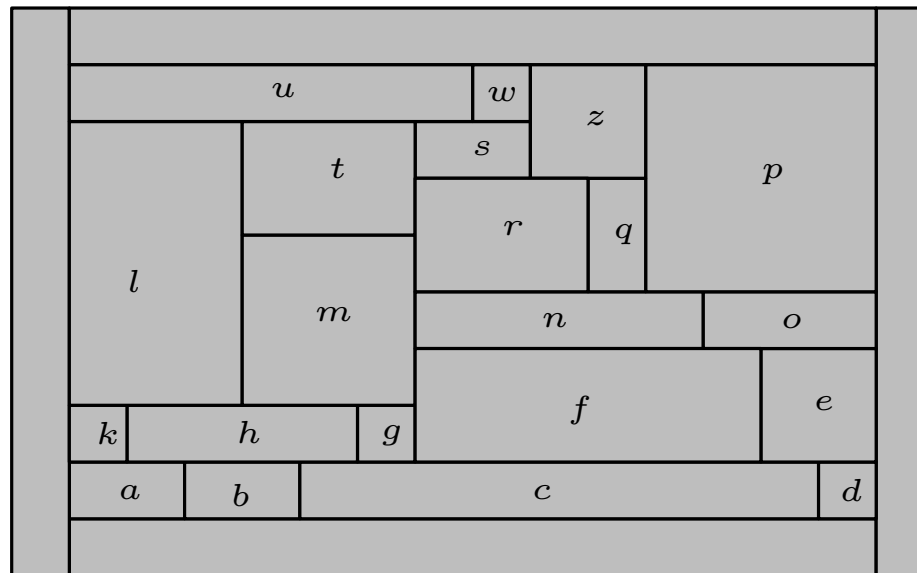


Algorithms for graph visualization

Contact representations of planar graphs.

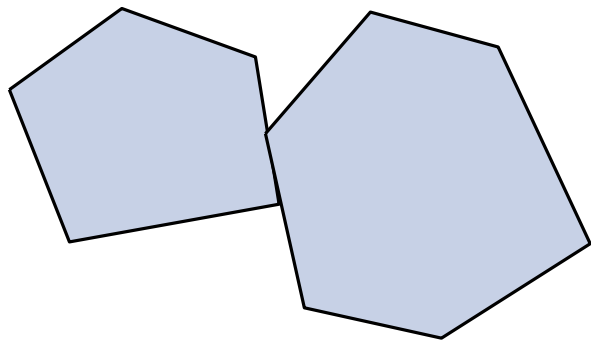
WINTER SEMESTER 2012/2013

Tamara Mchedlidze – MARTIN NÖLLENBURG – IGNAZ RUTTER

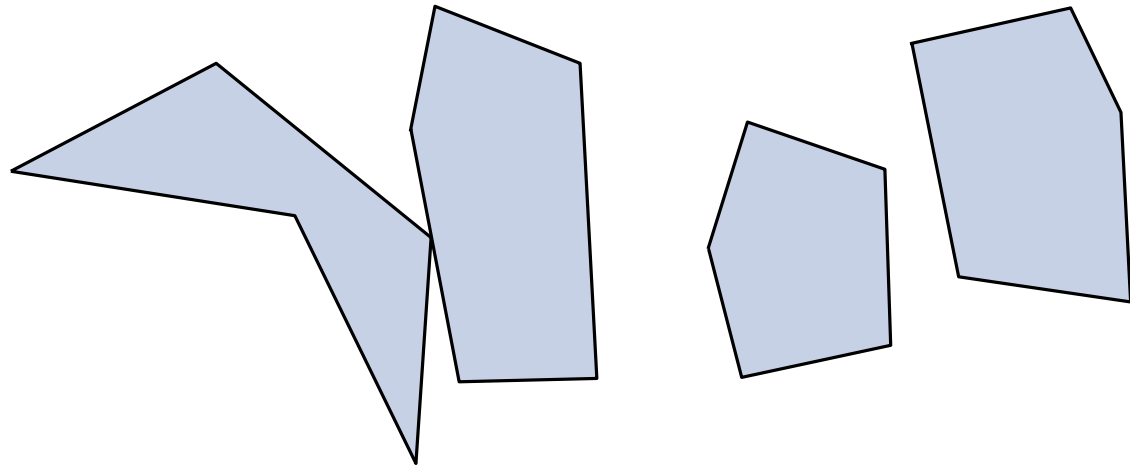


Contact representation

In a **contact representation** of a planar graph each vertex is represented as a geometrical object such that two objects touch if and only if the corresponding vertices are connected by an edge.



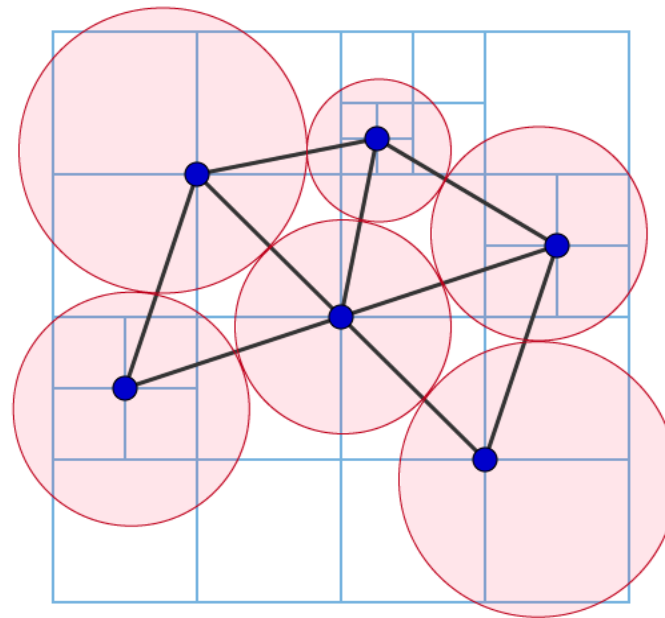
Touch



Do not touch

Contact representation

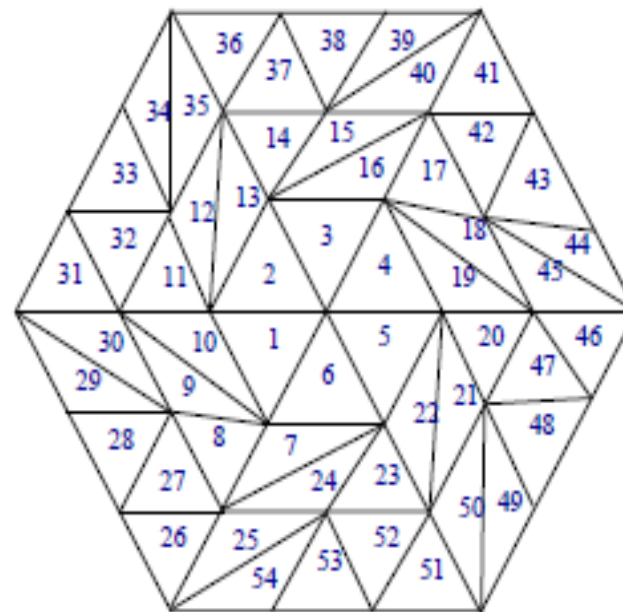
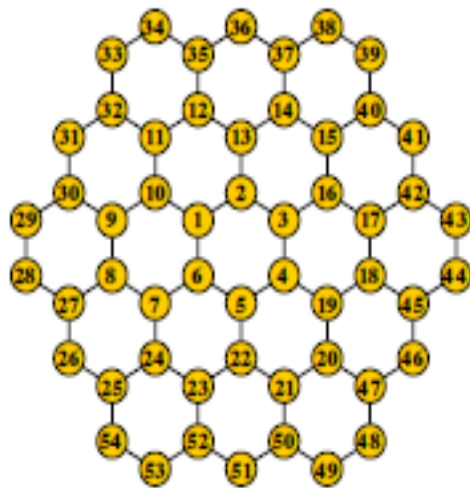
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Touching disk representation

Contact representation

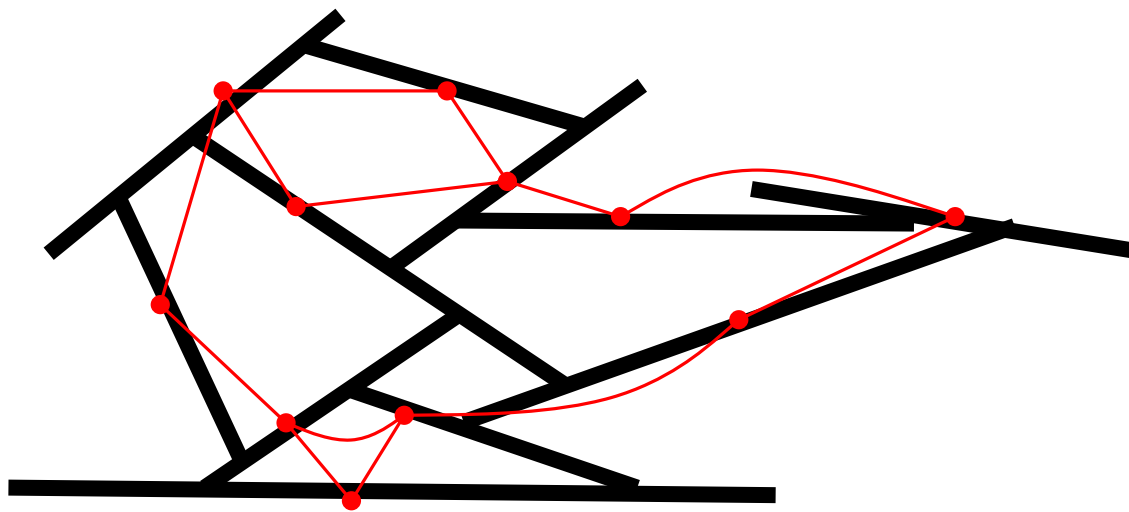
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Touching triangle representation

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Touching segment representation

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General idea for the construction of a contact representation of a planar graph using n -gons in worst case

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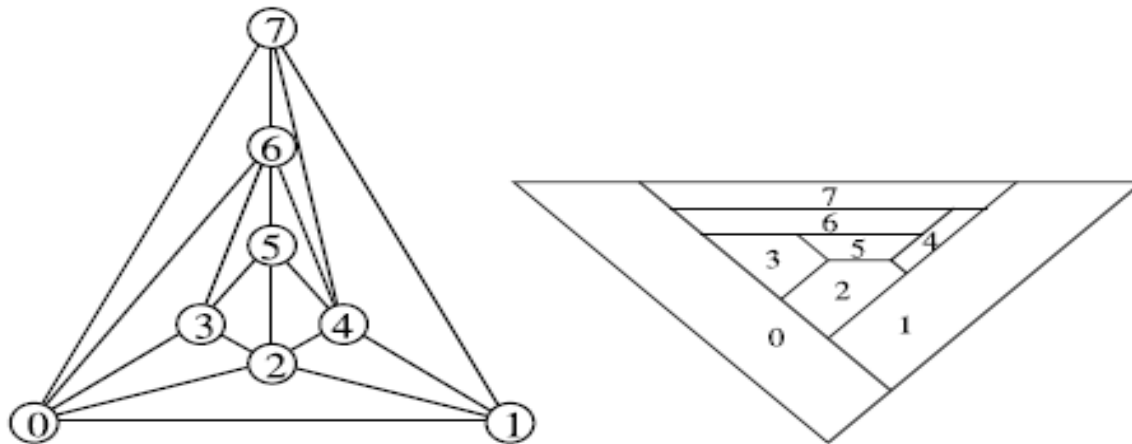
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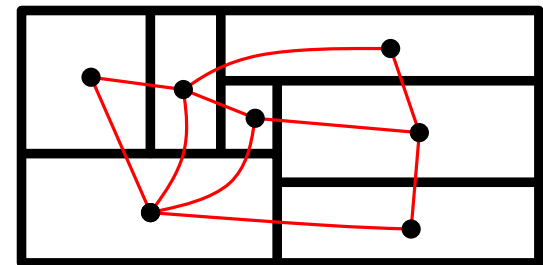
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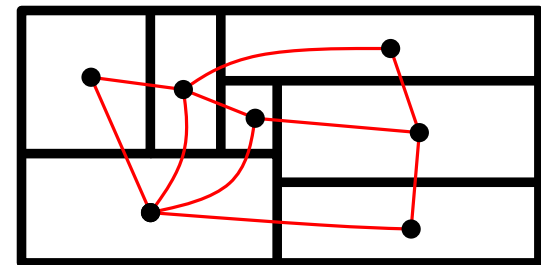
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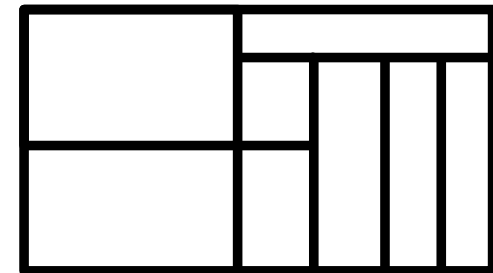
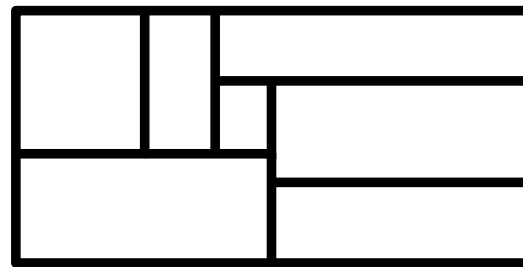
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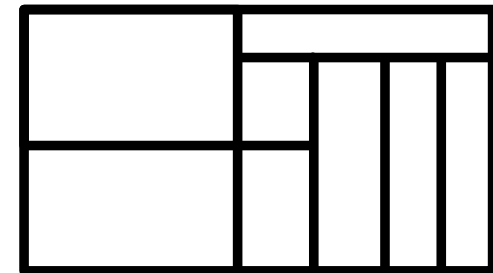
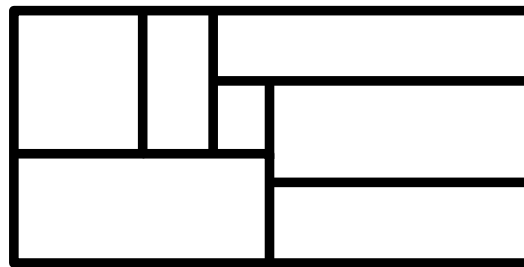
Rectangular Subdivision System

Let R be a rectangle. A **rectangular subdivision system** Φ of R is a partition of R into a set of non-intersecting smaller rectangles such that no four of them meet at the same point.



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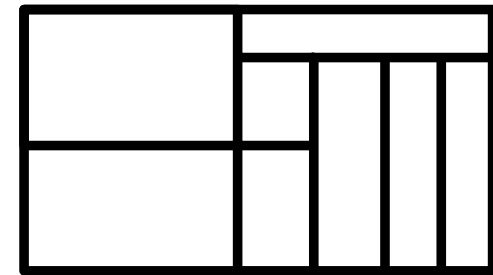
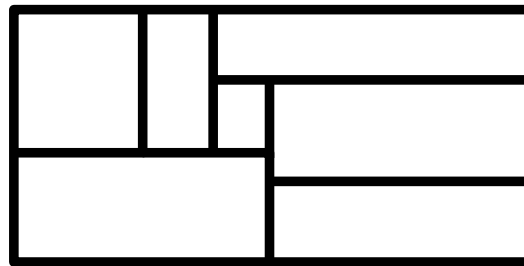
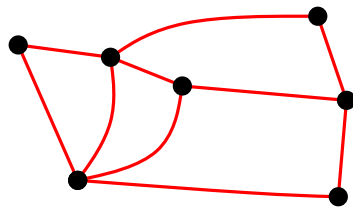


Rectangular Dual

A **rectangular dual** of a graph $G = (V, E)$ is a rectangular subdivision system Φ and a one-to-one correspondence $f : V \rightarrow \Phi$ such that $(u, v) \in E$ if and only if the rectangles $f(u)$ and $f(v)$, corresponding to u and v , share a common boundary.

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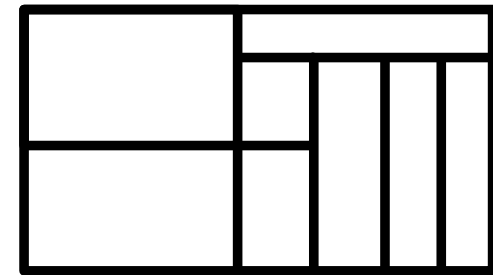
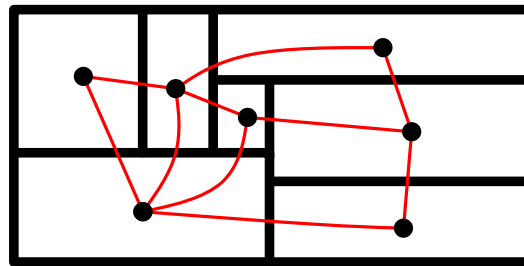
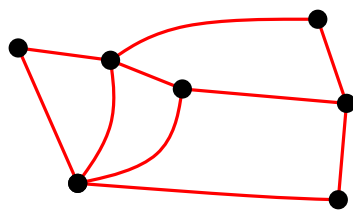


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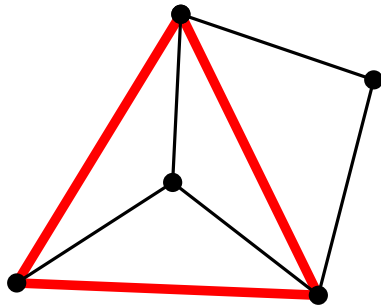


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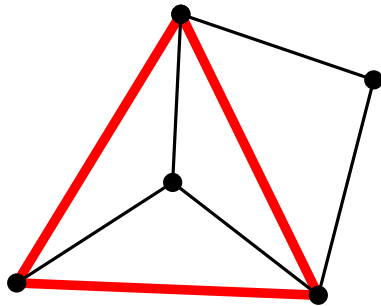
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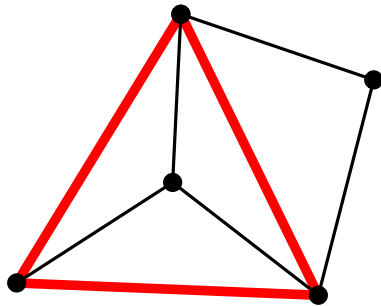
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← Does not have a rectangular dual!
(In order to enclose an area we need at least four boxes)

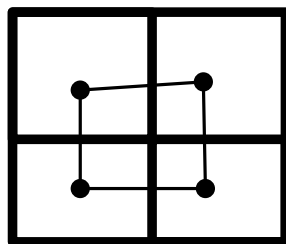
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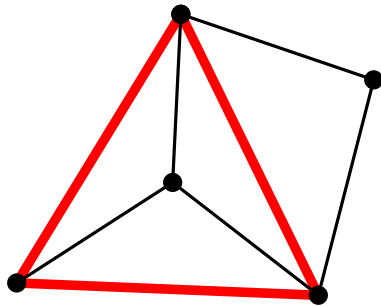
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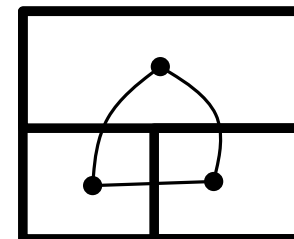
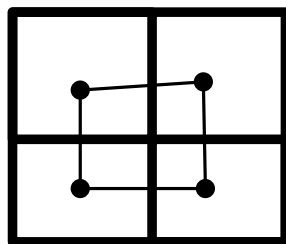
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Necessary conditions for a planar graph G to have a rectangular dual:

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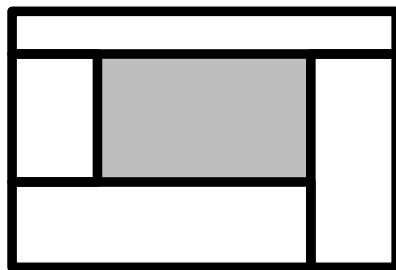
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Theorem [1985,1986,1987,1988,1990,1993 by Xin He]

A planar graph $G = (V, E)$ has a rectangular dual R with four rectangles on the boundary of R if and only if the following conditions hold:

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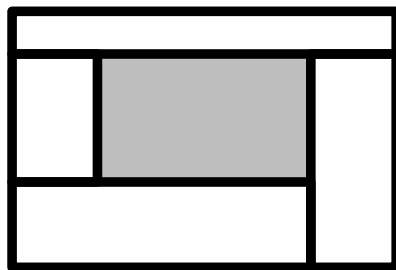
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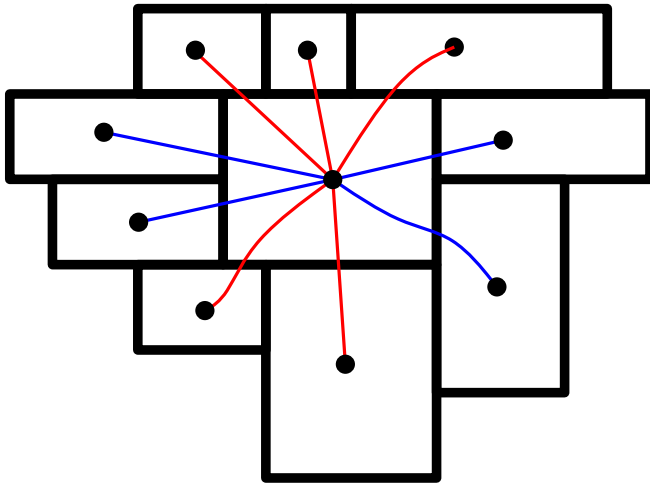
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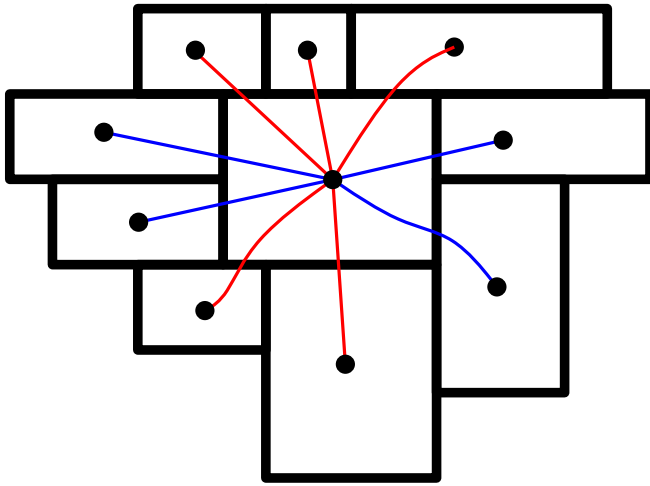
Proper Triangular Planar Graph (PTP)

Rectangular Dual



In order to construct a rectangular dual we need to partition our edges on **vertical** and **horizontal**. **Regular edge labeling** (REL, for short) is a tool for that.

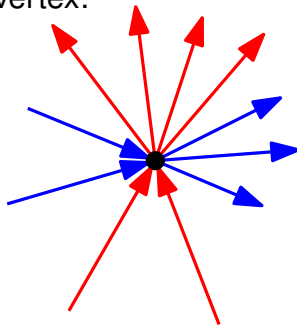
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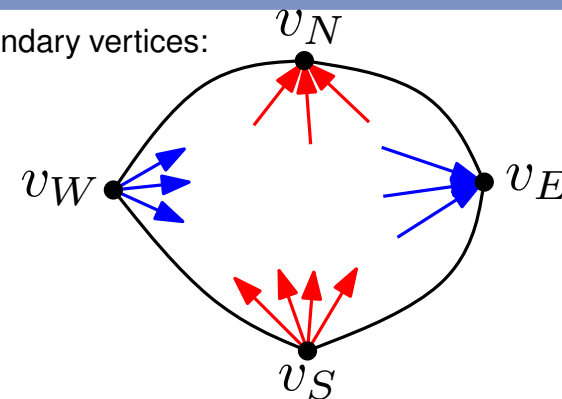
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Regular edge labeling

For each internal vertex:



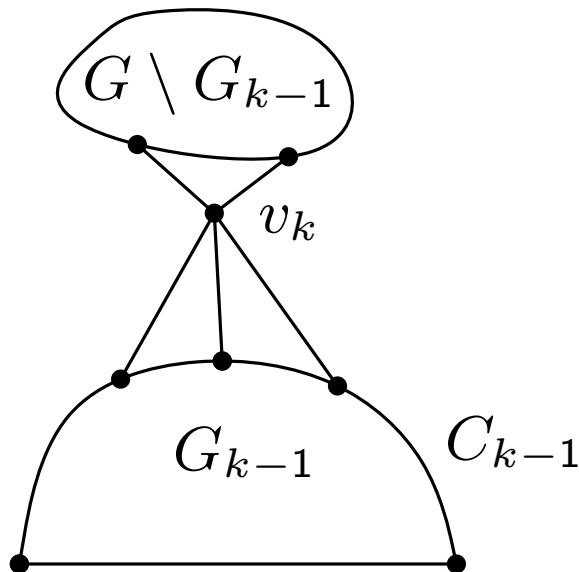
For the boundary vertices:



Theorem

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \dots, v_n = v_N$ such that for every $4 \leq k \leq n$:

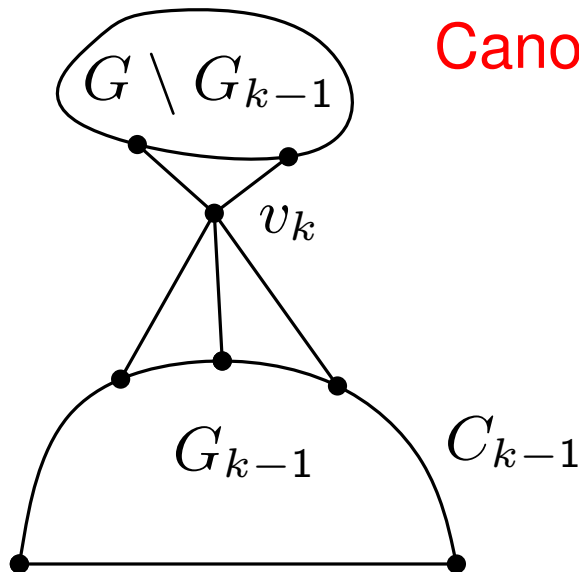
- The subgraph G_{k-1} induced by v_1, \dots, v_{k-1} is biconnected and boundary C_{k-1} of G_{k-1} contains the edge (v_S, v_W) .
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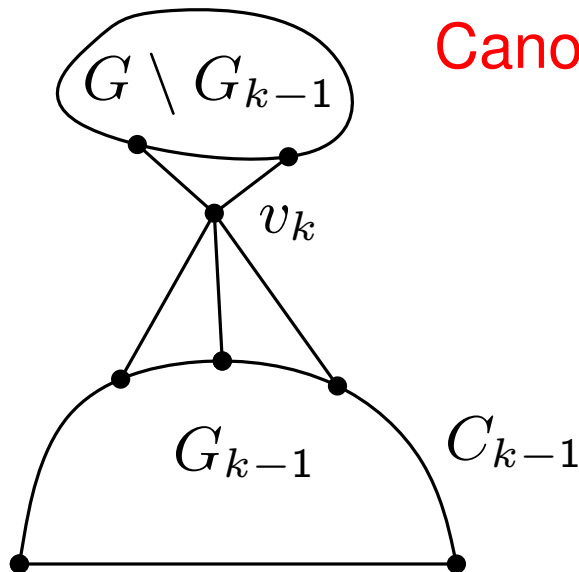


Canonical ordering with extra condition on v_k !

Theorem (Refined canonical ordering)

Let $G = (V, E)$ be a PTP graph. There exists a labeling of the vertices of G $v_1 = v_S, v_2 = v_W, v_3, \dots, v_n = v_N$ such that for every $4 \leq k \leq n$:

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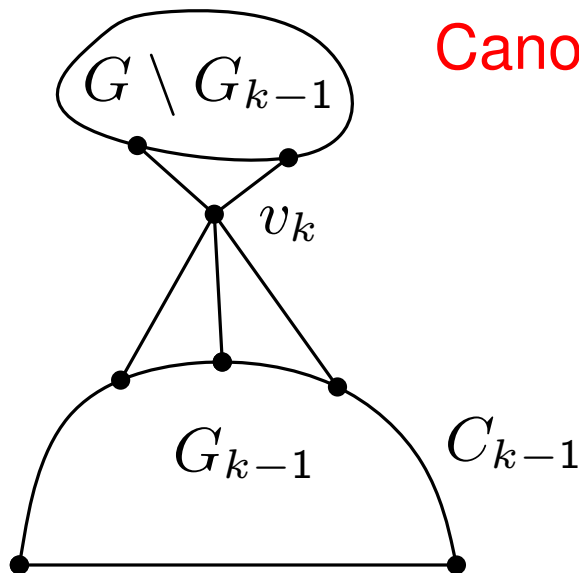


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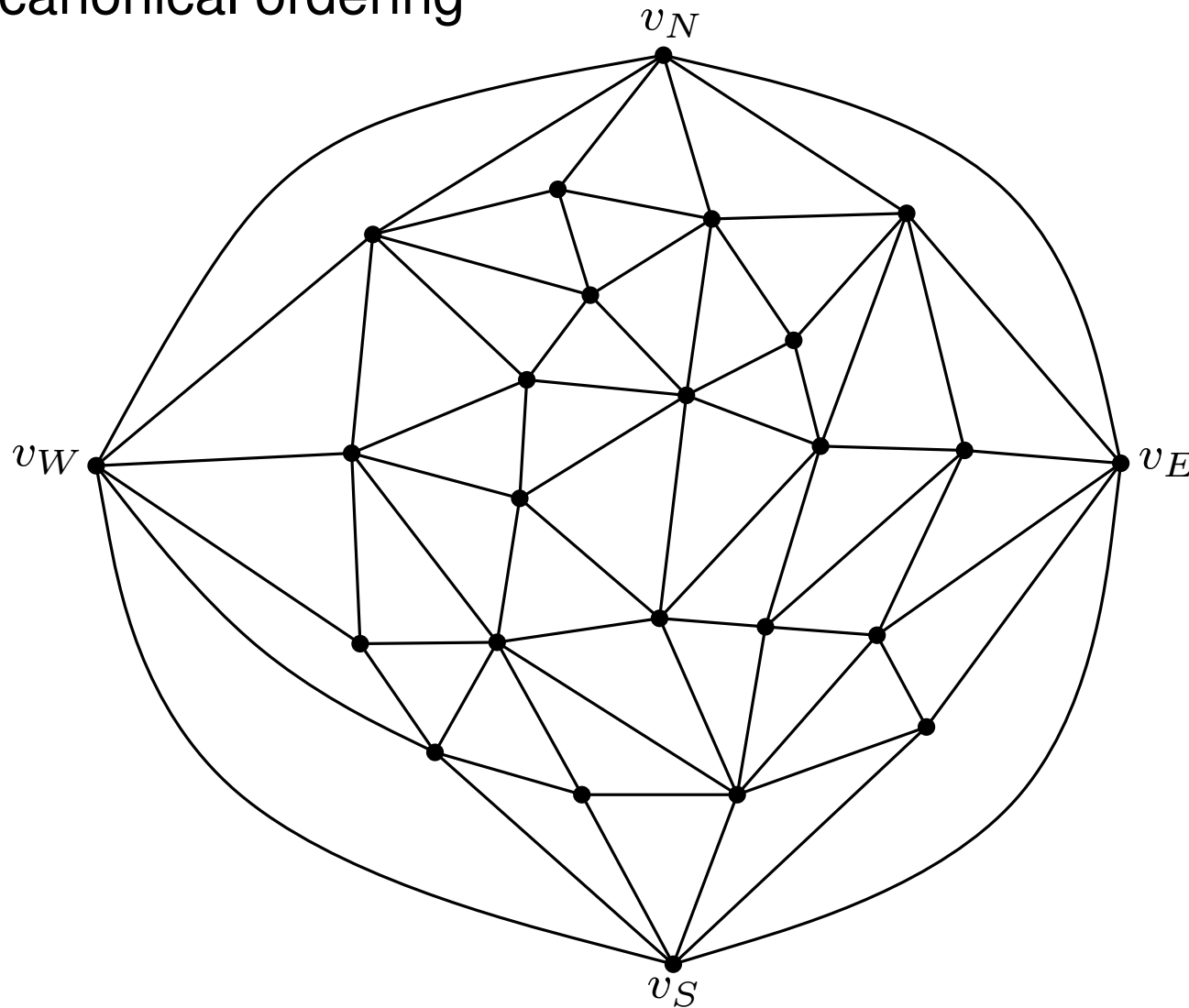


Canonical ordering with extra condition on v_k !

Home task!

