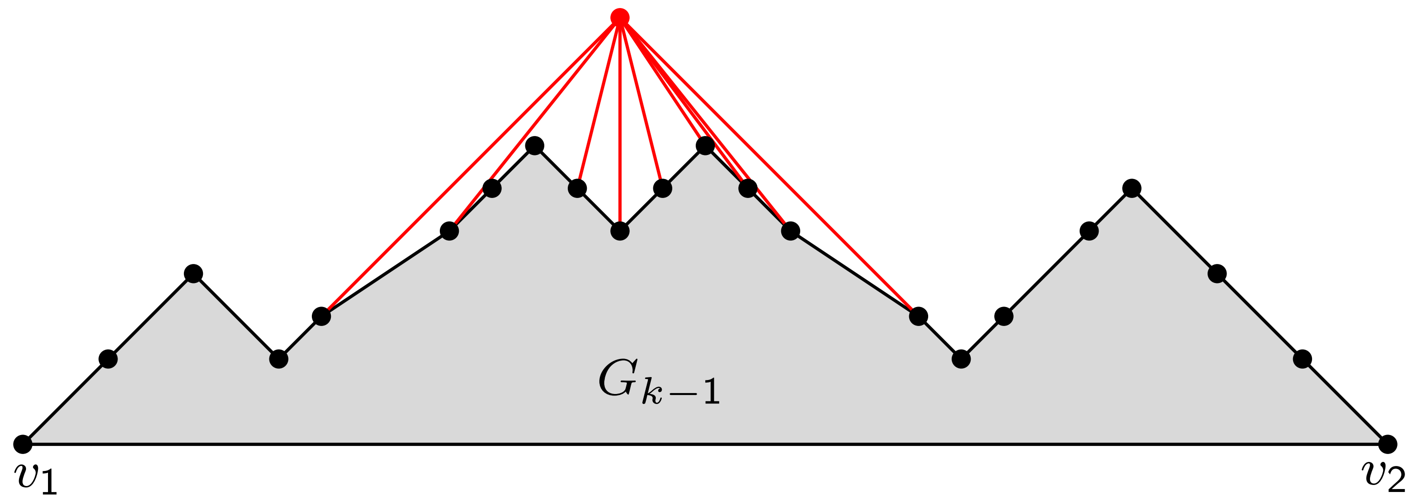

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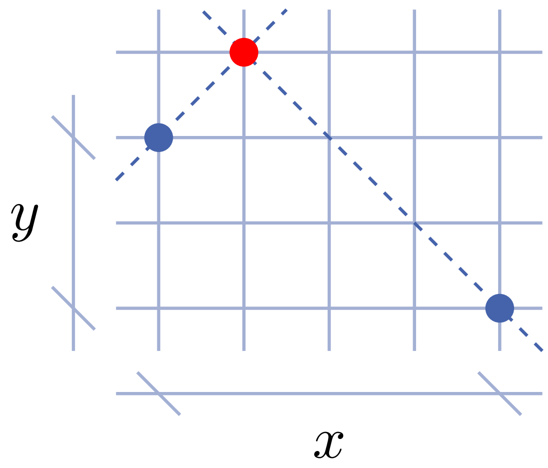

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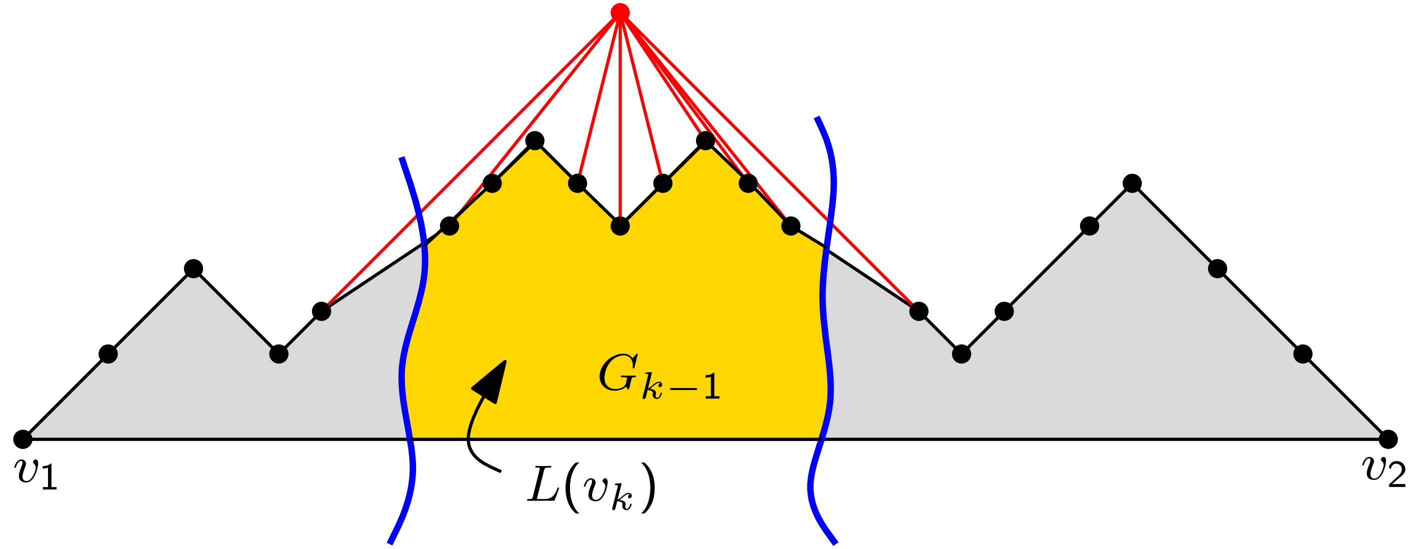

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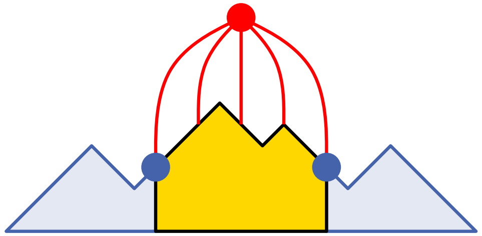

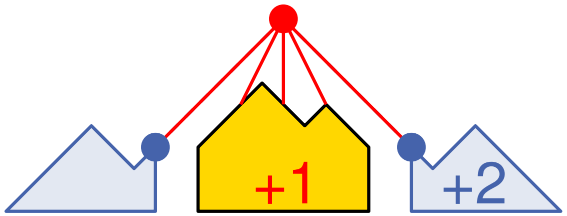

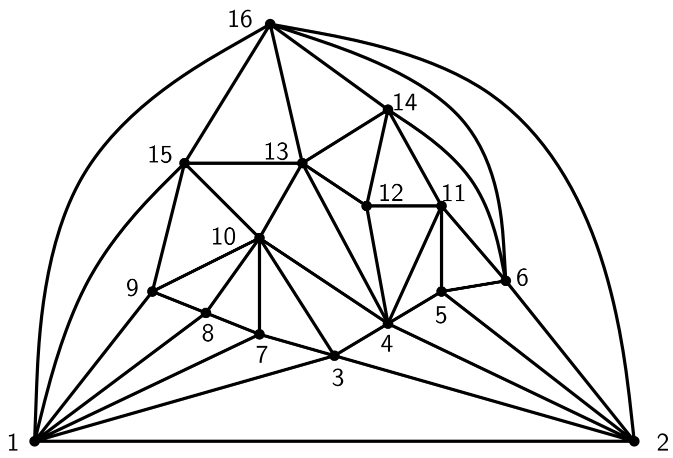

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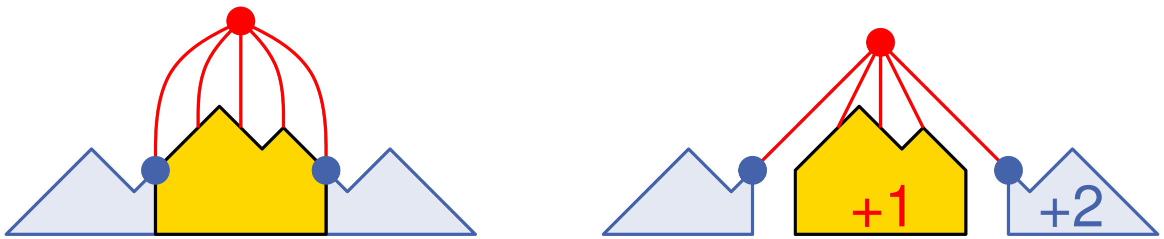

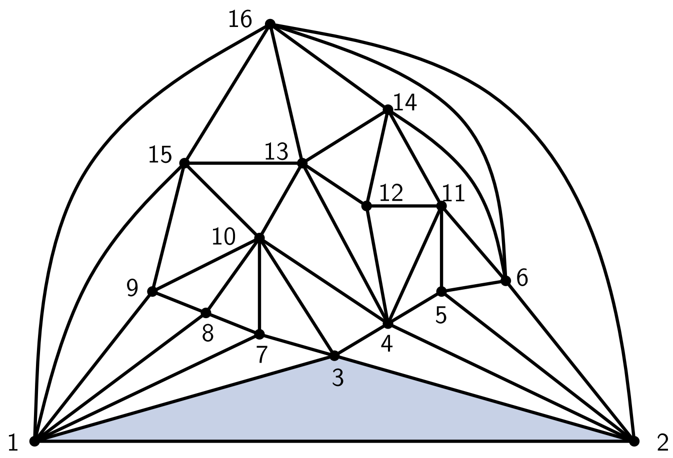

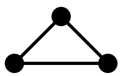

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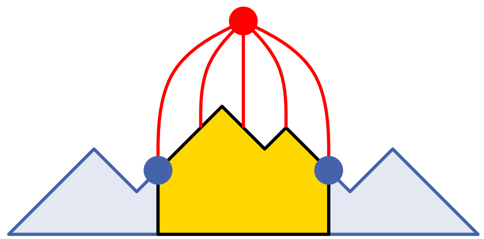

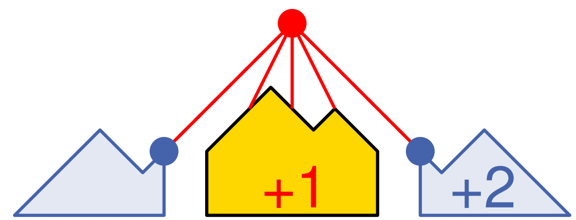

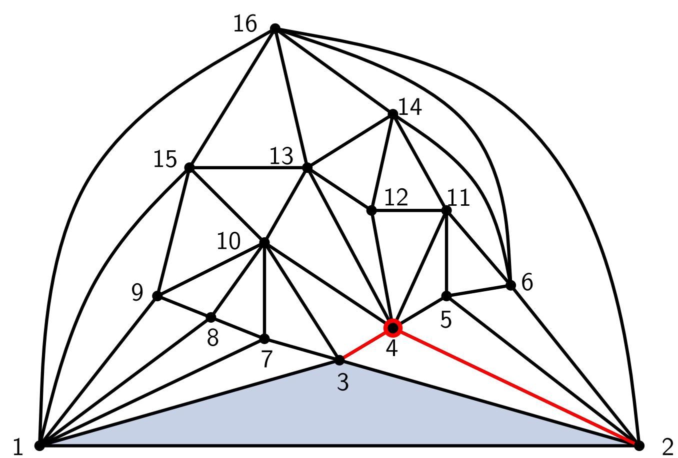

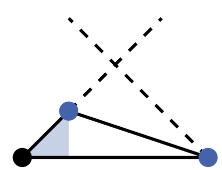

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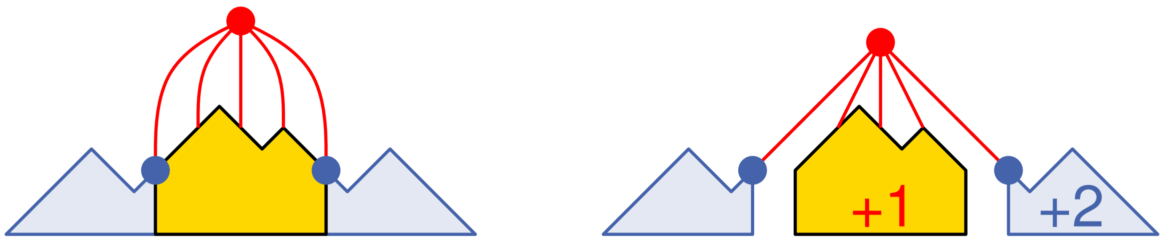

