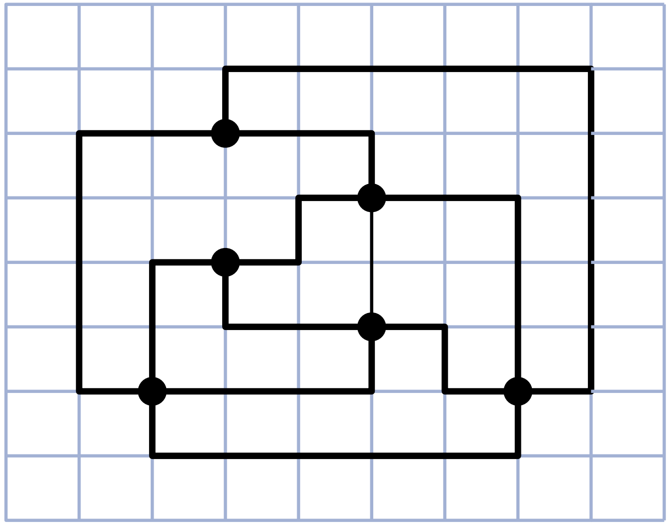
Karlsruhe Institute of Technology

Algorithms for graph visualization

Incremental algorithms. Orthogonal drawing.

WINTER SEMESTER 2012/2013

Tamara Mchedlidze – MARTIN NÖLLENBURG – IGNAZ RUTTER



Definition

Definition: Orthogonal Drawing

A drawing Γ of a graph $G = (V, E)$ is called orthogonal if its veritices are drawn as points and each edge is represented as a sequence of alternating horizontal and vertical segments.

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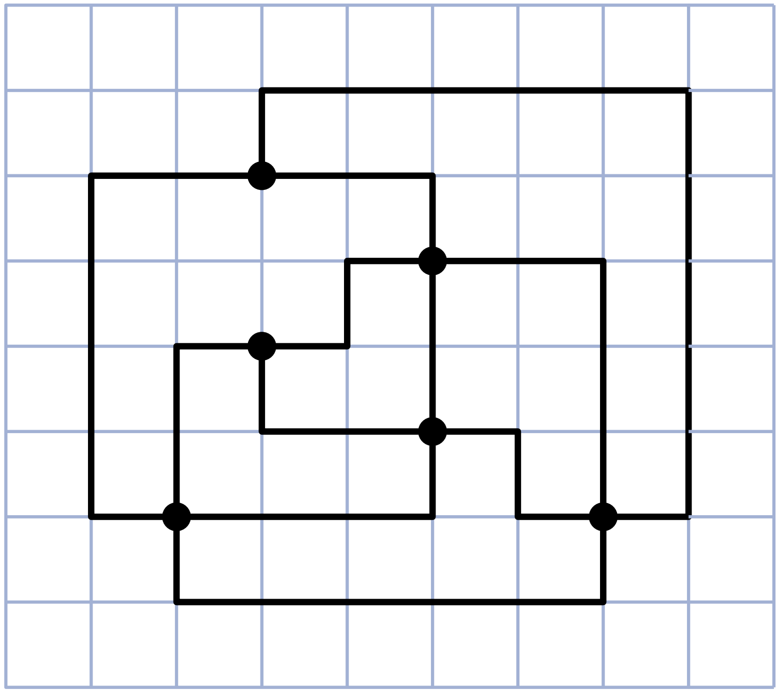

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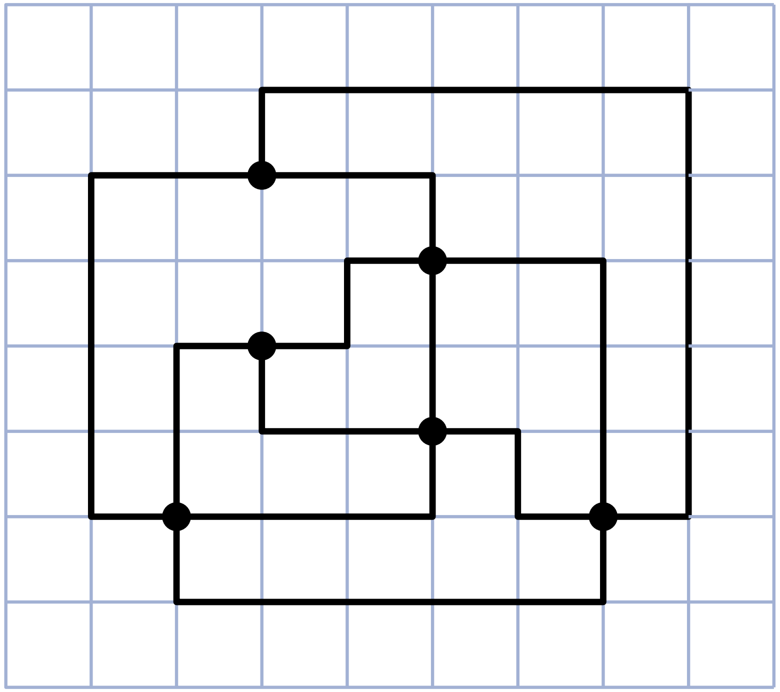