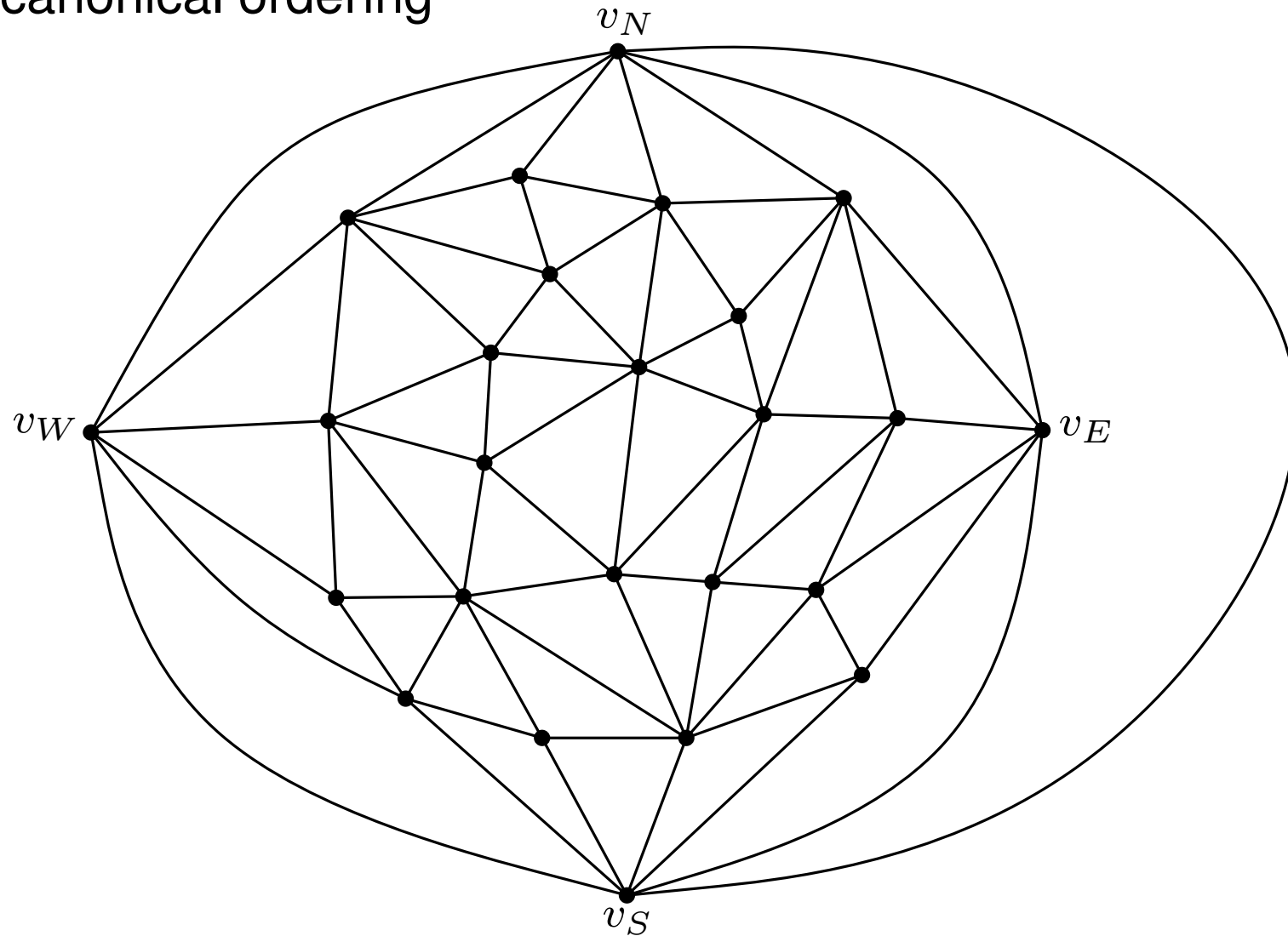
Rectangular Dual

Refined canonical ordering



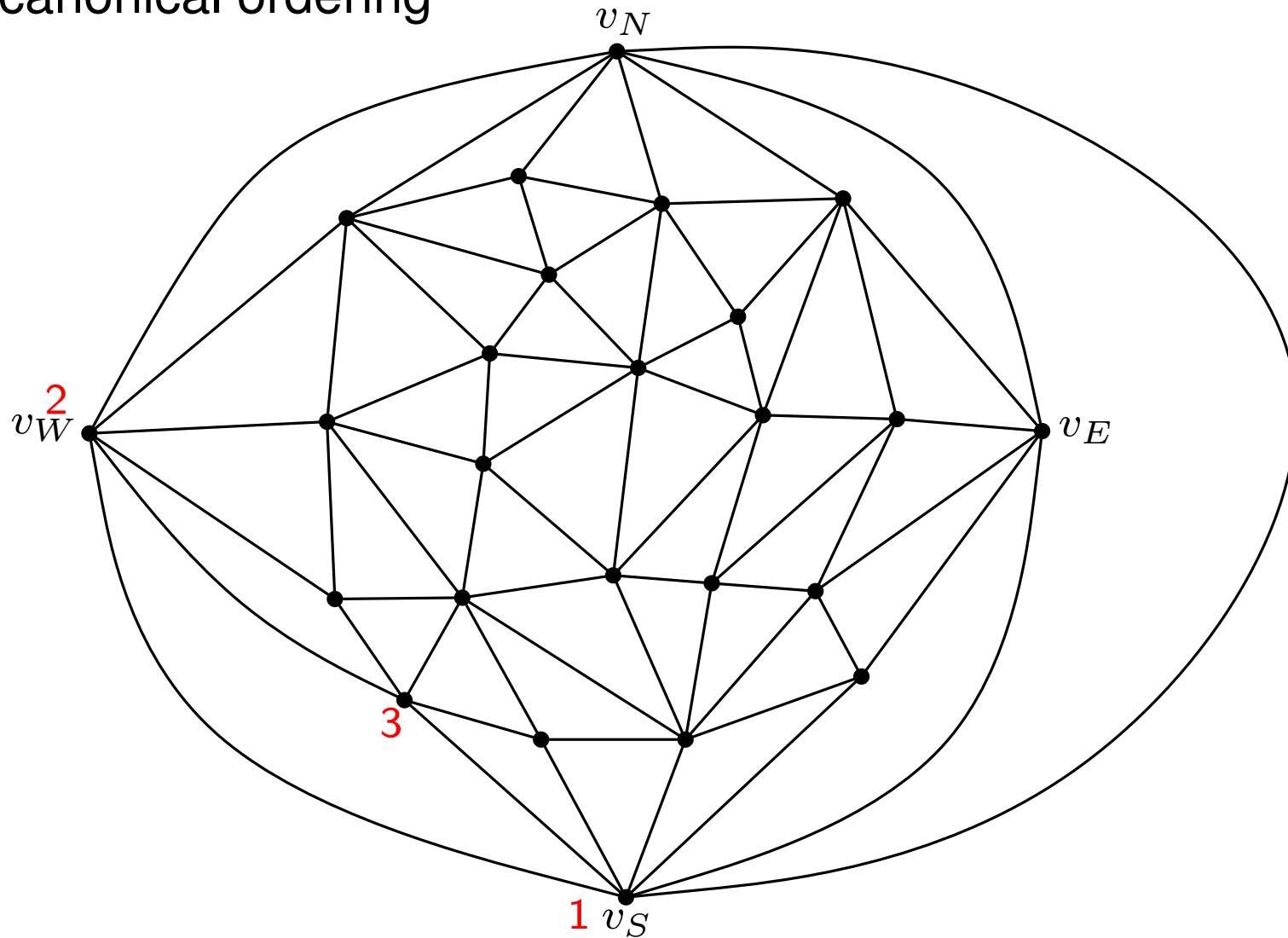
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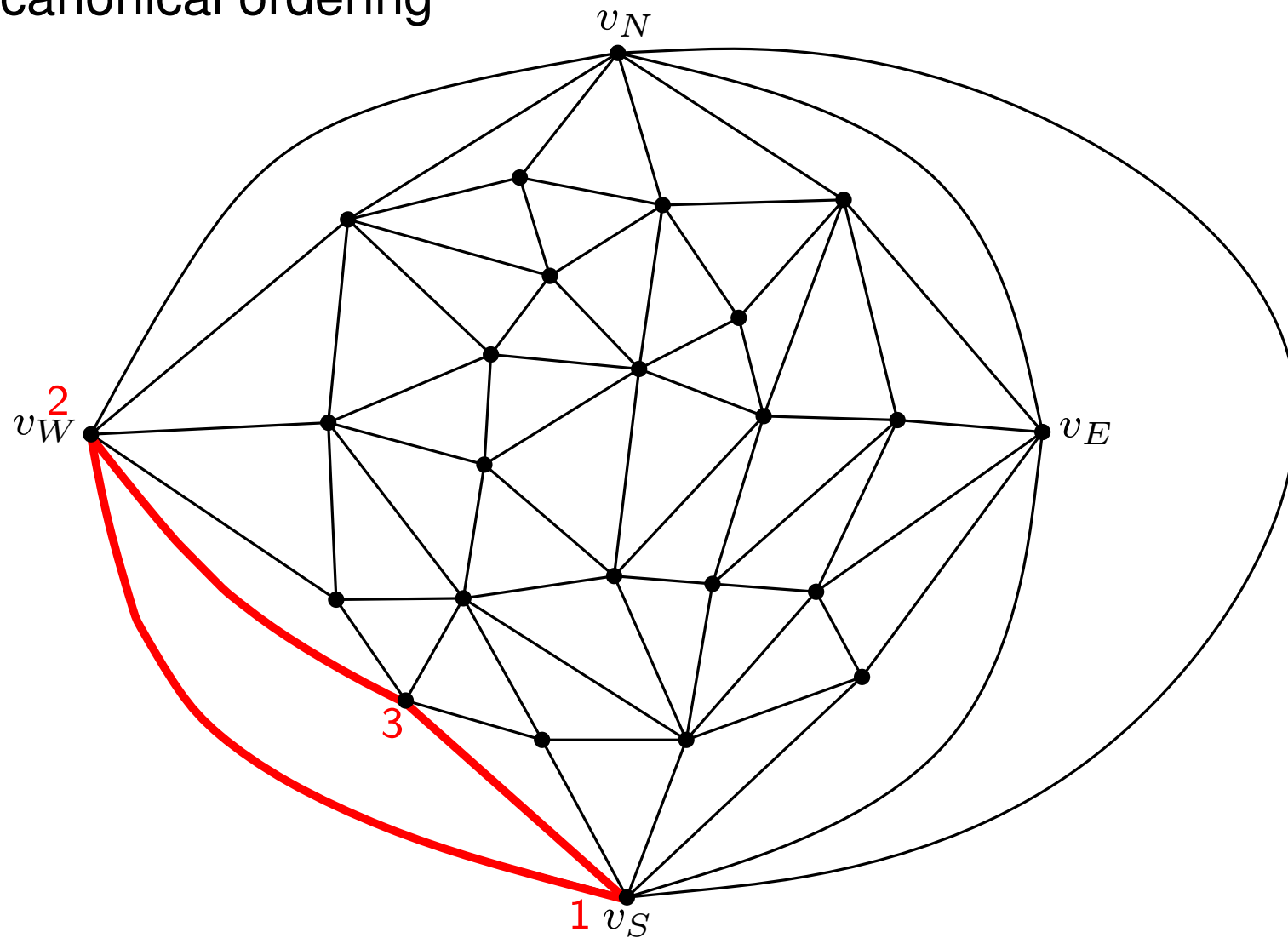
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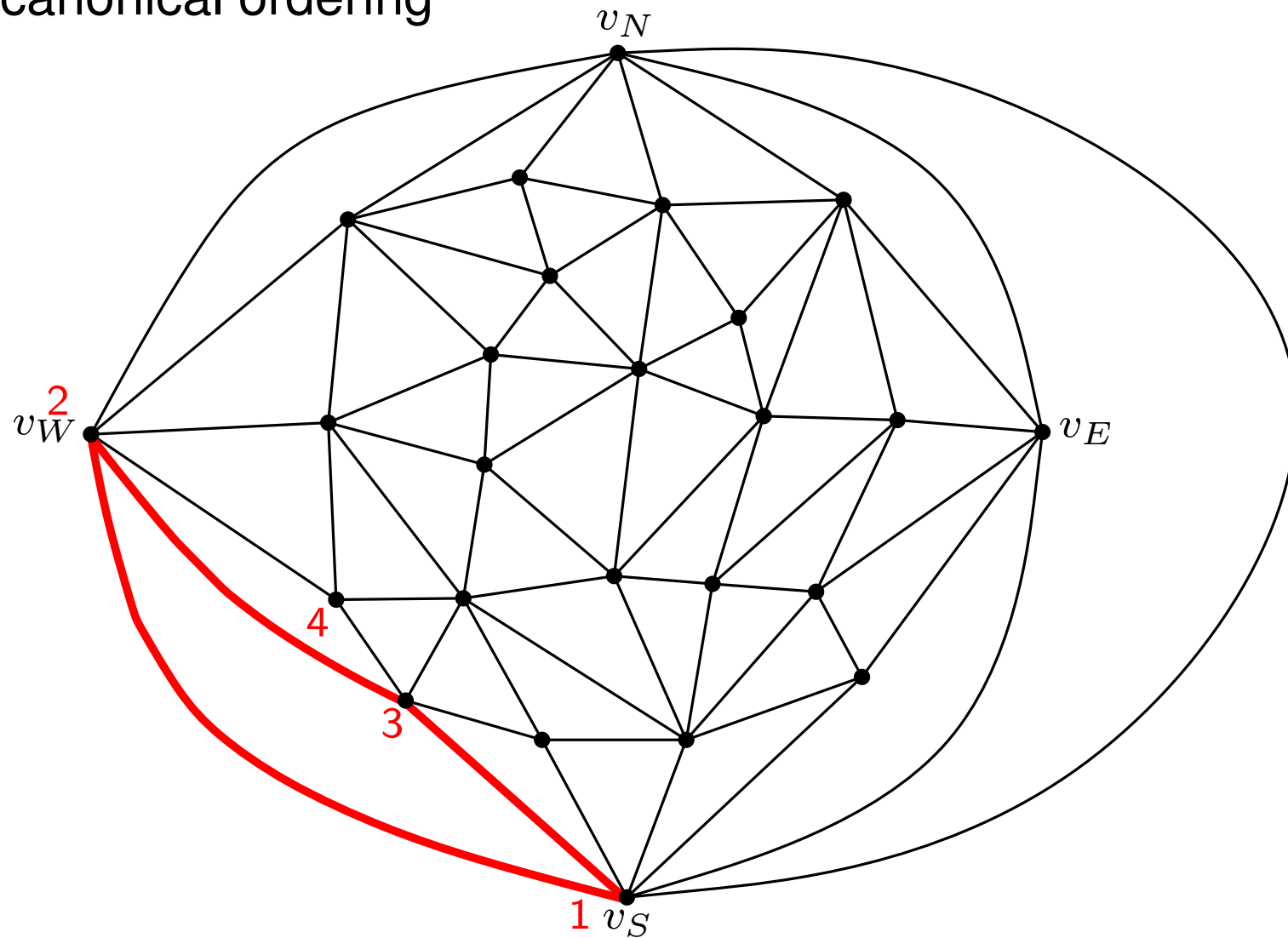
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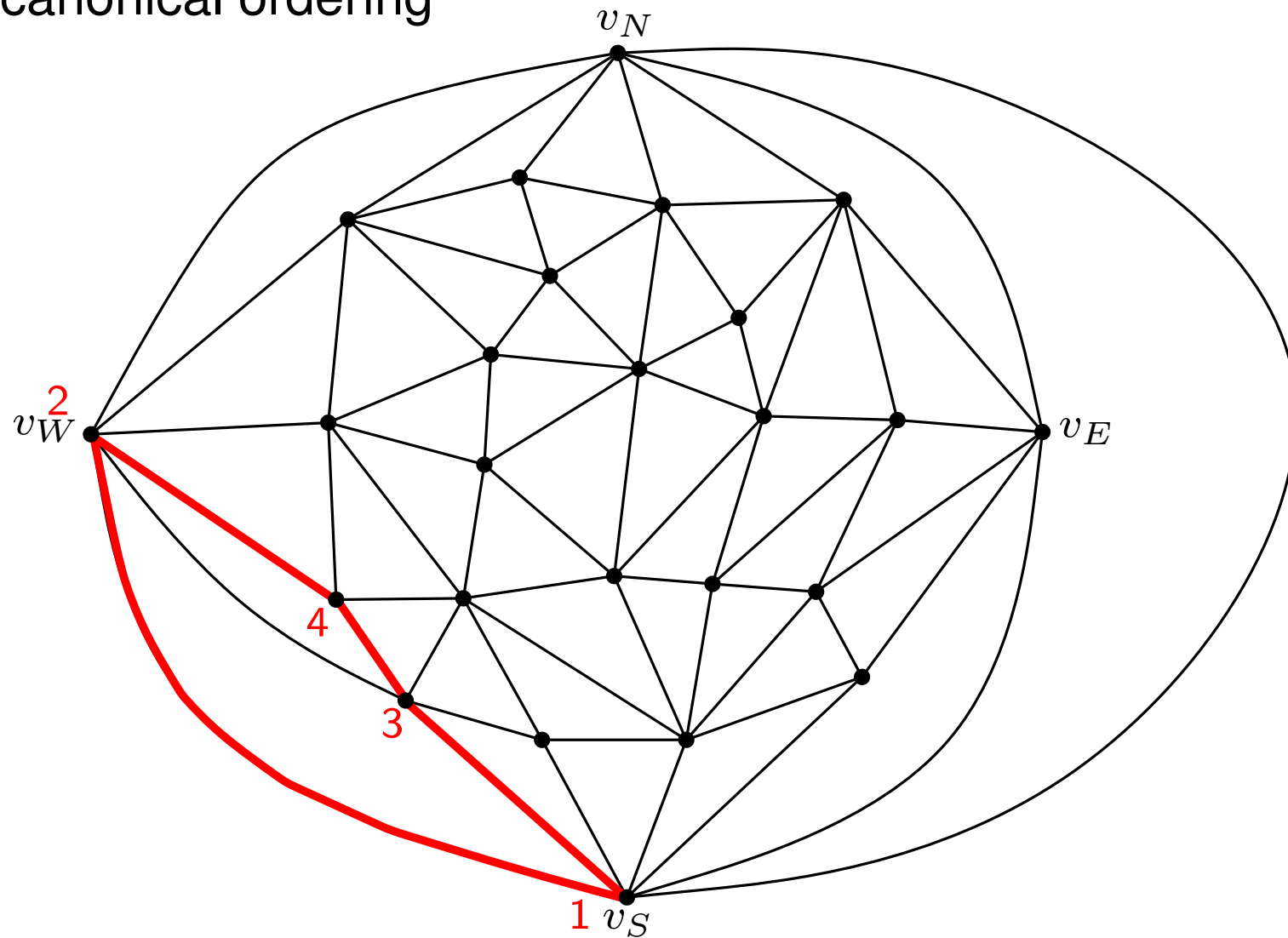
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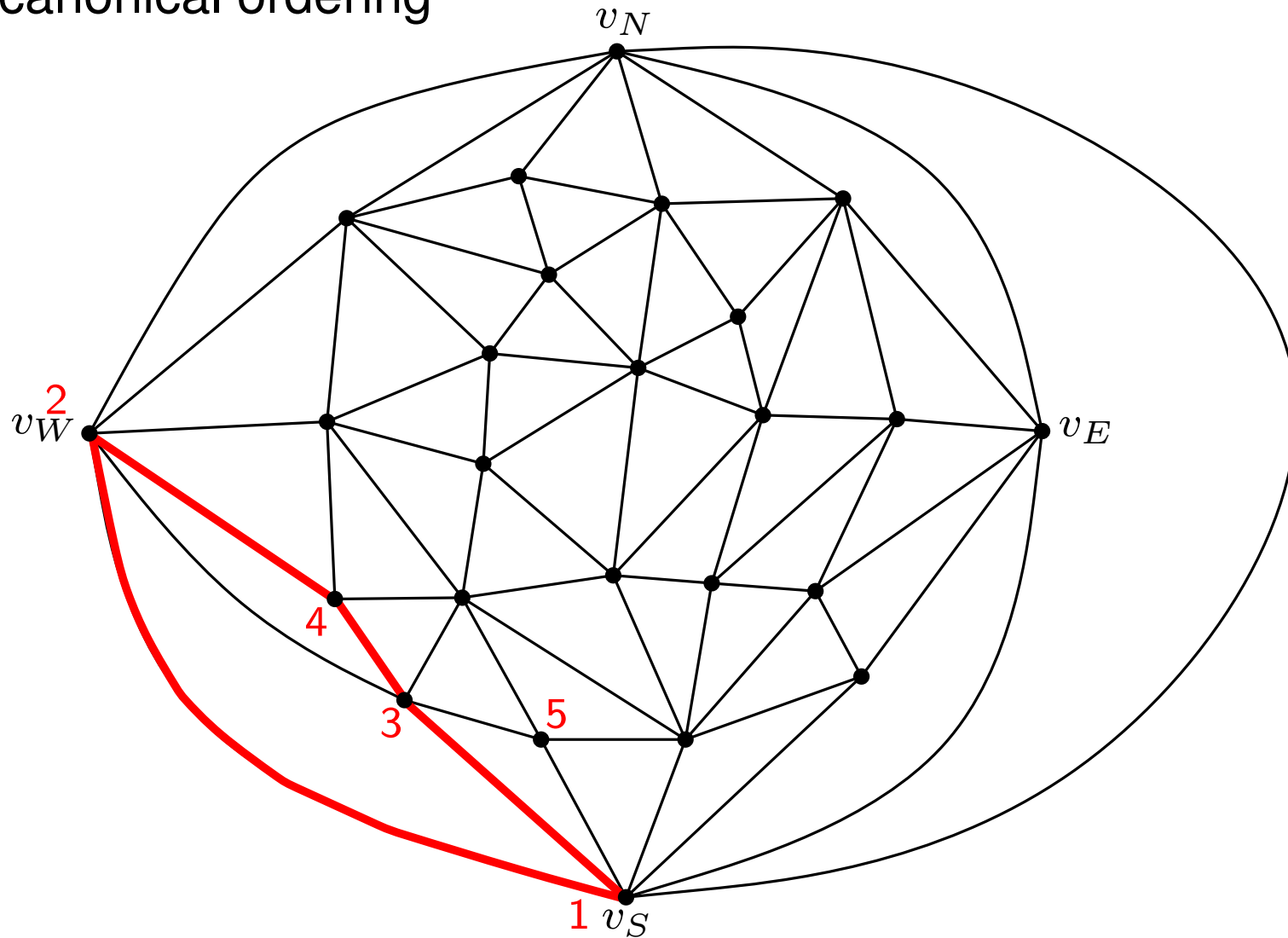
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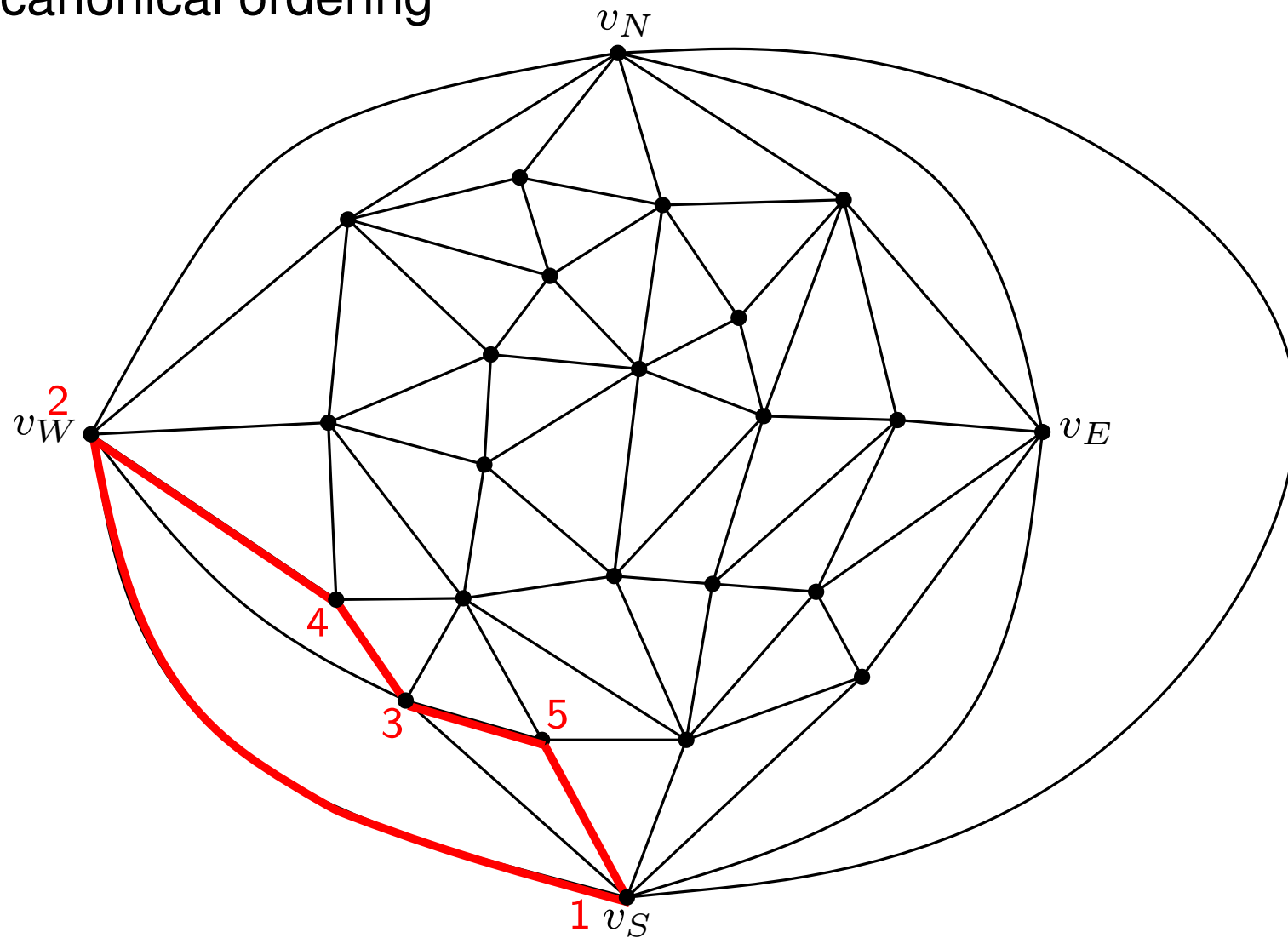
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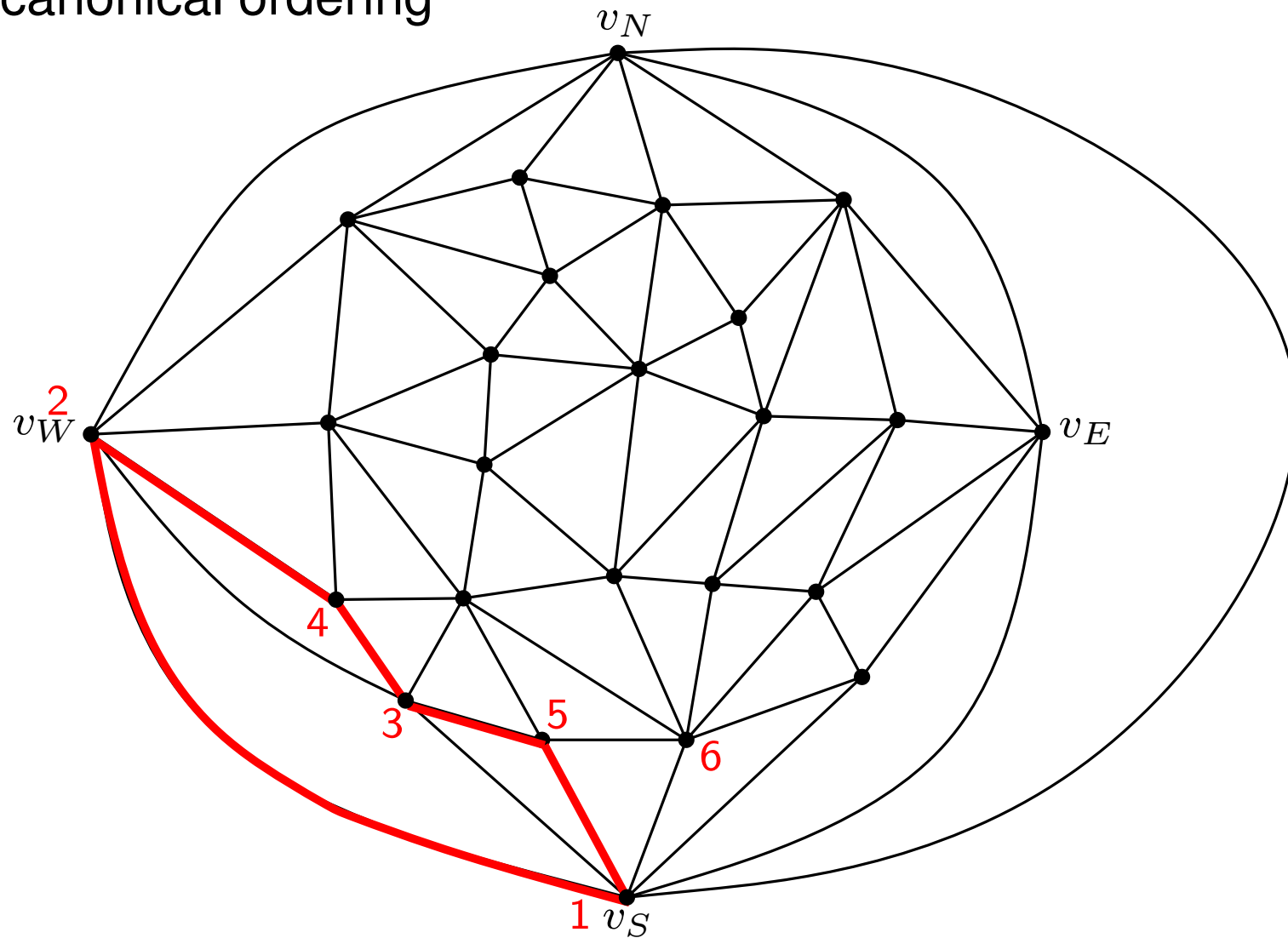
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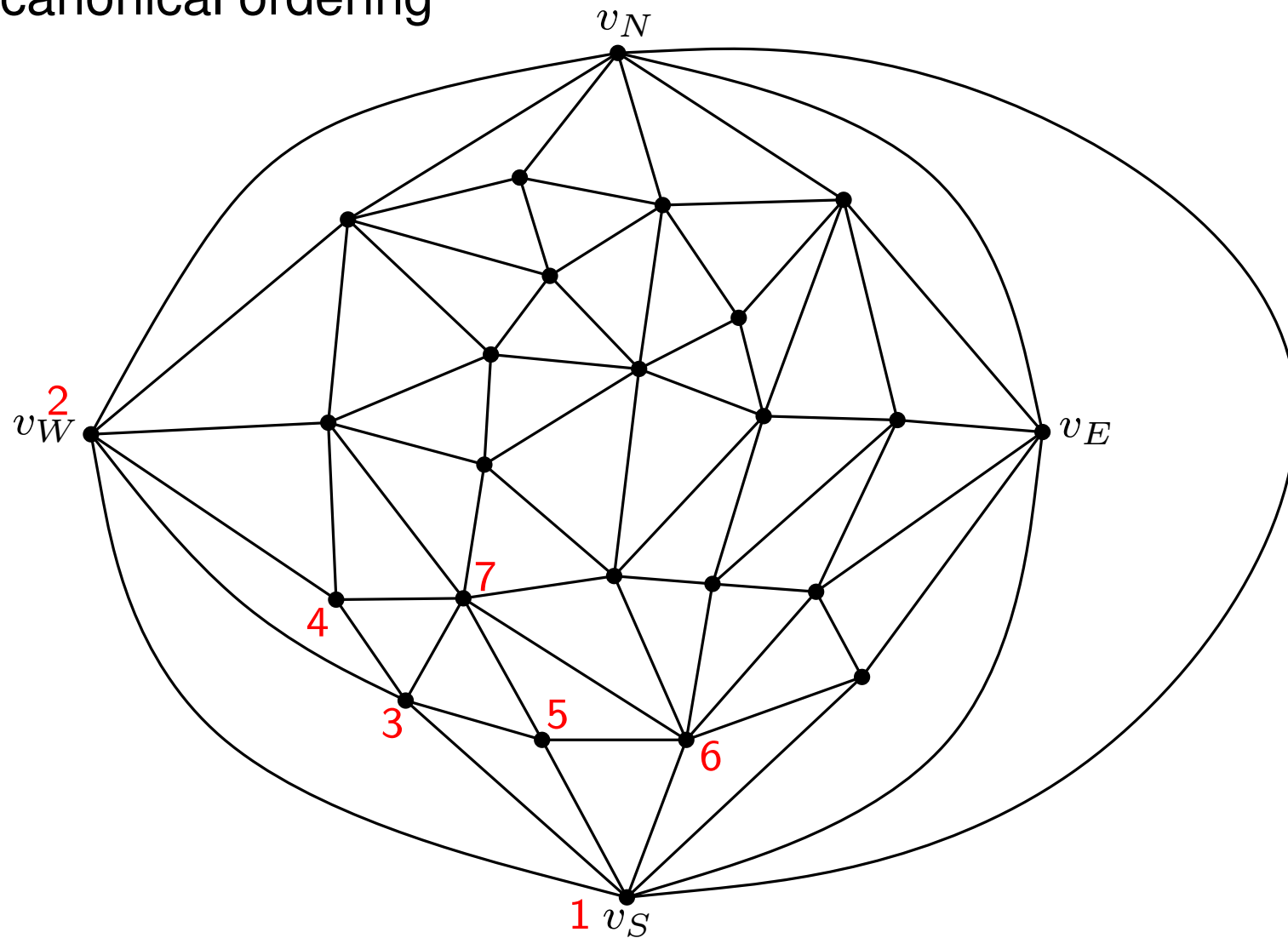
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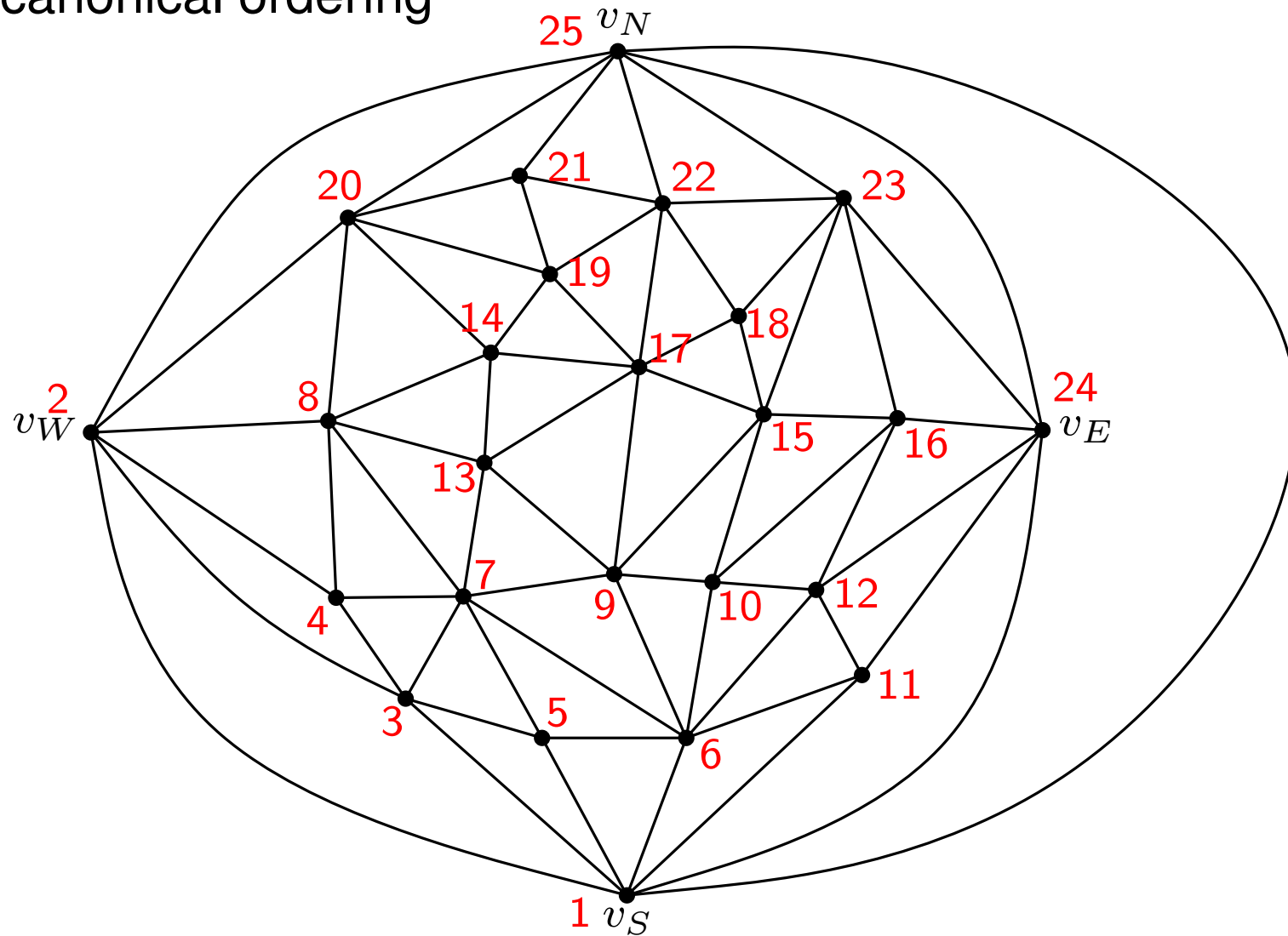
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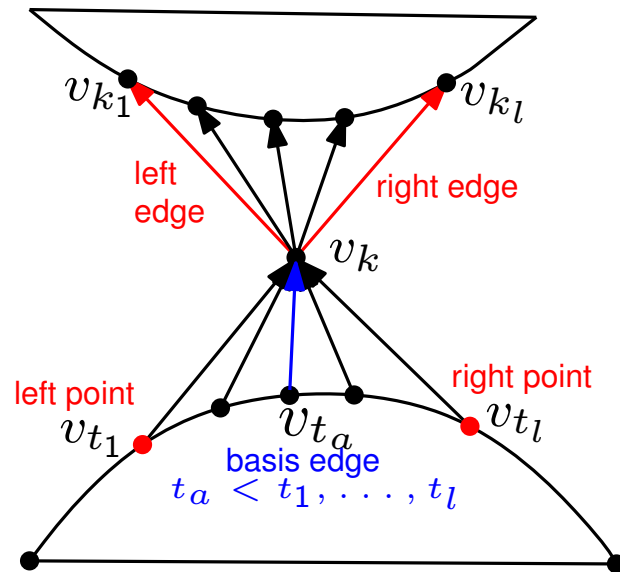
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Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

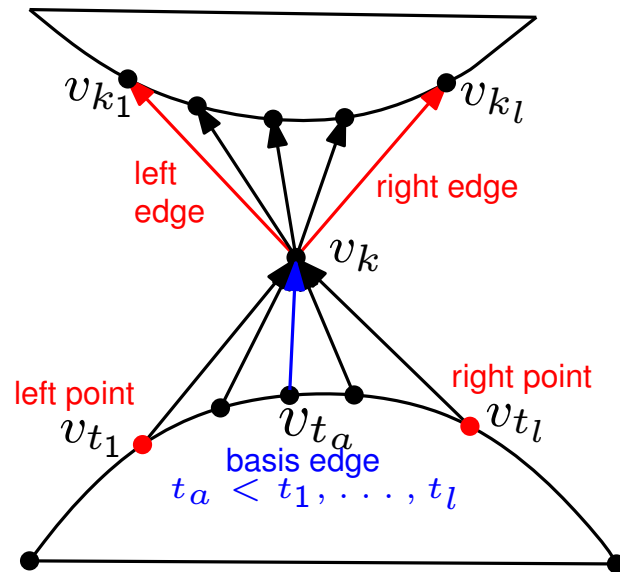
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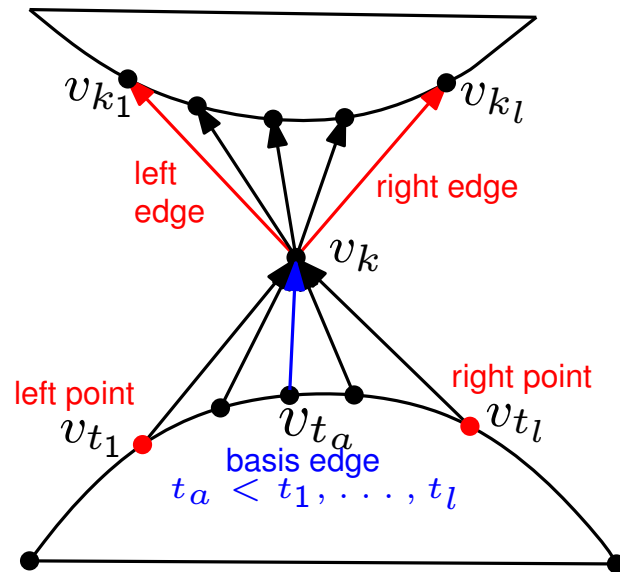
Lemma 1

A basis edge can not be a left edge or right edge.

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- If v_{k_1}, \dots, v_{k_l} are higher numbered neighbors of v_k , we call (v_k, v_{k_1}) **left edge** and (v_k, v_{k_l}) **right edge**.



Lemma 1

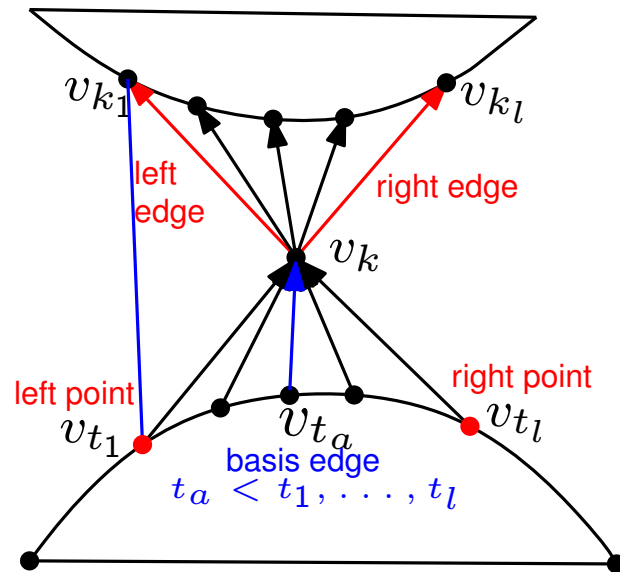
A basis edge can not be a left edge or right edge.

Proof: Assume that left edge (v_k, v_{k_1}) is the base edge of v_{k_1} .

Rectangular Dual

Given a refined canonical ordering of G we construct a REL as follows:

- For each (v_i, v_j) orient it from v_i to v_j , for $i < j$;
- **Base edge** of v_k is (v_l, v_k) , where $l < k$ is minimal.
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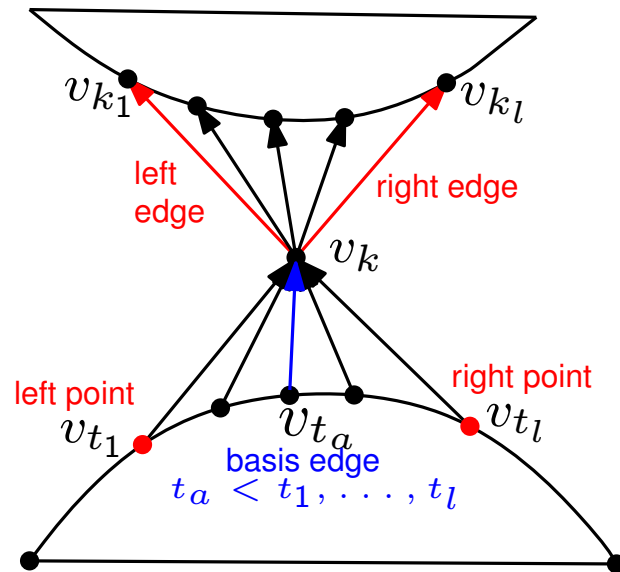
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Lemma 2

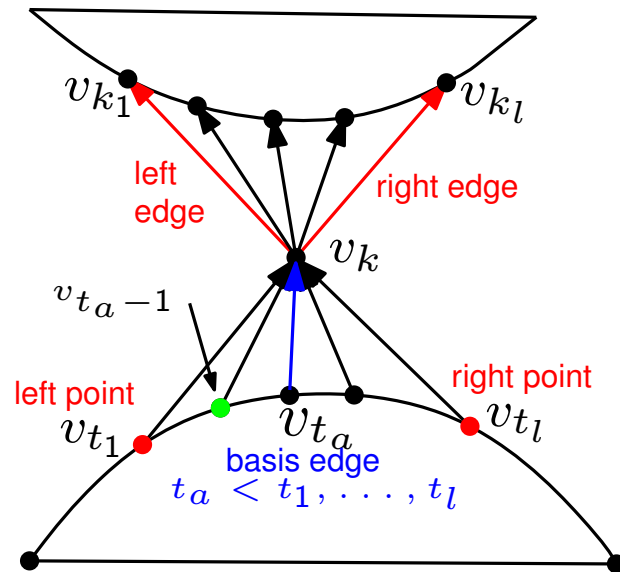
An edge is either a left edge, a right edge or a basis edge.

Proof:

- The exclusive “or” follows from Lemma 1.
- Let (v_{t_a}, v_k) be basis edge of v_k .
- v_{t_a} is right point of $v_{t_{a-1}}$, $v_{t_{a-1}}$ is right point of $v_{t_{a-2}}$, generally $v_{t_{i+1}}$ is right point of v_{t_i} , $1 \leq i < a - 1$
- Edges (v_k, v_{t_i}) , $1 \leq i < a - 1$, are right edges;
- Similarly we prove that edges (v_k, v_{t_i}) , $a + 1 \leq i < l$, are left edges;

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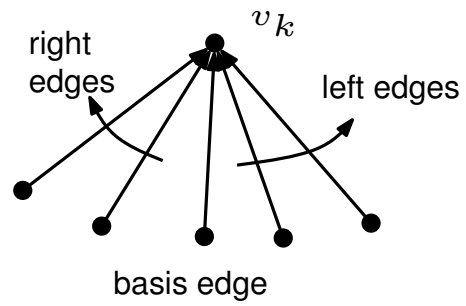
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An edge is either a left edge, a right edge or a basis edge.