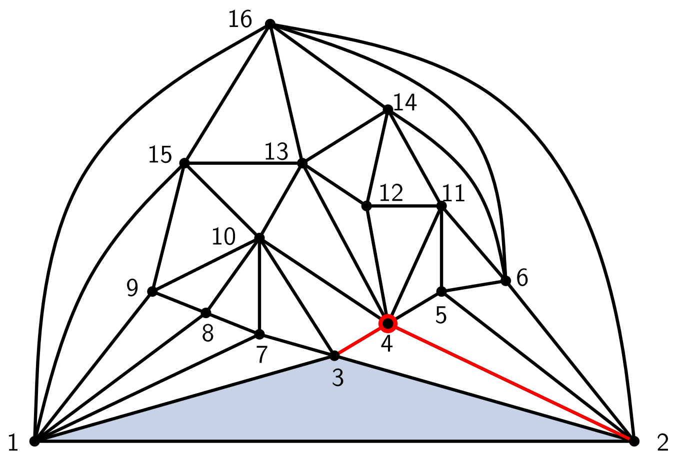

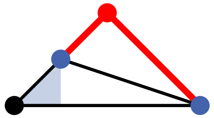

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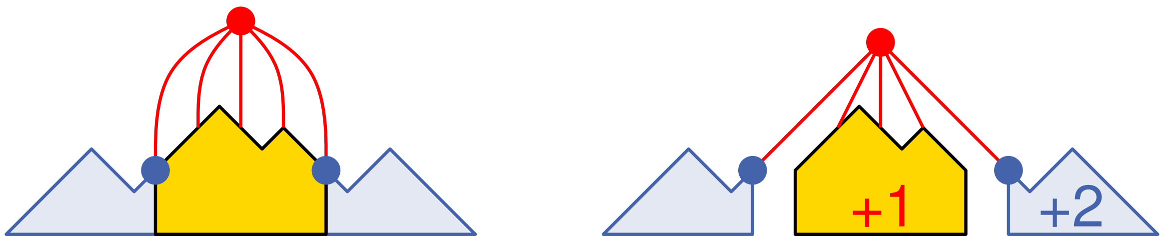

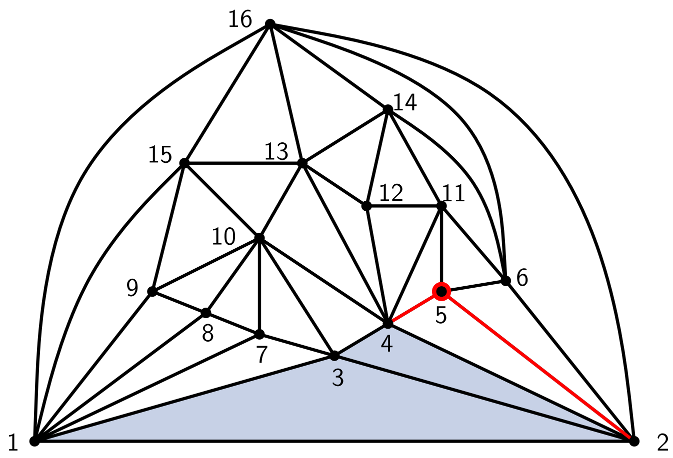

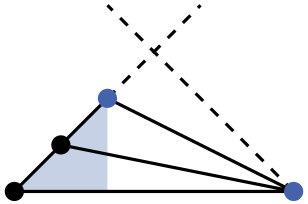

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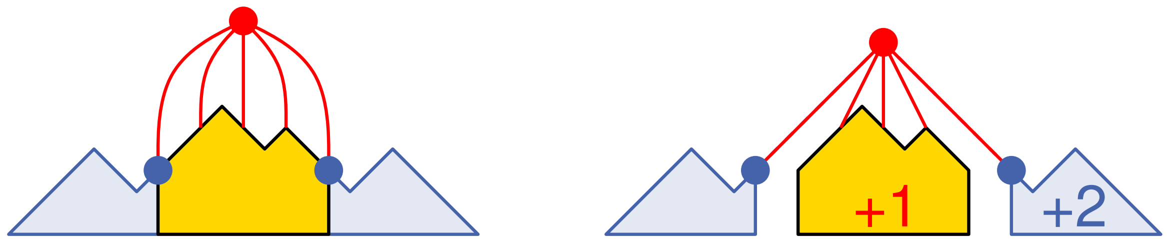

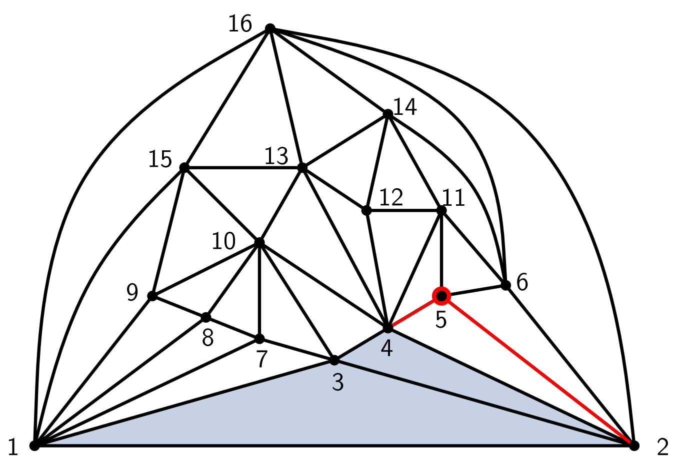

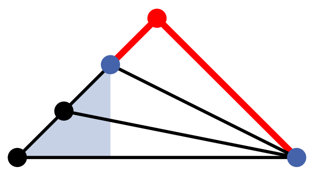

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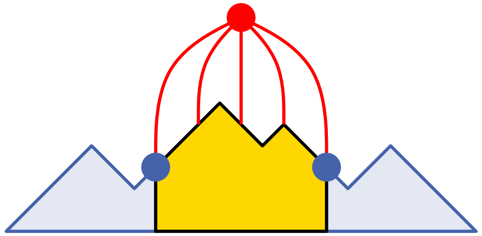

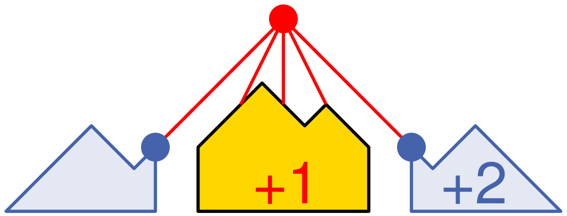

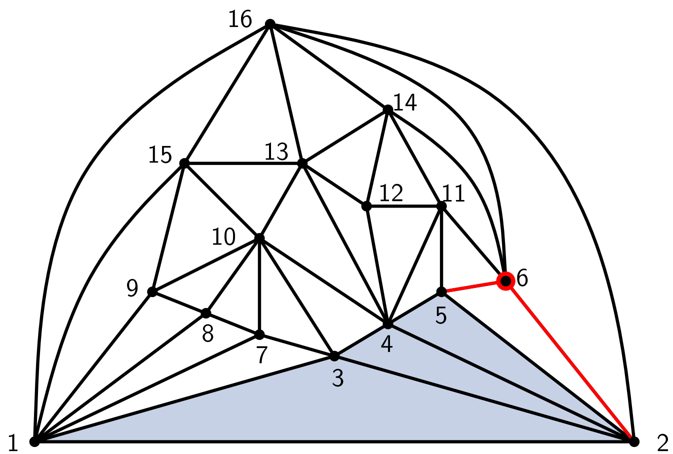

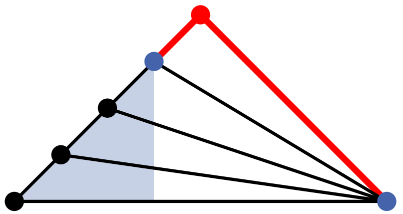

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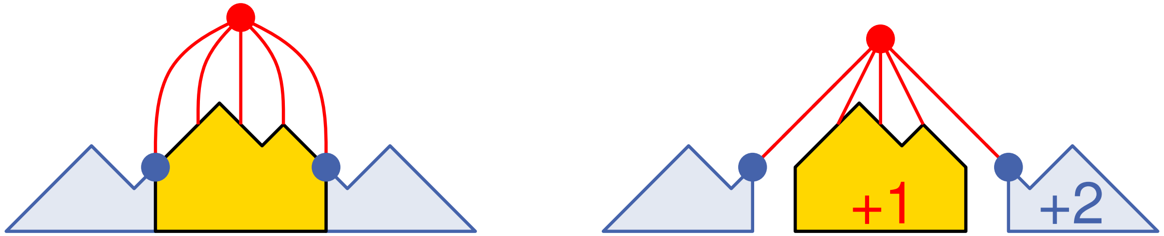

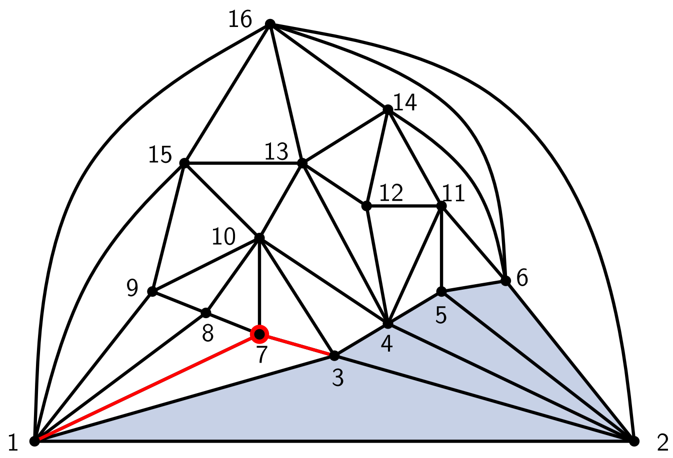

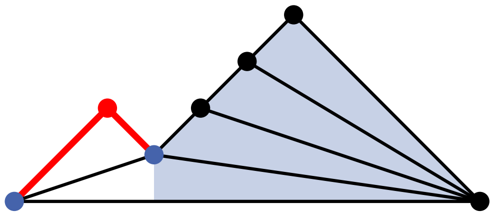

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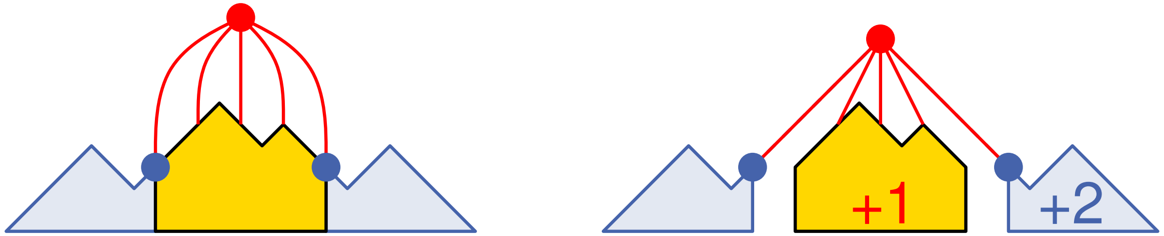

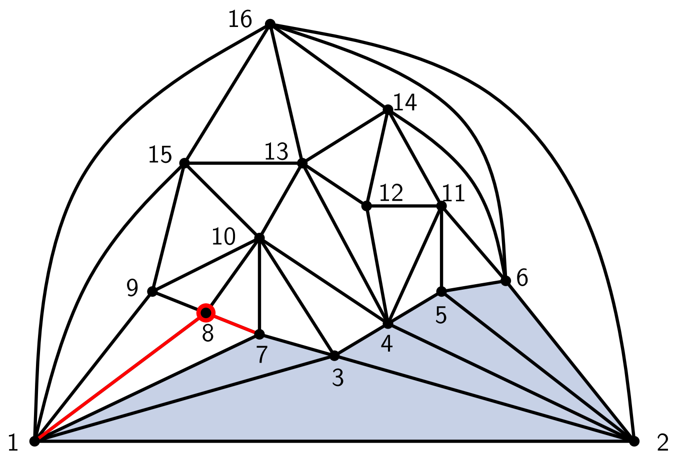

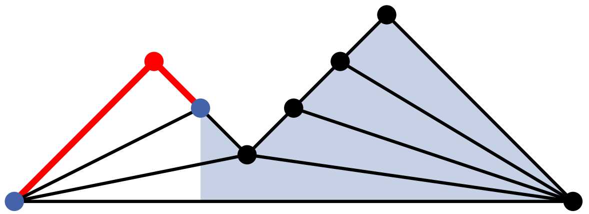

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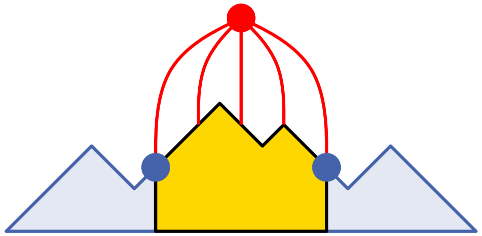