- degree of each vertex has to be at most 4

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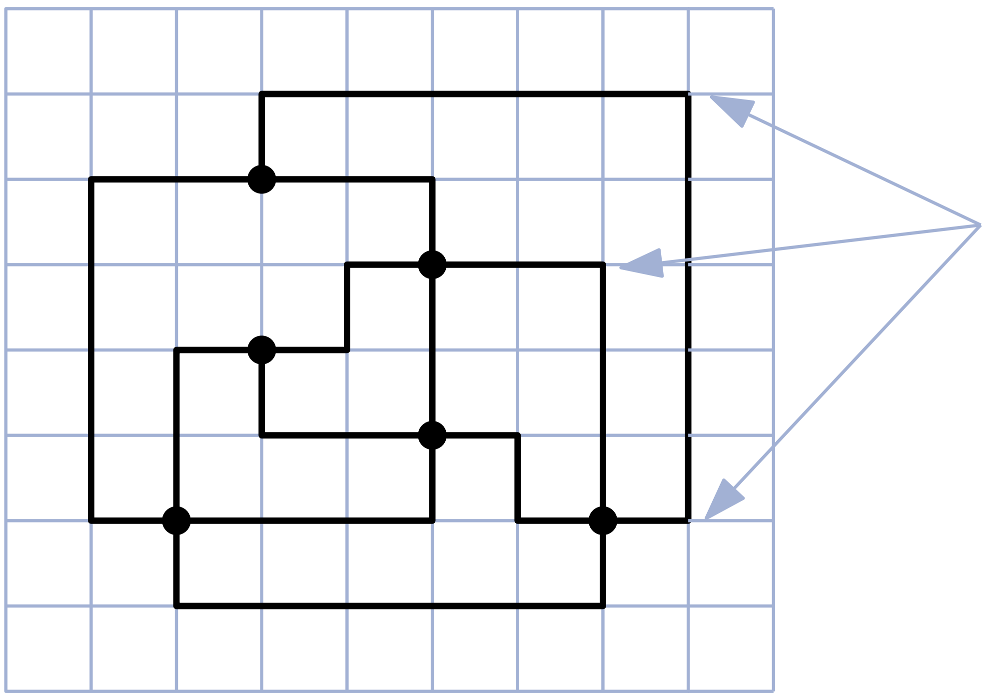

- degree of each vertex has to be at most 4
- bends on edges

st-ordering

Definition: st-ordering

An st-ordering of a graph $G = (V, E)$ is an ordering of the vertices $\{v_1, v_2, \ldots, v_n\}$, such that for each j, $2 \leq j \leq n - 1$, vertex v_j has at least one neighbour v_i with $i < j$, and at least one neighbour v_k with $k > j$.

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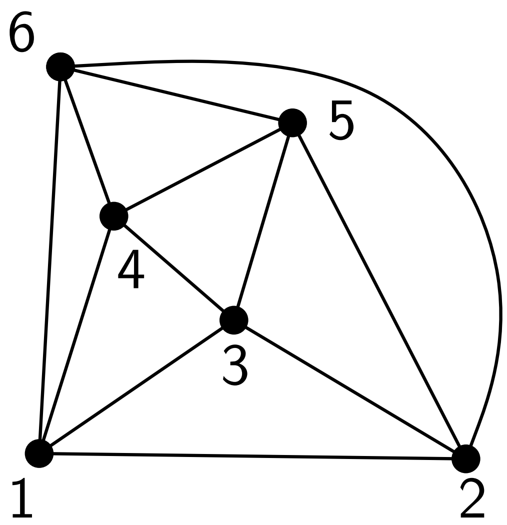
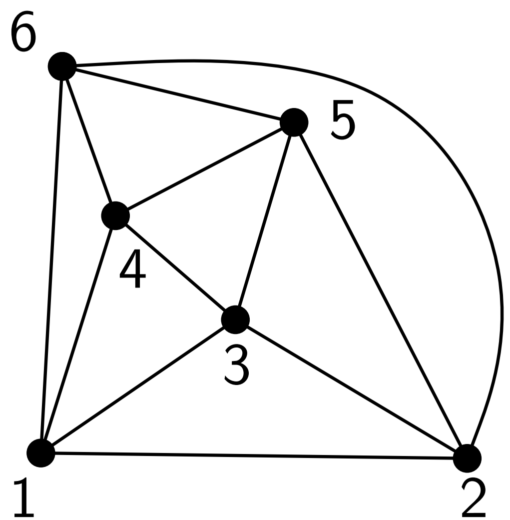
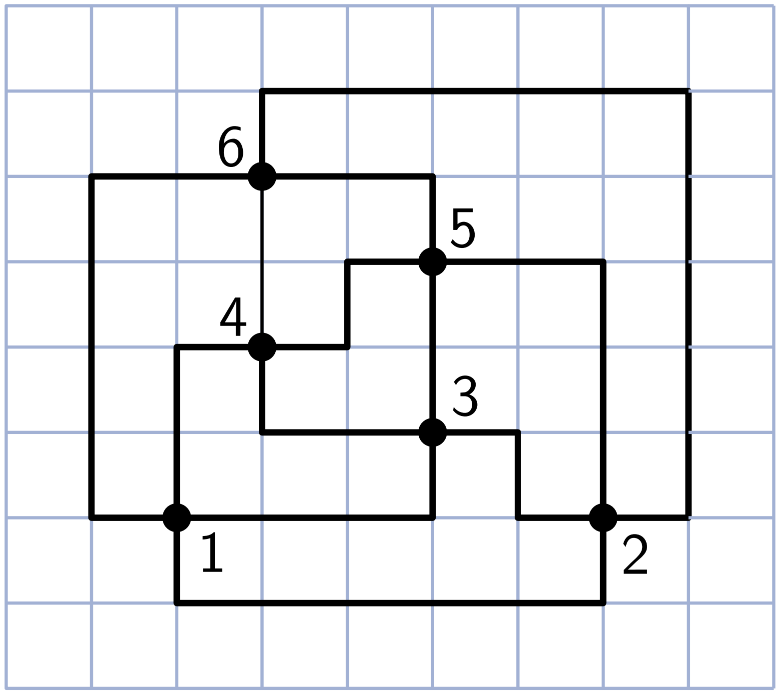