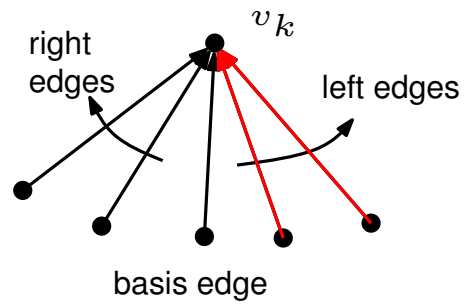
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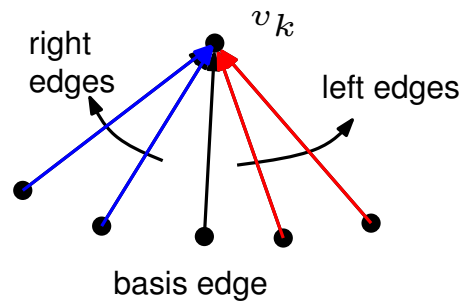
Rectangular Dual



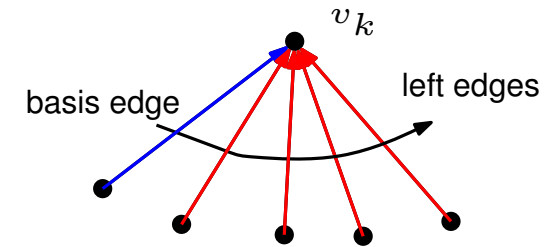
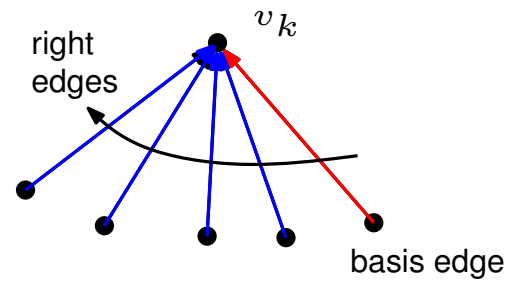
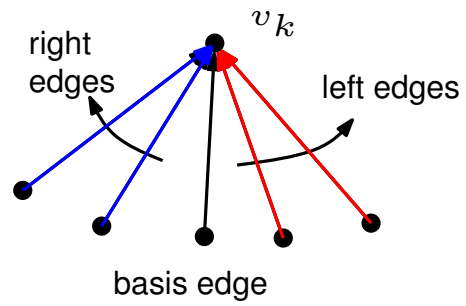
Rectangular Dual



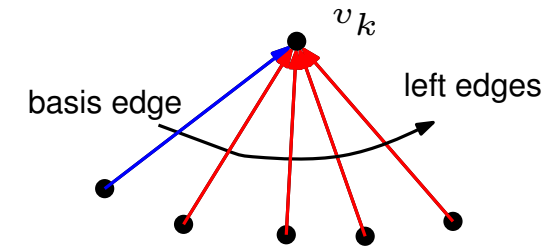
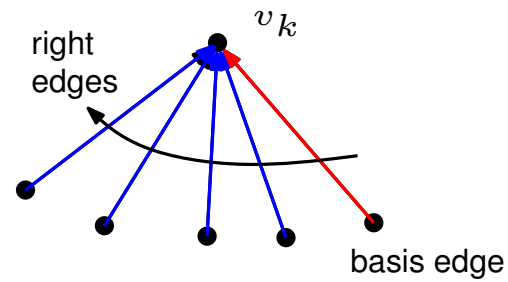
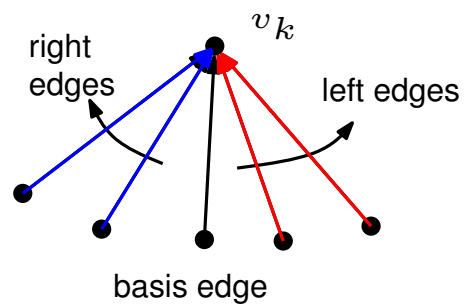
Rectangular Dual



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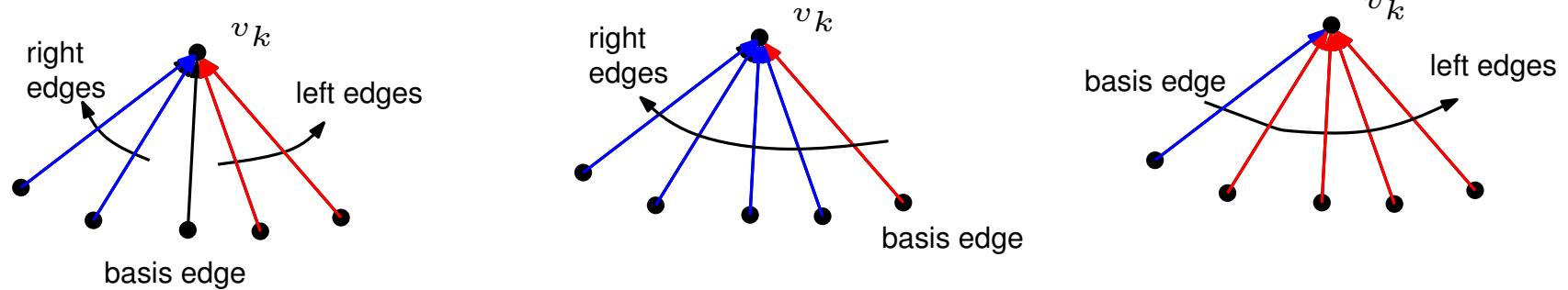


Rectangular Dual



We call T_b blue edges and T_r red edges.

Rectangular Dual

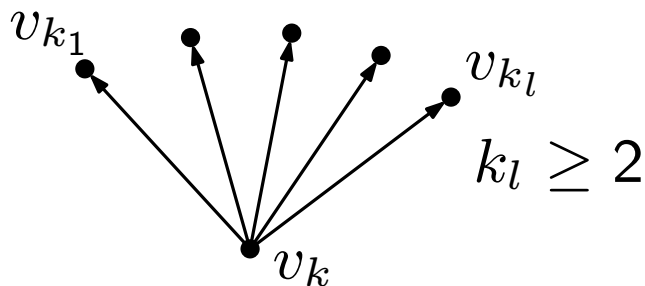


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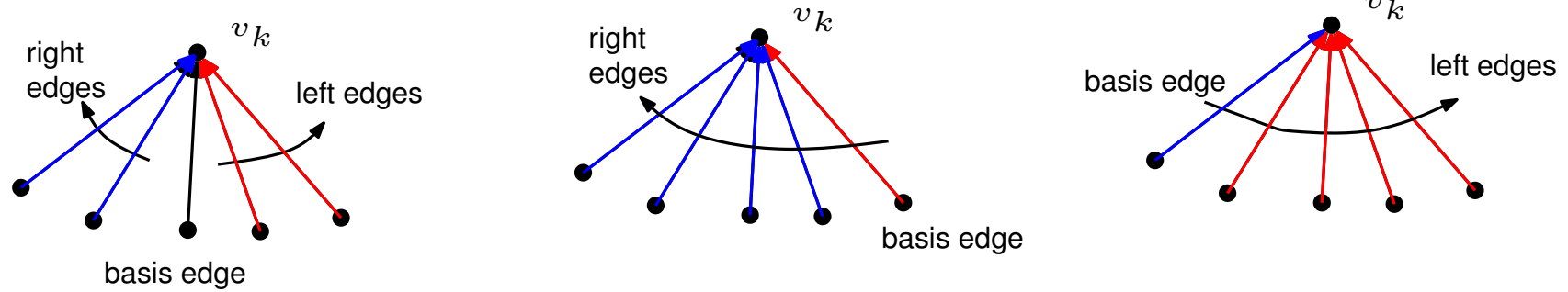
Lemma 3

$\{T_r, T_b\}$ is a regular edge labeling.

Proof:



Rectangular Dual

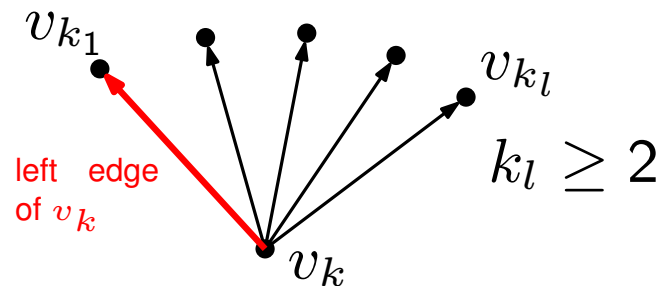


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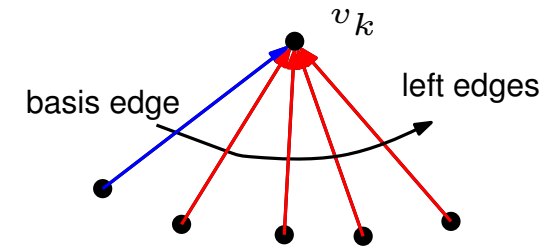
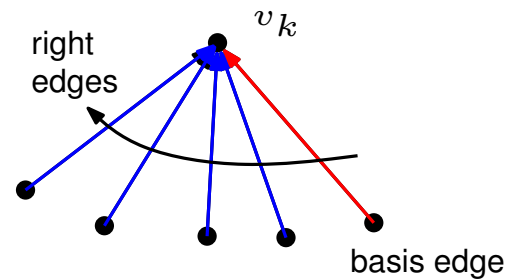
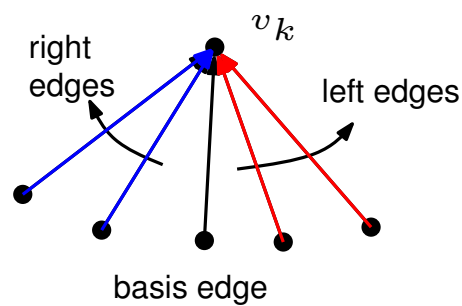
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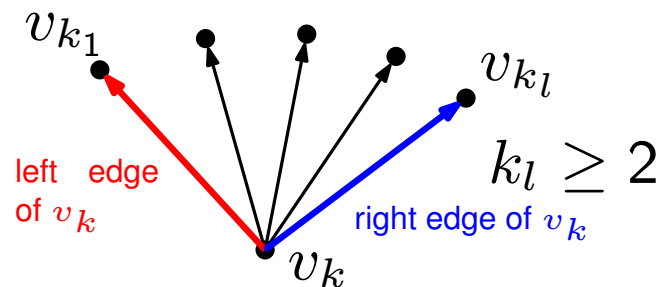


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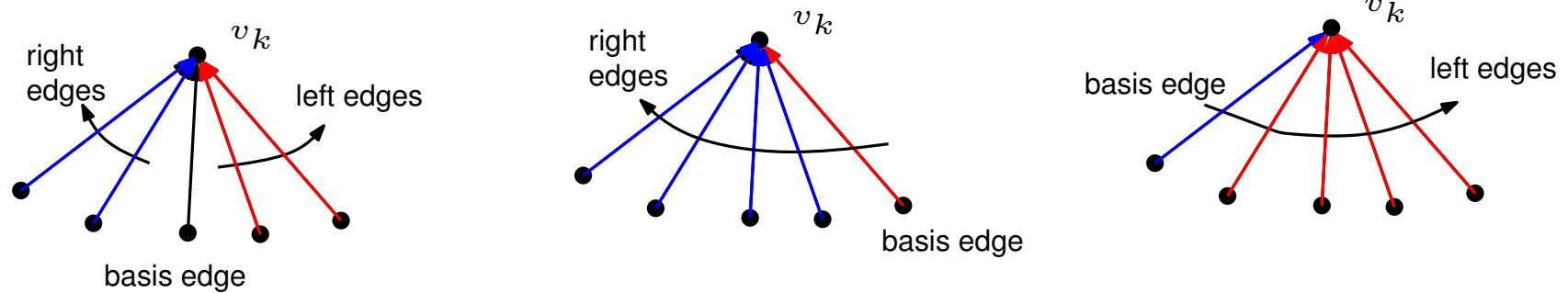
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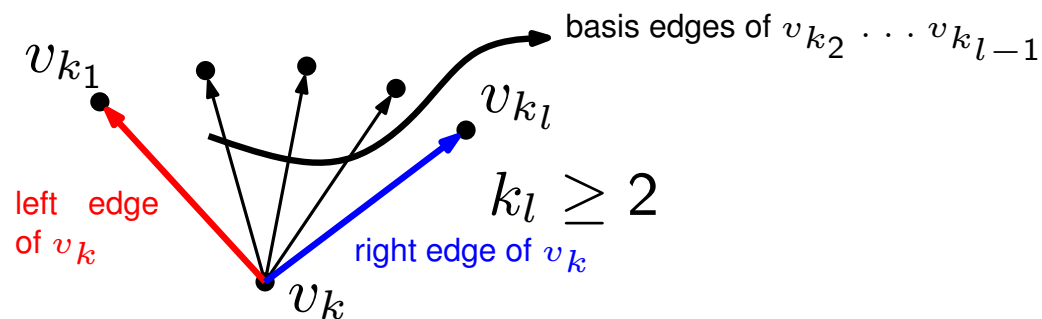


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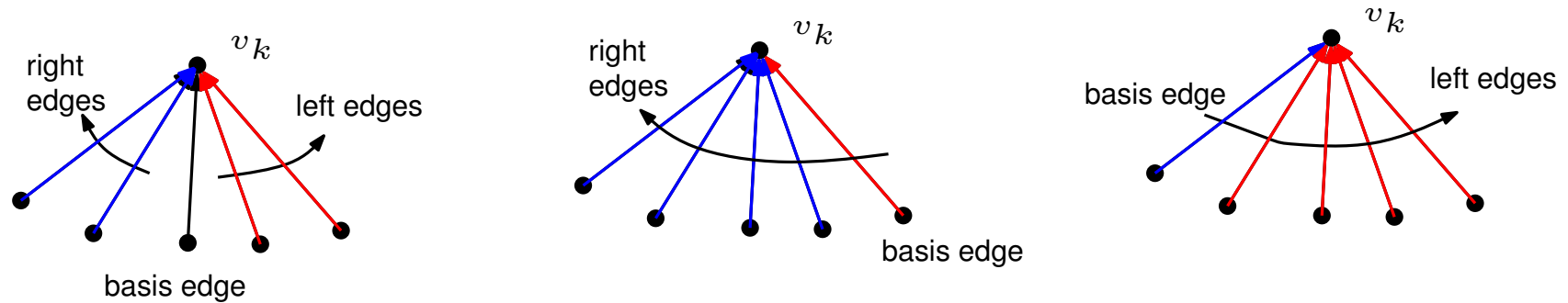
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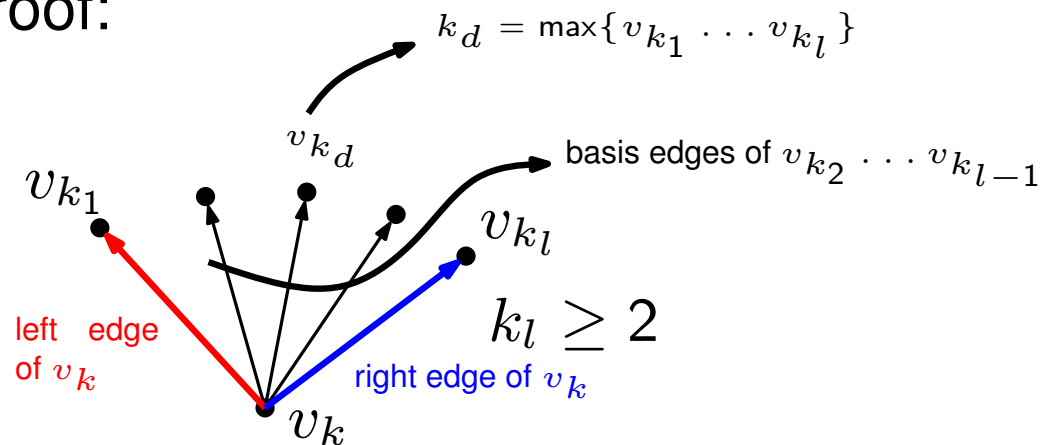


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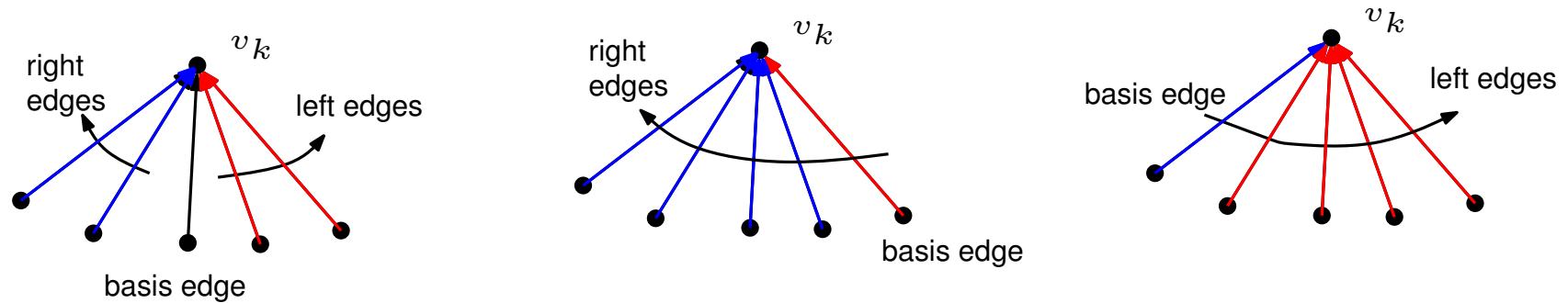
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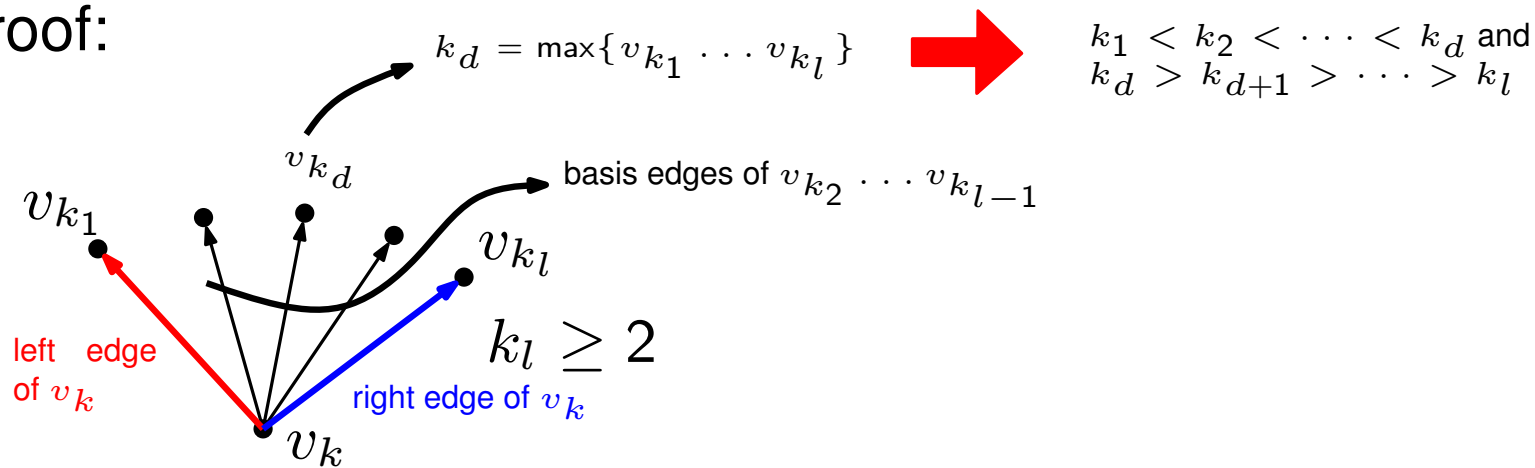
Rectangular Dual



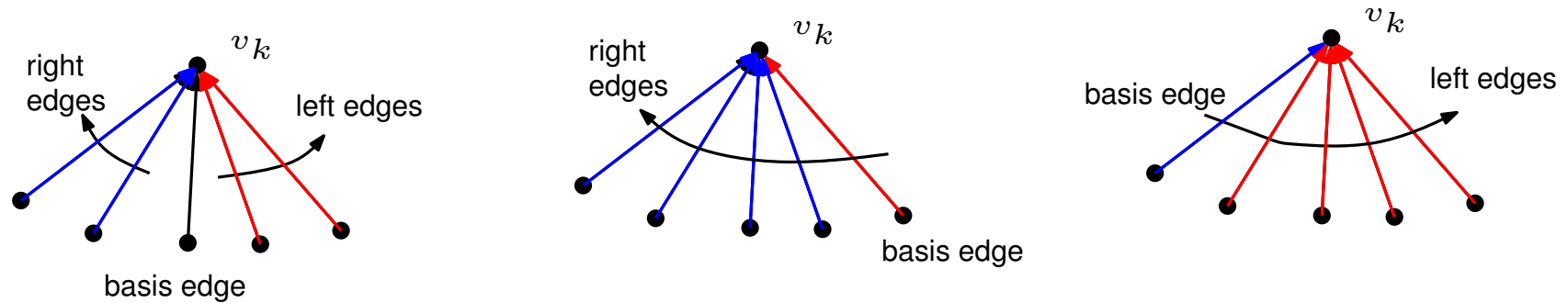
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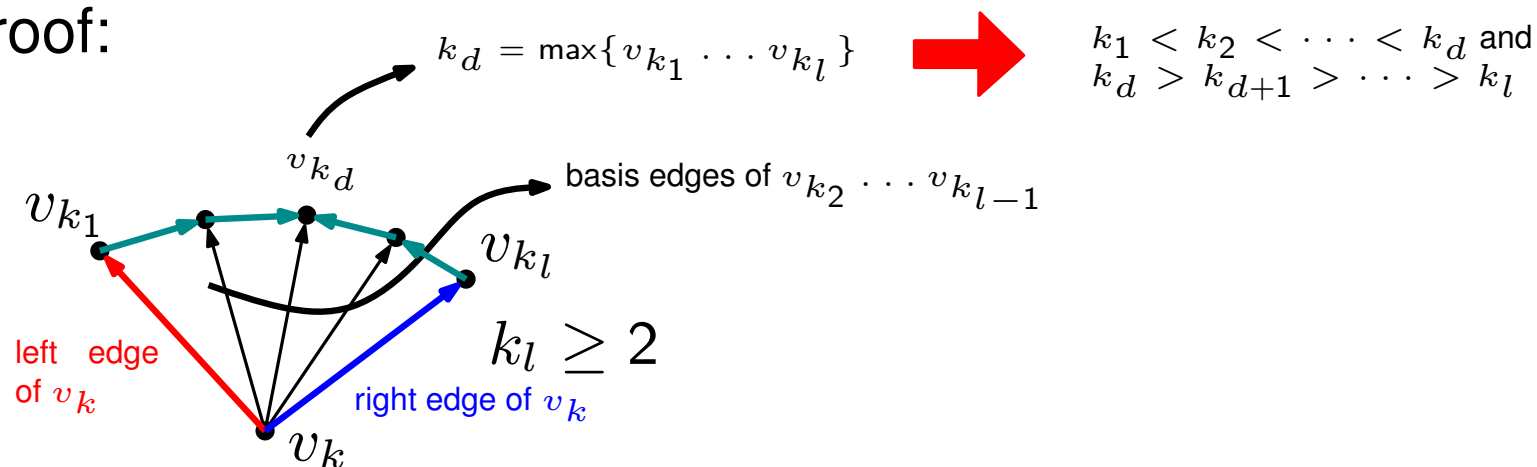


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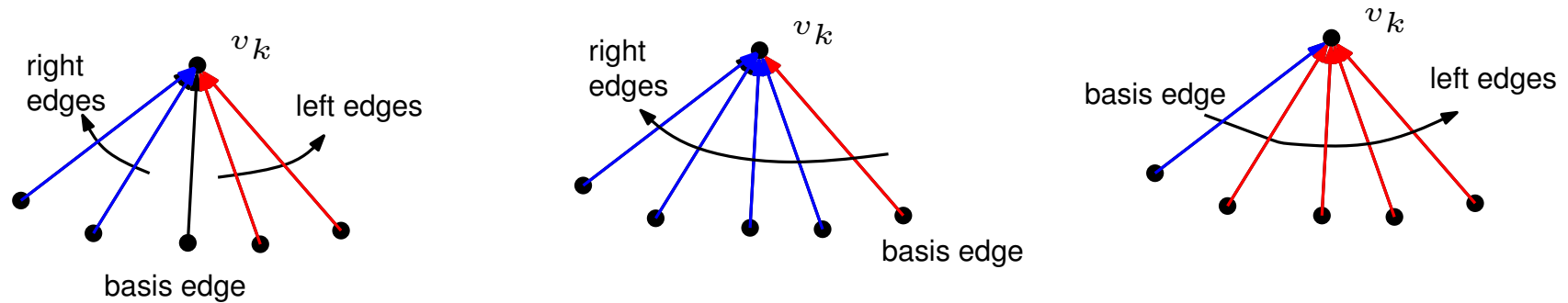
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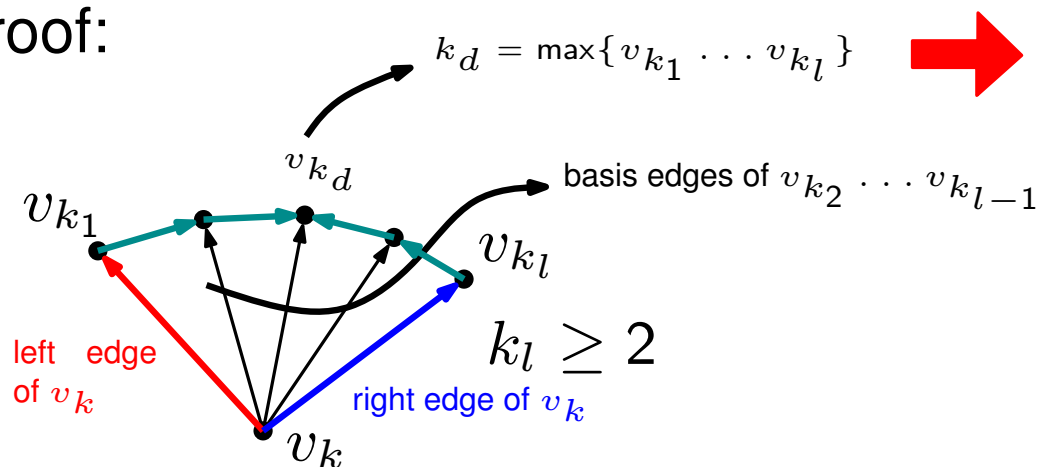
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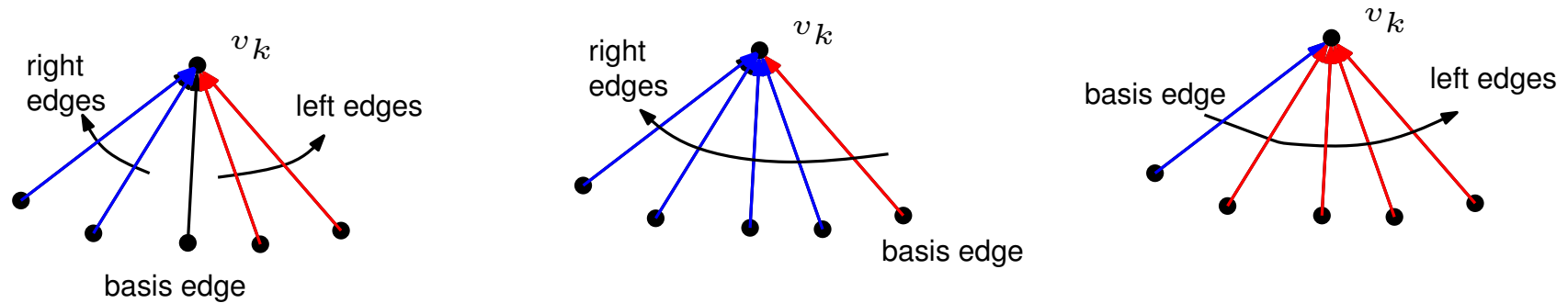


$$k_1 < k_2 < \dots < k_d \text{ and } k_d > k_{d+1} > \dots > k_l$$



$(v_k, v_{k_i}), 2 \leq i \leq d - 1$ are red
 $(v_k, v_{k_i}), d + 1 \leq i \leq l - 1$ are blue
 edge (v_k, v_{k_d}) is either red or blue

Rectangular Dual

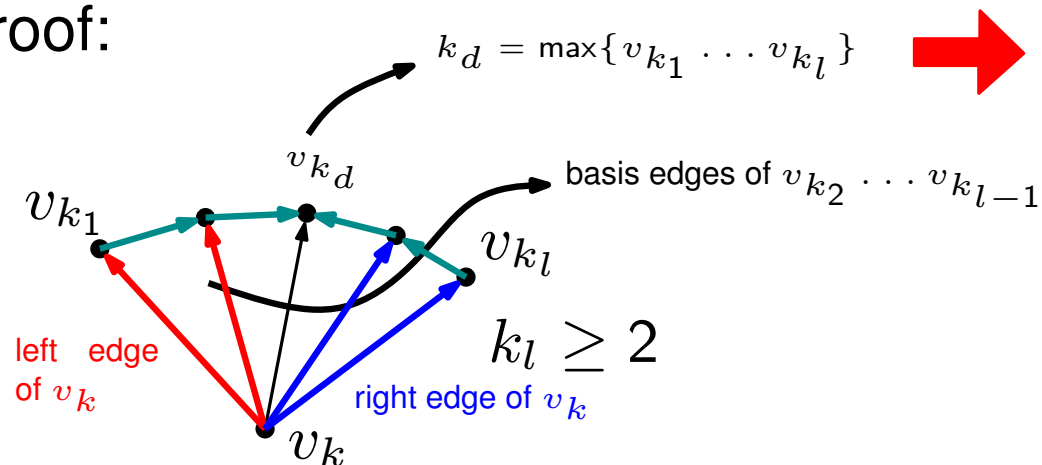


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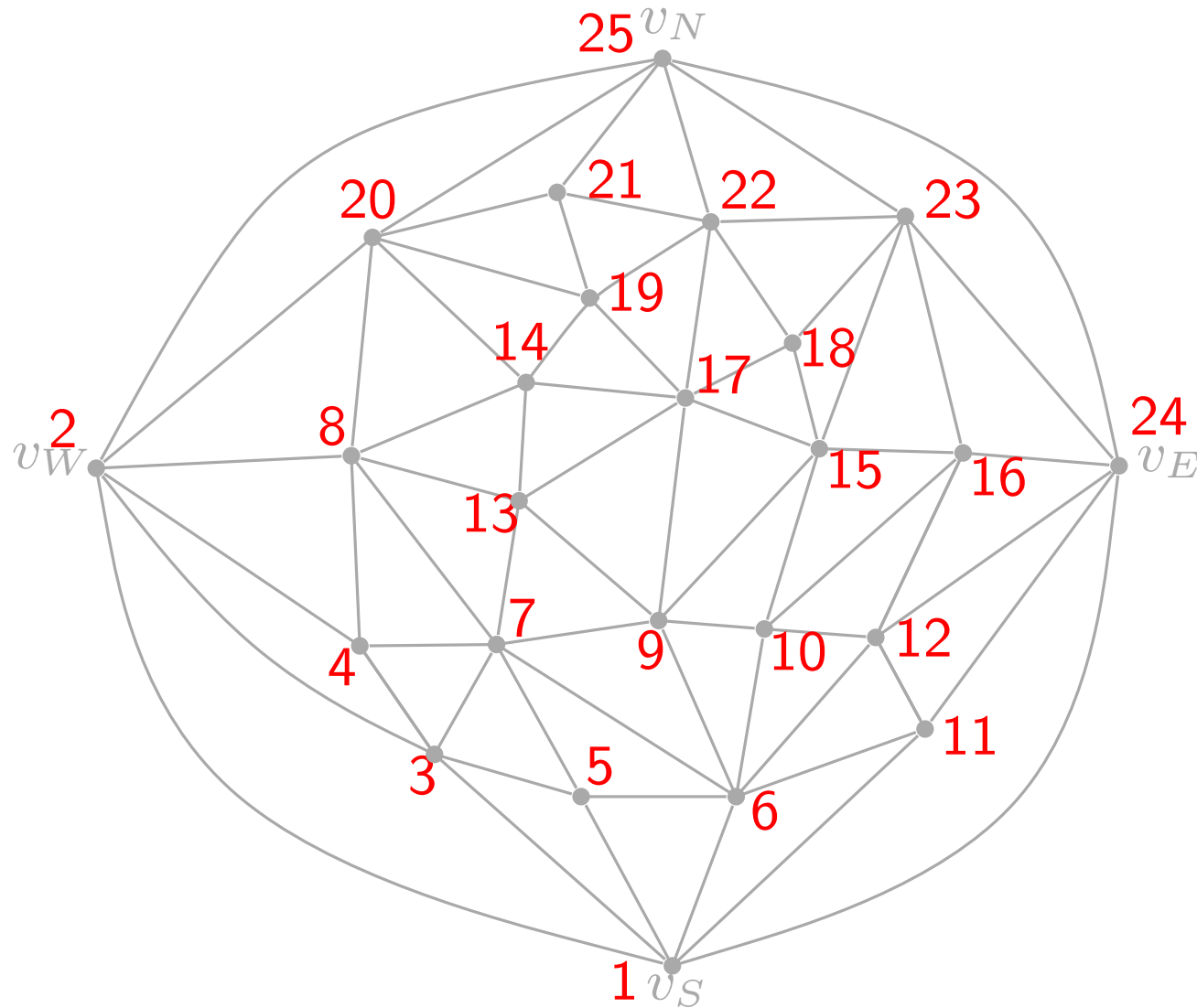
Proof:



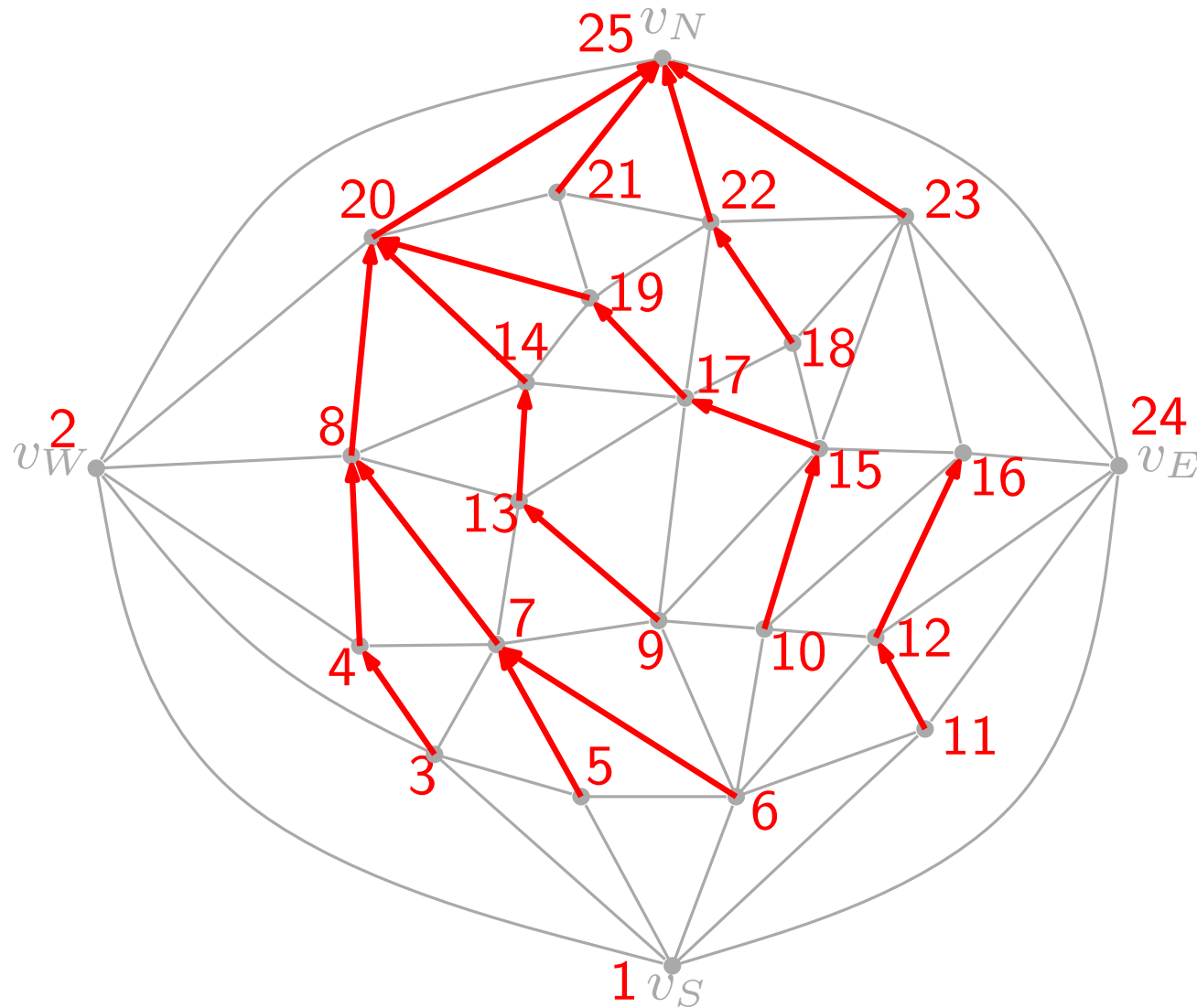
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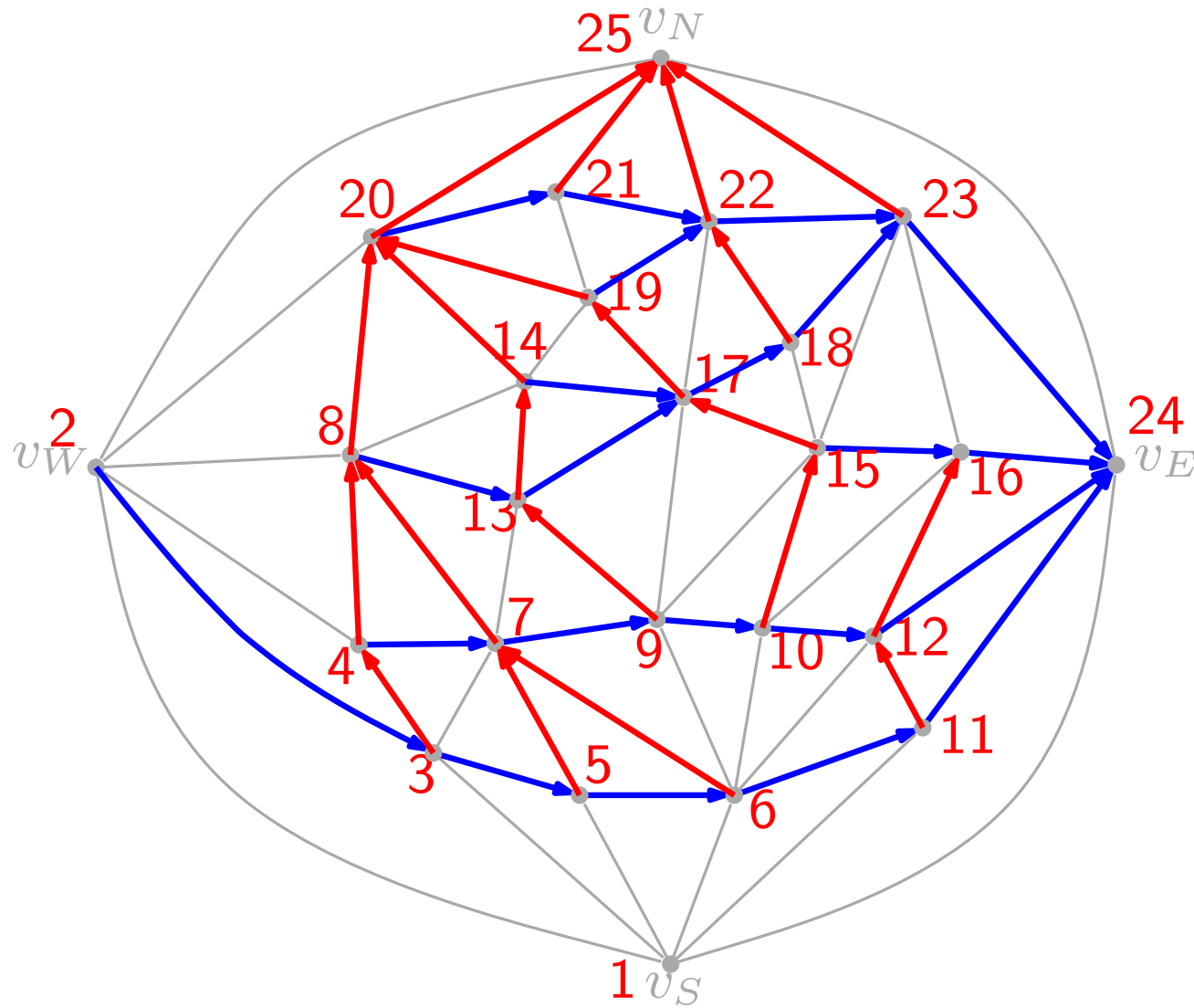
Rectangular Dual