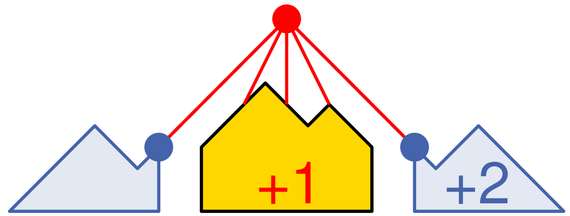

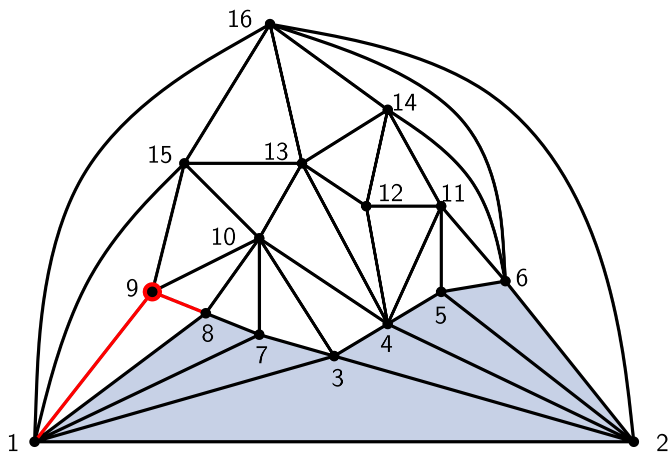

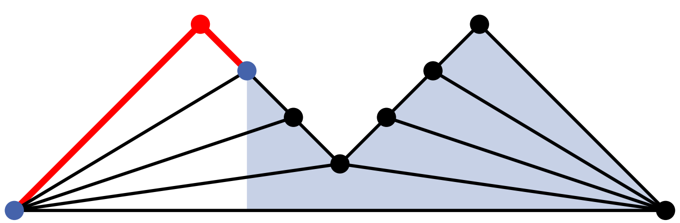

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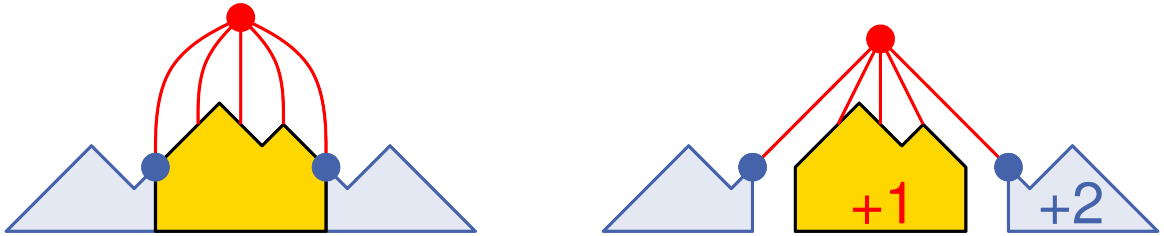

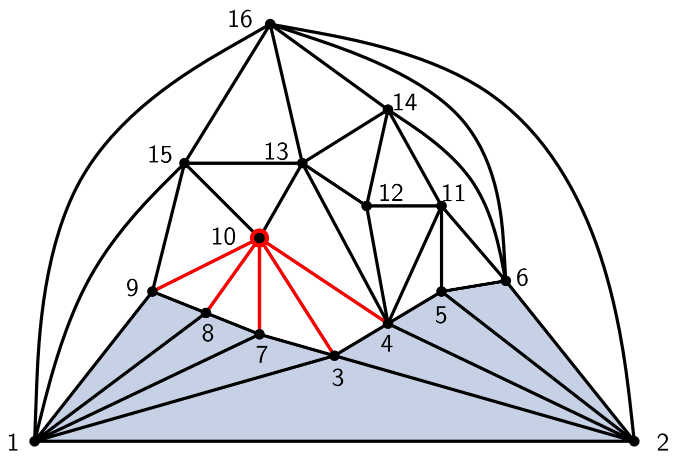

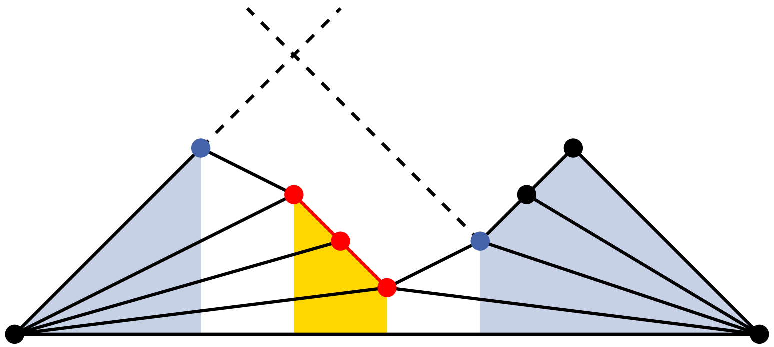

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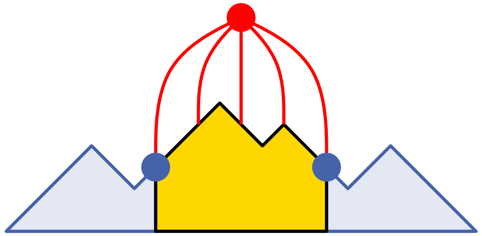

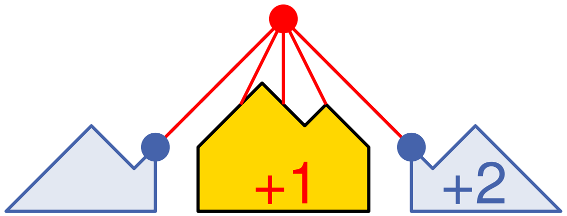

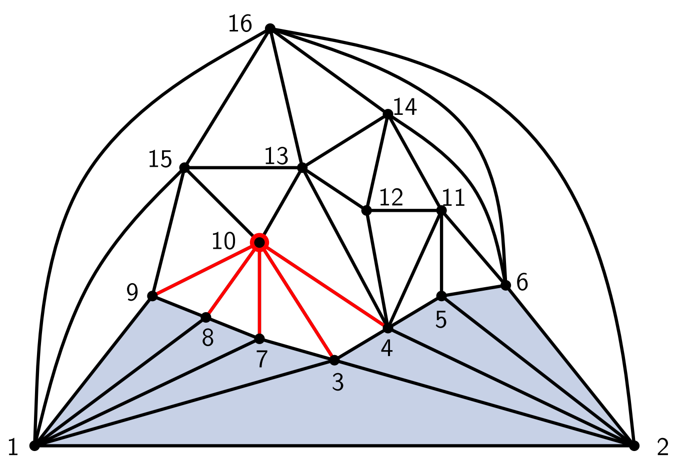

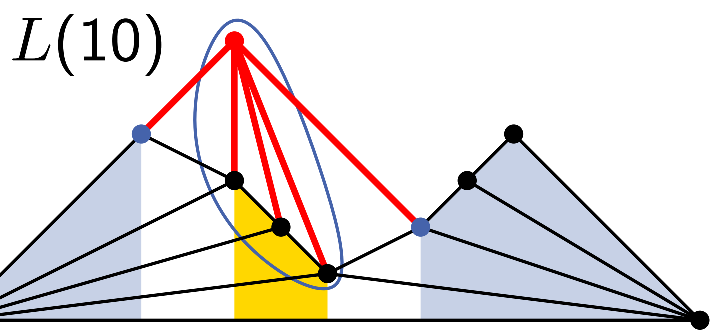

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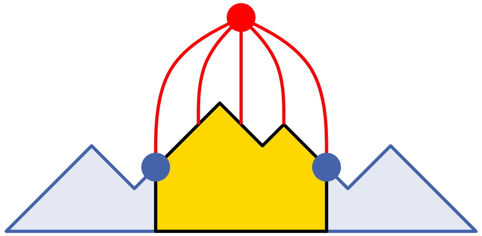

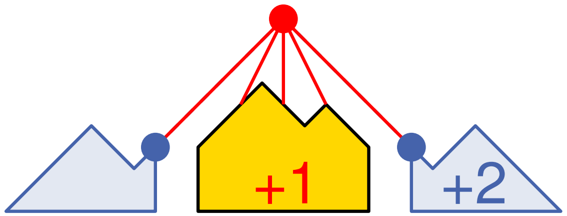

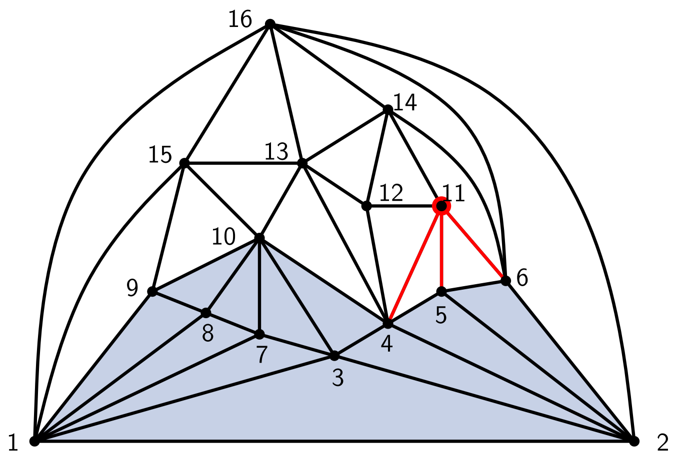

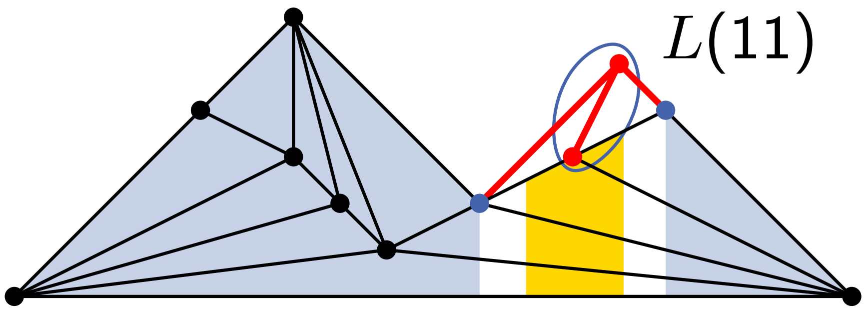

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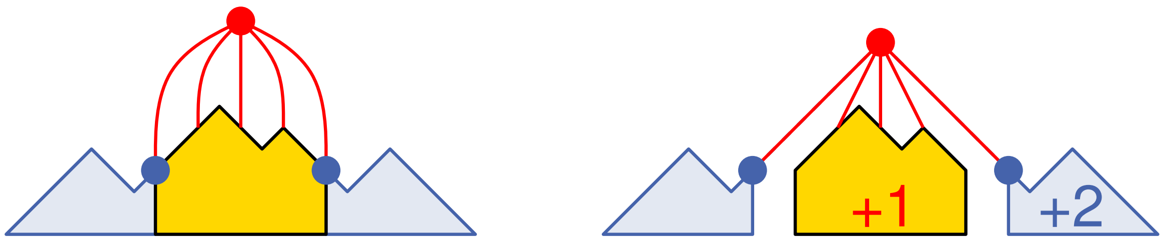
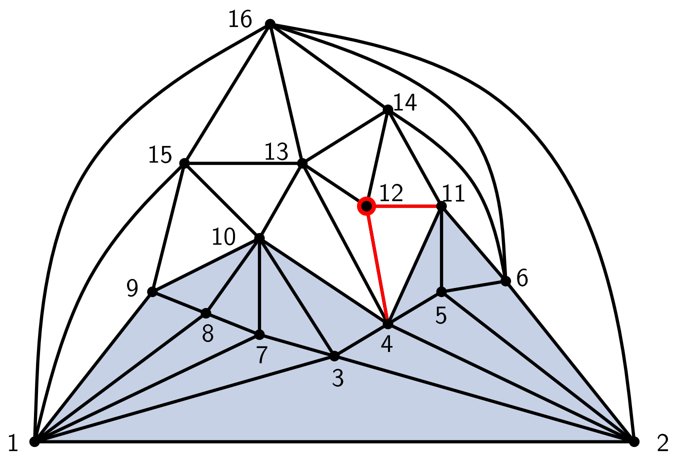
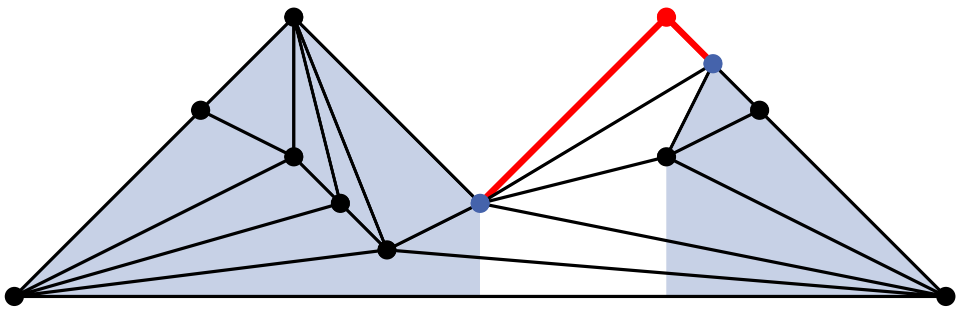
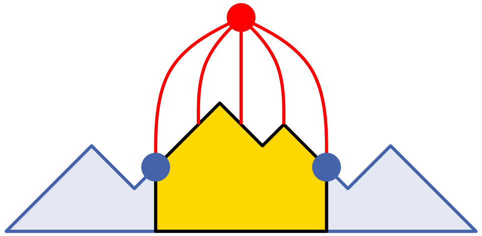
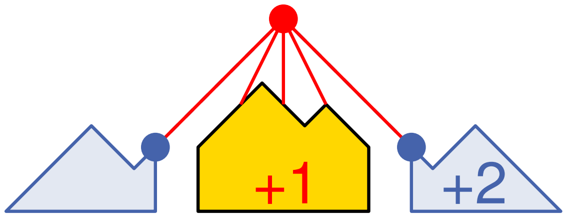


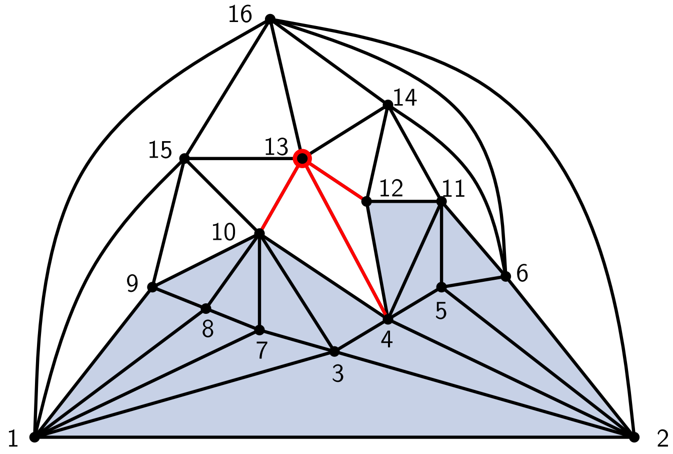


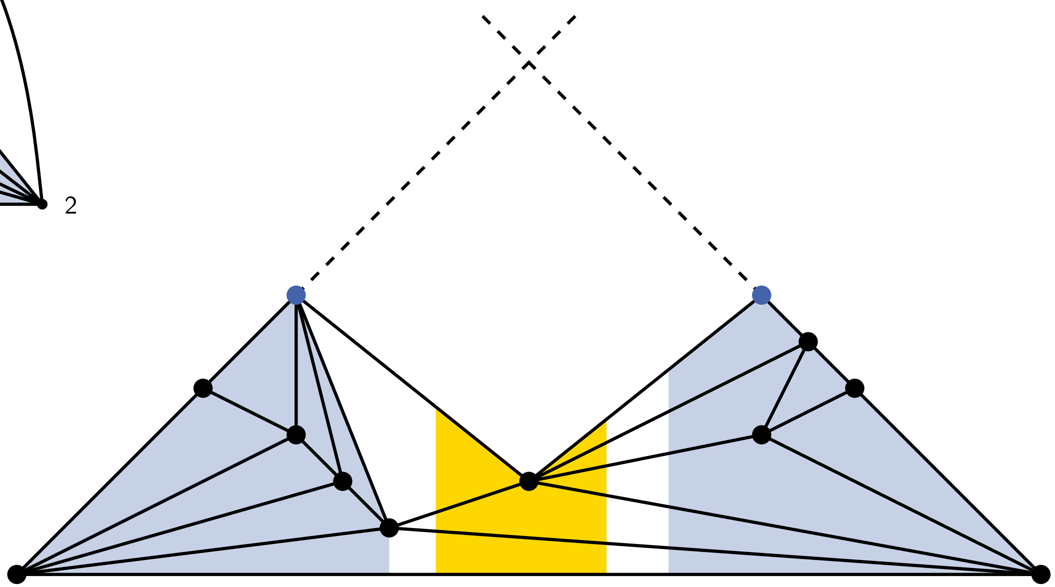


De Fraysseix Pach Pollack (Shift) Algorithm









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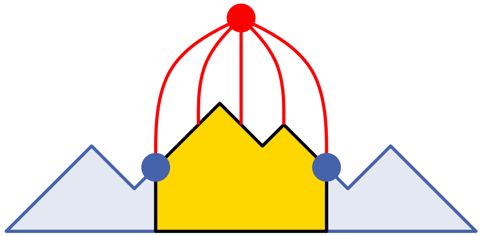

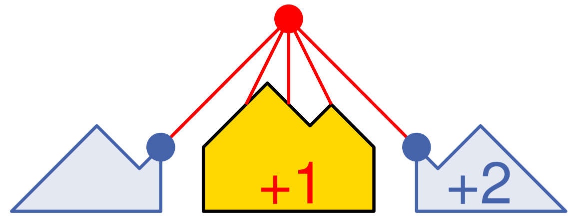

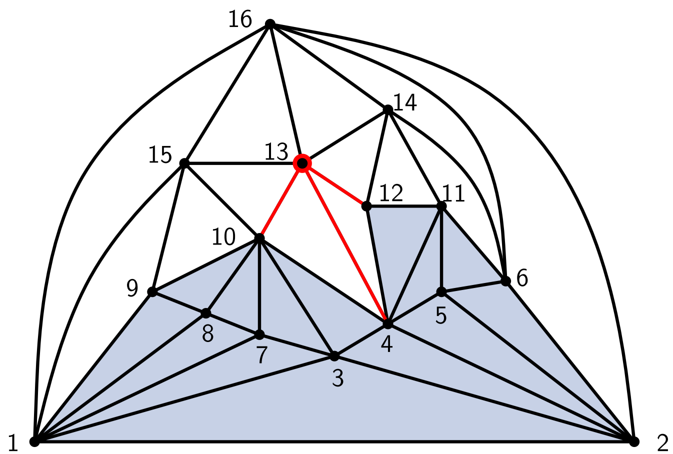

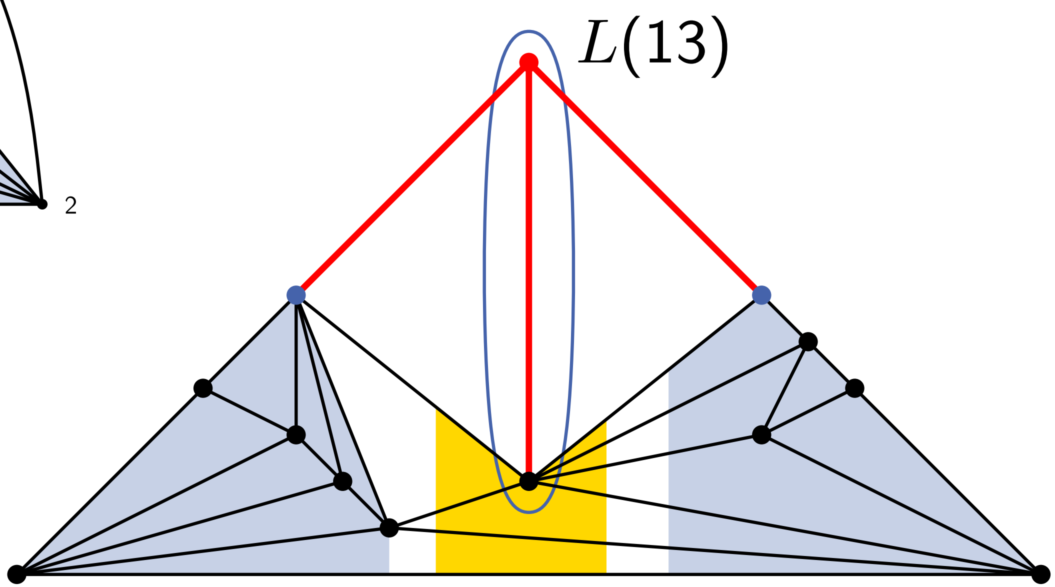

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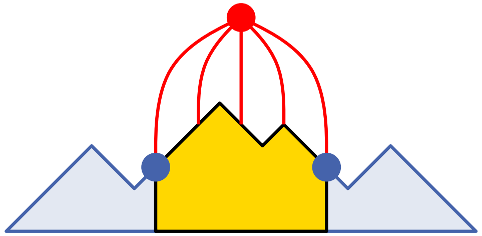

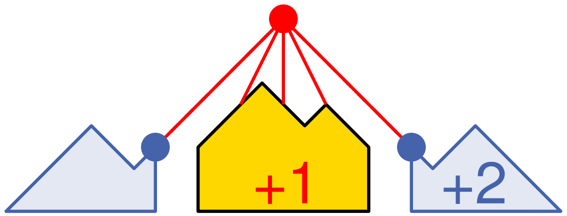

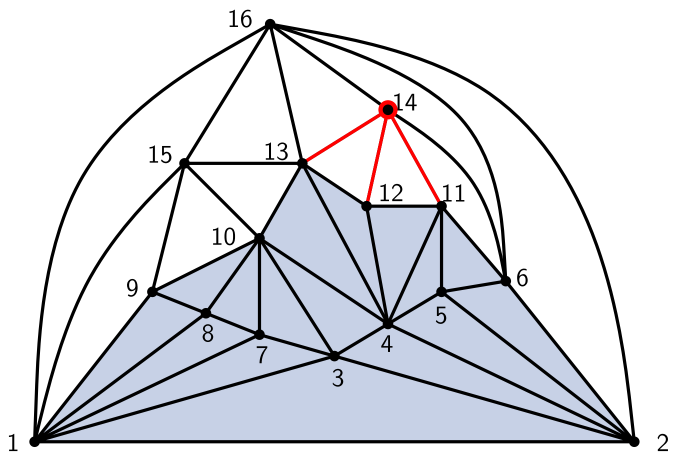

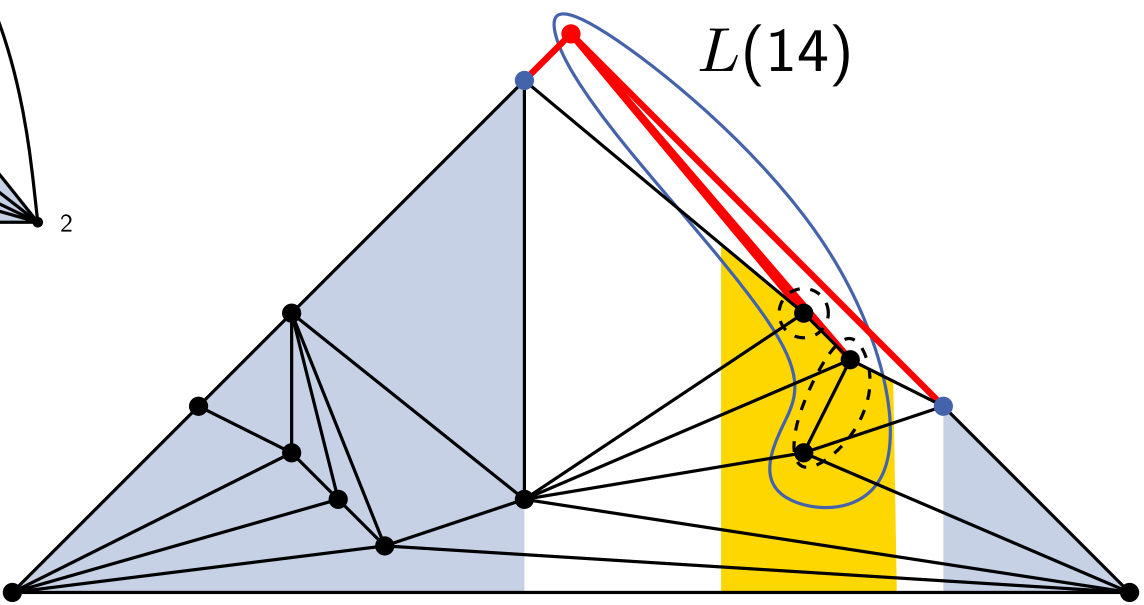

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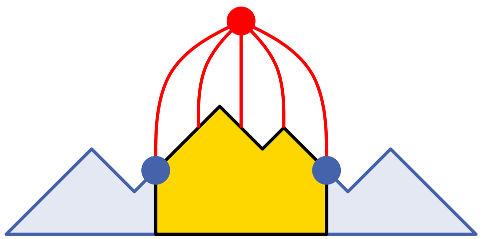