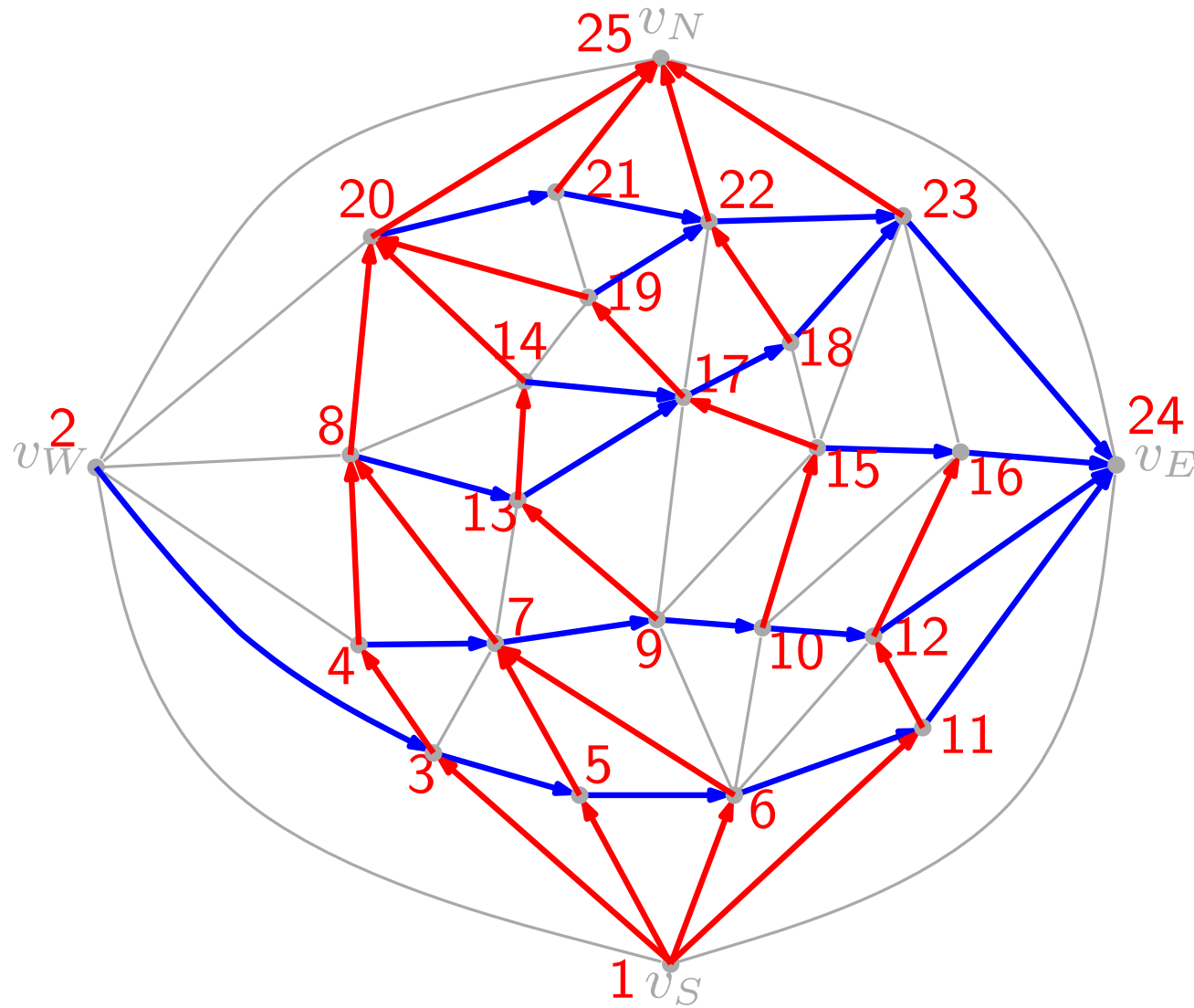
Rectangular Dual



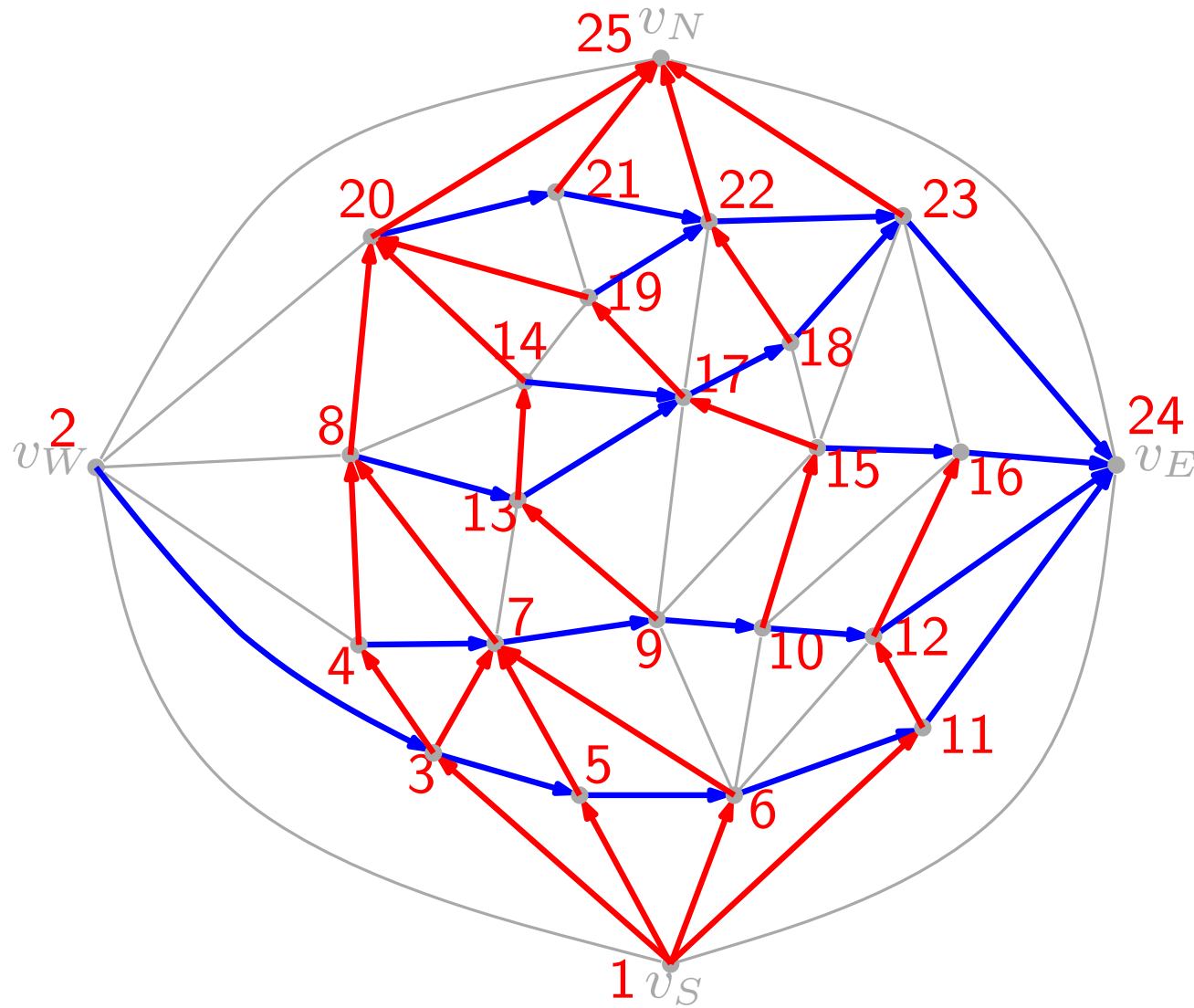
Rectangular Dual



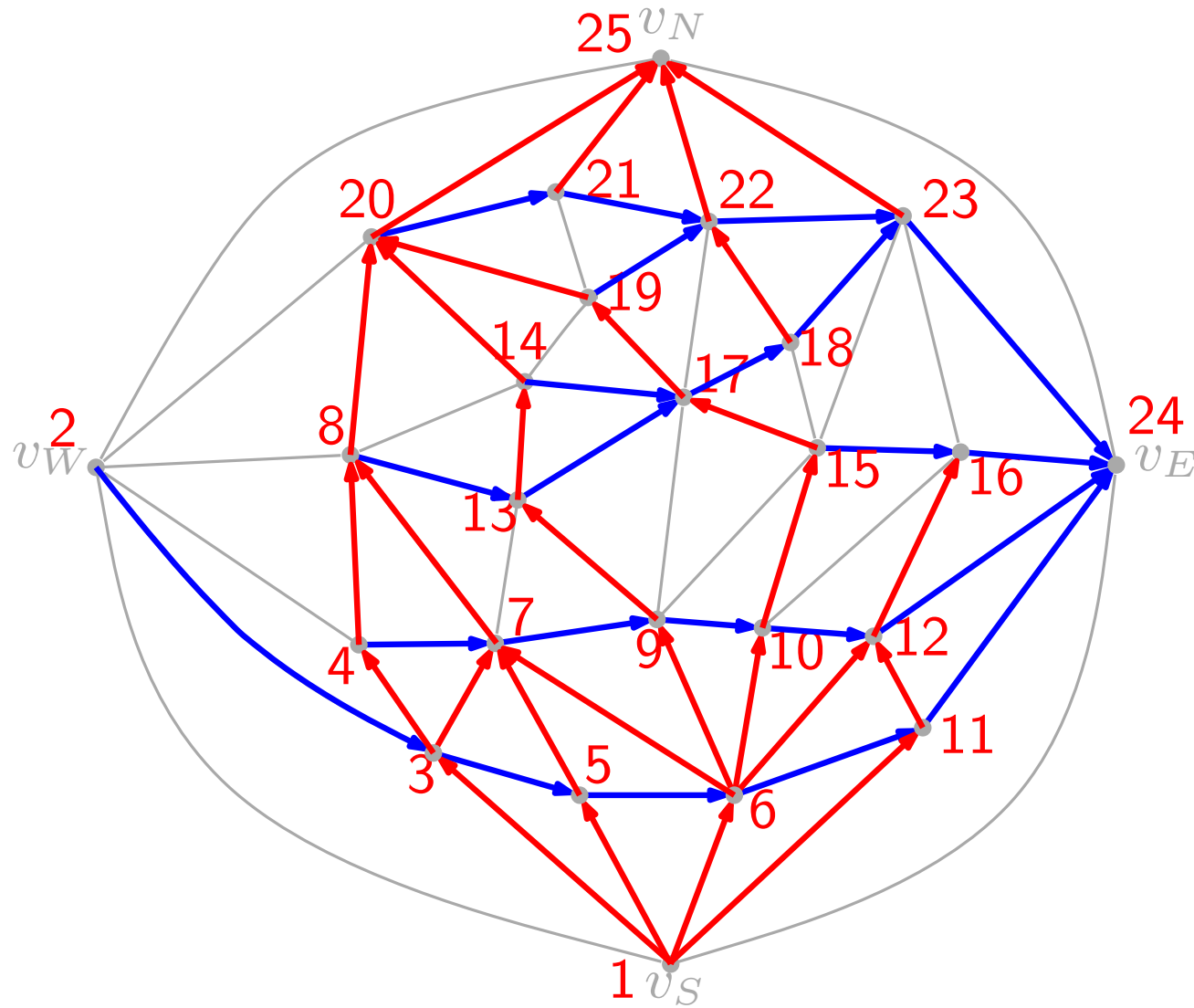
Rectangular Dual



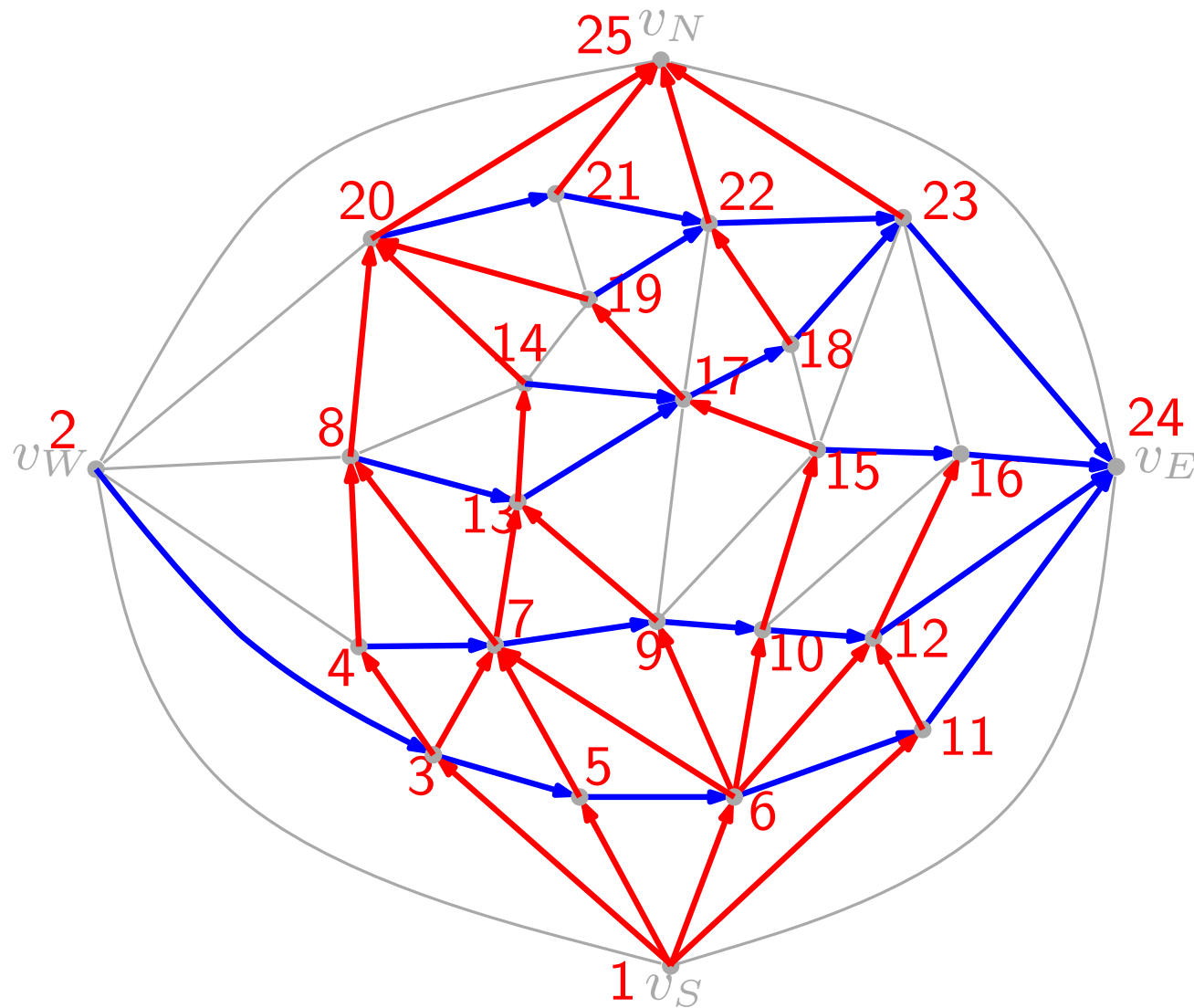
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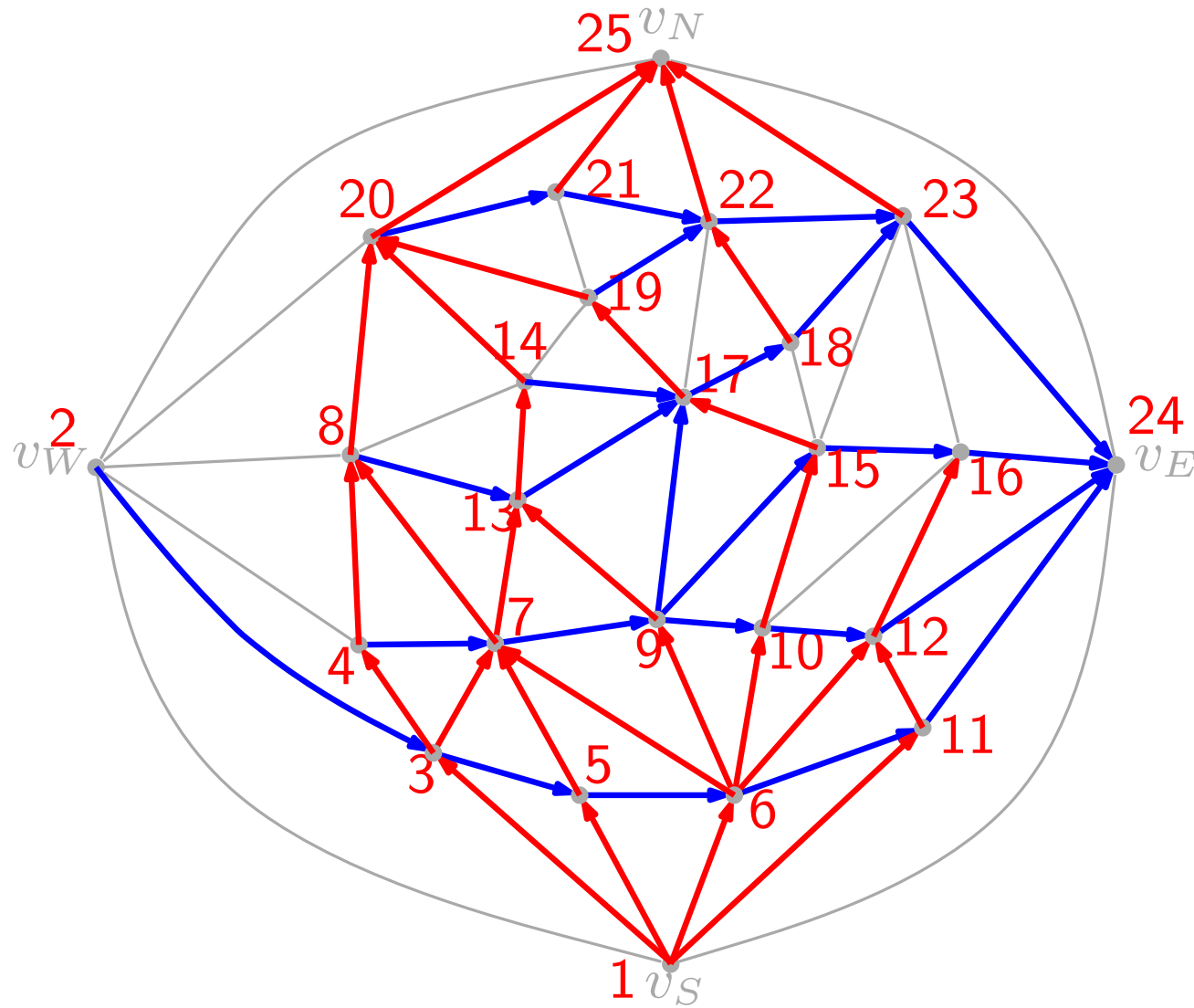
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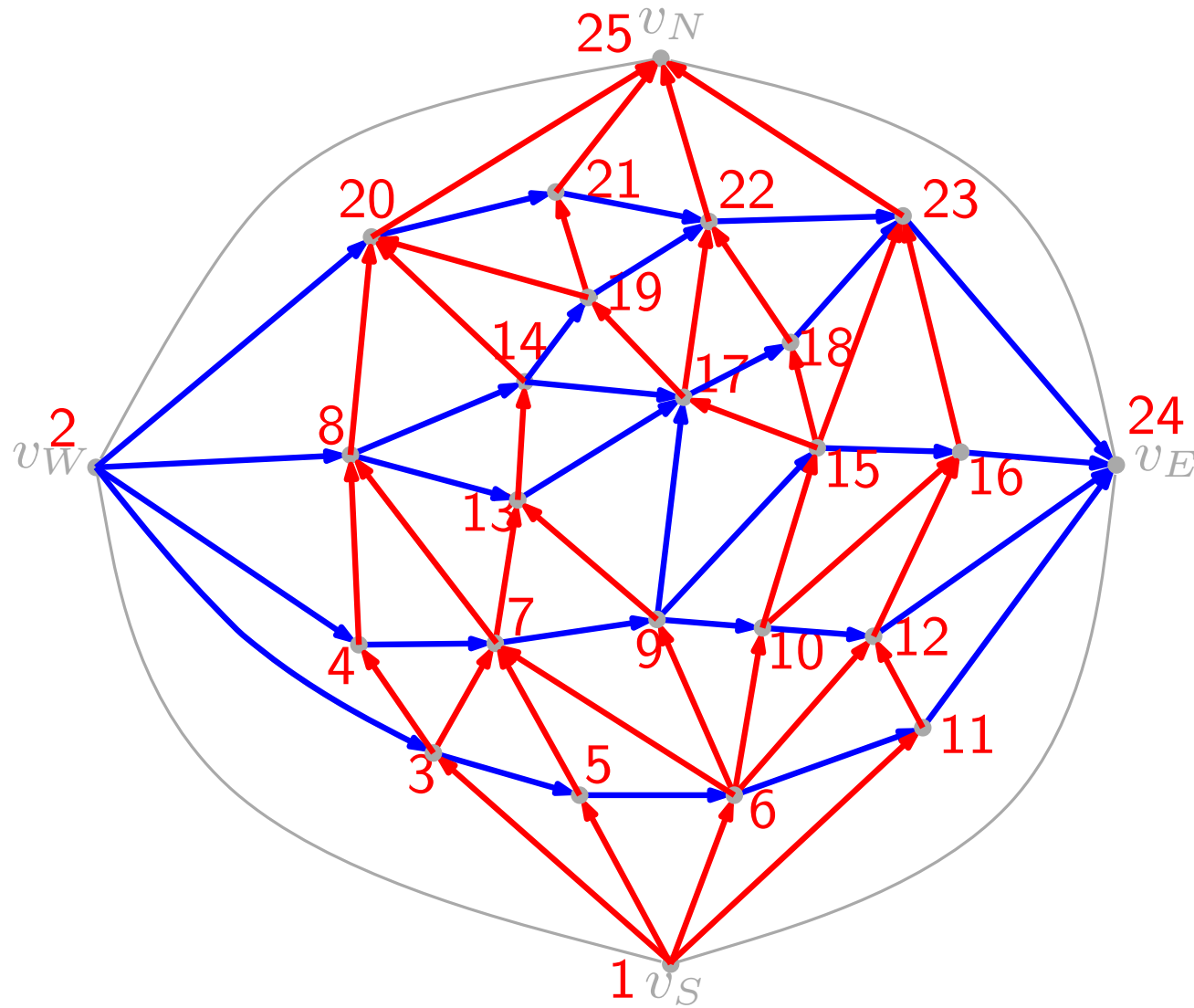
Rectangular Dual



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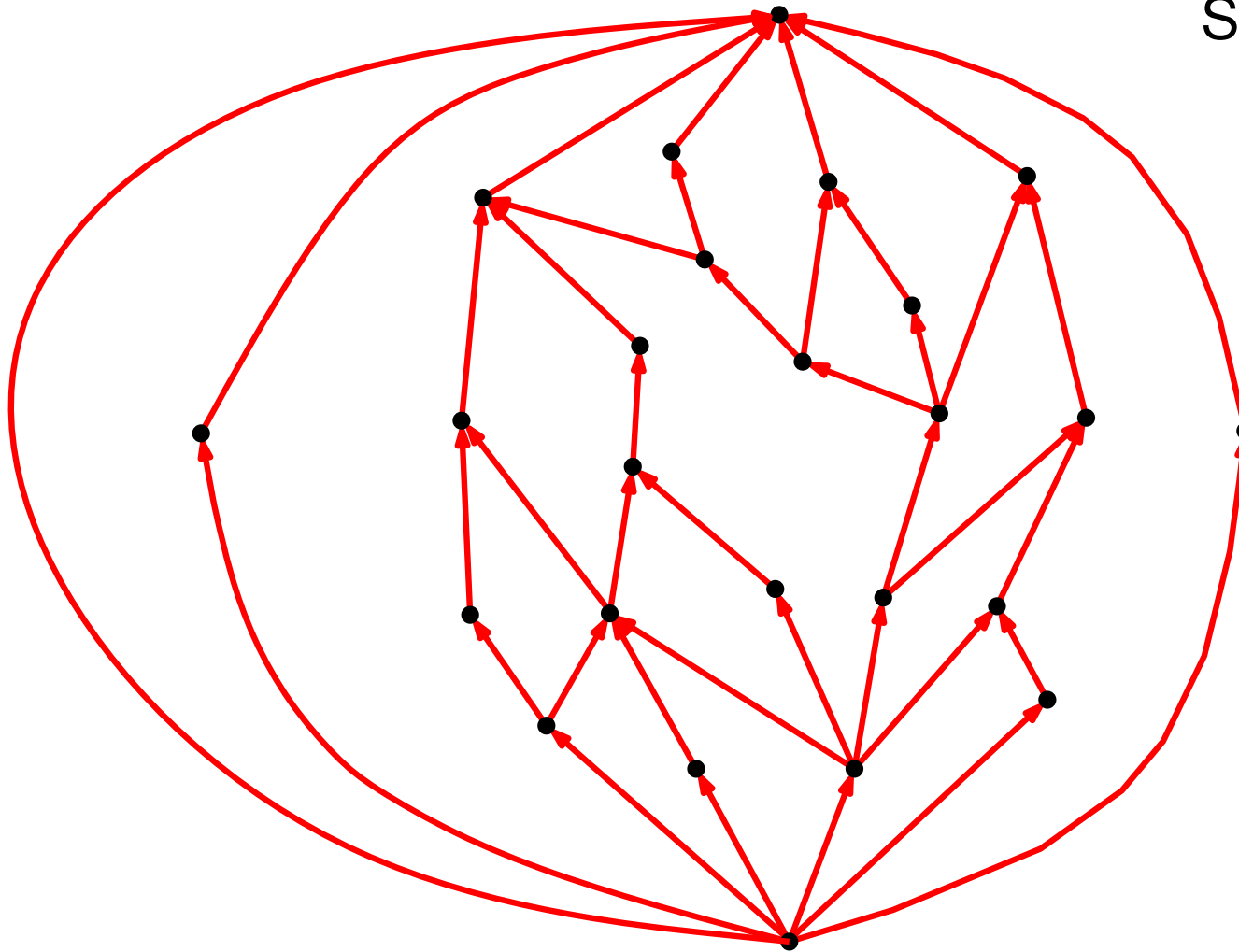