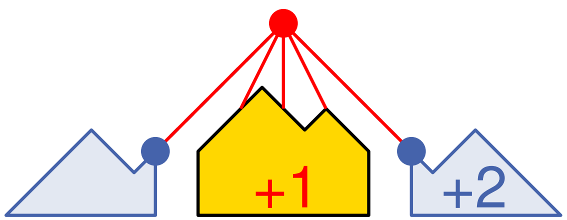

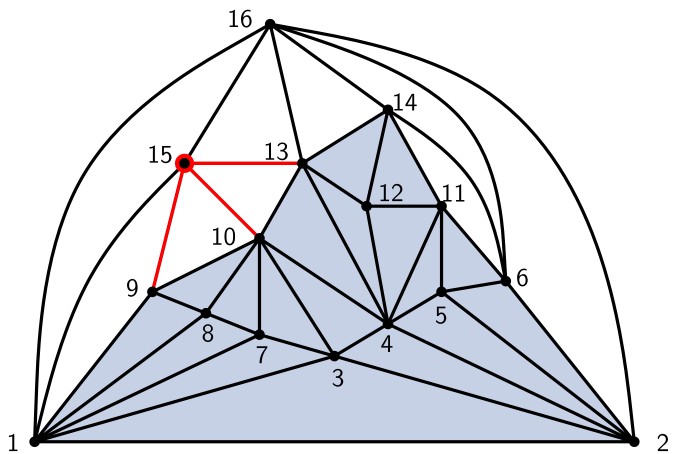

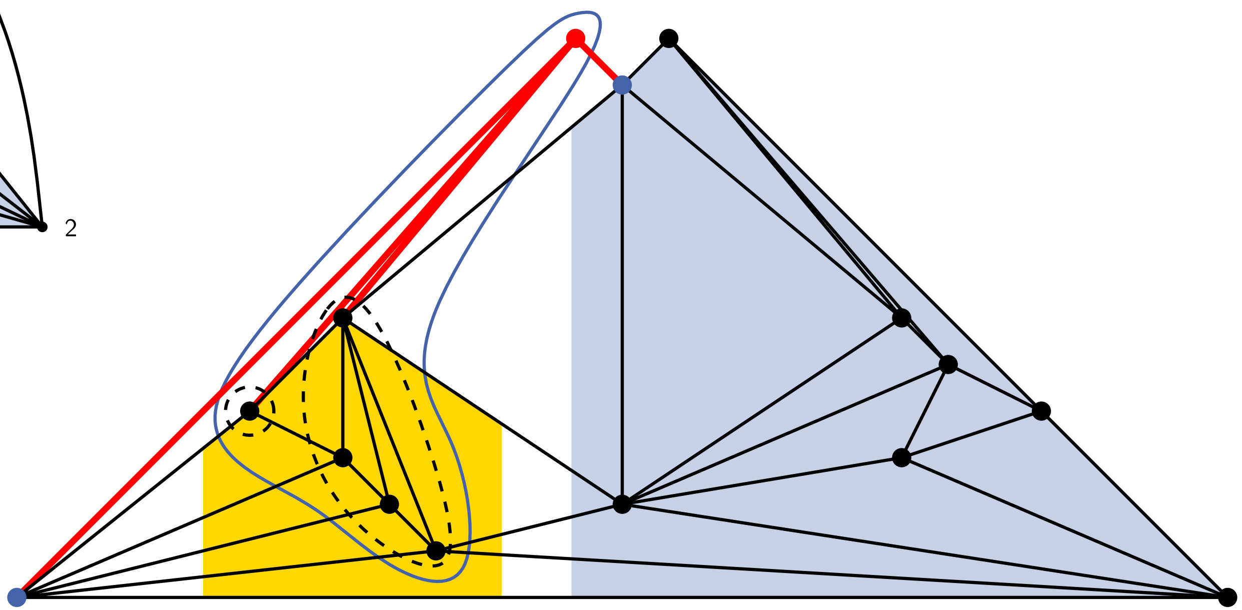

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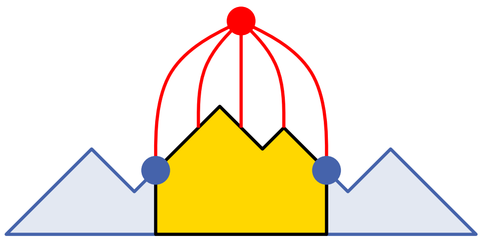

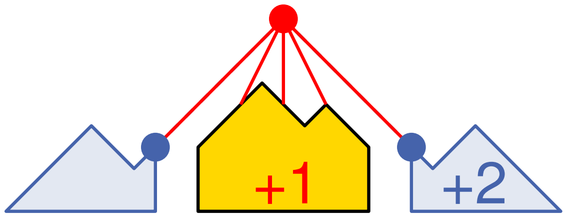

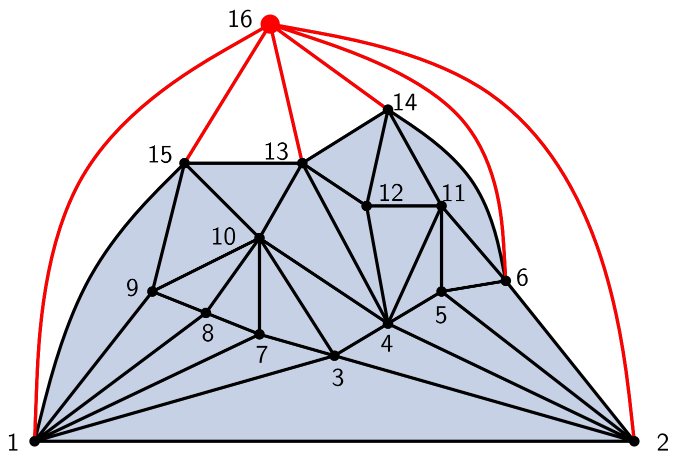

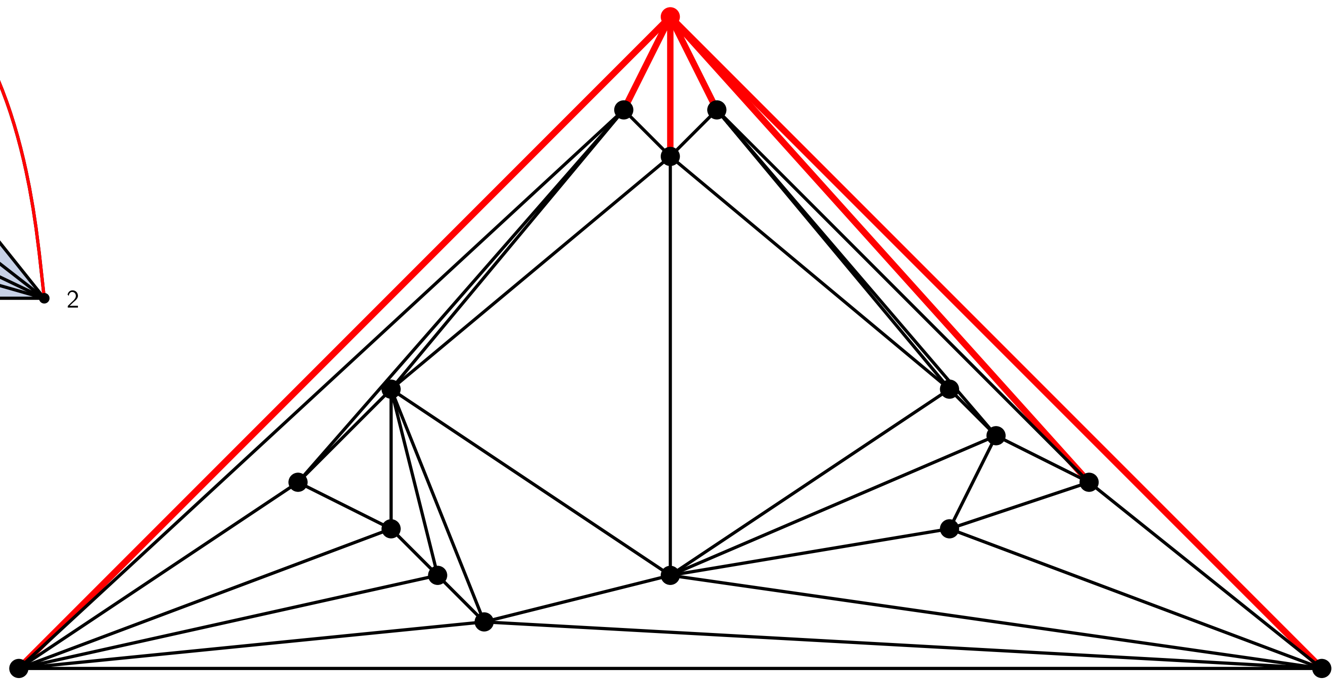

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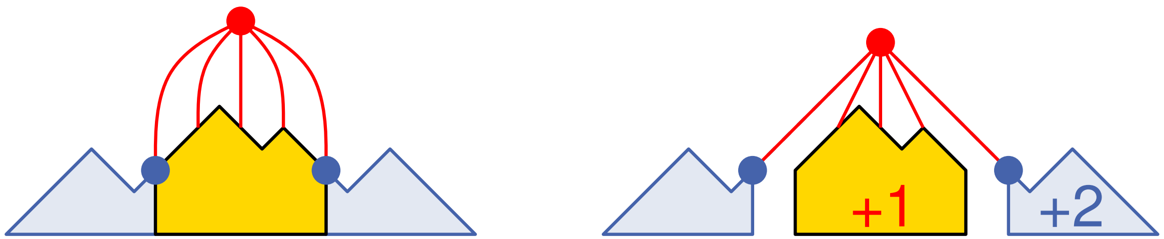

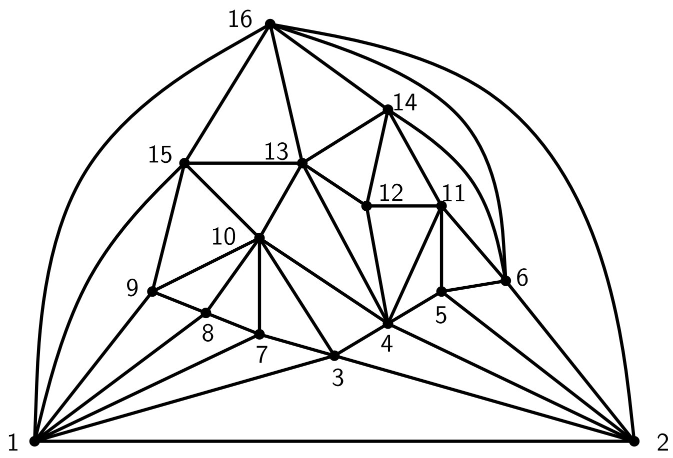

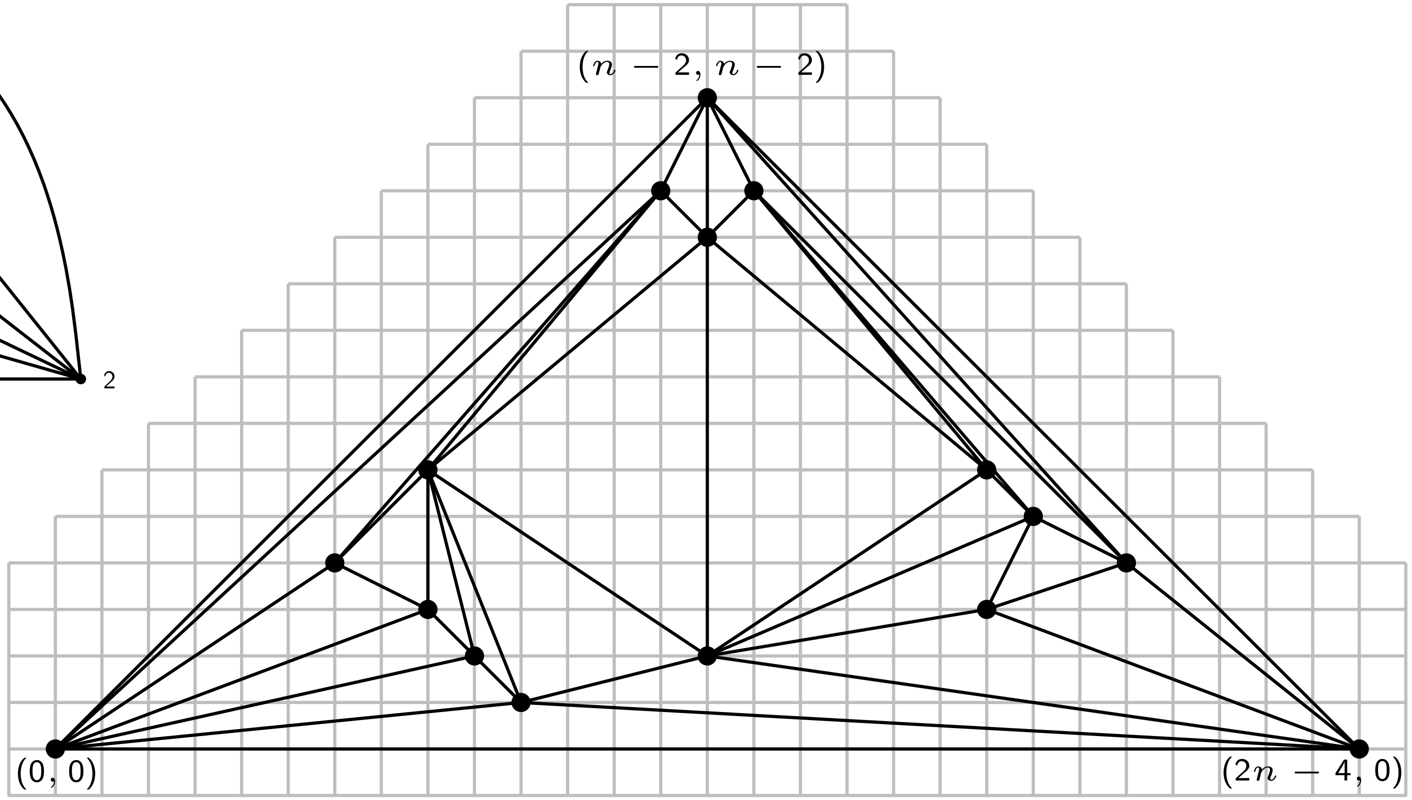

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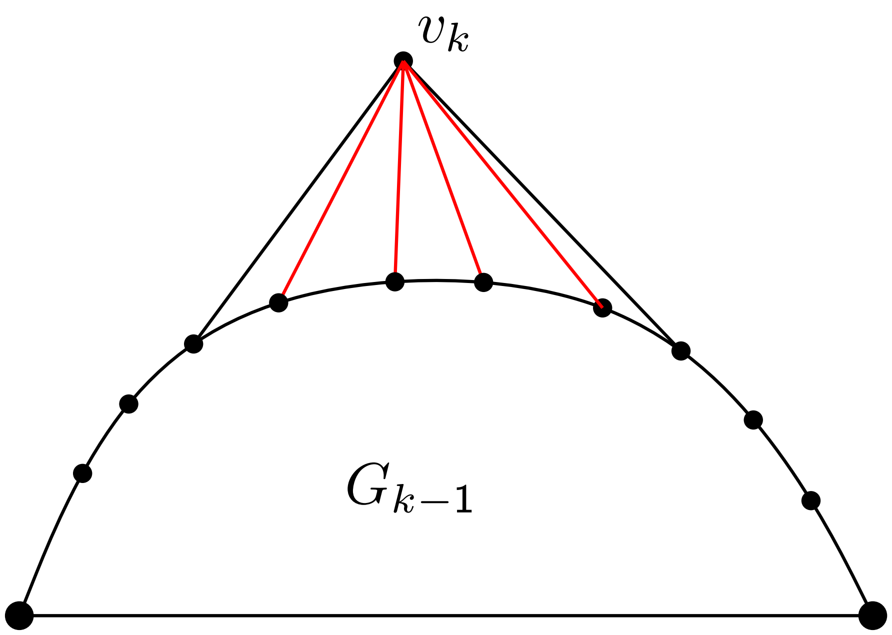

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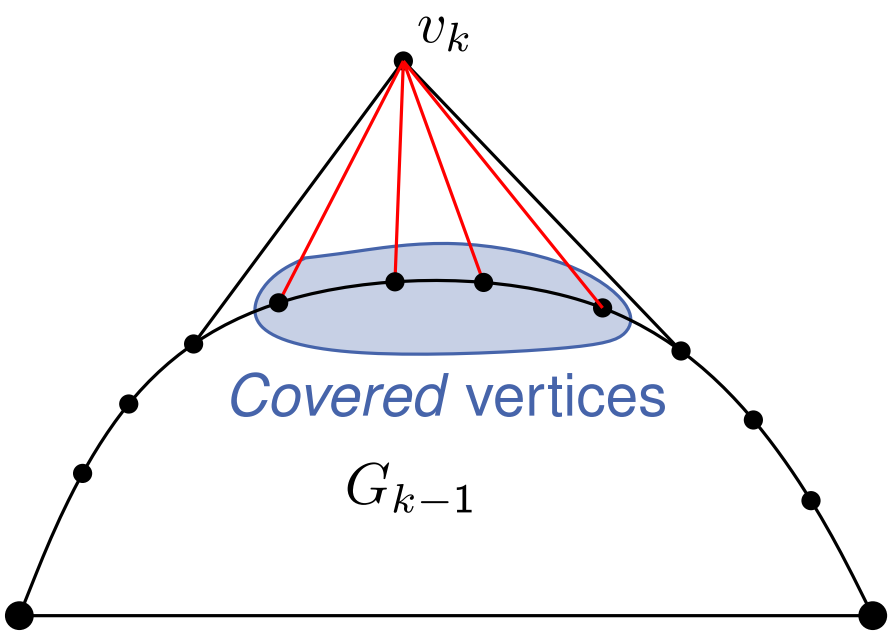

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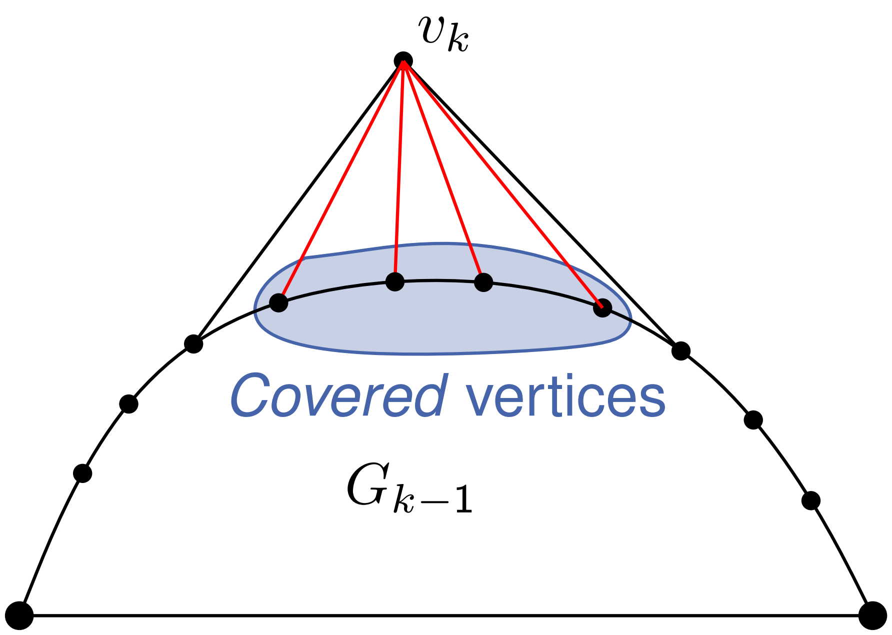

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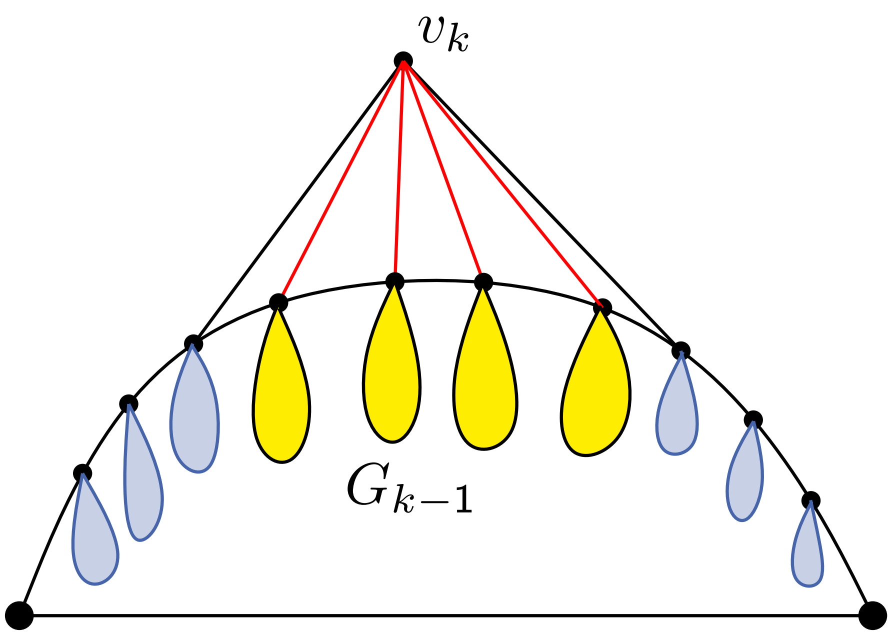

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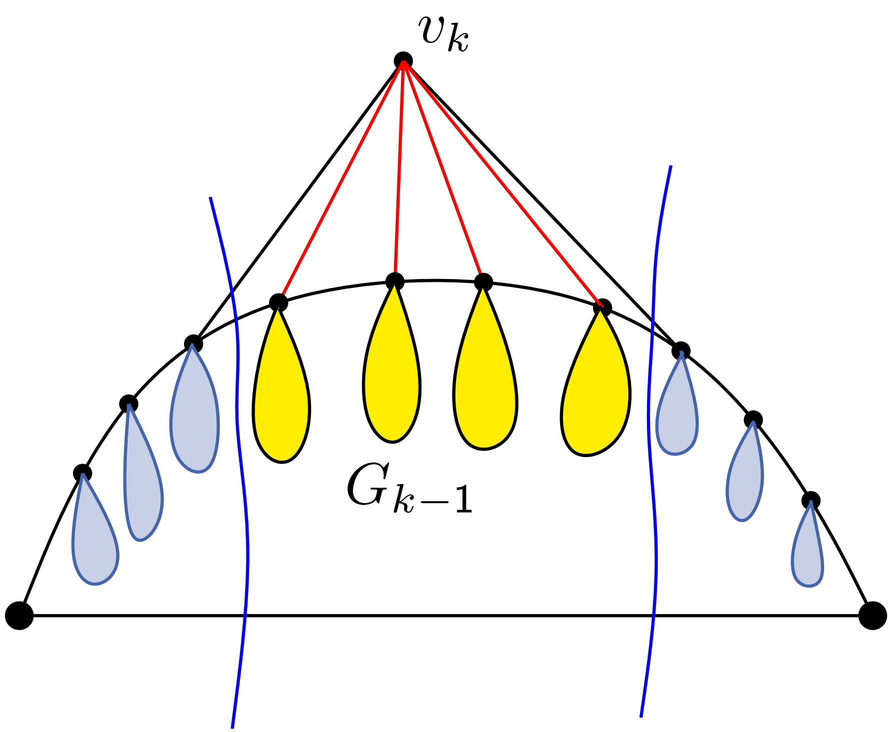

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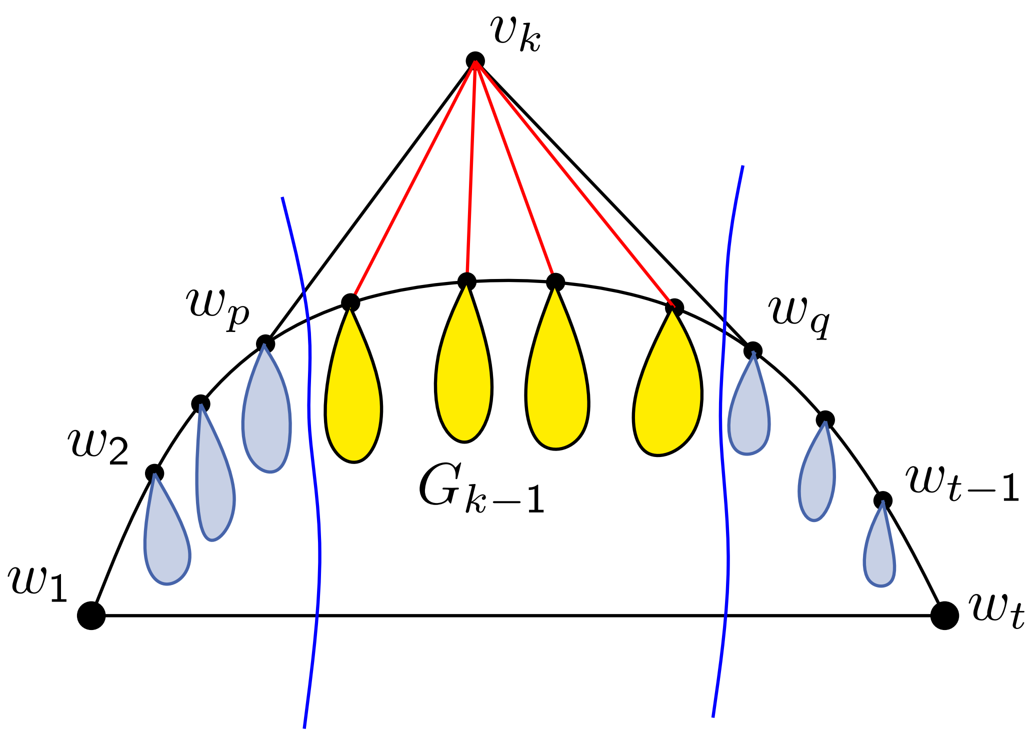

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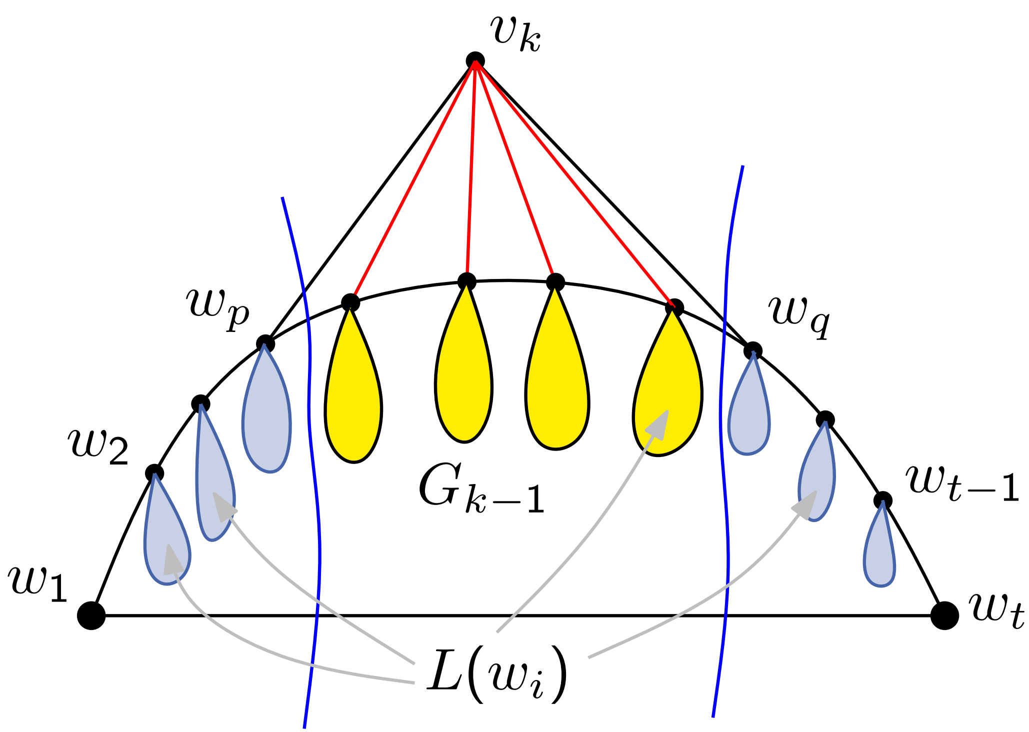