Rectangular Dual



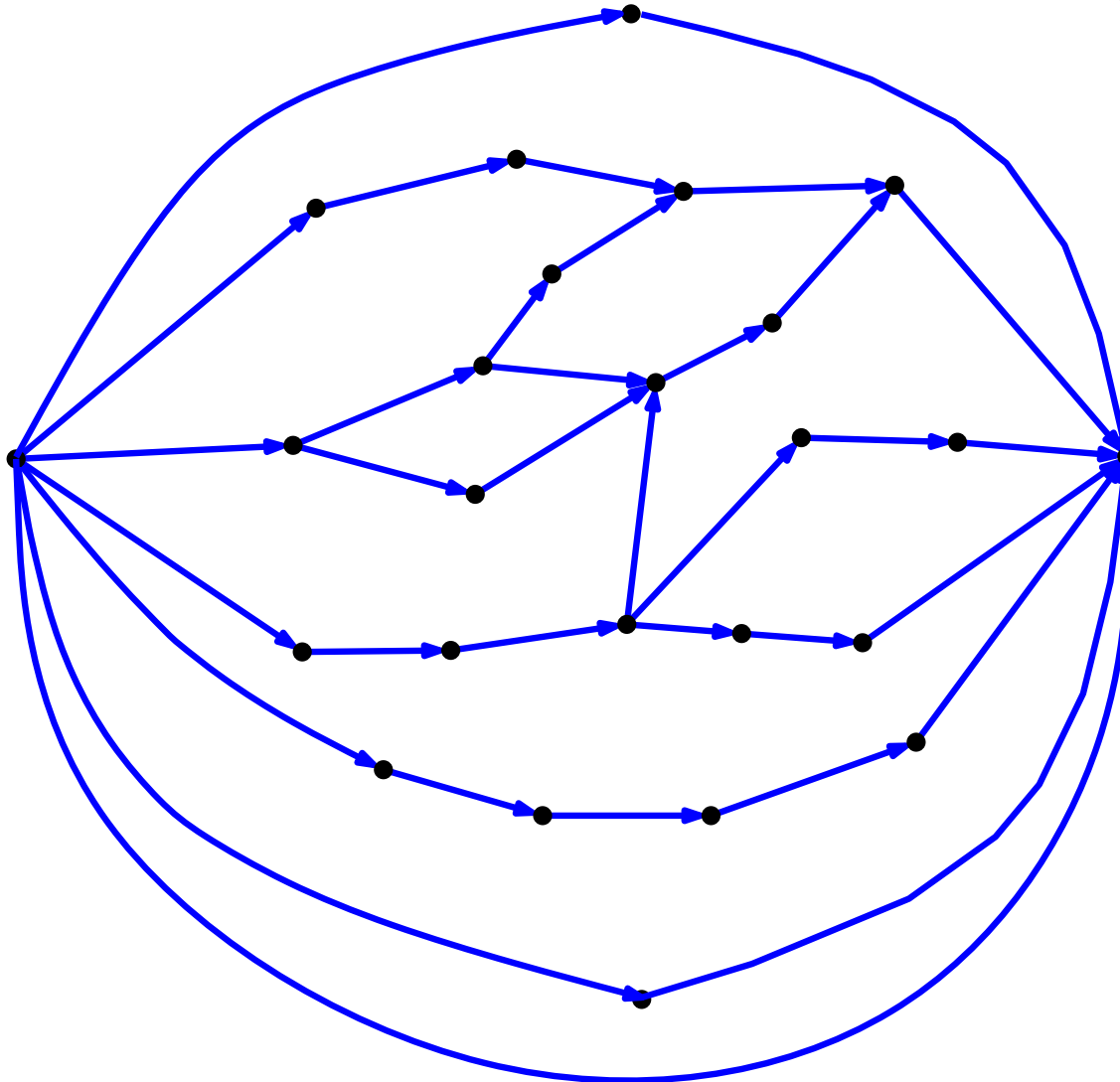
Rectangular Dual

S-N net G_{S-N}



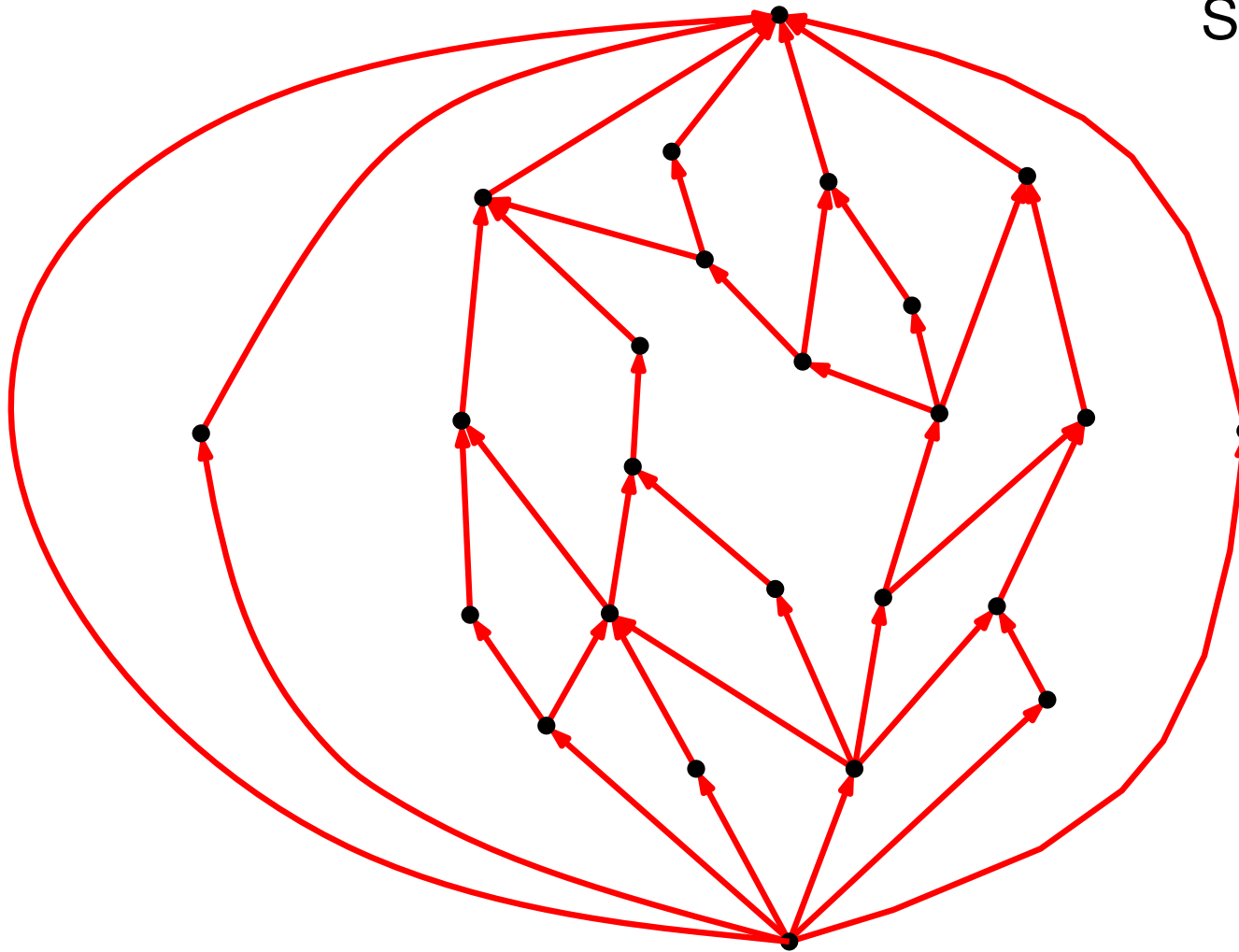
Rectangular Dual

W-E net G_{W-E}



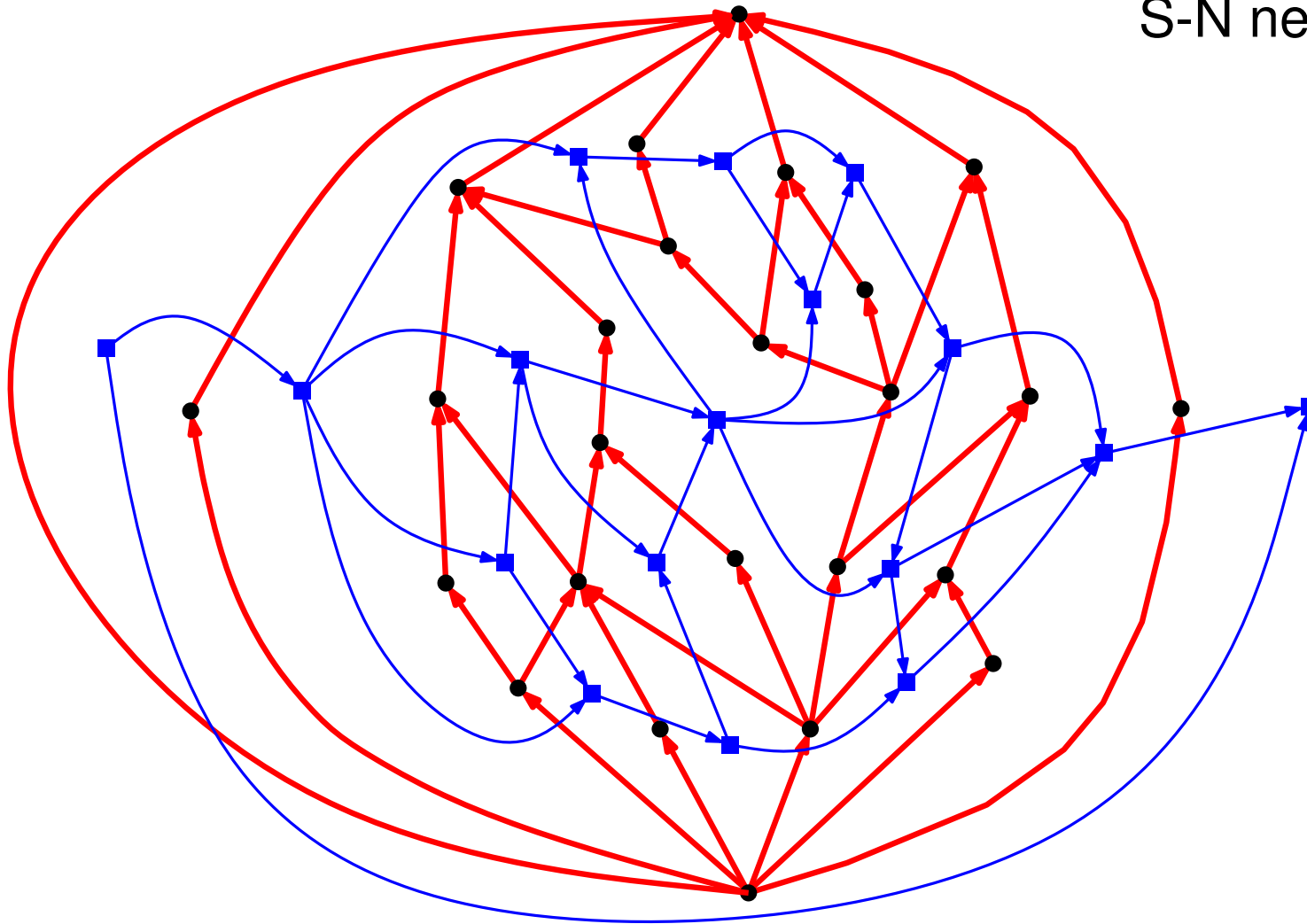
Rectangular Dual

S-N net G_{S-N}

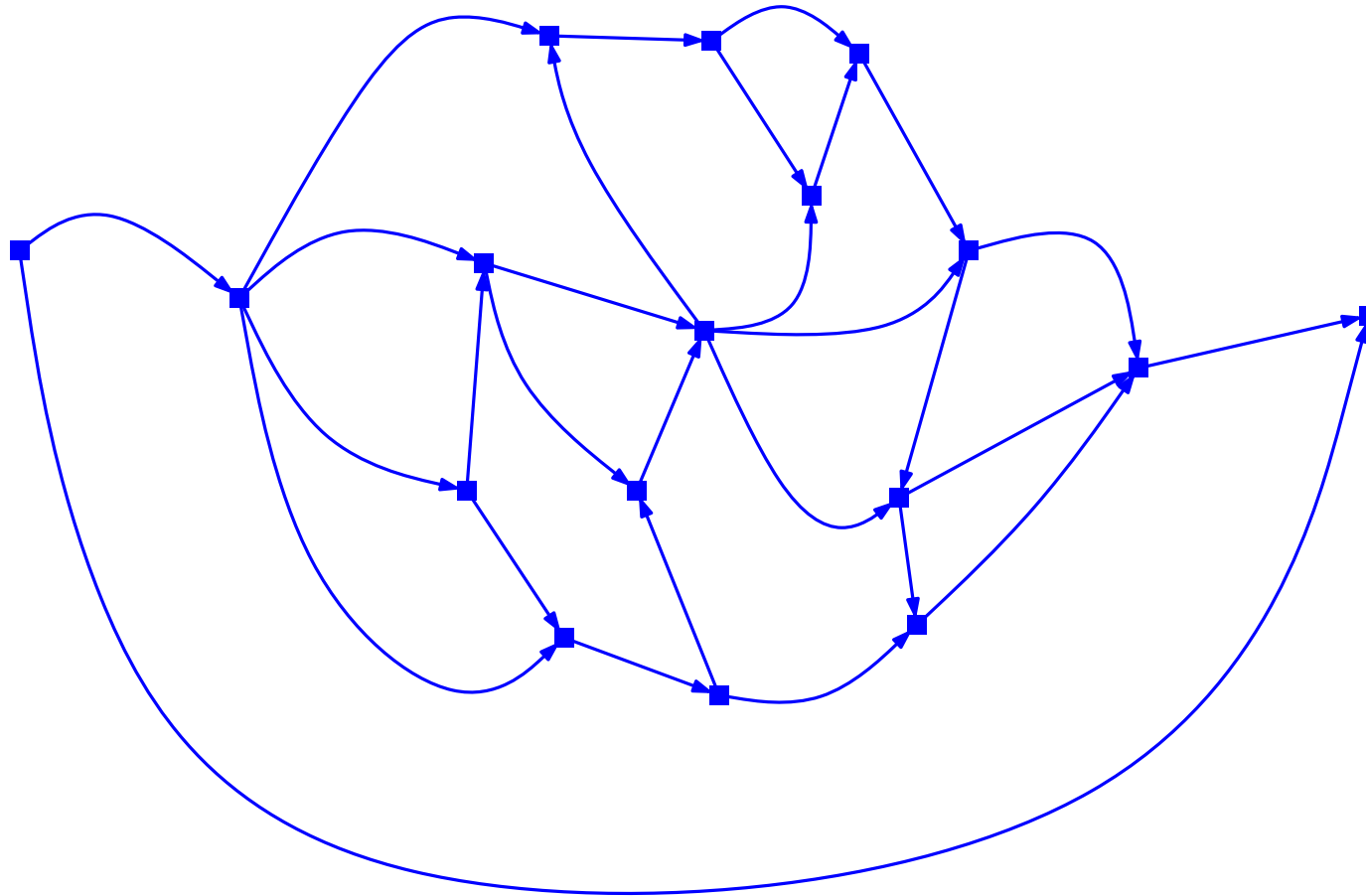


Rectangular Dual

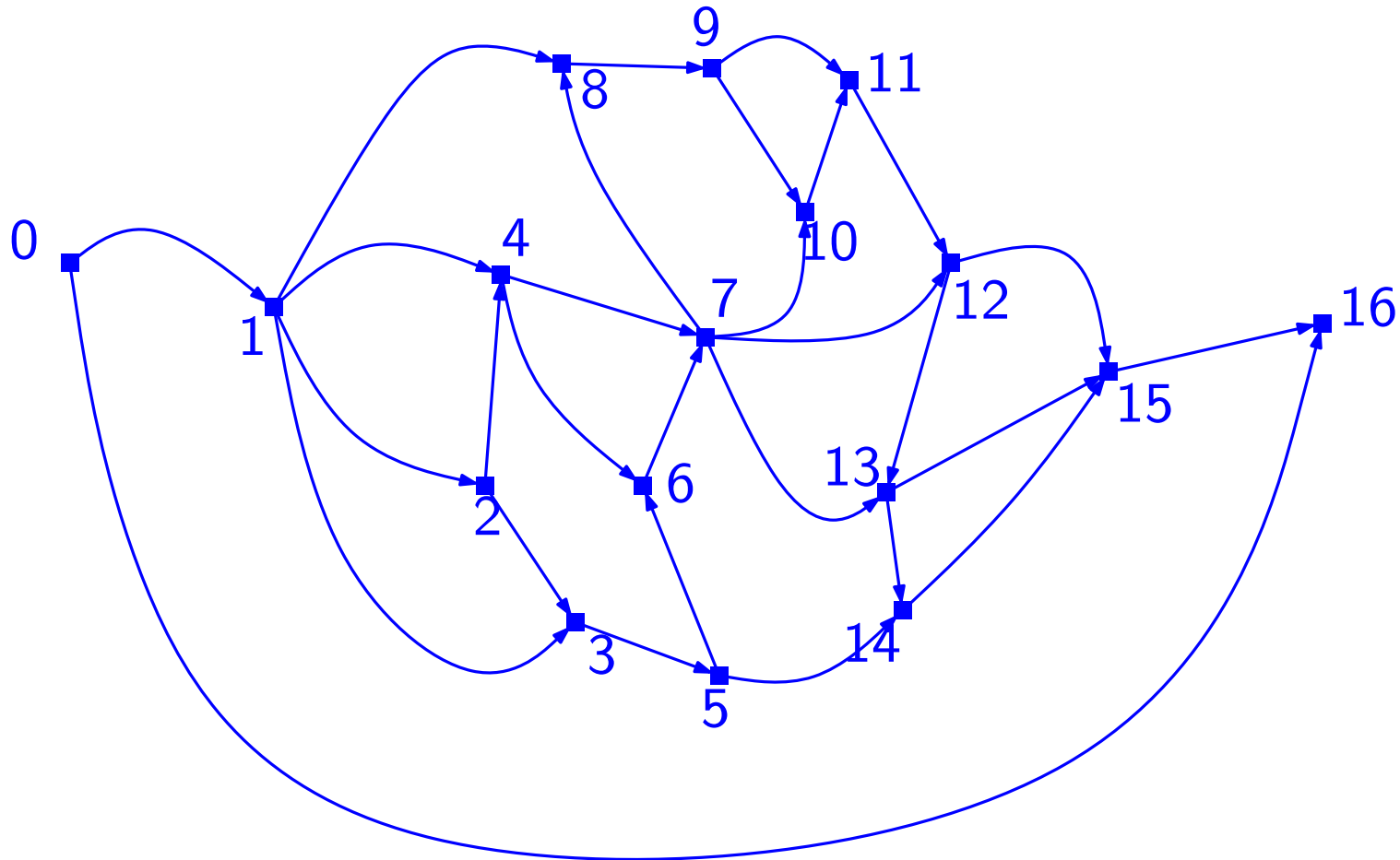
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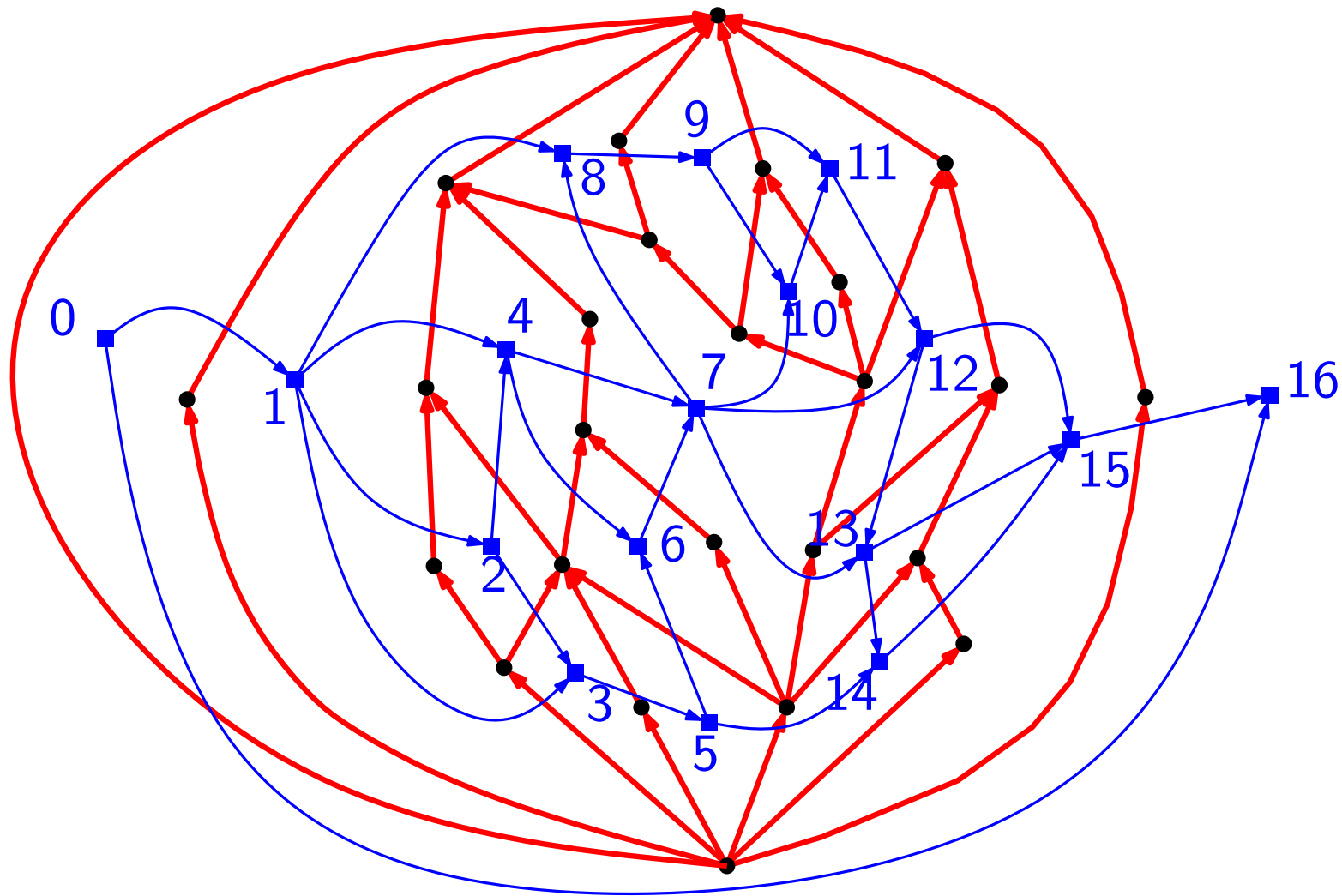
Rectangular Dual



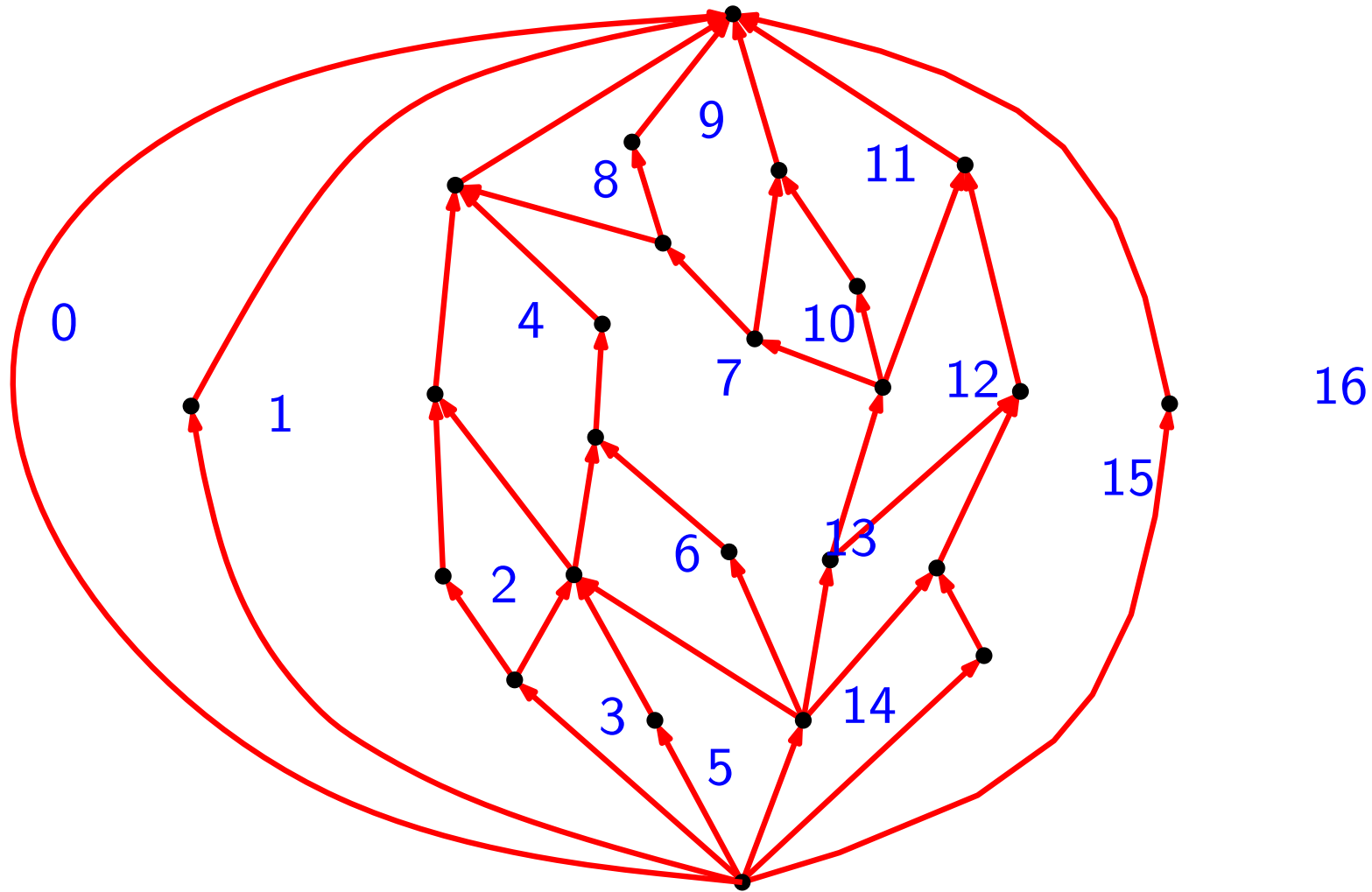
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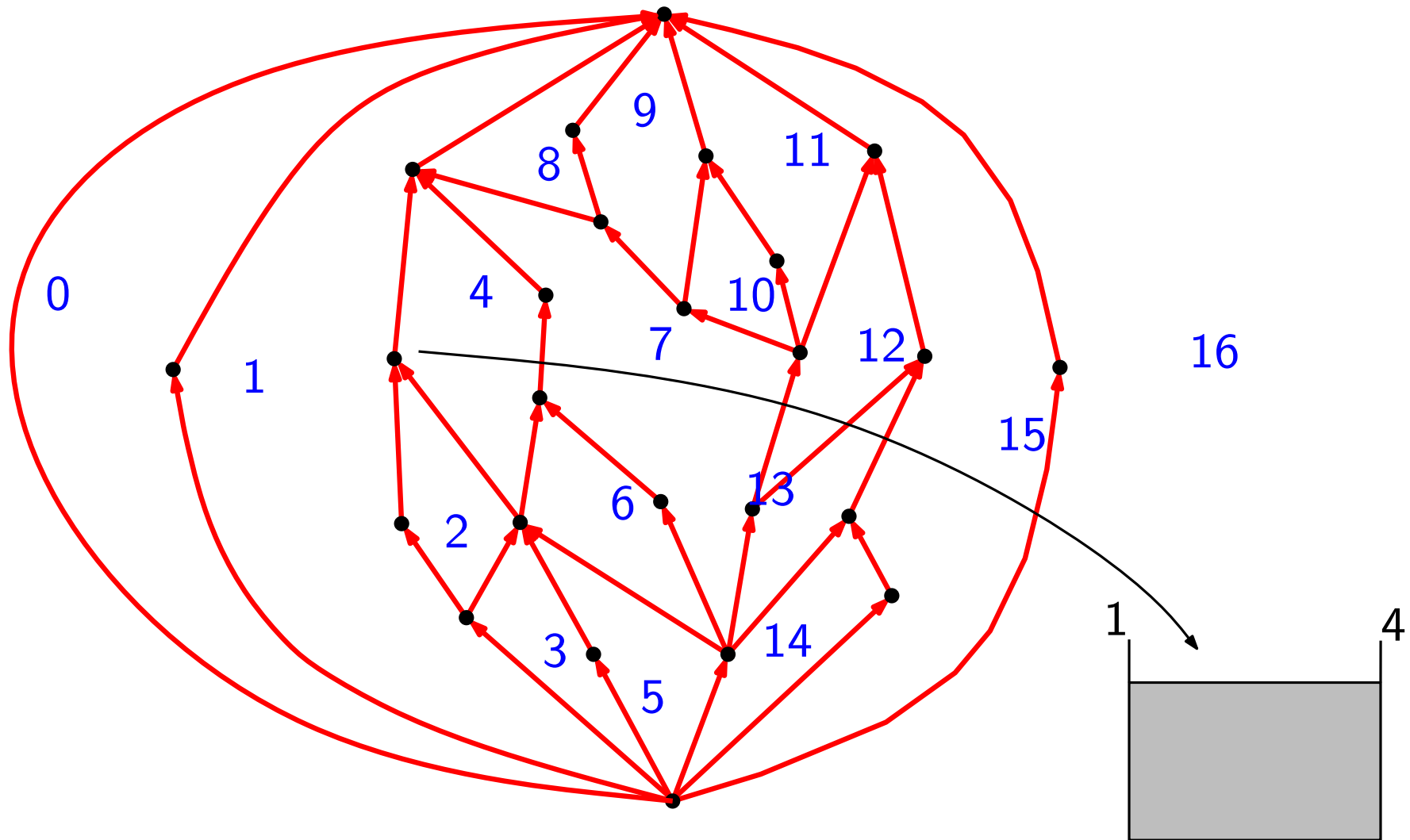
Rectangular Dual



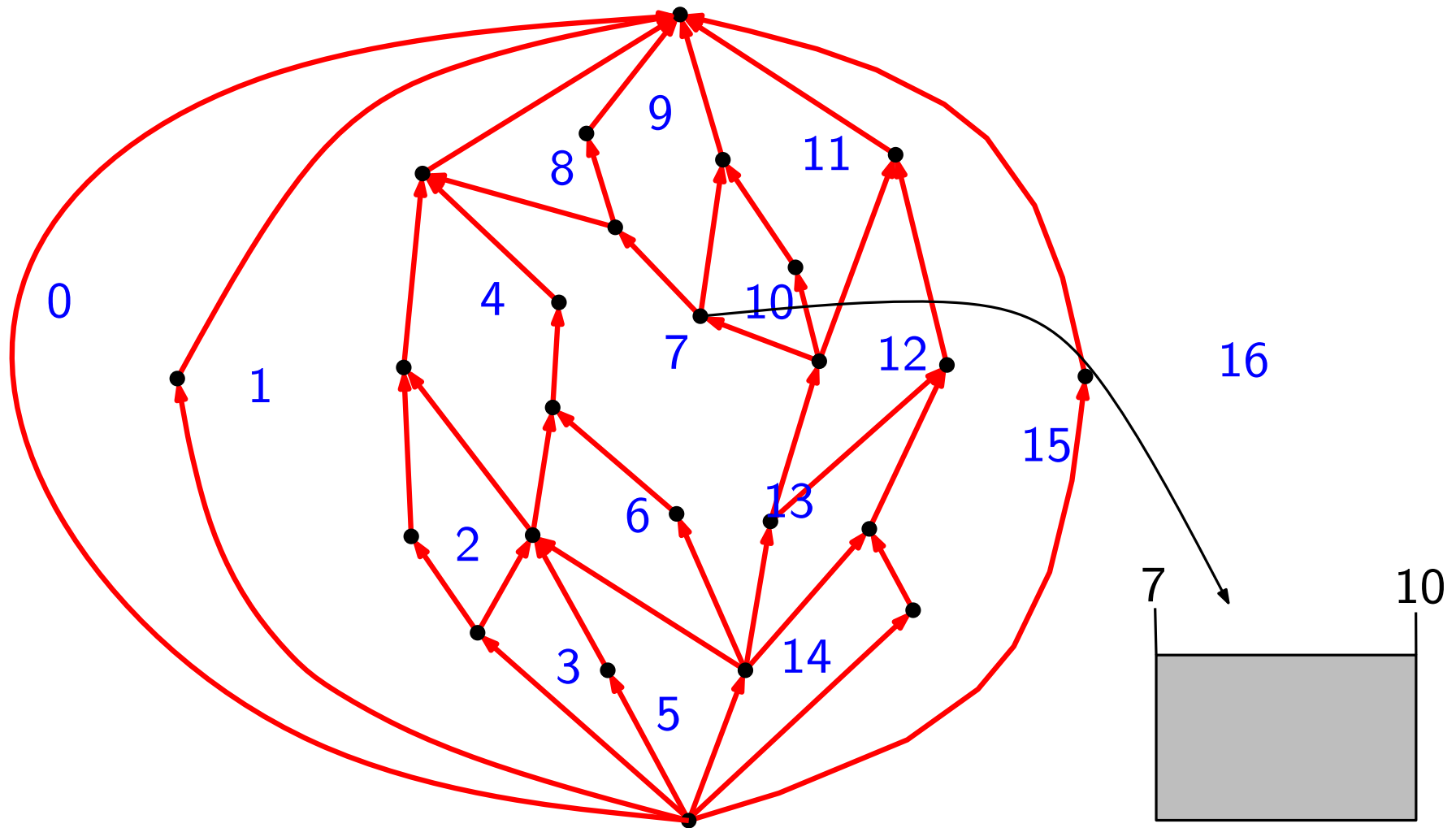
Rectangular Dual



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Rectangular Dual

Algorithm Rectangular dual

Input: A PTP graph $G = (V, E)$

- Find a REL T_r, T_b of G ;
- Construct a S-N net G_{S-N} of G (consists of T_r plus outer edges)
- Construct the dual G_{S-N}^* of G_{S-N} and compute an st -ordering f_{sn} of G_{S-N}^*
- For each vertex $v \in V$, let f and g be the face on the left and face on the right of v . Set $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$.
- Define $x_1(v_N) = x_1(v_S) = 1$ and $x_2(v_N) = x_2(v_S) = \max f_{sn} - 1$

Rectangular Dual

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Rectangular Dual

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- Find a REL T_r, T_b of G ;
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- For each vertex $v \in V$, let f and g be the face and face v . Set = $f_{sn}(f)$ and = $f_{sn}(g)$.
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- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$

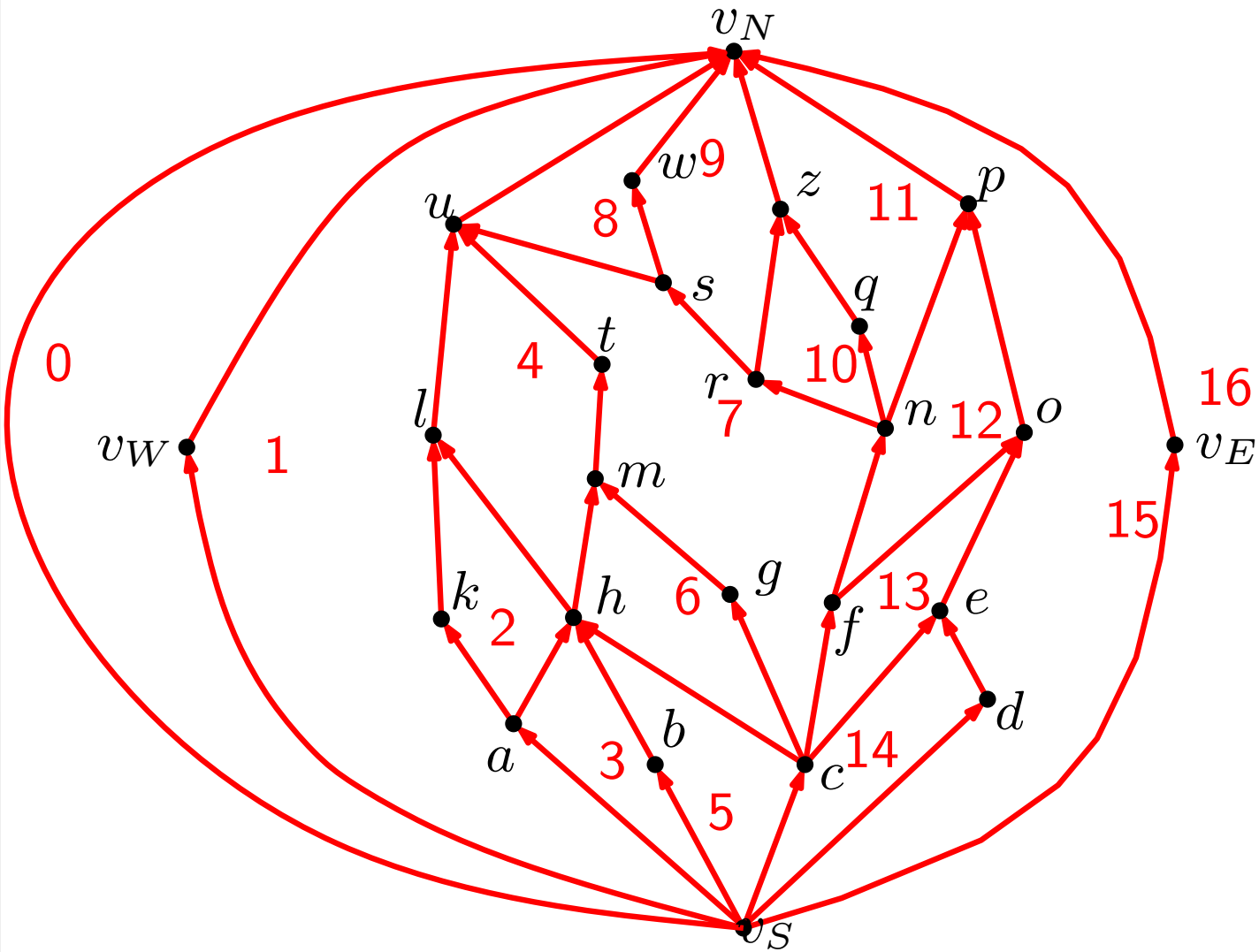
Rectangular Dual

Algorithm Rectangular dual

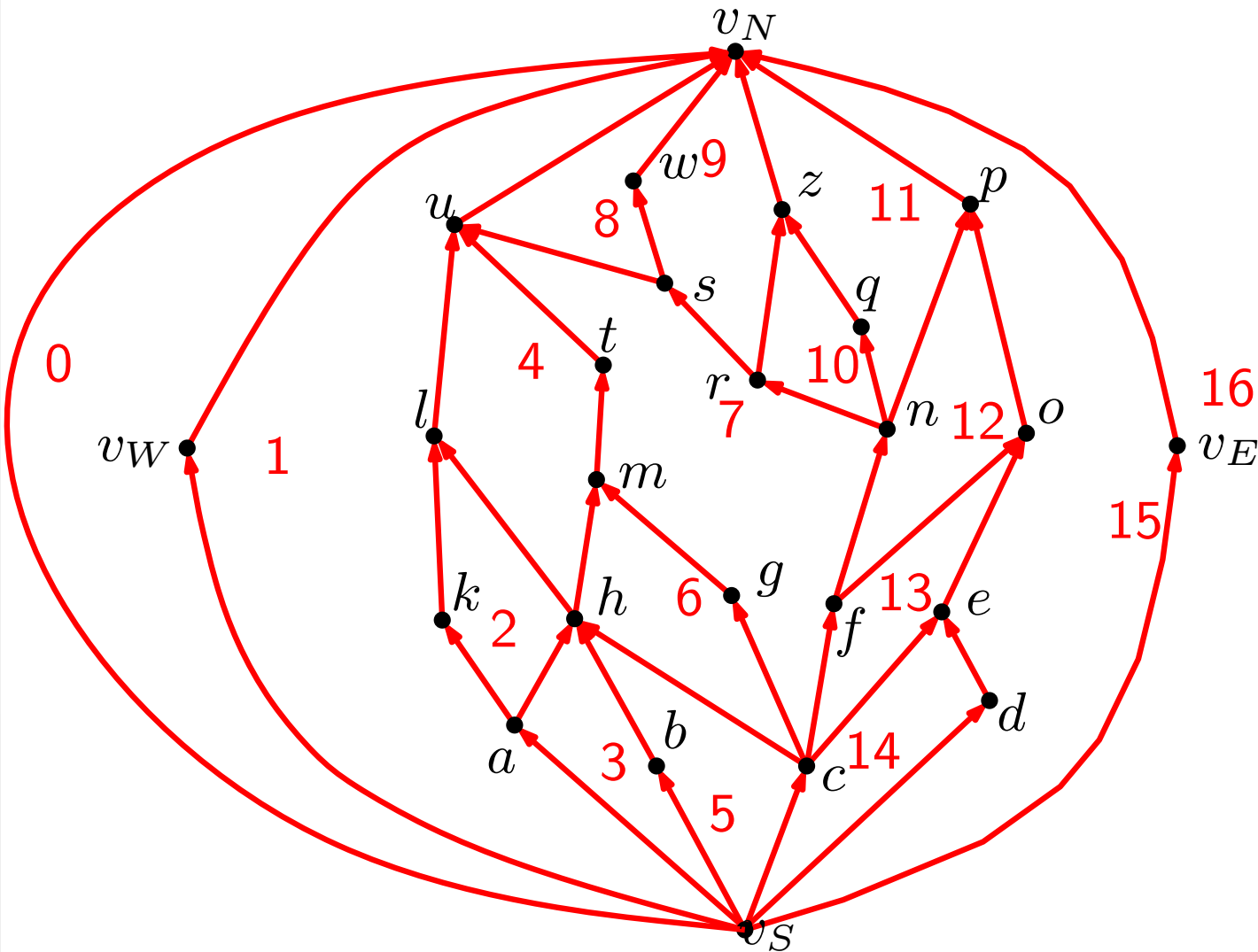
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- Define $y_1(v_W) = y_1(s_E) = 0$ and $y_1(v_W) = y_1(s_E) = \max f_{we}$
- For each $v \in V$, assign a rectangle $R(v)$ bounded by x-coordinates $x_1(v)$, $x_2(v)$ and y-coordinates $y_1(v)$, $y_2(v)$.

Rectangular Dual

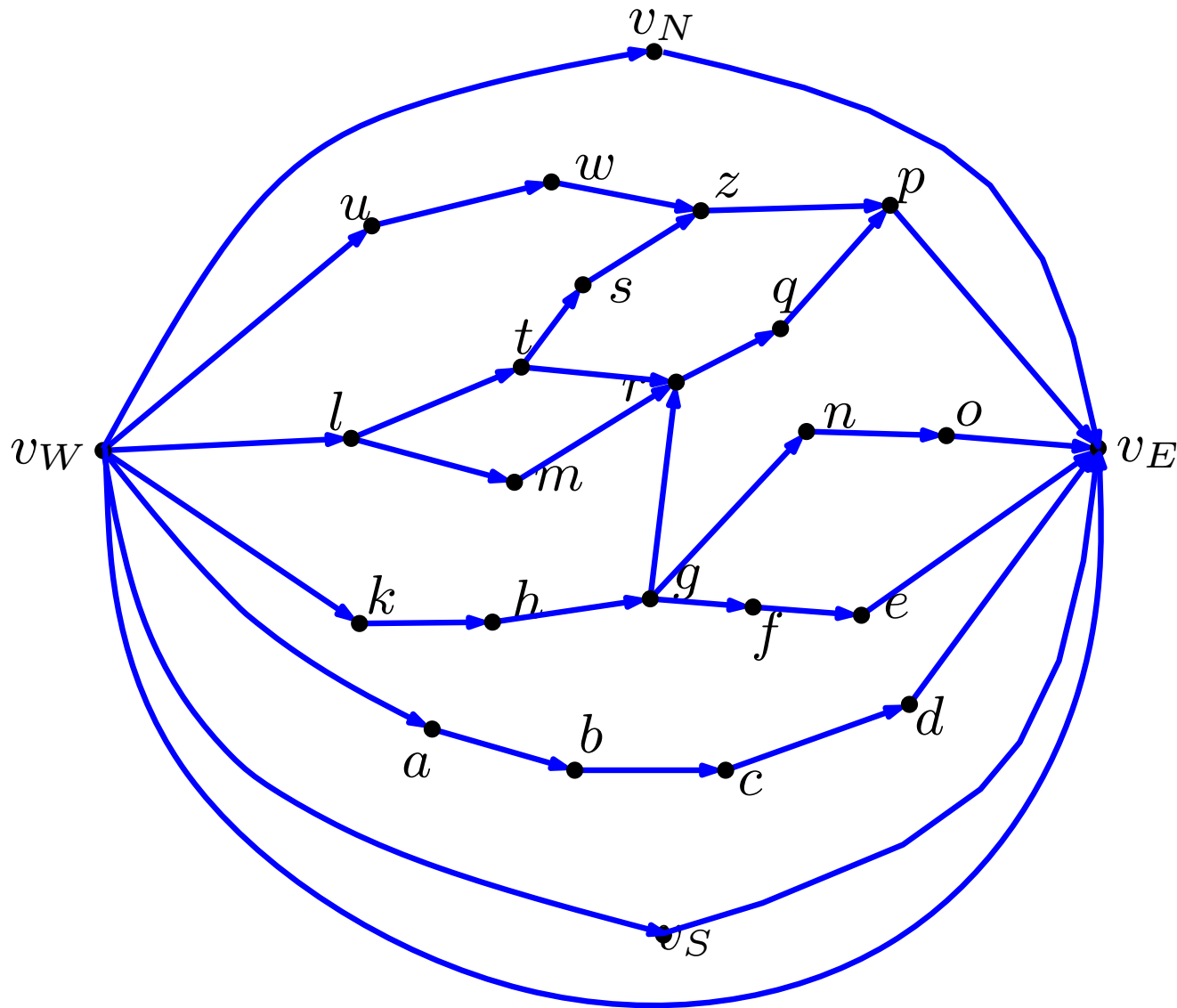


Rectangular Dual



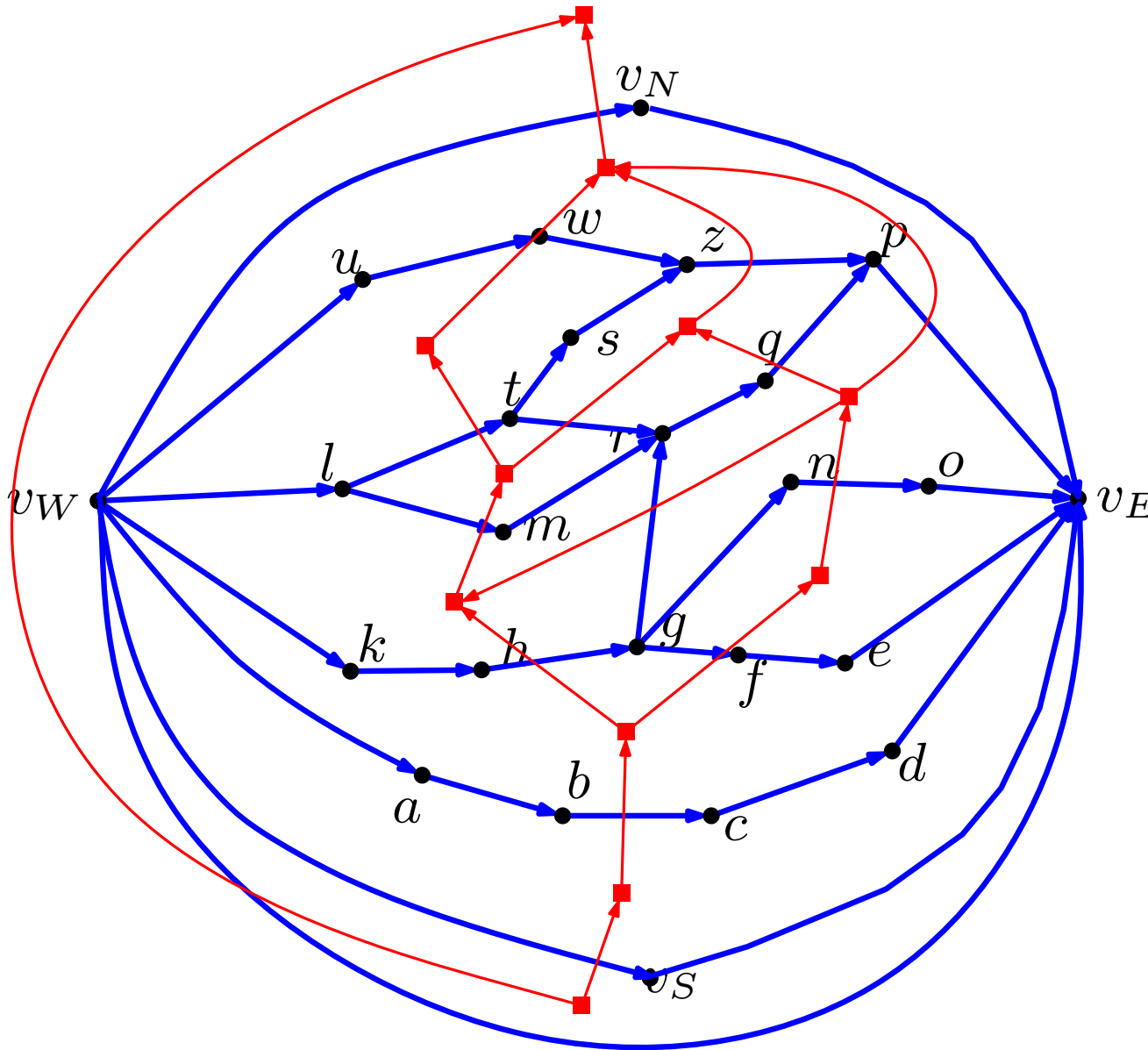
$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