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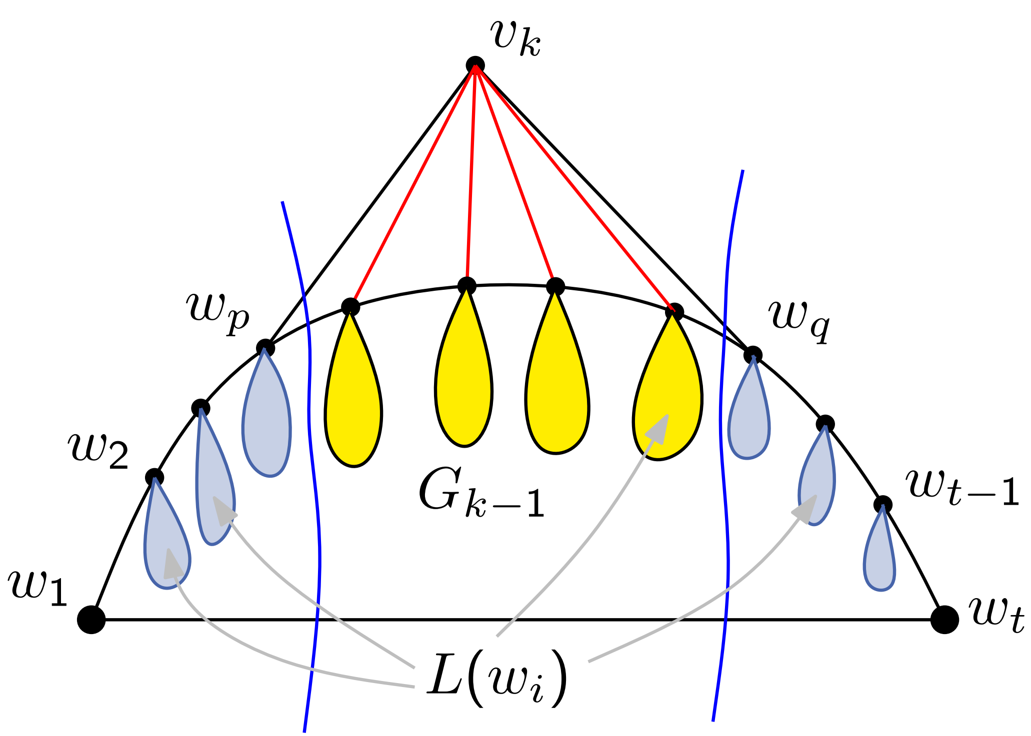

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De Fraysseix Pach Pollack (Shift) Algorithm

Karlsruhe Institute of Technology

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- Let $\delta(w_1) \leq \cdots \leq \delta(w_p) \leq \delta(v_k) \leq \delta(w_q) \leq \cdots \leq \delta(w_t)$.
- We set $\delta'(w_i) = \delta(w_i)$ for $1 \leq i \leq p$,
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- We can complete the drawing by placing v_k.

De Fraysseix Pach Pollack (Shift) Algorithm

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Algorithm Shift

Let $v_1, \ldots, v_n$ be a canonical ordering of G

for $i = 1$ **to** n **do**

  $L(v_i) \leftarrow \{v_i\}$;

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$; $P(v_3) \leftarrow (1,1)$;

for $i = 4$ **to** n **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1};
  and let $w_p, \ldots, w_q$ be the neighbors v_i;

  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 1$;

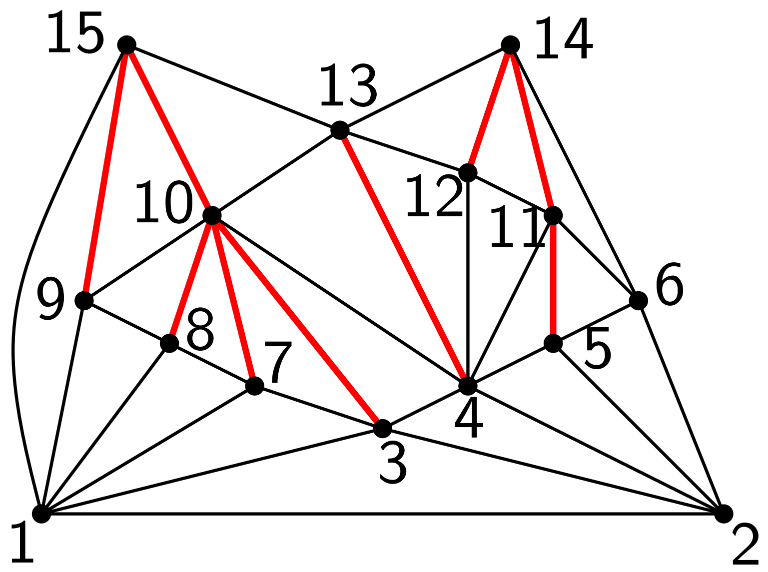
  for $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 2$;

  $P(v_i) \leftarrow$ intersection of $+1$ and -1 edges from $P(w_p)$ and $P(w_q)$;

  $L(v_i) = \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$;

Linear Time Implementation of Shift Algorithm



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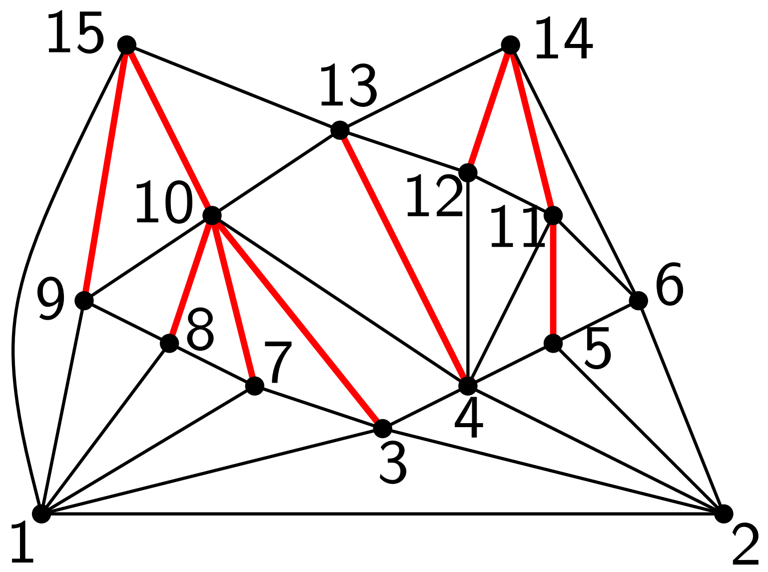

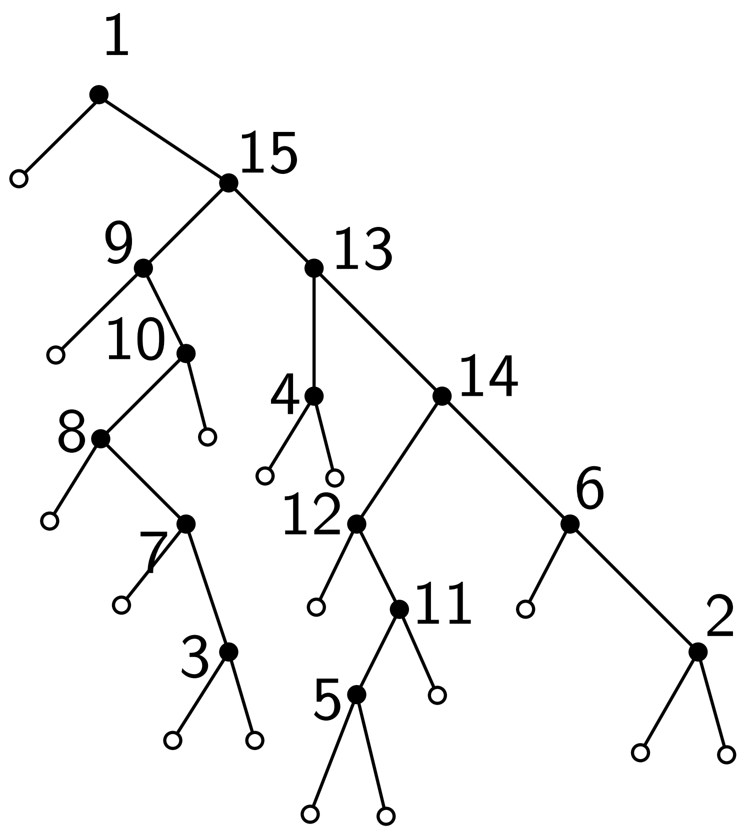

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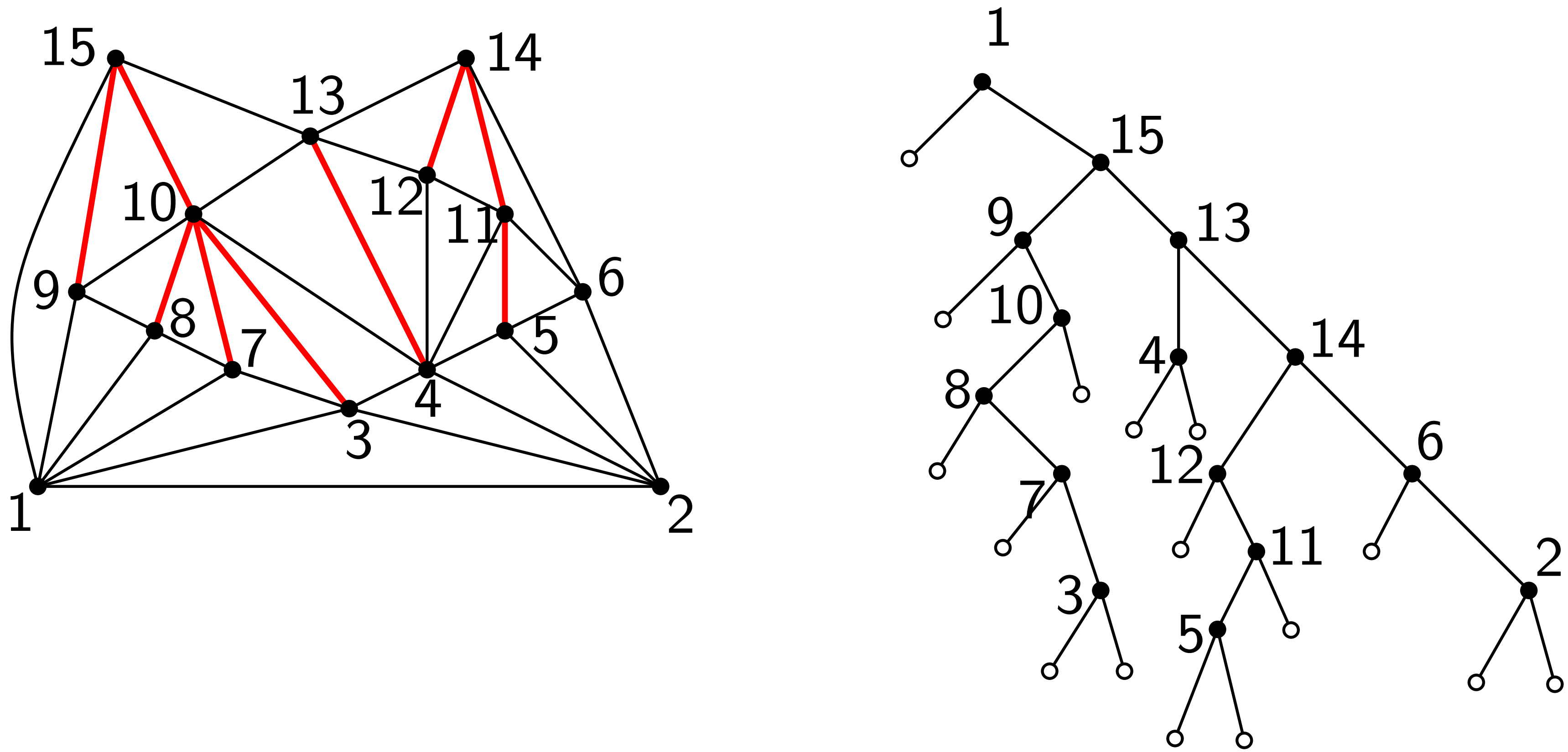

- $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$ (1)
- $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$ (2)
- $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$ (3)

Linear Time Implementation of Shift Algorithm

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- If we know the y-coordinates of w_p and w_q and the difference $x(w_p) - x(w_q)$, we can compute the relative distance of v_k and w_p.
- In the binary tree which we construct we keep the relative x-distance of each node from its parent.