Rectangular Dual



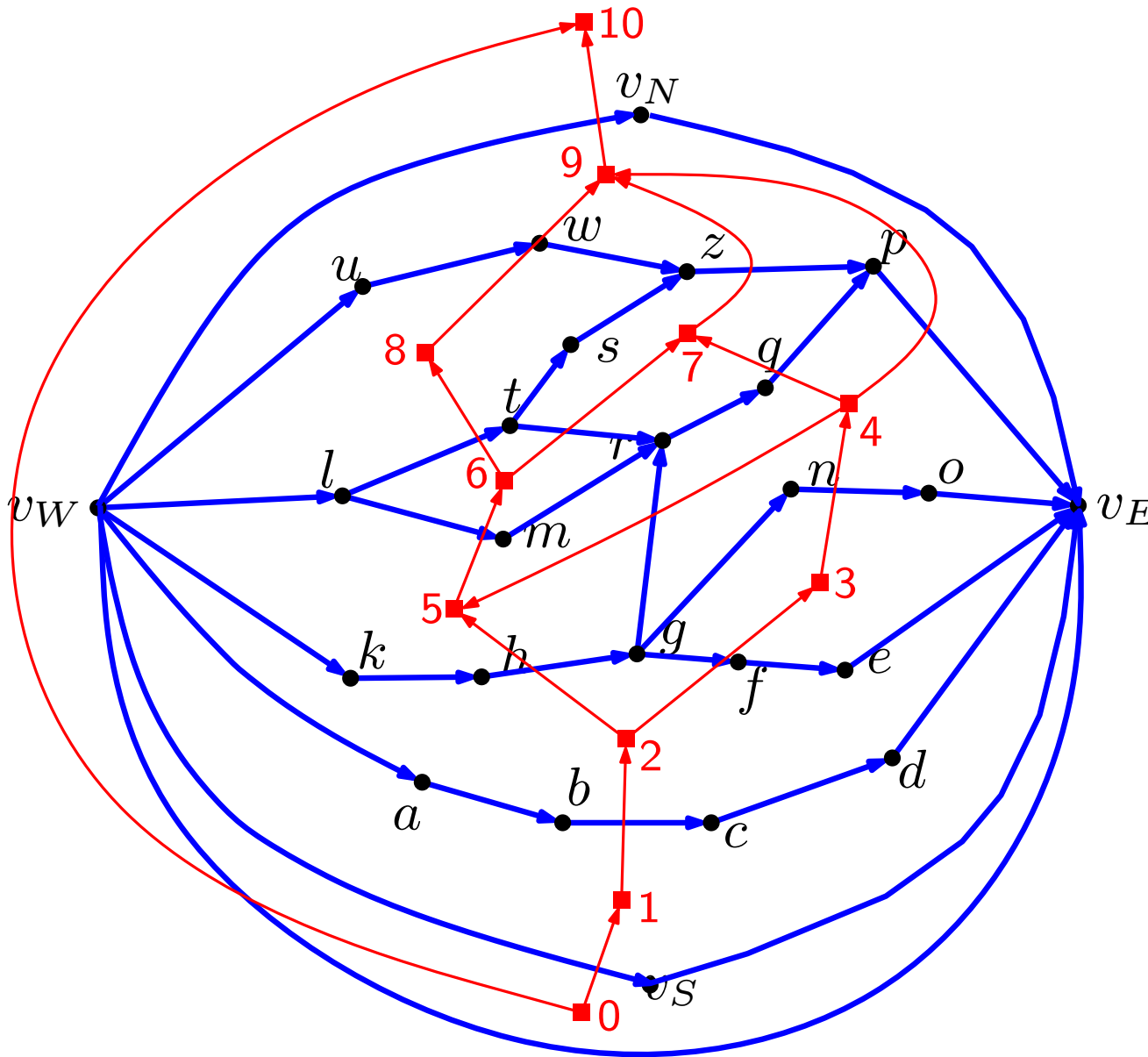
$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

Rectangular Dual



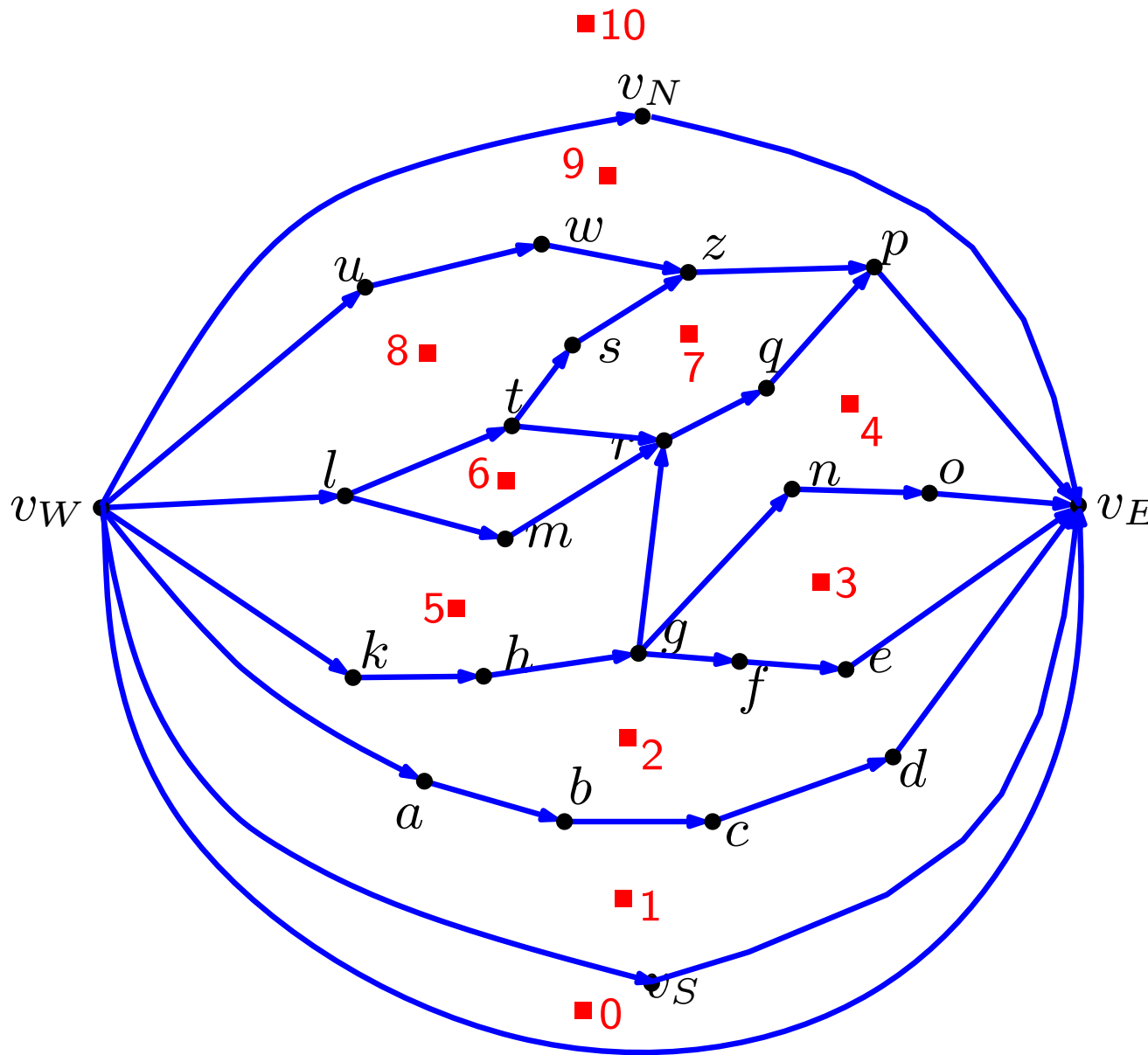
$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

Rectangular Dual



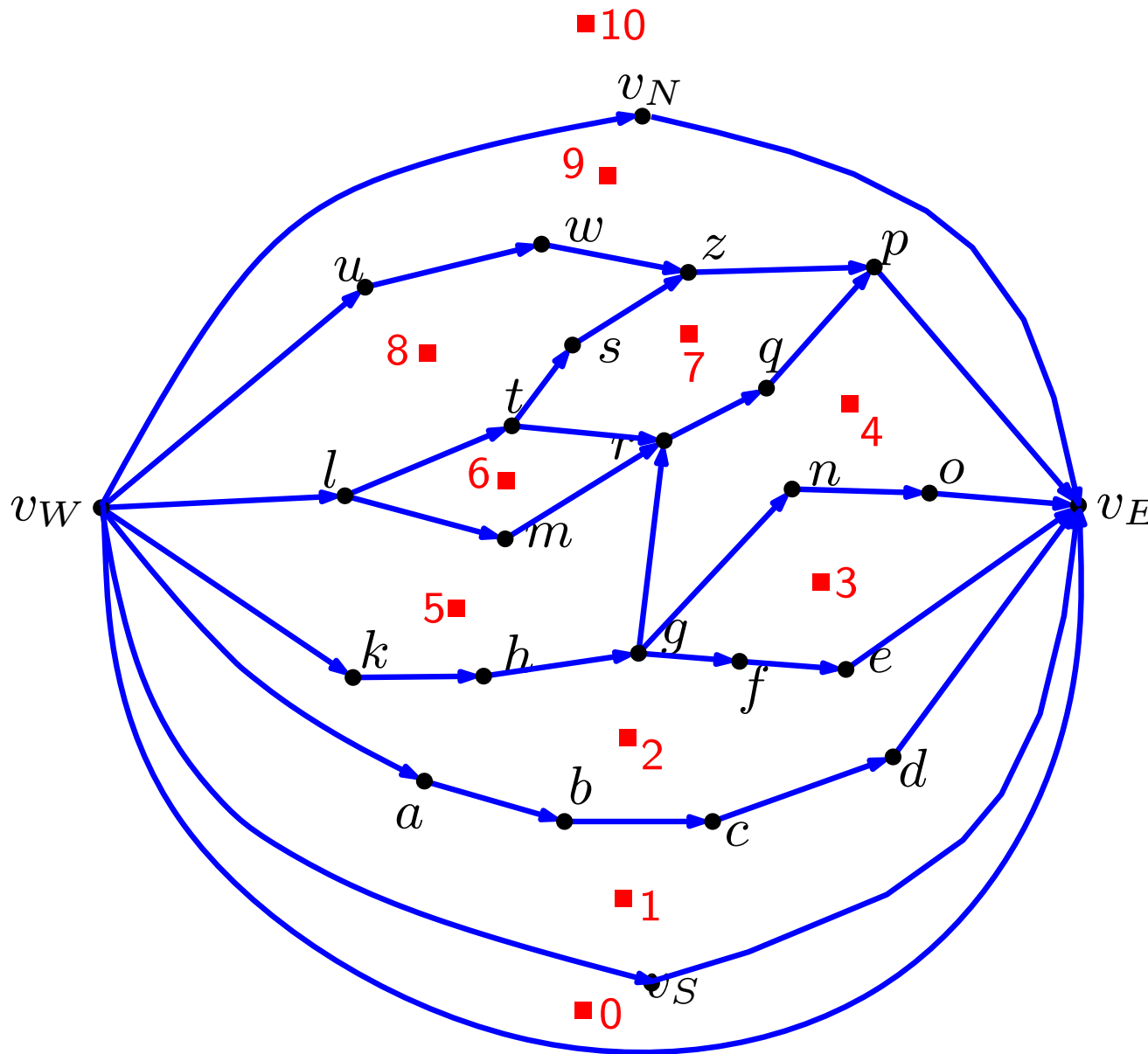
$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

$$\begin{aligned}
 y_1(v_W) &= 0, & y_2(v_W) &= 10 \\
 y_1(v_E) &= 0, & y_2(v_E) &= 10 \\
 y_1(v_N) &= 9, & y_2(v_N) &= 10 \\
 y_1(v_S) &= 0, & y_2(v_S) &= 1 \\
 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
 y_1(d) &= 1, & y_2(d) &= 2 \\
 y_1(e) &= 2, & y_2(e) &= 3
 \end{aligned}$$

Rectangular Dual

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$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

$$\begin{aligned}y_1(v_W) &= 0, & y_2(v_W) &= 10 \\y_1(v_E) &= 0, & y_2(v_E) &= 10 \\y_1(v_N) &= 9, & y_2(v_N) &= 10 \\y_1(v_S) &= 0, & y_2(v_S) &= 1 \\y_1(a) &= 1, & y_2(a) &= 2 \\y_1(b) &= 1, & y_2(b) &= 2 \\y_1(c) &= 1, & y_2(c) &= 2 \\y_1(d) &= 1, & y_2(d) &= 2 \\y_1(e) &= 2, & y_2(e) &= 3\end{aligned}$$

Rectangular Dual

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$$x_1(v_N) = 1, x_2(v_N) = 15$$

$$x_1(v_S) = 1, x_2(v_S) = 15$$


$$x_1(v_W) = 0, x_2(v_W) = 1$$

$$x_1(v_E) = 15, x_2(v_E) = 16$$


$$x_1(a) = 1, x_2(a) = 3$$

$$x_1(b) = 3, x_2(b) = 5$$

$$x_1(c) = 5, x_2(c) = 14$$

$$x_1(d) = 14, x_2(d) = 15$$

$$x_1(e) = 13, x_2(e) = 15$$


$$y_1(v_W) = 0, y_2(v_W) = 10$$

$$y_1(v_E) = 0, y_2(v_E) = 10$$

$$y_1(v_N) = 9, y_2(v_N) = 10$$

$$y_1(v_S) = 0, y_2(v_S) = 1$$

$$y_1(a) = 1, y_2(a) = 2$$

$$y_1(b) = 1, y_2(b) = 2$$

$$y_1(c) = 1, y_2(c) = 2$$

$$y_1(d) = 1, y_2(d) = 2$$

$$y_1(e) = 2, y_2(e) = 3$$

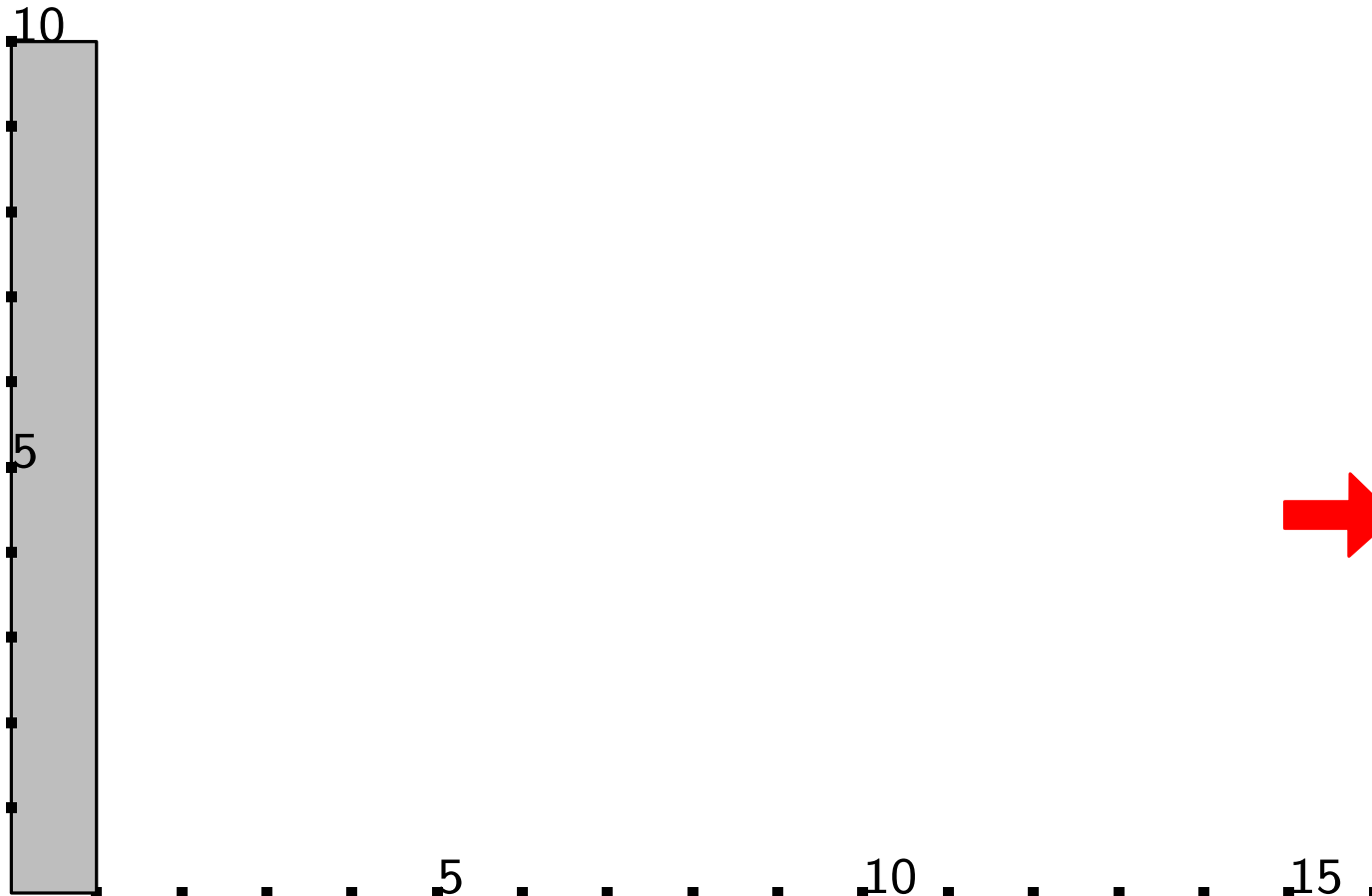
Rectangular Dual



$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\ \rightarrow x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

$$\begin{aligned}\rightarrow y_1(v_W) &= 0, & y_2(v_W) &= 10 \\y_1(v_E) &= 0, & y_2(v_E) &= 10 \\y_1(v_N) &= 9, & y_2(v_N) &= 10 \\y_1(v_S) &= 0, & y_2(v_S) &= 1 \\y_1(a) &= 1, & y_2(a) &= 2 \\y_1(b) &= 1, & y_2(b) &= 2 \\y_1(c) &= 1, & y_2(c) &= 2 \\y_1(d) &= 1, & y_2(d) &= 2 \\y_1(e) &= 2, & y_2(e) &= 3\end{aligned}$$

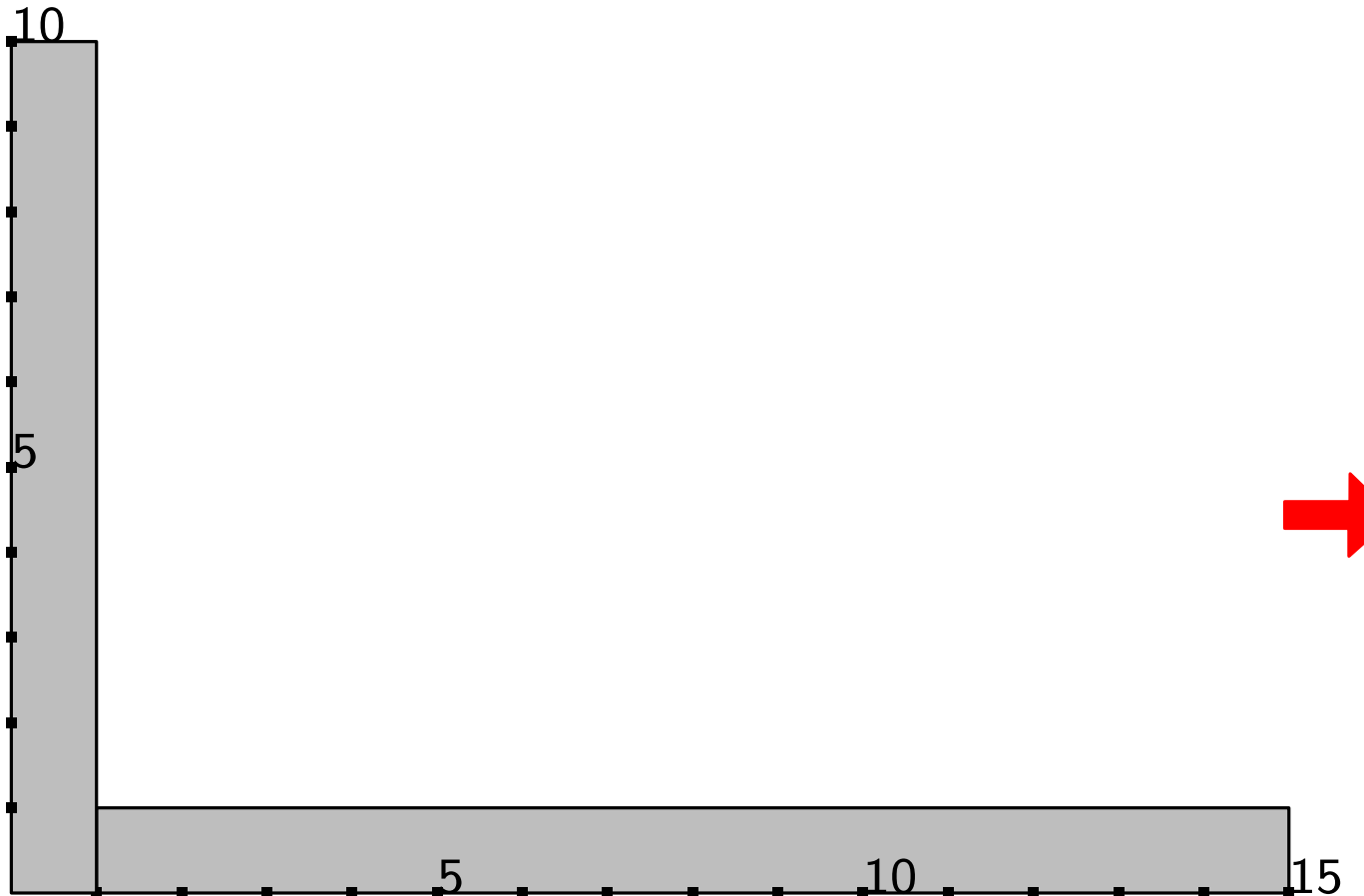
Rectangular Dual



→ $x_1(v_N) = 1, x_2(v_N) = 15$
 $x_1(v_S) = 1, x_2(v_S) = 15$
 $x_1(v_W) = 0, x_2(v_W) = 1$
 $x_1(v_E) = 15, x_2(v_E) = 16$
 $x_1(a) = 1, x_2(a) = 3$
 $x_1(b) = 3, x_2(b) = 5$
 $x_1(c) = 5, x_2(c) = 14$
 $x_1(d) = 14, x_2(d) = 15$
 $x_1(e) = 13, x_2(e) = 15$

$y_1(v_W) = 0, y_2(v_W) = 10$
 $y_1(v_E) = 0, y_2(v_E) = 10$
 $y_1(v_N) = 9, y_2(v_N) = 10$
→ $y_1(v_S) = 0, y_2(v_S) = 1$
 $y_1(a) = 1, y_2(a) = 2$
 $y_1(b) = 1, y_2(b) = 2$
 $y_1(c) = 1, y_2(c) = 2$
 $y_1(d) = 1, y_2(d) = 2$
 $y_1(e) = 2, y_2(e) = 3$

Rectangular Dual



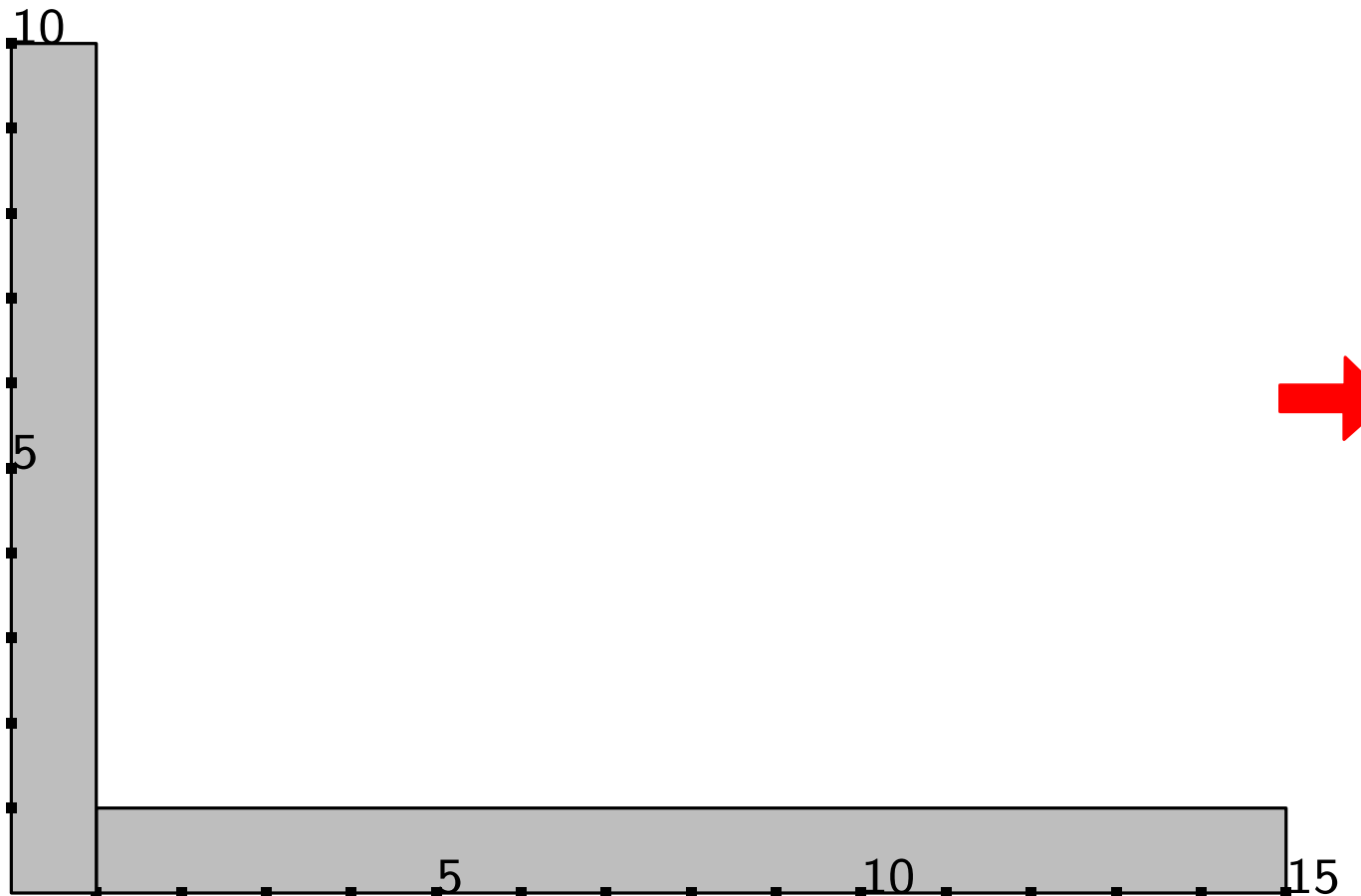
→

$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

$$\begin{aligned}y_1(v_W) &= 0, & y_2(v_W) &= 10 \\y_1(v_E) &= 0, & y_2(v_E) &= 10 \\y_1(v_N) &= 9, & y_2(v_N) &= 10 \\y_1(v_S) &= 0, & y_2(v_S) &= 1 \\y_1(a) &= 1, & y_2(a) &= 2 \\y_1(b) &= 1, & y_2(b) &= 2 \\y_1(c) &= 1, & y_2(c) &= 2 \\y_1(d) &= 1, & y_2(d) &= 2 \\y_1(e) &= 2, & y_2(e) &= 3\end{aligned}$$

→

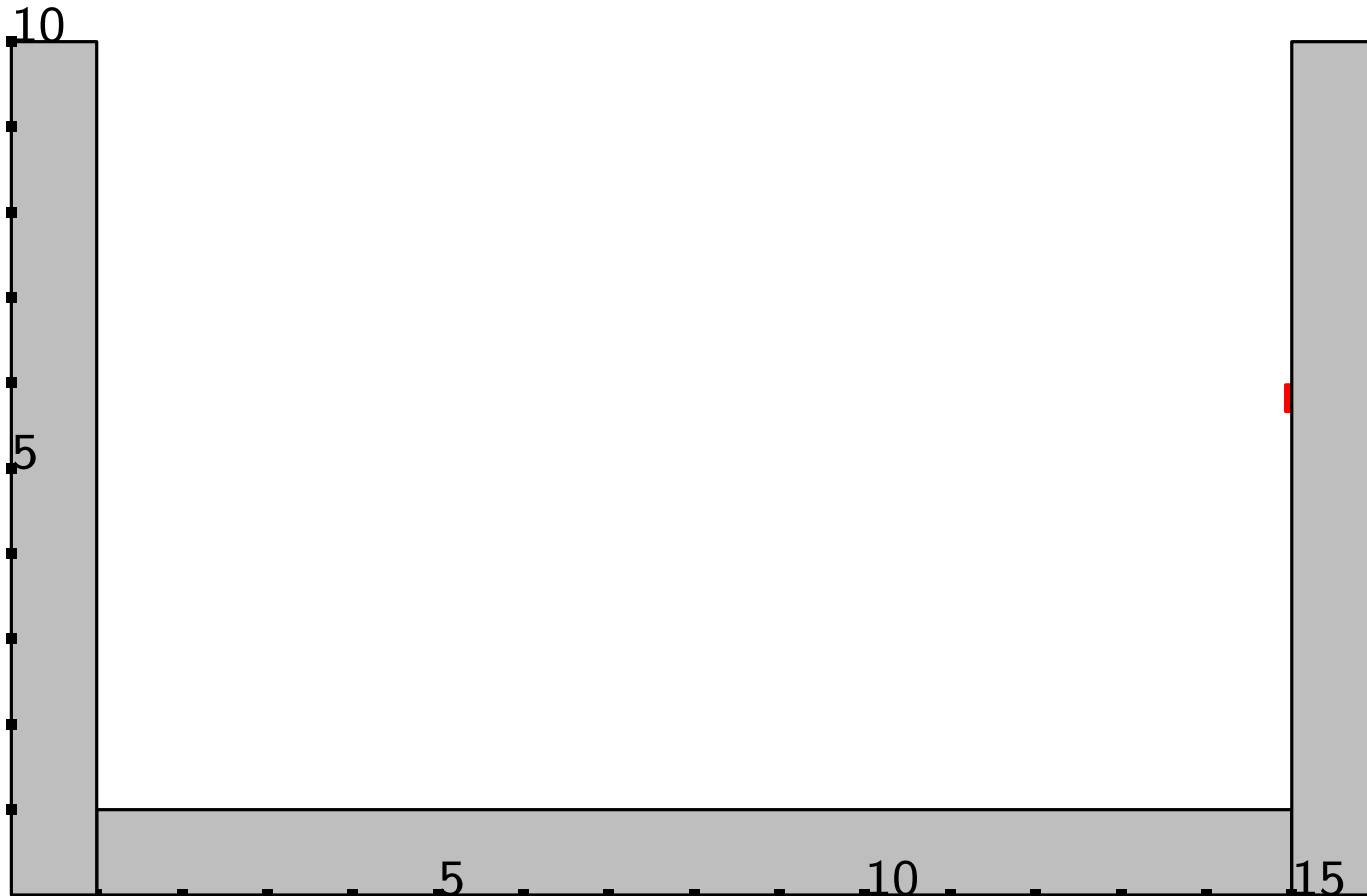
Rectangular Dual



$x_1(v_N) = 1, x_2(v_N) = 15$
 $x_1(v_S) = 1, x_2(v_S) = 15$
 $x_1(v_W) = 0, x_2(v_W) = 1$
→ $x_1(v_E) = 15, x_2(v_E) = 16$
 $x_1(a) = 1, x_2(a) = 3$
 $x_1(b) = 3, x_2(b) = 5$
 $x_1(c) = 5, x_2(c) = 14$
 $x_1(d) = 14, x_2(d) = 15$
 $x_1(e) = 13, x_2(e) = 15$

→ $y_1(v_W) = 0, y_2(v_W) = 10$
 $y_1(v_E) = 0, y_2(v_E) = 10$
 $y_1(v_N) = 9, y_2(v_N) = 10$
 $y_1(v_S) = 0, y_2(v_S) = 1$
 $y_1(a) = 1, y_2(a) = 2$
 $y_1(b) = 1, y_2(b) = 2$
 $y_1(c) = 1, y_2(c) = 2$
 $y_1(d) = 1, y_2(d) = 2$
 $y_1(e) = 2, y_2(e) = 3$

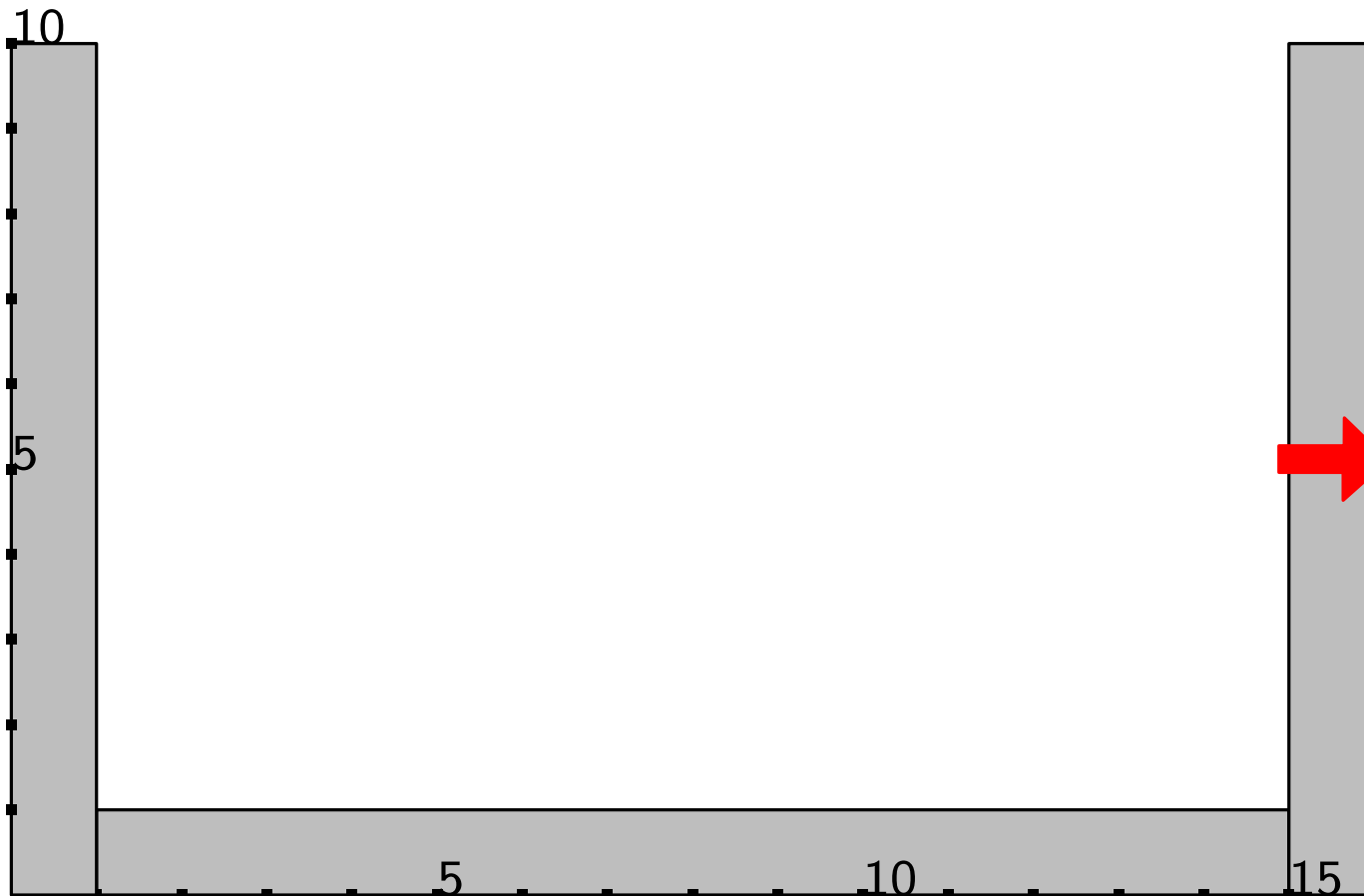
Rectangular Dual



$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

$$\begin{aligned}y_1(v_W) &= 0, & y_2(v_W) &= 10 \\y_1(v_E) &= 0, & y_2(v_E) &= 10 \\y_1(v_N) &= 9, & y_2(v_N) &= 10 \\y_1(v_S) &= 0, & y_2(v_S) &= 1 \\y_1(a) &= 1, & y_2(a) &= 2 \\y_1(b) &= 1, & y_2(b) &= 2 \\y_1(c) &= 1, & y_2(c) &= 2 \\y_1(d) &= 1, & y_2(d) &= 2 \\y_1(e) &= 2, & y_2(e) &= 3\end{aligned}$$

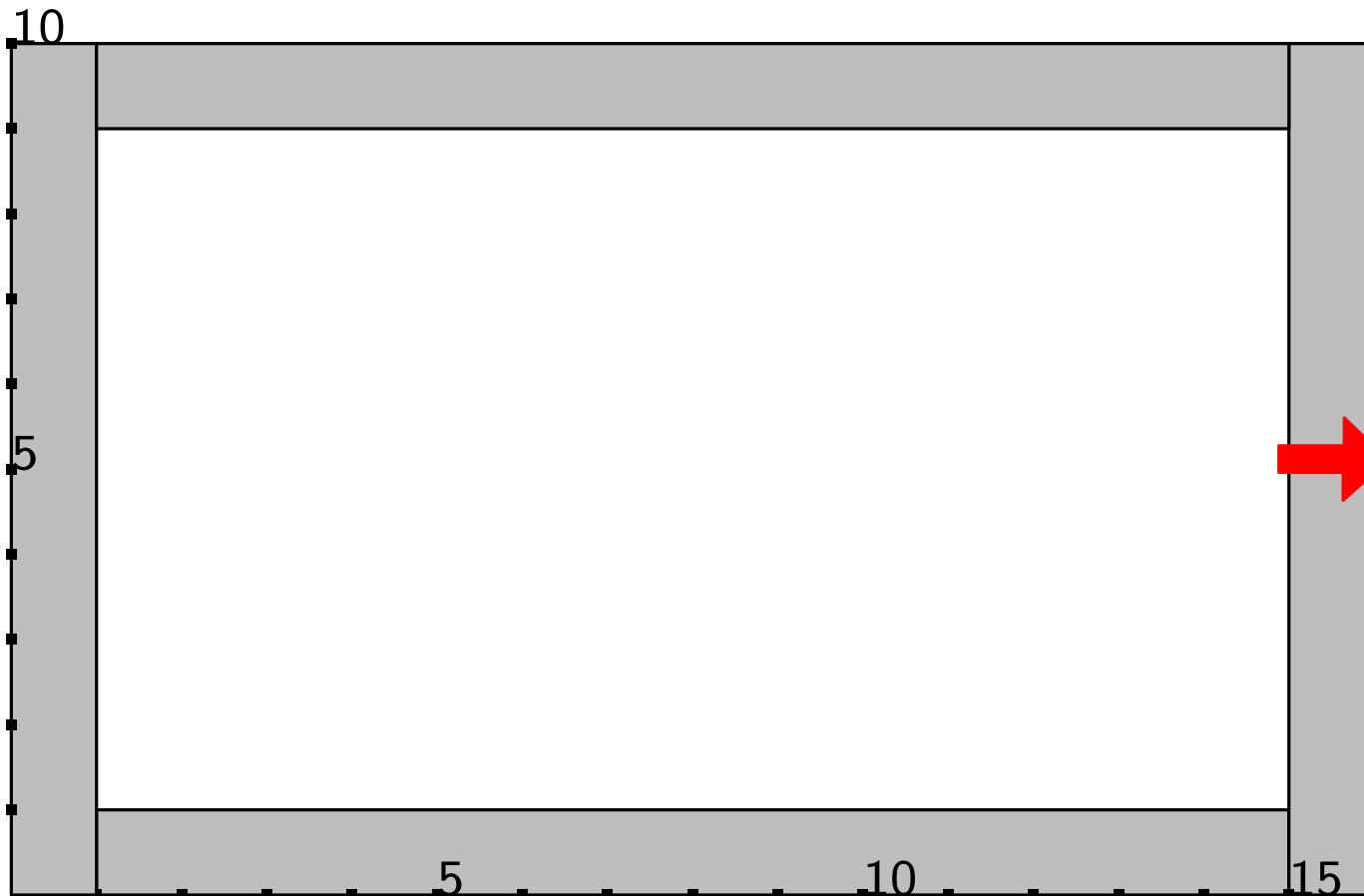
Rectangular Dual



→ $x_1(v_N) = 1, x_2(v_N) = 15$
 $x_1(v_S) = 1, x_2(v_S) = 15$
 $x_1(v_W) = 0, x_2(v_W) = 1$
 $x_1(v_E) = 15, x_2(v_E) = 16$
 $x_1(a) = 1, x_2(a) = 3$
 $x_1(b) = 3, x_2(b) = 5$
 $x_1(c) = 5, x_2(c) = 14$
 $x_1(d) = 14, x_2(d) = 15$
 $x_1(e) = 13, x_2(e) = 15$

→ $y_1(v_W) = 0, y_2(v_W) = 10$
 $y_1(v_E) = 0, y_2(v_E) = 10$
 $y_1(v_N) = 9, y_2(v_N) = 10$
 $y_1(v_S) = 0, y_2(v_S) = 1$
 $y_1(a) = 1, y_2(a) = 2$
 $y_1(b) = 1, y_2(b) = 2$
 $y_1(c) = 1, y_2(c) = 2$
 $y_1(d) = 1, y_2(d) = 2$
 $y_1(e) = 2, y_2(e) = 3$

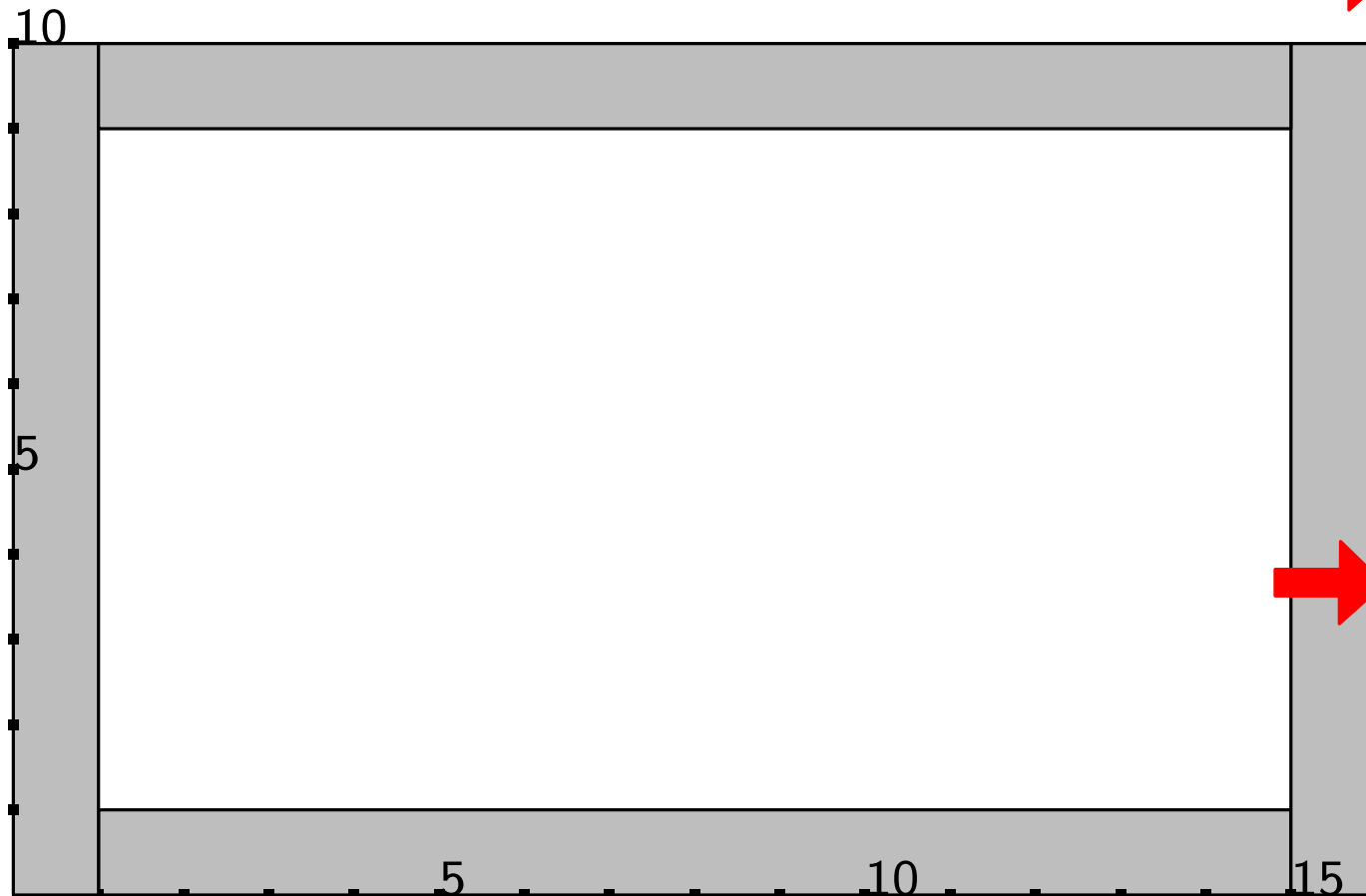
Rectangular Dual



→ $x_1(v_N) = 1, x_2(v_N) = 15$
 $x_1(v_S) = 1, x_2(v_S) = 15$
 $x_1(v_W) = 0, x_2(v_W) = 1$
 $x_1(v_E) = 15, x_2(v_E) = 16$
 $x_1(a) = 1, x_2(a) = 3$
 $x_1(b) = 3, x_2(b) = 5$
 $x_1(c) = 5, x_2(c) = 14$
 $x_1(d) = 14, x_2(d) = 15$
 $x_1(e) = 13, x_2(e) = 15$

→ $y_1(v_W) = 0, y_2(v_W) = 10$
 $y_1(v_E) = 0, y_2(v_E) = 10$
 $y_1(v_N) = 9, y_2(v_N) = 10$
 $y_1(v_S) = 0, y_2(v_S) = 1$
 $y_1(a) = 1, y_2(a) = 2$
 $y_1(b) = 1, y_2(b) = 2$
 $y_1(c) = 1, y_2(c) = 2$
 $y_1(d) = 1, y_2(d) = 2$
 $y_1(e) = 2, y_2(e) = 3$

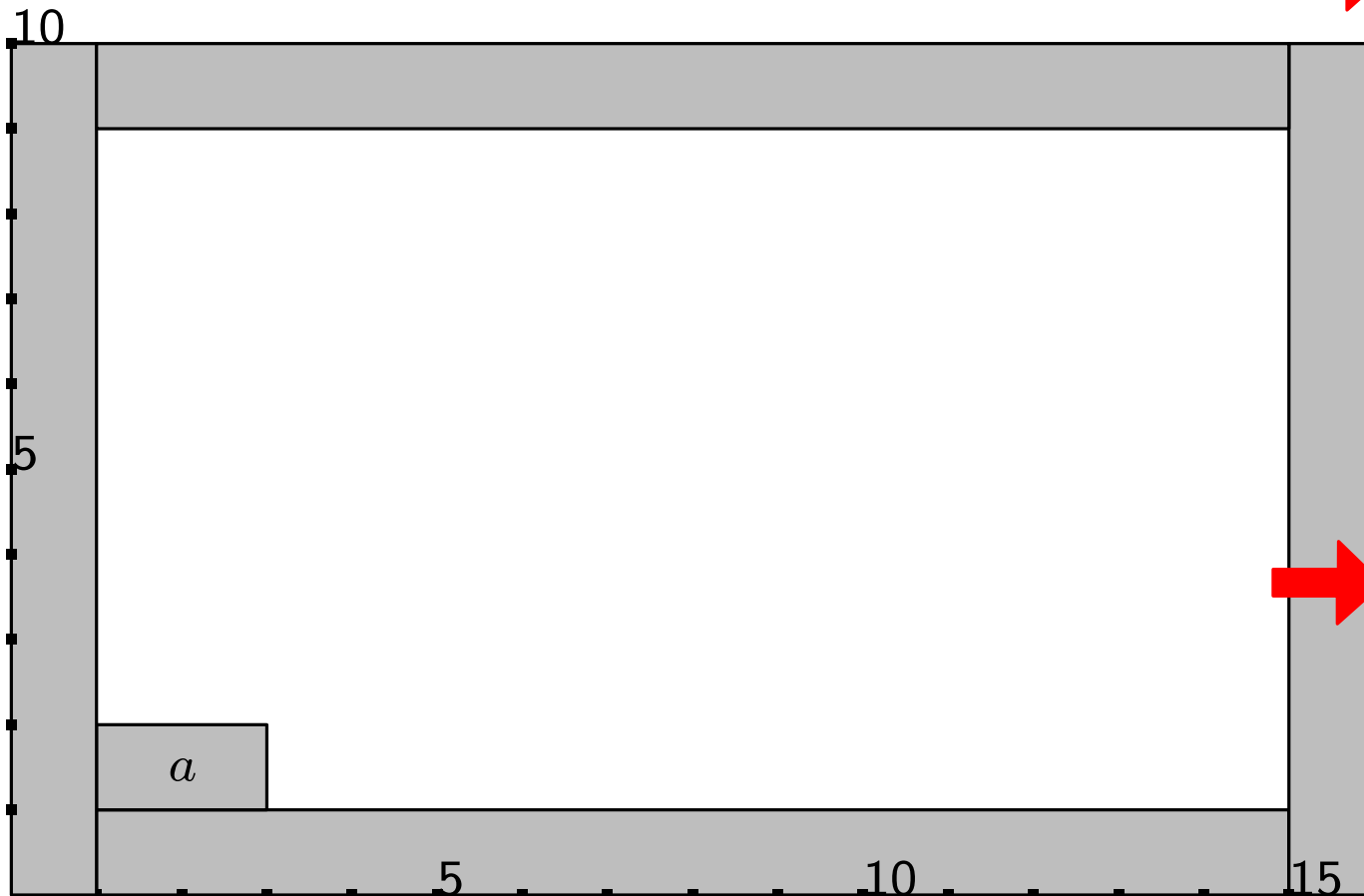
Rectangular Dual



$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

$$\begin{aligned}y_1(v_W) &= 0, & y_2(v_W) &= 10 \\y_1(v_E) &= 0, & y_2(v_E) &= 10 \\y_1(v_N) &= 9, & y_2(v_N) &= 10 \\y_1(v_S) &= 0, & y_2(v_S) &= 1 \\y_1(a) &= 1, & y_2(a) &= 2 \\y_1(b) &= 1, & y_2(b) &= 2 \\y_1(c) &= 1, & y_2(c) &= 2 \\y_1(d) &= 1, & y_2(d) &= 2 \\y_1(e) &= 2, & y_2(e) &= 3\end{aligned}$$

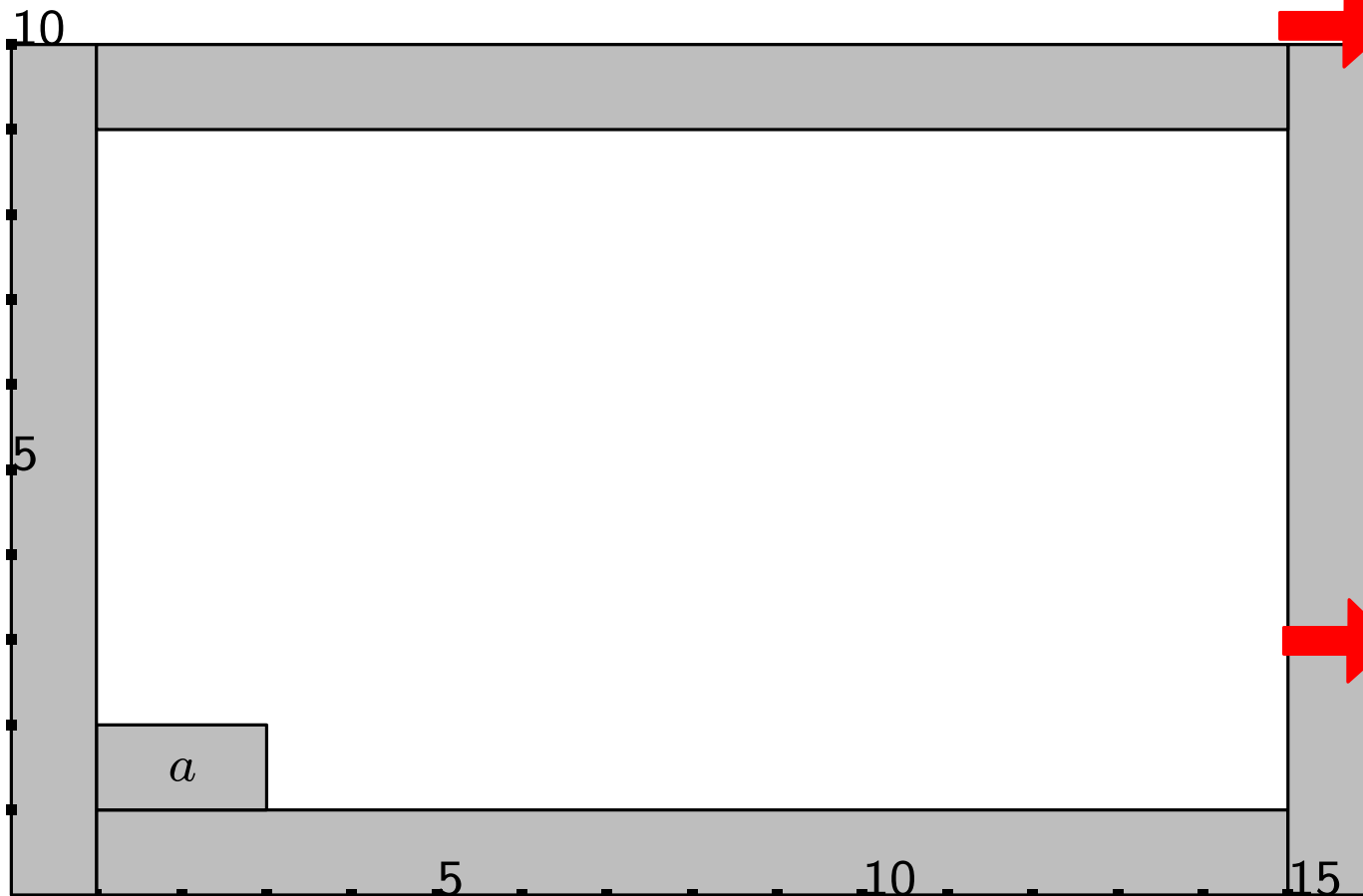
Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

$$\begin{aligned}
 y_1(v_W) &= 0, & y_2(v_W) &= 10 \\
 y_1(v_E) &= 0, & y_2(v_E) &= 10 \\
 y_1(v_N) &= 9, & y_2(v_N) &= 10 \\
 y_1(v_S) &= 0, & y_2(v_S) &= 1 \\
 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
 y_1(d) &= 1, & y_2(d) &= 2 \\
 y_1(e) &= 2, & y_2(e) &= 3
 \end{aligned}$$

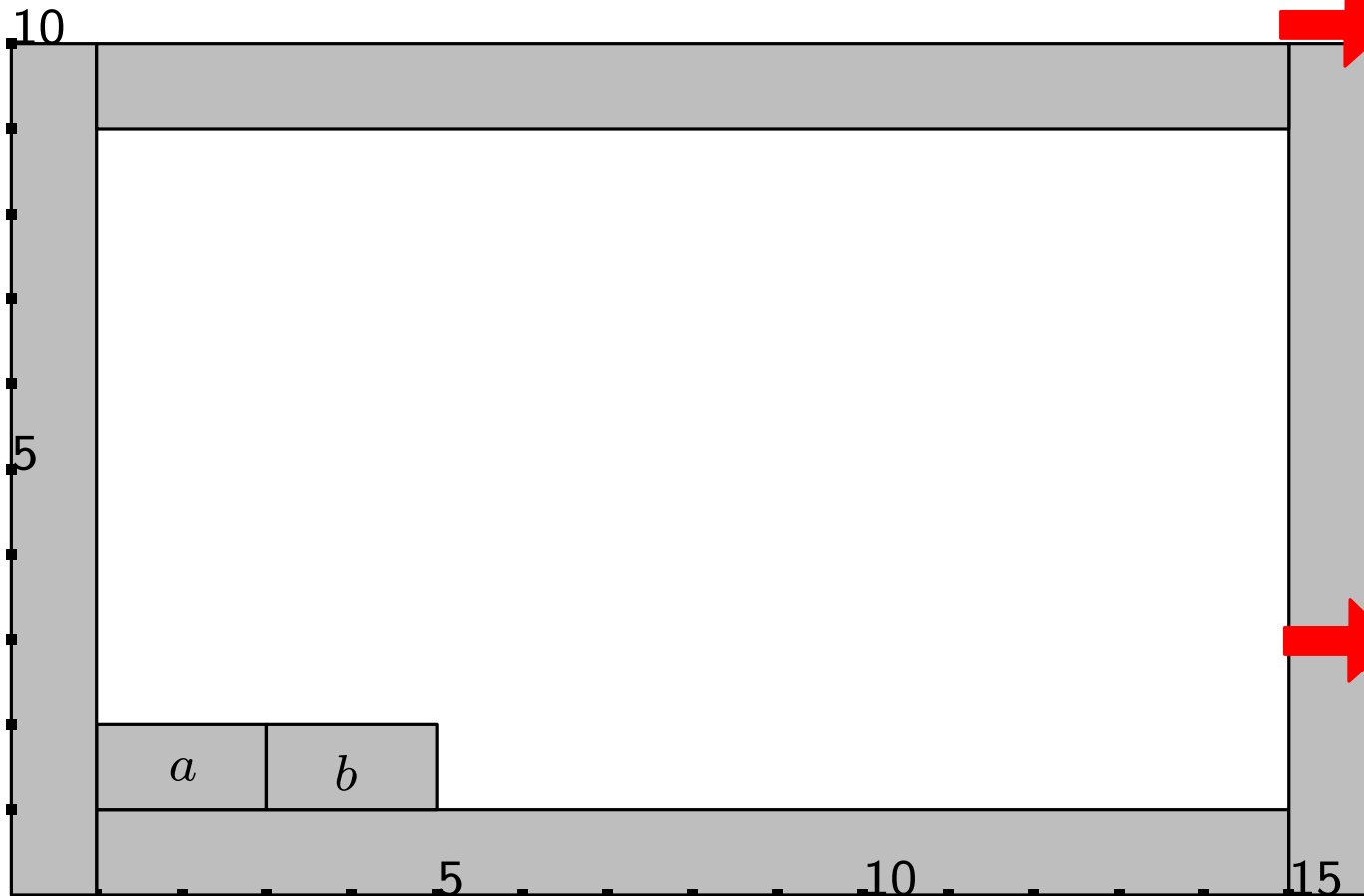
Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

$$\begin{aligned}
 y_1(v_W) &= 0, & y_2(v_W) &= 10 \\
 y_1(v_E) &= 0, & y_2(v_E) &= 10 \\
 y_1(v_N) &= 9, & y_2(v_N) &= 10 \\
 y_1(v_S) &= 0, & y_2(v_S) &= 1 \\
 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
 y_1(d) &= 1, & y_2(d) &= 2 \\
 y_1(e) &= 2, & y_2(e) &= 3
 \end{aligned}$$

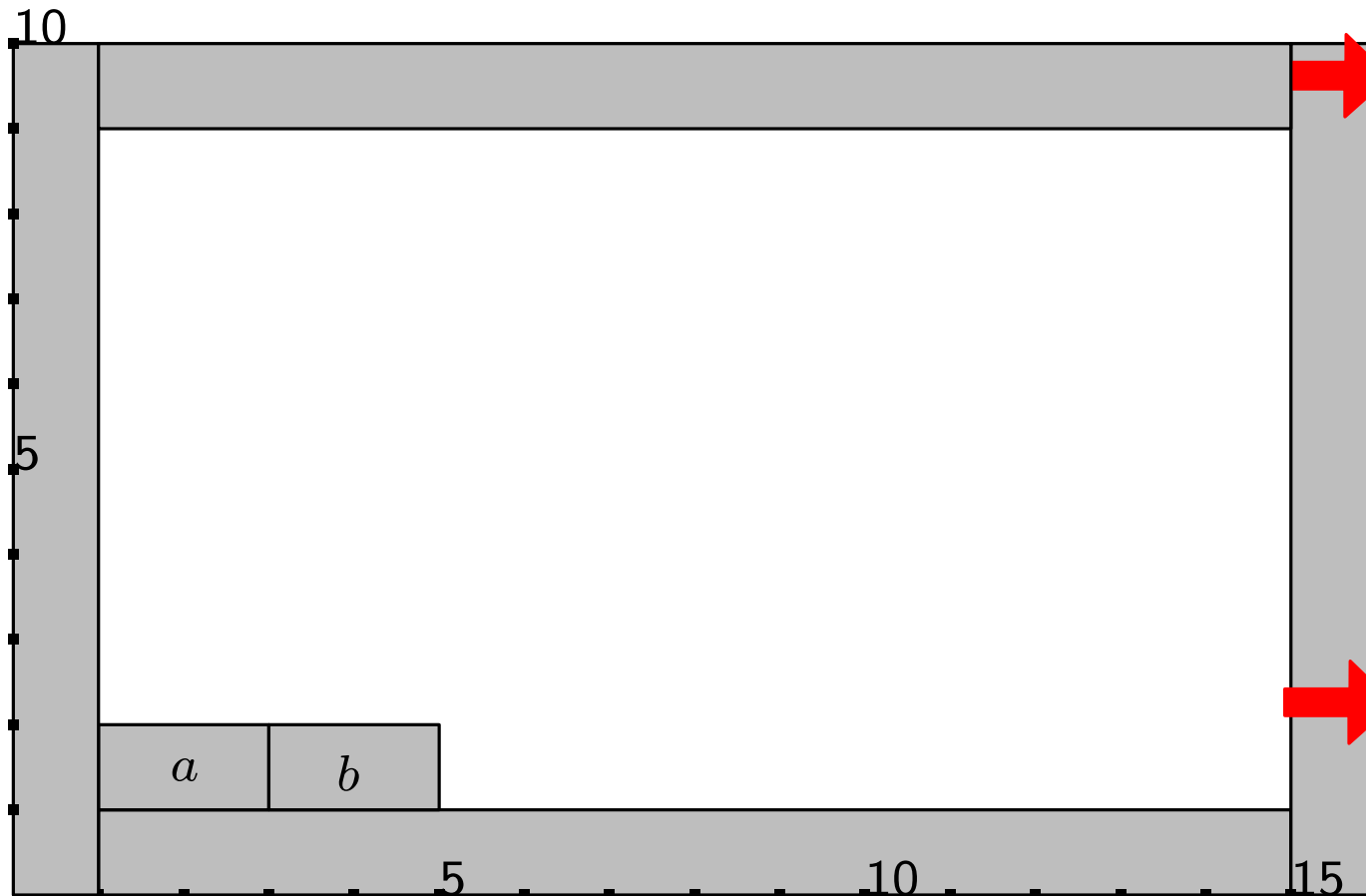
Rectangular Dual



$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

$$\begin{aligned}y_1(v_W) &= 0, & y_2(v_W) &= 10 \\y_1(v_E) &= 0, & y_2(v_E) &= 10 \\y_1(v_N) &= 9, & y_2(v_N) &= 10 \\y_1(v_S) &= 0, & y_2(v_S) &= 1 \\y_1(a) &= 1, & y_2(a) &= 2 \\y_1(b) &= 1, & y_2(b) &= 2 \\y_1(c) &= 1, & y_2(c) &= 2 \\y_1(d) &= 1, & y_2(d) &= 2 \\y_1(e) &= 2, & y_2(e) &= 3\end{aligned}$$

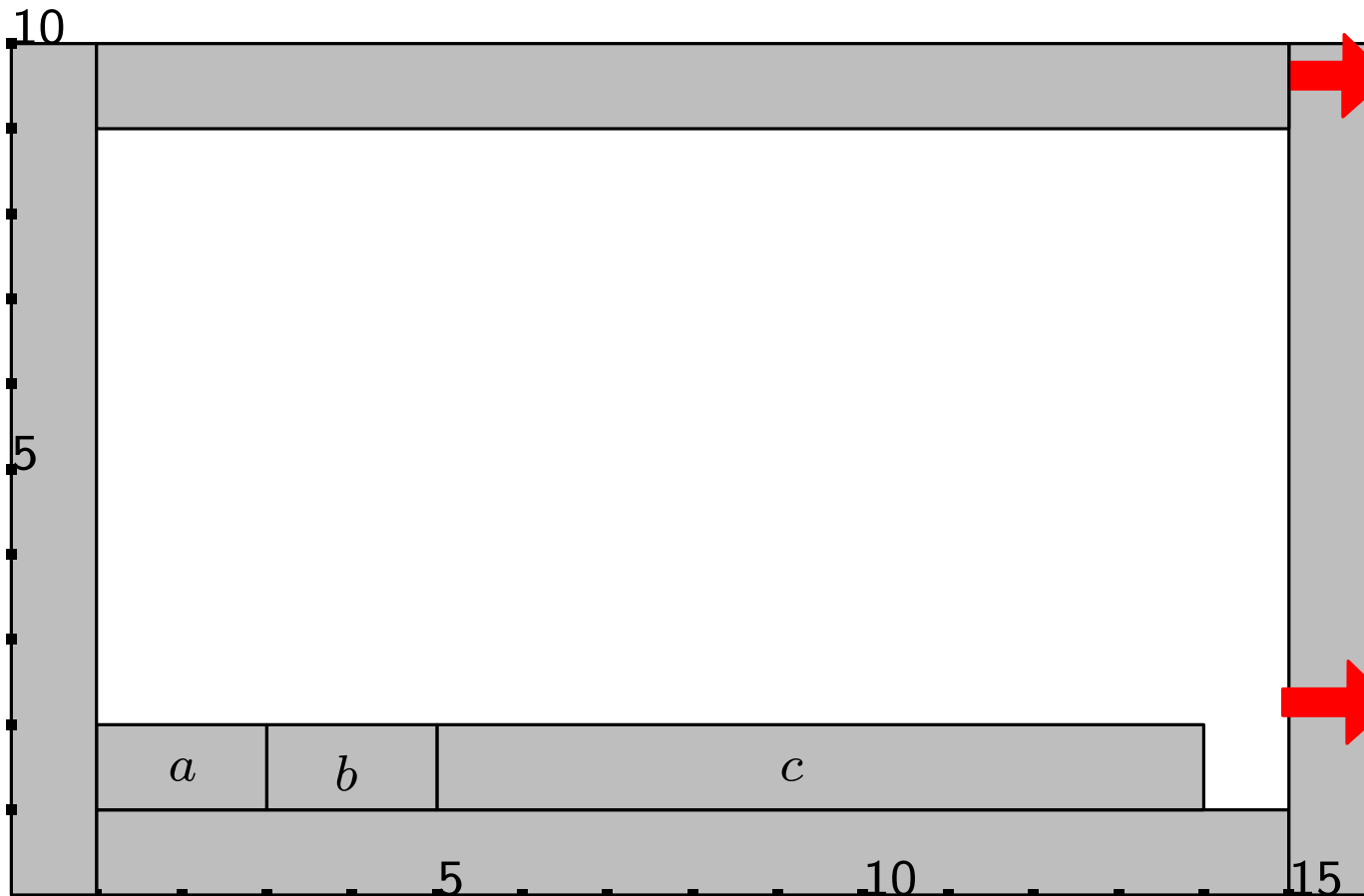
Rectangular Dual



$$\begin{aligned} x_1(v_N) &= 1, & x_2(v_N) &= 15 \\ x_1(v_S) &= 1, & x_2(v_S) &= 15 \\ x_1(v_W) &= 0, & x_2(v_W) &= 1 \\ x_1(v_E) &= 15, & x_2(v_E) &= 16 \\ x_1(a) &= 1, & x_2(a) &= 3 \\ x_1(b) &= 3, & x_2(b) &= 5 \\ x_1(c) &= 5, & x_2(c) &= 14 \\ x_1(d) &= 14, & x_2(d) &= 15 \\ x_1(e) &= 13, & x_2(e) &= 15 \end{aligned}$$

$$\begin{aligned} y_1(v_W) &= 0, & y_2(v_W) &= 10 \\ y_1(v_E) &= 0, & y_2(v_E) &= 10 \\ y_1(v_N) &= 9, & y_2(v_N) &= 10 \\ y_1(v_S) &= 0, & y_2(v_S) &= 1 \\ y_1(a) &= 1, & y_2(a) &= 2 \\ y_1(b) &= 1, & y_2(b) &= 2 \\ y_1(c) &= 1, & y_2(c) &= 2 \\ y_1(d) &= 1, & y_2(d) &= 2 \\ y_1(e) &= 2, & y_2(e) &= 3 \end{aligned}$$

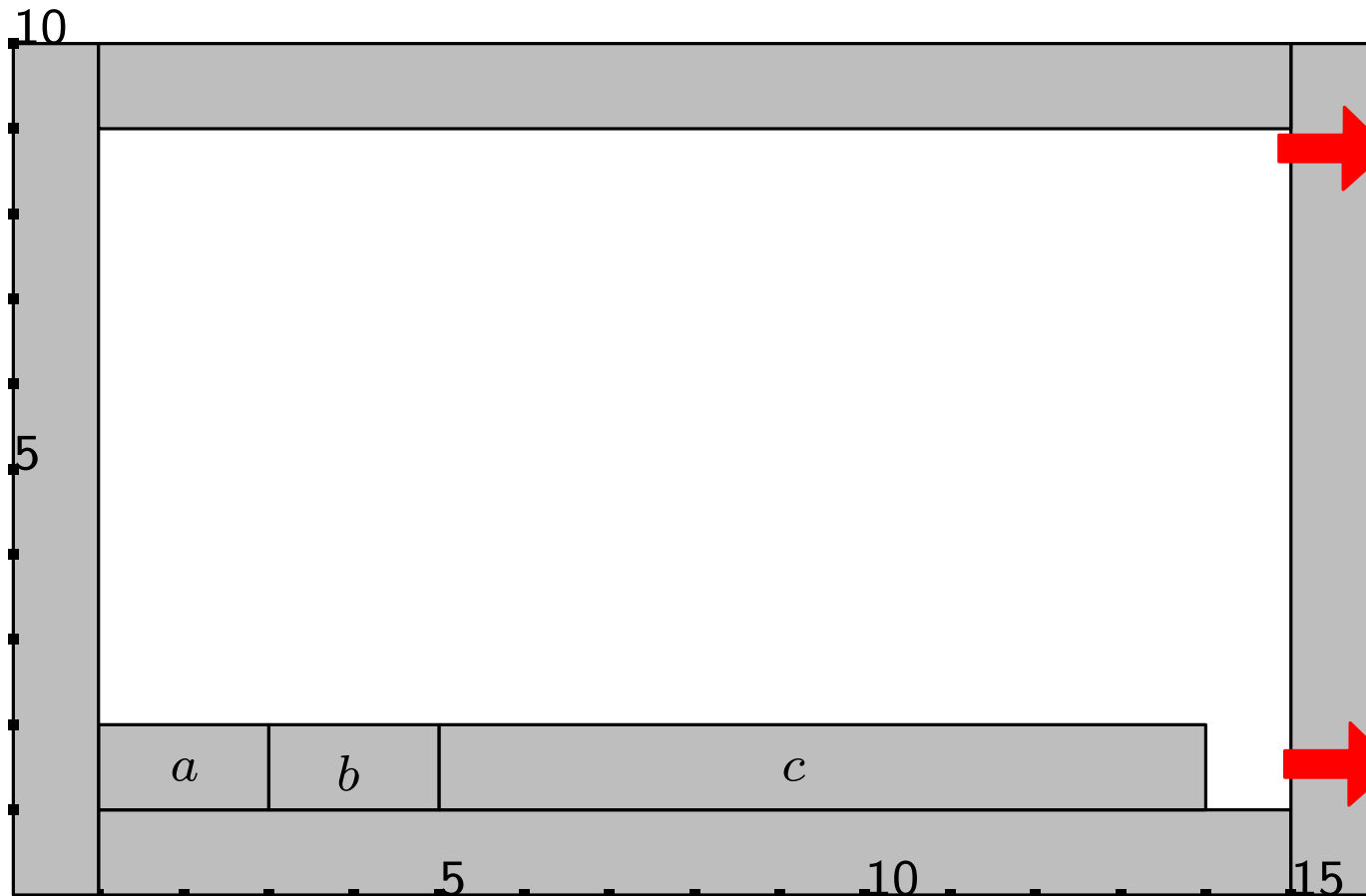
Rectangular Dual



$$\begin{aligned}x_1(v_N) &= 1, & x_2(v_N) &= 15 \\x_1(v_S) &= 1, & x_2(v_S) &= 15 \\x_1(v_W) &= 0, & x_2(v_W) &= 1 \\x_1(v_E) &= 15, & x_2(v_E) &= 16 \\x_1(a) &= 1, & x_2(a) &= 3 \\x_1(b) &= 3, & x_2(b) &= 5 \\x_1(c) &= 5, & x_2(c) &= 14 \\x_1(d) &= 14, & x_2(d) &= 15 \\x_1(e) &= 13, & x_2(e) &= 15\end{aligned}$$

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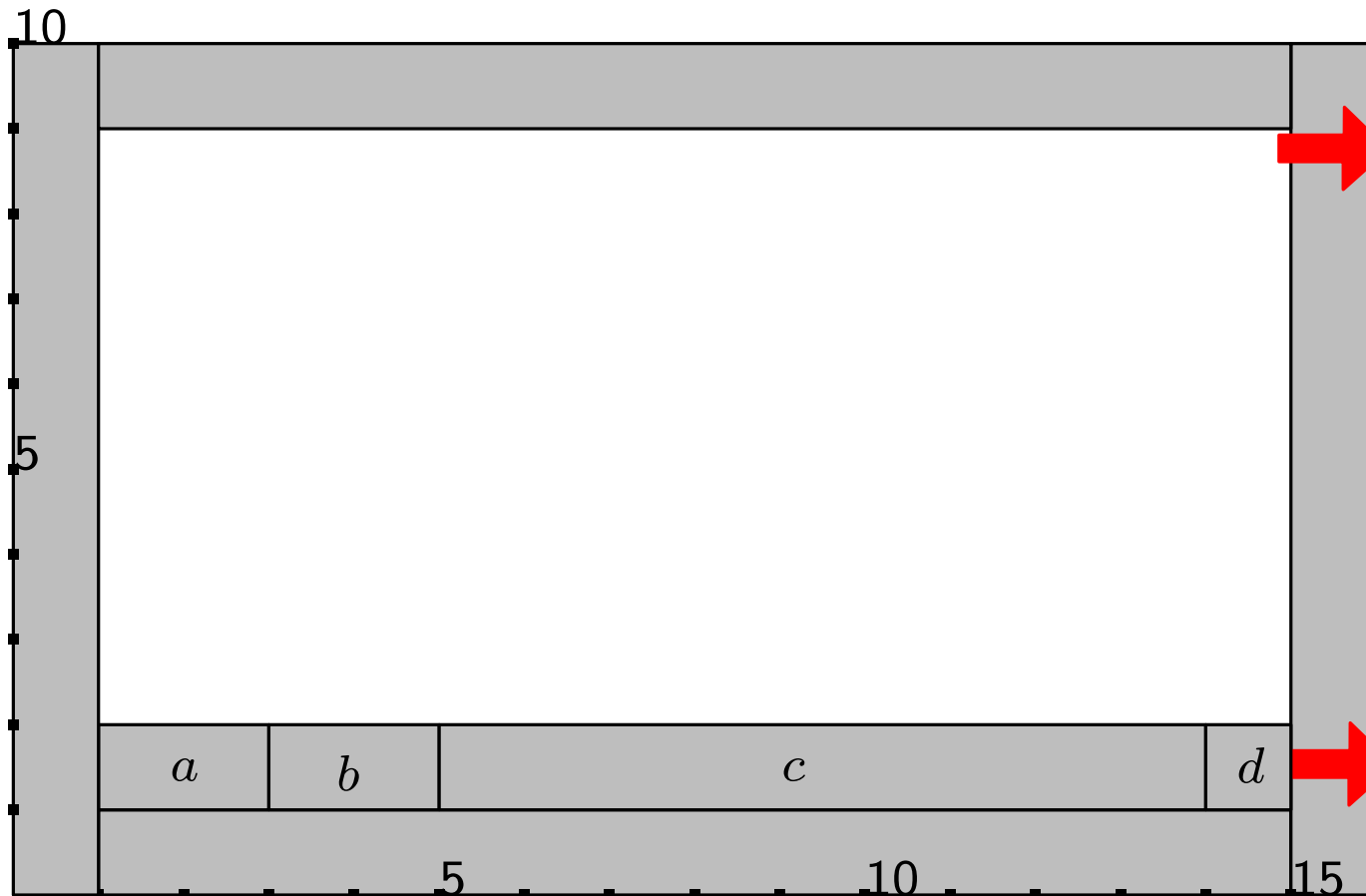
Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
 x_1(e) &= 13, & x_2(e) &= 15
 \end{aligned}$$

$$\begin{aligned}
 y_1(v_W) &= 0, & y_2(v_W) &= 10 \\
 y_1(v_E) &= 0, & y_2(v_E) &= 10 \\
 y_1(v_N) &= 9, & y_2(v_N) &= 10 \\
 y_1(v_S) &= 0, & y_2(v_S) &= 1 \\
 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
 y_1(d) &= 1, & y_2(d) &= 2 \\
 y_1(e) &= 2, & y_2(e) &= 3
 \end{aligned}$$

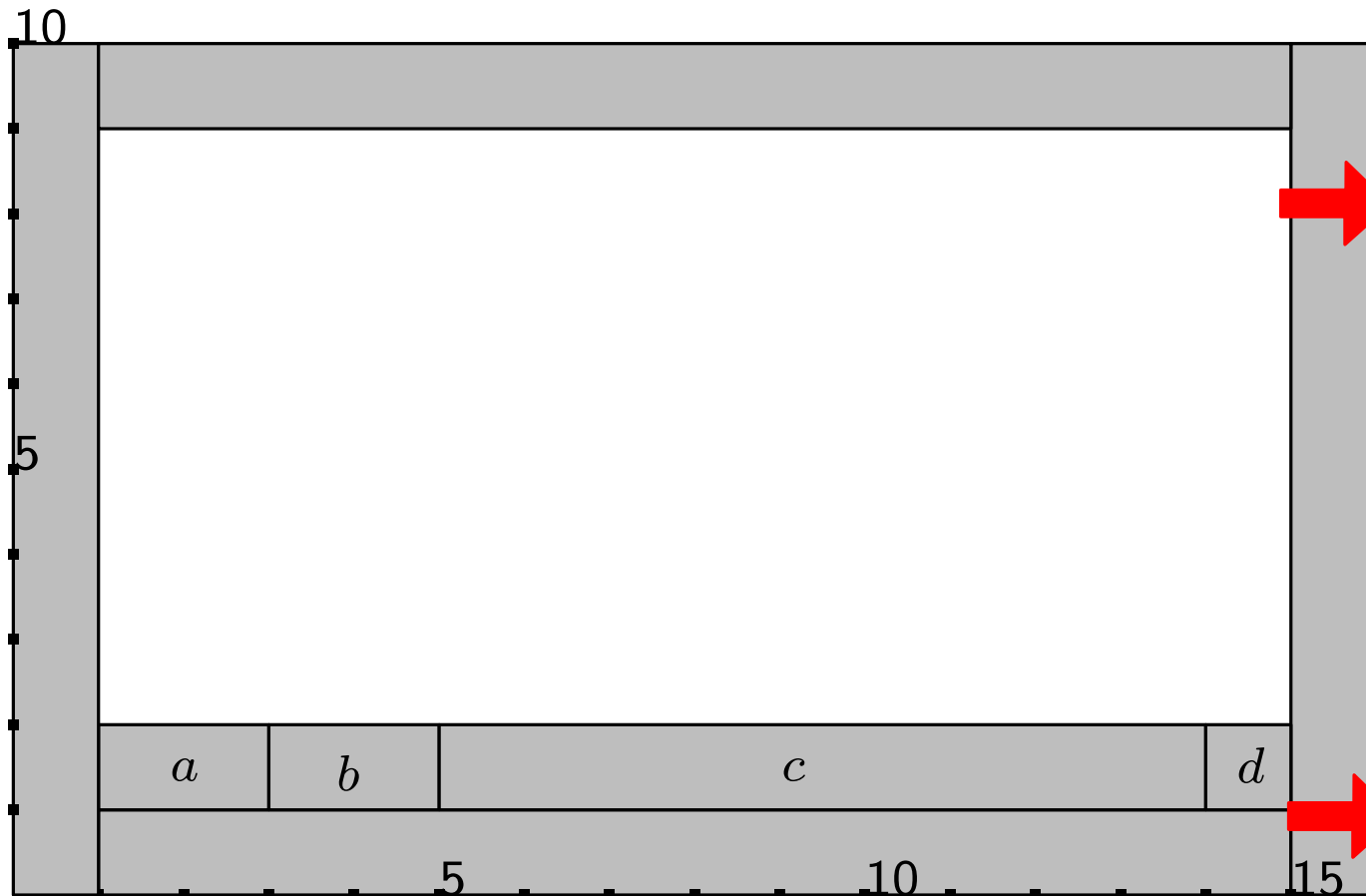
Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
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 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
 y_1(d) &= 1, & y_2(d) &= 2 \\
 y_1(e) &= 2, & y_2(e) &= 3
 \end{aligned}$$