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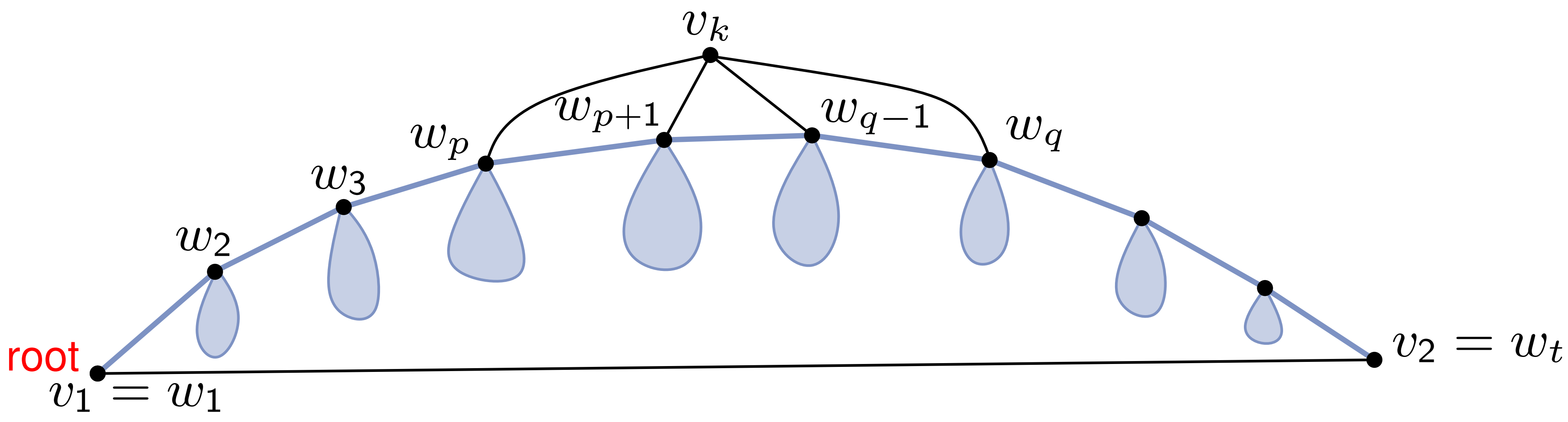

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- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \cdots + \Delta_x(w_q)$
- Calculate $\Delta_x(v_k)$ by eq. (3)
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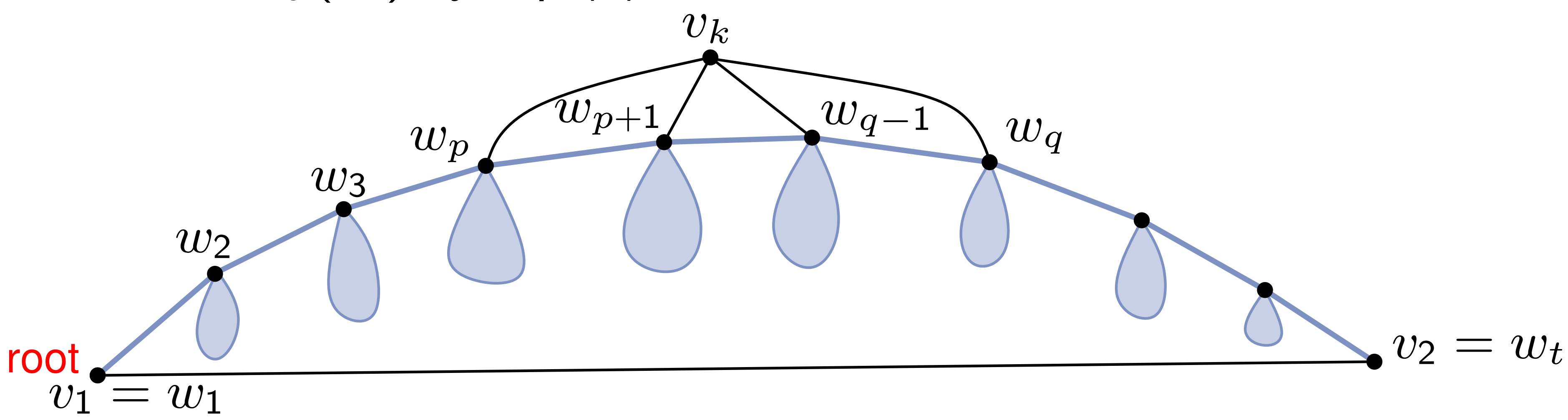

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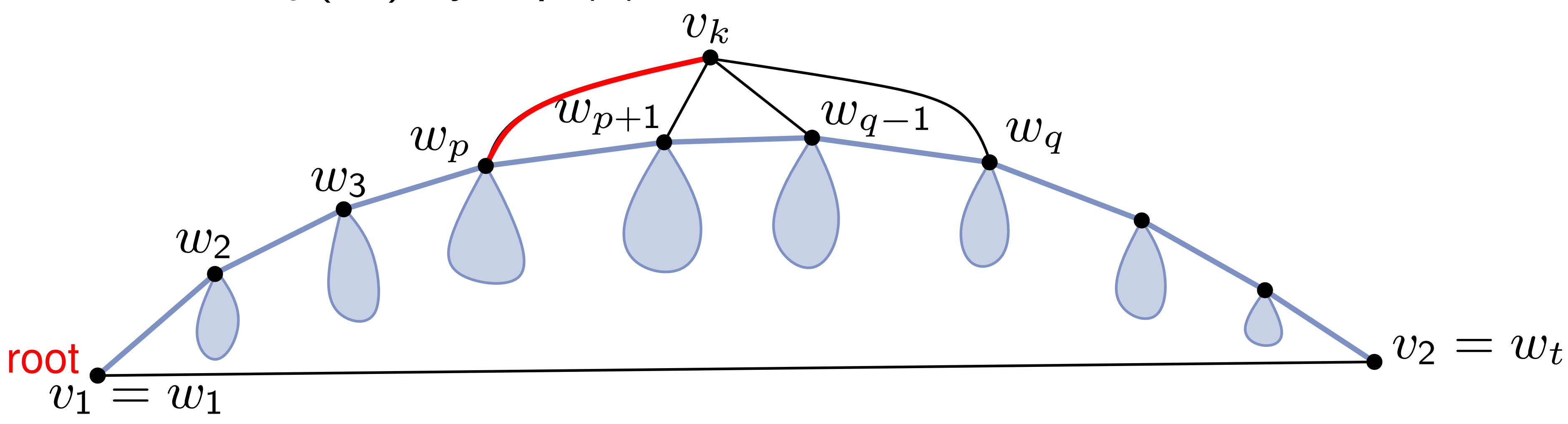

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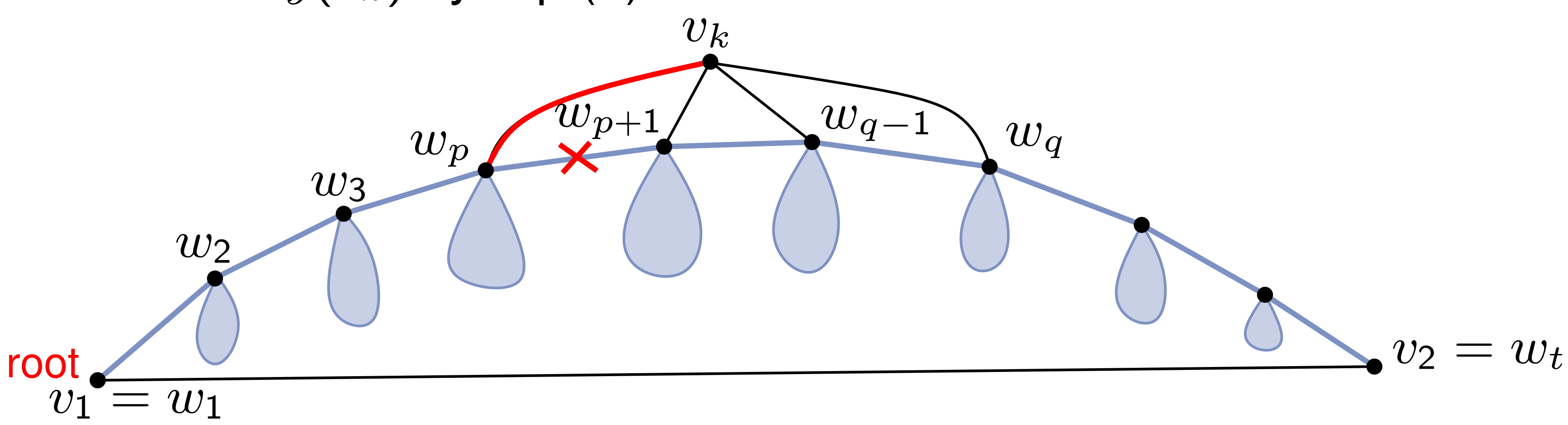

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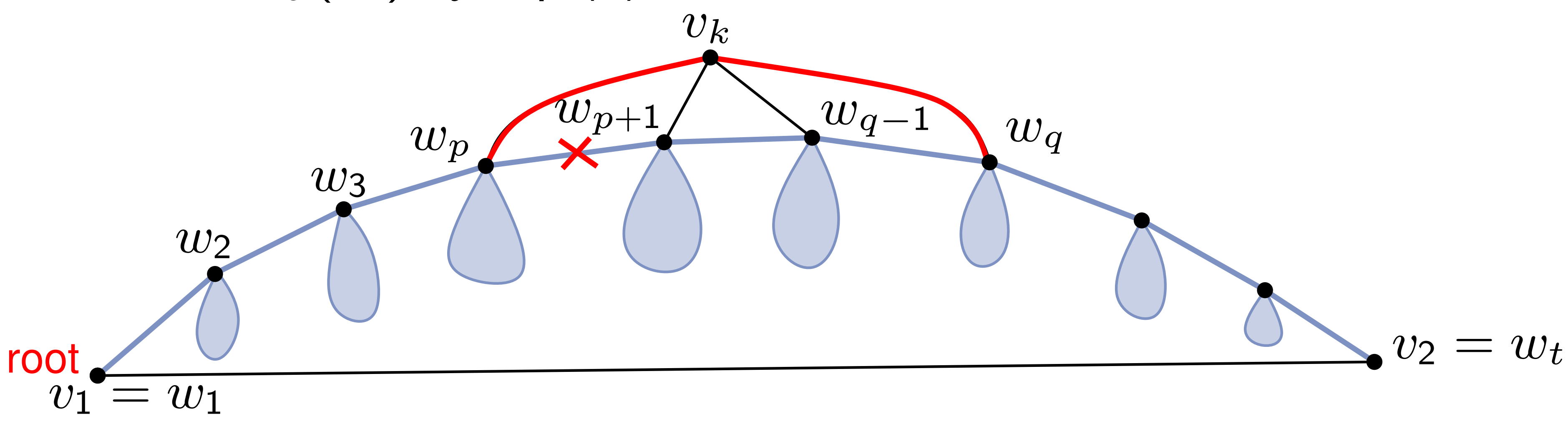

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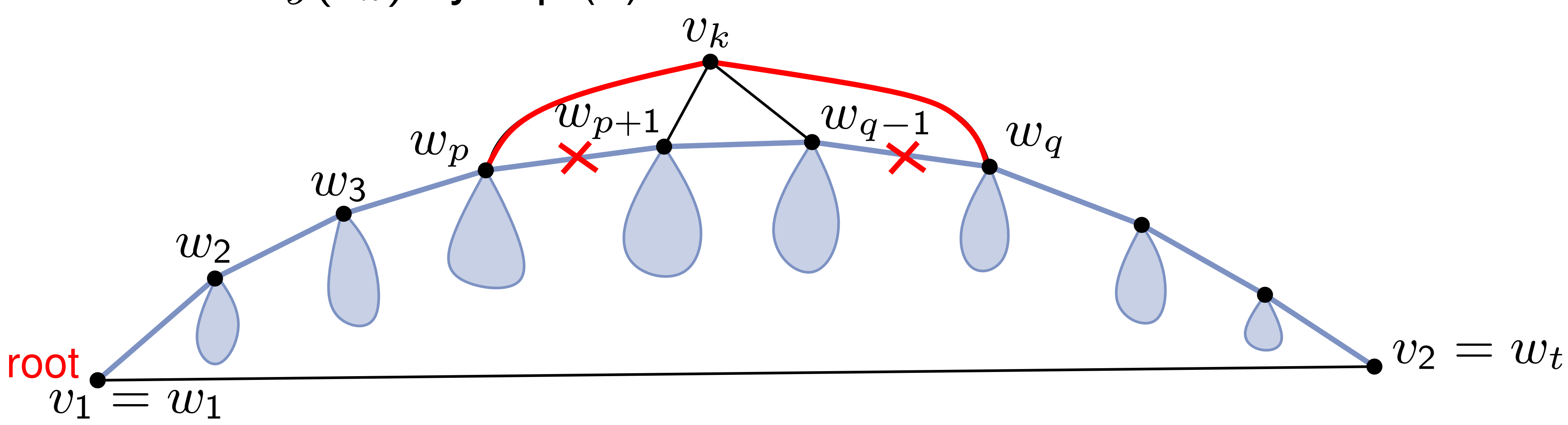
- If we know the y-coordinates of w_p and w_q and the difference $x(w_p) - x(w_q)$, we can compute the relative distance of v_k and w_p.
- In the binary tree which we construct we keep the relative x-distance of each node from its parent.
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \cdots + \Delta_x(w_q)$
- Calculate $\Delta_x(v_k)$ by eq. (3)
- Calculate $y(v_k)$ by eq. (2)



Linear Time Implementation of Shift Algorithm

- If we know the y-coordinates of w_p and w_q and the difference $x(w_p) - x(w_q)$, we can compute the relative distance of v_k and w_p.
- In the binary tree which we construct we keep the relative x-distance of each node from its parent.
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \cdots + \Delta_x(w_q)$
- Calculate $\Delta_x(v_k)$ by eq. (3)
- Calculate $y(v_k)$ by eq. (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$