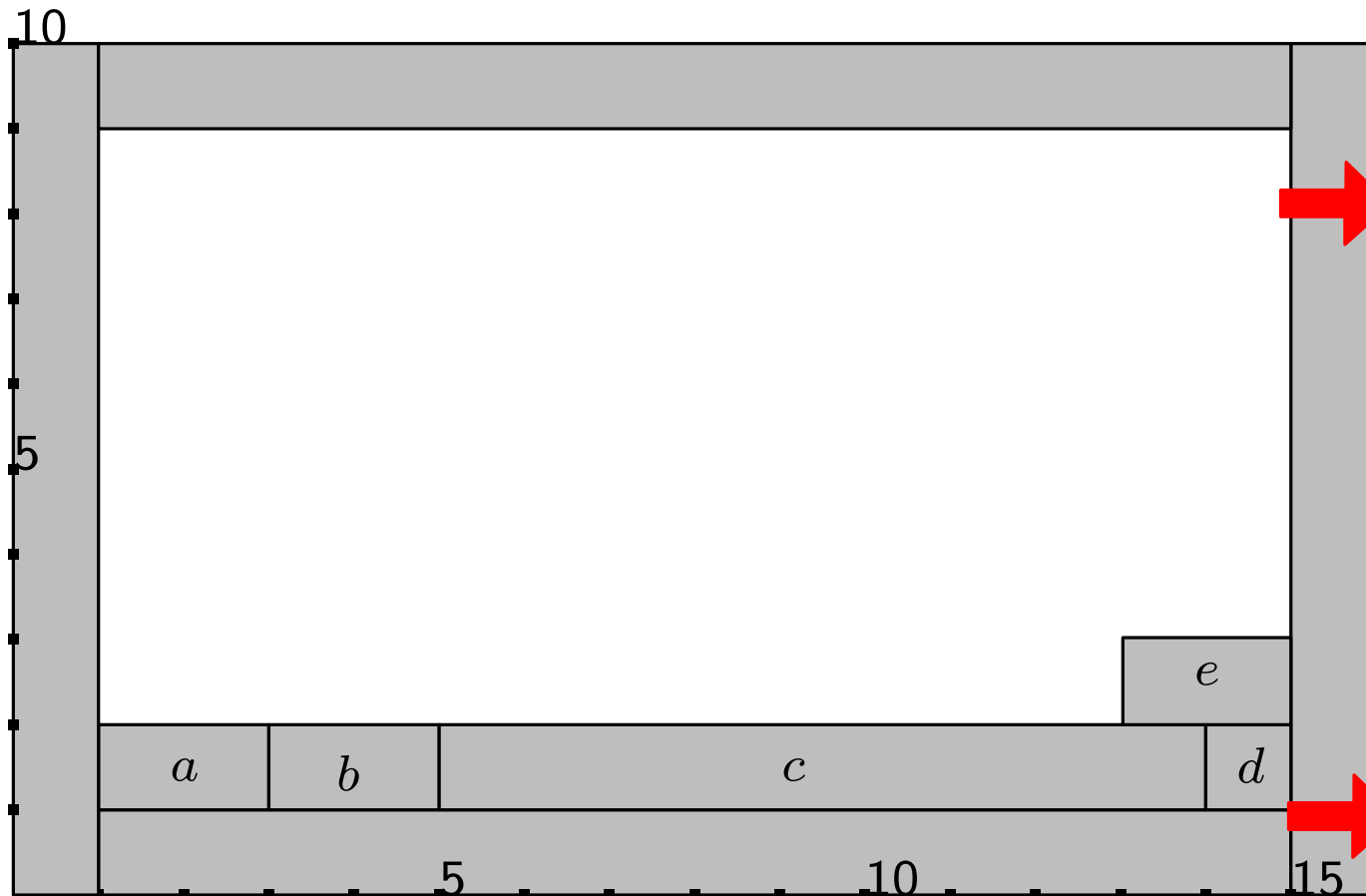
Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
 x_1(b) &= 3, & x_2(b) &= 5 \\
 x_1(c) &= 5, & x_2(c) &= 14 \\
 x_1(d) &= 14, & x_2(d) &= 15 \\
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 \end{aligned}$$

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 y_1(v_W) &= 0, & y_2(v_W) &= 10 \\
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 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
 y_1(d) &= 1, & y_2(d) &= 2 \\
 y_1(e) &= 2, & y_2(e) &= 3
 \end{aligned}$$

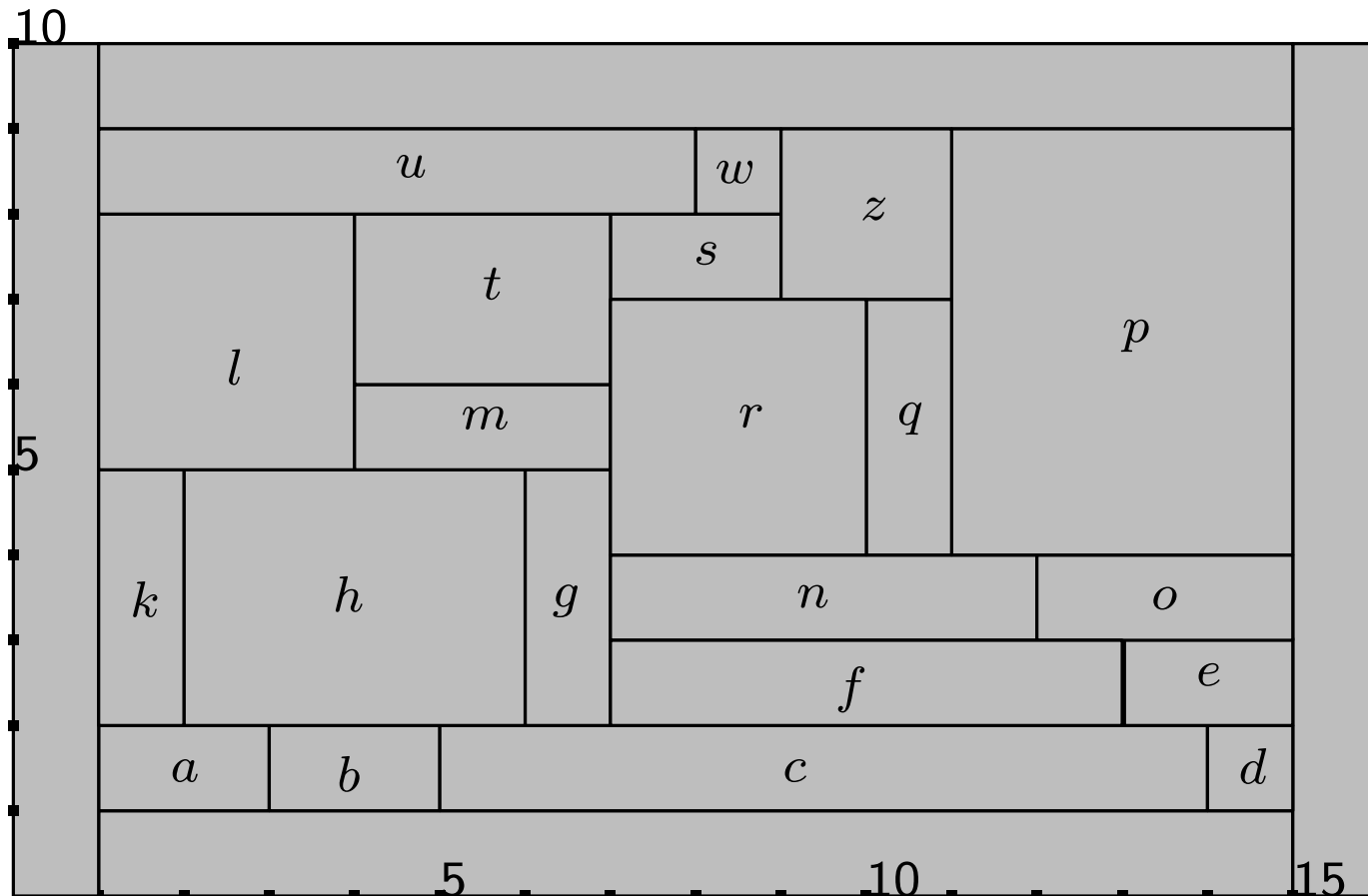
Rectangular Dual



$$\begin{aligned}
 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
 x_1(v_S) &= 1, & x_2(v_S) &= 15 \\
 x_1(v_W) &= 0, & x_2(v_W) &= 1 \\
 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
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 x_1(c) &= 5, & x_2(c) &= 14 \\
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 y_1(v_S) &= 0, & y_2(v_S) &= 1 \\
 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
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 x_1(v_N) &= 1, & x_2(v_N) &= 15 \\
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 x_1(v_E) &= 15, & x_2(v_E) &= 16 \\
 x_1(a) &= 1, & x_2(a) &= 3 \\
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 y_1(v_W) &= 0, & y_2(v_W) &= 10 \\
 y_1(v_E) &= 0, & y_2(v_E) &= 10 \\
 y_1(v_N) &= 9, & y_2(v_N) &= 10 \\
 y_1(v_S) &= 0, & y_2(v_S) &= 1 \\
 y_1(a) &= 1, & y_2(a) &= 2 \\
 y_1(b) &= 1, & y_2(b) &= 2 \\
 y_1(c) &= 1, & y_2(c) &= 2 \\
 y_1(d) &= 1, & y_2(d) &= 2 \\
 y_1(e) &= 2, & y_2(e) &= 3
 \end{aligned}$$

Rectangular Dual

- Let G be a PTP graph and let G_{S-N} (resp. G_{W-E}) be its S-N net, let G_{S-N}^* (resp. G_{W-E}^*) be the dual of G_{S-N} (resp. G_{W-E})
- Let f_1, \dots, f_k be the faces of G_{S-N}^* (resp. G_{W-E}^*), enumerated according to st -numbering f_{sn} (resp. f_{we})
- Let G_{S-N}^i (resp. G_{W-E}^i) denote the subgraph of G that is induced by vertices and edges of f_1, \dots, f_i
- We denote P_i (resp. Q_i) the right (resp. top) boundary of G_{S-N}^i (resp. G_{W-E}^i).
- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) cross at exactly one vertex.

- Let G be a PTP graph and let G_{S-N} (resp. G_{W-E}) be its S-N net, let G_{S-N}^* (resp. G_{W-E}^*) be the dual of G_{S-N} (resp. G_{W-E})
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Lemma 4

Let $v \in V$, f and g are the left and the right face of v . Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

- Let G be a PTP graph and let G_{S-N} (resp. G_{W-E}) be its S-N net, let G_{S-N}^* (resp. G_{W-E}^*) be the dual of G_{S-N} (resp. G_{W-E})
- Let f_1, \dots, f_k be the faces of G_{S-N}^* (resp. G_{W-E}^*), enumerated according to st -numbering f_{sn} (resp. f_{we})
- Let G_{S-N}^i (resp. G_{W-E}^i) denote the subgraph of G that is induced by vertices and edges of f_1, \dots, f_i
- We denote P_i (reps. Q_i) the right (resp. top) boundary of G_{S-N}^i (resp. G_{W-E}^i).
- Paths P_i and Q_j for any i, j (except for (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d) $i = \max f_{sn} - 1, j = \max f_{we} - 1$) cross at exactly one vertex.

Lemma 4

Let $v \in V$, f and g are the left and the right face of v . Let $x_1(v) = f_{sn}(f)$ and $x_2(v) = f_{sn}(g)$. Vertex v belongs to path P_i if and only if $x_1(v) \leq i \leq x_2(v) - 1$.

Lemma 5

Let $v \in V$, f and g are the faces below and above v in G_{W-E} . Let $y_1(v) = f_{we}(f)$ and $y_2(v) = f_{we}(g)$. Vertex v belongs to path Q_j if and only if $y_1(v) \leq j \leq y_2(v) - 1$.

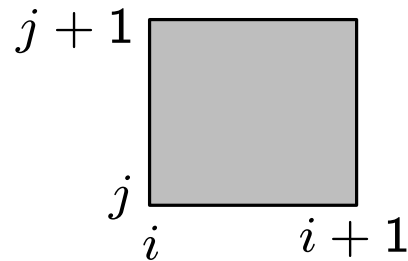
Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

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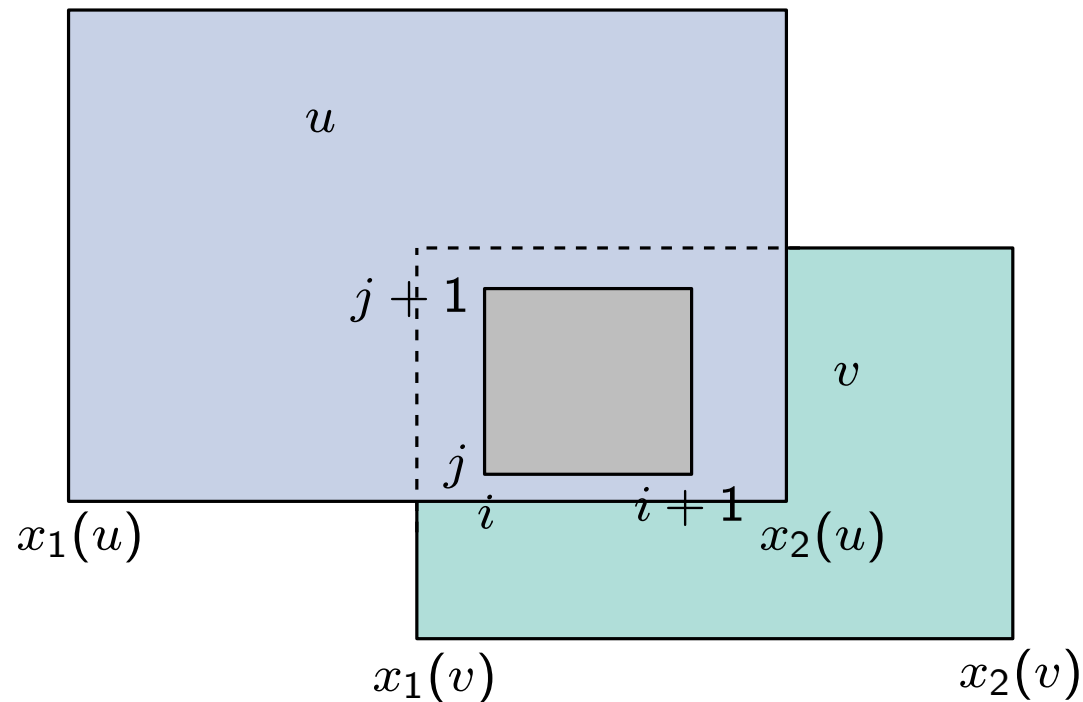
Proof:



Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:

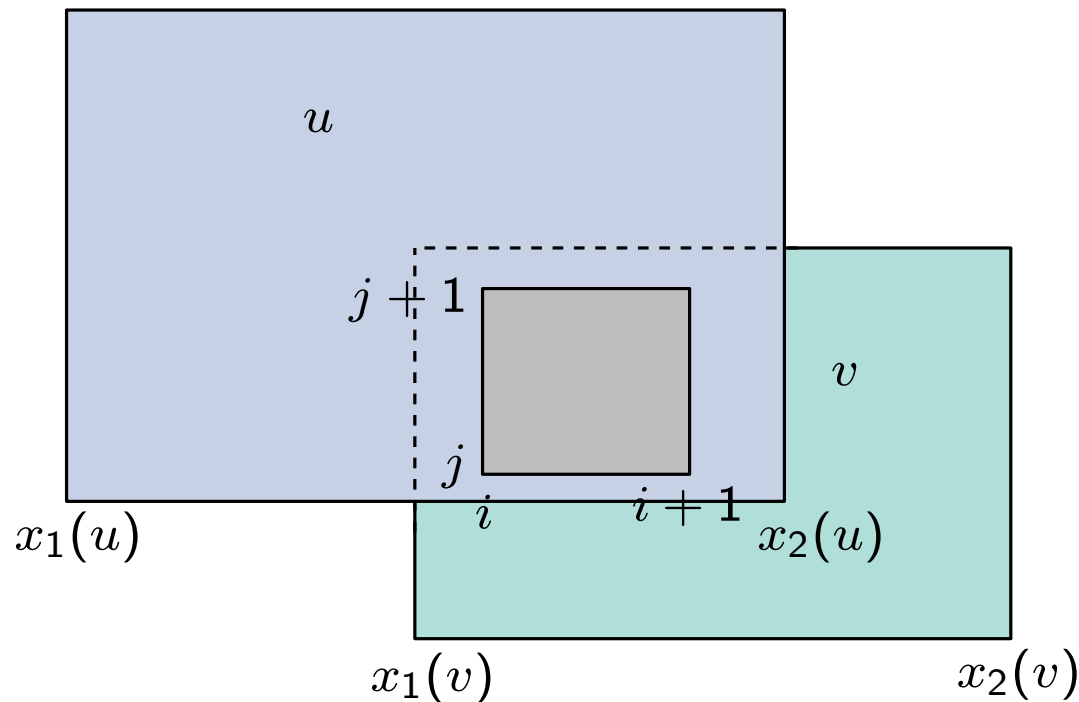


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The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:

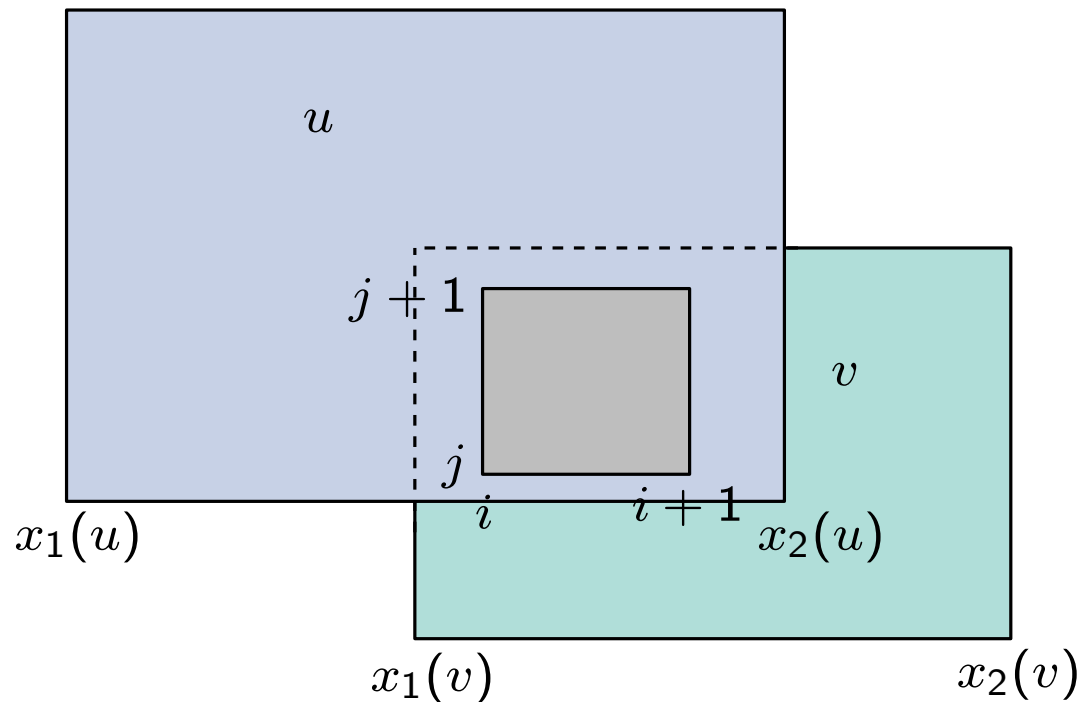
$$x_1(u) \leq i \text{ and } i+1 \leq x_2(u)$$



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The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:



$$x_1(u) \leq i \text{ and } i+1 \leq x_2(u)$$



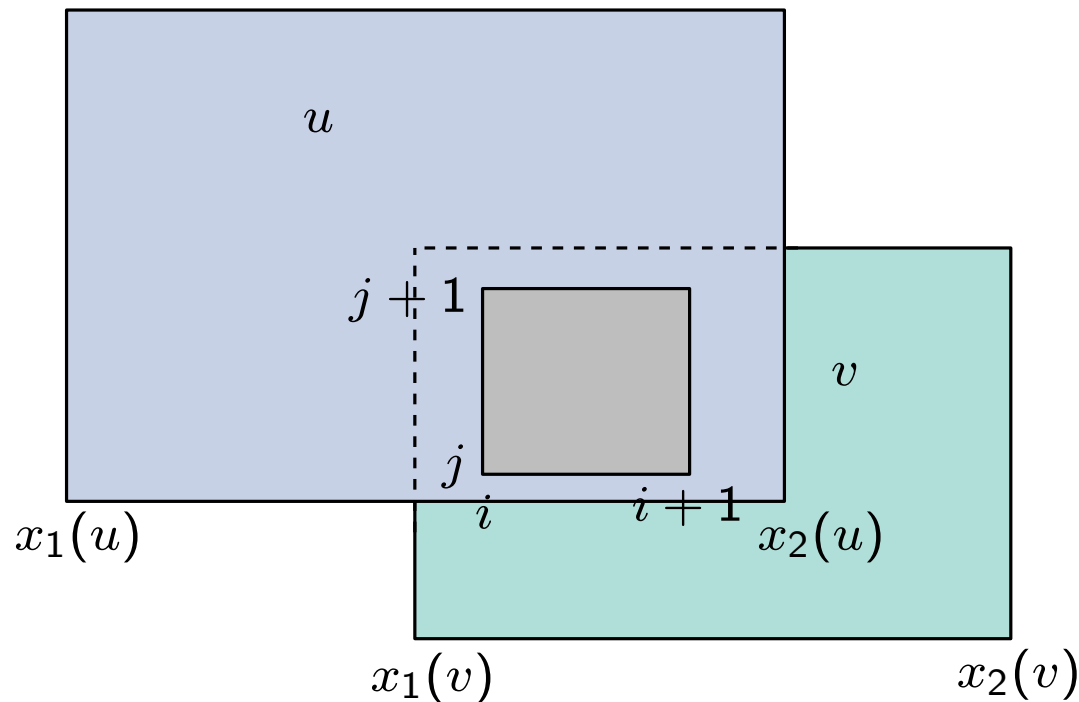
(Lemma 4)

v belongs to P_i

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Proof:



$$x_1(u) \leq i \text{ and } i+1 \leq x_2(u)$$



(Lemma 4)

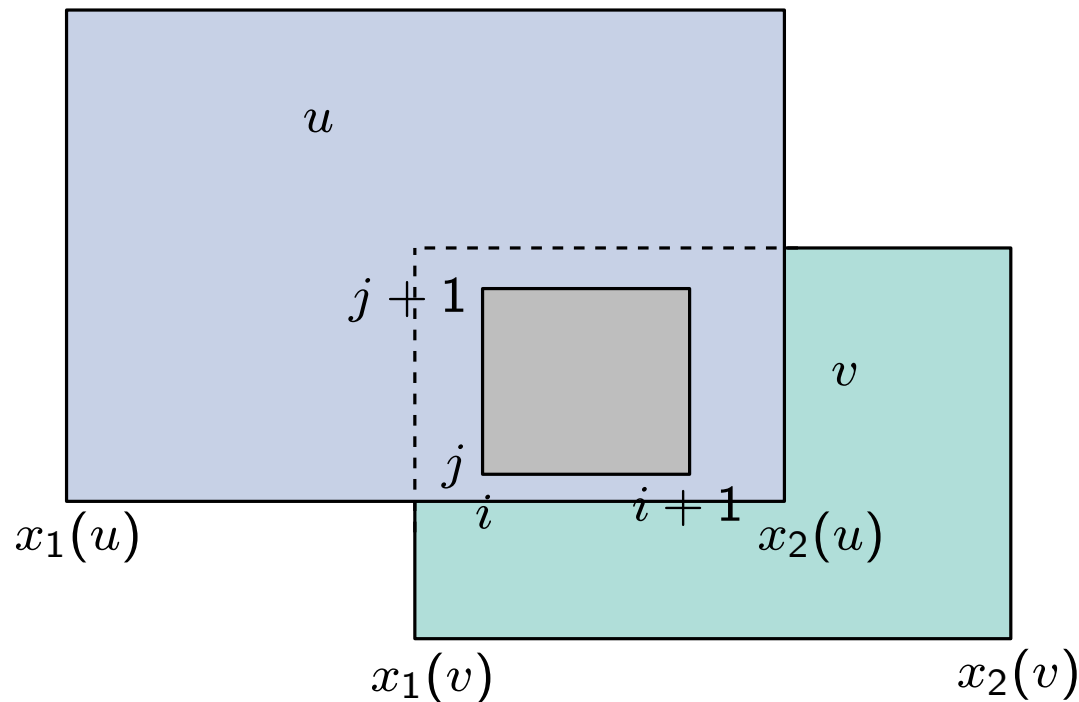
v belongs to P_i

Similarly: $v \in P_i, u \in Q_j, v \in Q_j$.

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:



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Similarly: $v \in P_i, u \in Q_j, v \in Q_j$.

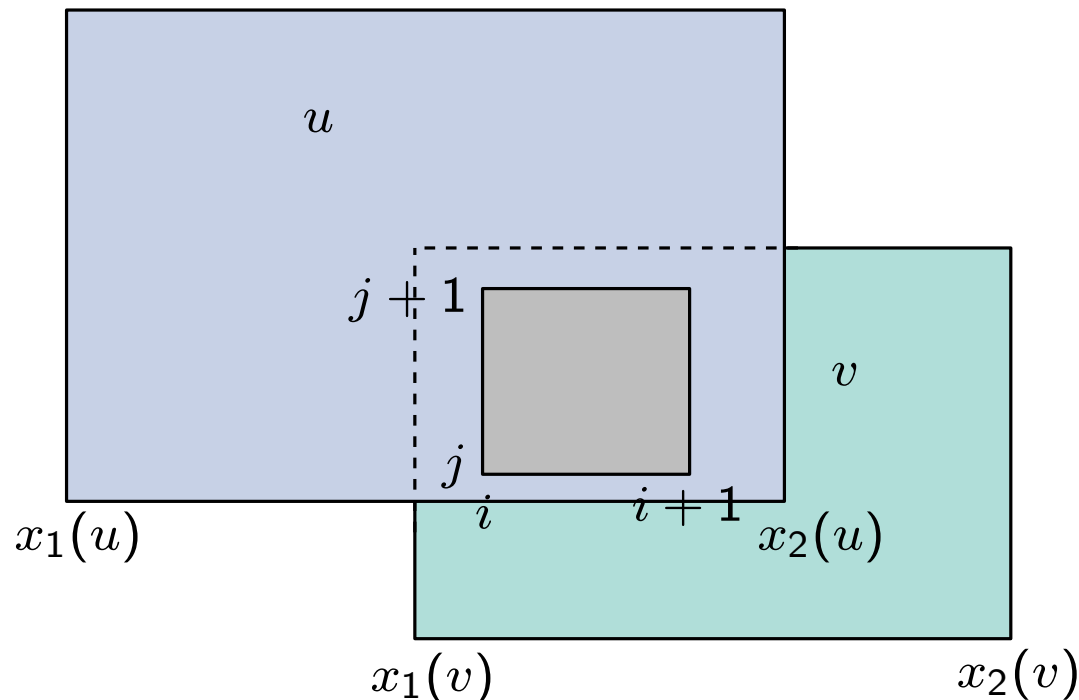


Paths P_i and Q_j intersect at two vertices u and v .

Lemma 6

The assignment provided by the algorithm do not produce neither gaps nor overlapping rectangles.

Proof:



$$x_1(u) \leq i \text{ and } i+1 \leq x_2(u)$$



(Lemma 4)

v belongs to P_i

Similarly: $v \in P_i, u \in Q_j, v \in Q_j$.



Paths P_i and Q_j intersect at two vertices u and v .



Which is a contradiction to the property of paths P_i, Q_j except for the cases when:
 (a) $i = 0, j = 0$, (b) $i = \max f_{sn} - 1, j = 0$, (c) $i = 0, j = \max f_{we} - 1$, (d)
 $i = \max f_{sn} - 1, j = \max f_{we} - 1$ (corner boxes).

Lemma 7

Let G_{S-N} and G_{W-E} . The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
- If there exist a directed path from u to v in G_{W-E} containing at least two edges, then $x_2(u) < x_1(v)$
- If $(u, v) \in G_{S-N}$ then $y_2(u) = y_1(v)$;
- If there exist a directed path from u to v in G_{S-N} containing at least two edges, then $y_2(u) < y_1(v)$

Proof...

Lemma 7

Let G_{S-N} and G_{W-E} . The following are true:

- If $(u, v) \in G_{W-E}$ then $x_2(u) = x_1(v)$;
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Proof...

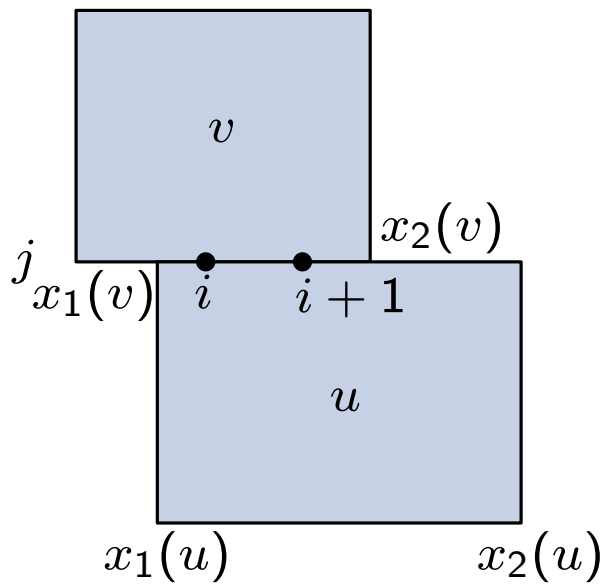
Lemma 8

The assignment provided by the algorithm has the following property: rectangles assigned to vertices u and v have a common segment if and only if there exists edge (u, v) in the graph.

Proof

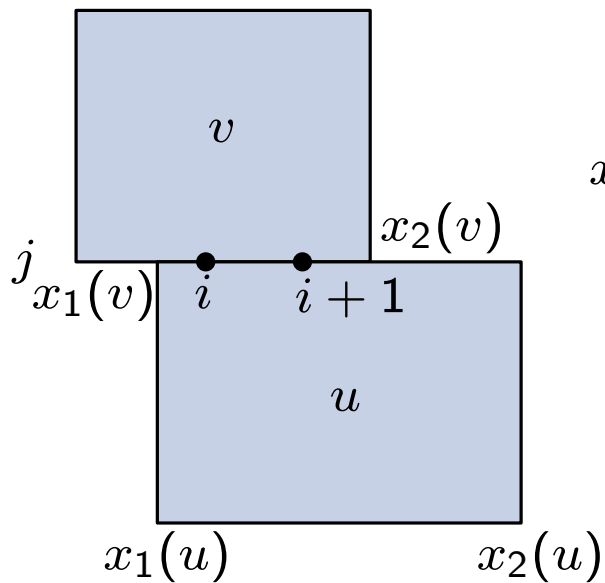
Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



Rectangular Dual

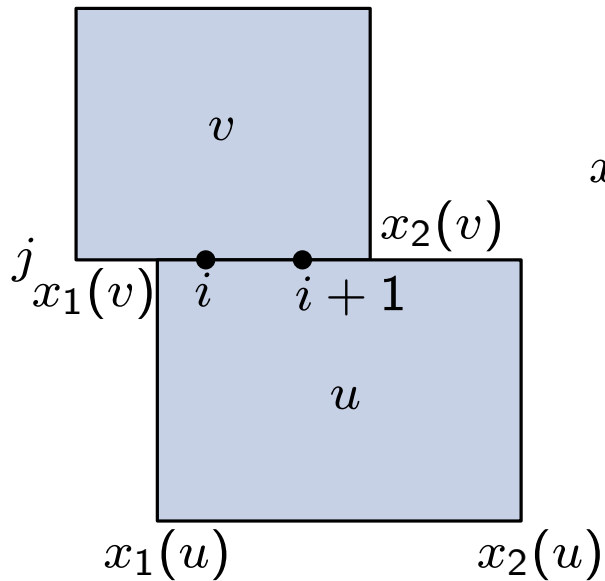
- Assume $R(u)$ and $R(v)$ have a common boundary.



$$x_1(v) \leq i, i + 1 \leq x_2(v) \text{ and } x_1(u) \leq i, i + 1 \leq x_2(u)$$

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$$x_1(v) \leq i, i + 1 \leq x_2(v) \text{ and } x_1(u) \leq i, i + 1 \leq x_2(u)$$

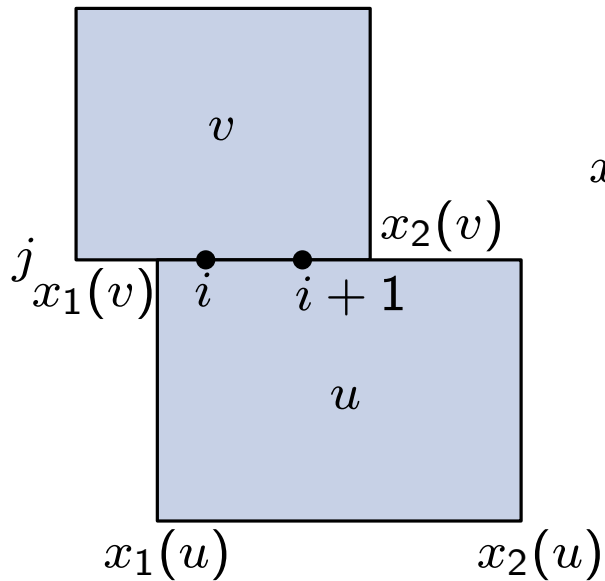


(Lemma 4)

u, v belong to P_i

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$$x_1(v) \leq i, i + 1 \leq x_2(v) \text{ and } x_1(u) \leq i, i + 1 \leq x_2(u)$$

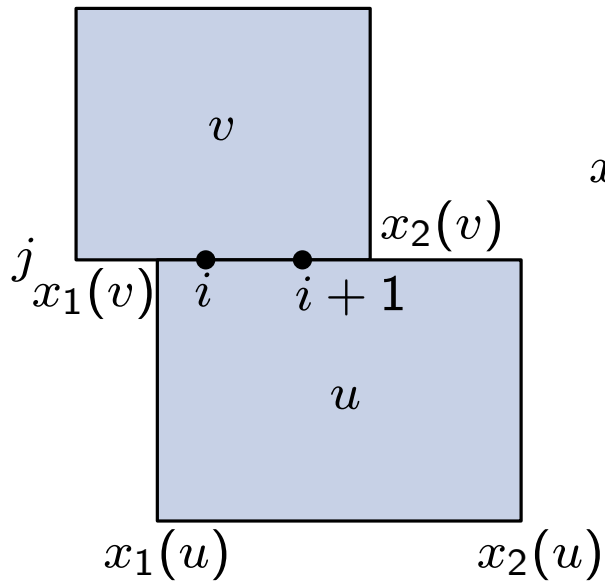
(Lemma 4)

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If path has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

Rectangular Dual

- Assume $R(u)$ and $R(v)$ have a common boundary.



$$x_1(v) \leq i, i + 1 \leq x_2(v) \text{ and } x_1(u) \leq i, i + 1 \leq x_2(u)$$

(Lemma 4)

u, v belong to P_i

If path has at least 2 edges, then by Lemma 7, $y_2(u) < y_1(v)$

A contradiction to the hypothesis!

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.



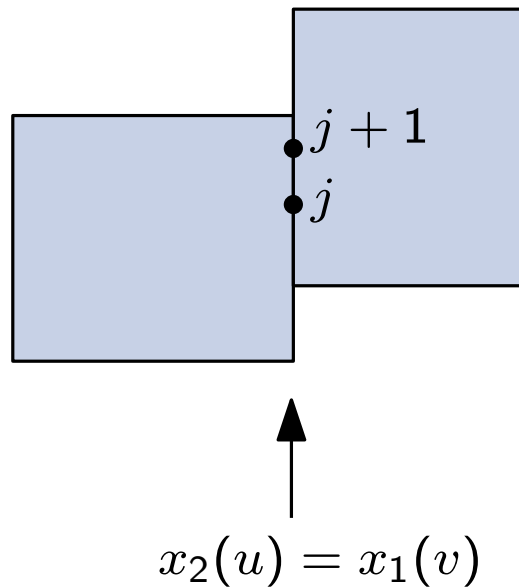
- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$

Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.



- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$

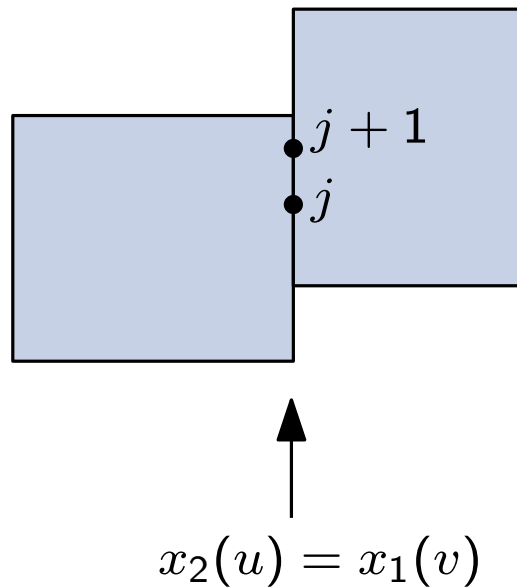


Rectangular Dual

- Assume there exists an edge $(u, v) \in G_{W-E}$.



- Let Q_j be the path of G_{W-E} where (u, v) belongs. By Lemma 5, $y_1(u) \leq j$, $j + 1 \leq y_2(u)$ and $y_1(v) \leq j$, $j + 1 \leq y_2(v)$
- By Lemma 7, $x_2(u) = x_1(v)$



Lemma 8 is proved!



Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

Theorem

Every PTP graph G has a rectangular dual which can be computed in linear time.

- Compute a planar embedding of G
- Compute a **revised canonical ordering** of G
- Traverse the graph and color the edges, construct G_{S-N} and G_{E-W}
- Construct the duals G_{S-N}^* and G_{E-W}^* of G_{S-N} and G_{E-W} , respectively
- Compute an st -numbering of G_{S-N}^* and G_{E-W}^*
- Assigning coordinates to the rectangles representing vertices.