Theorem [Lempel, Even, Cederbaum, 66]

Let G be a biconnected graph G and let s, t be vetices of G. G has an st-ordering such that s appears as the first and t as the last vertex in this ordering.

Biedl & Kant Orthogonal Drawing Algorithm





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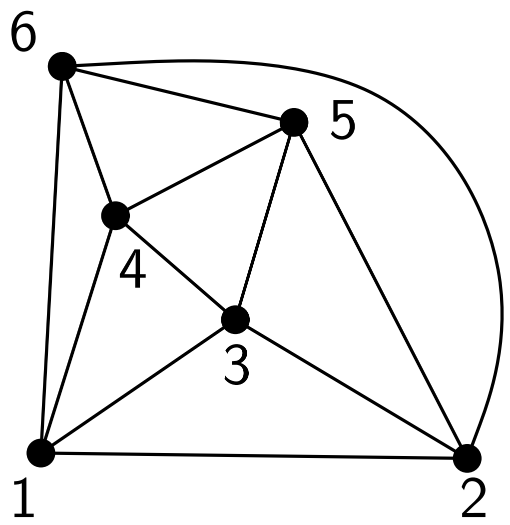
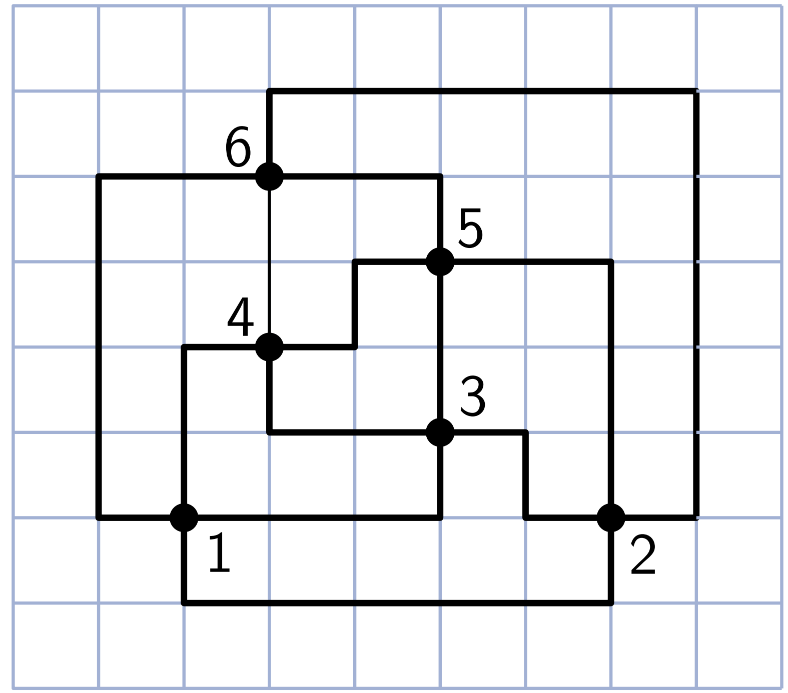
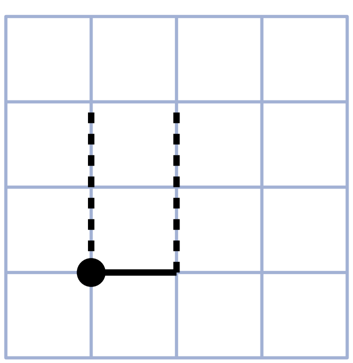




first vertex



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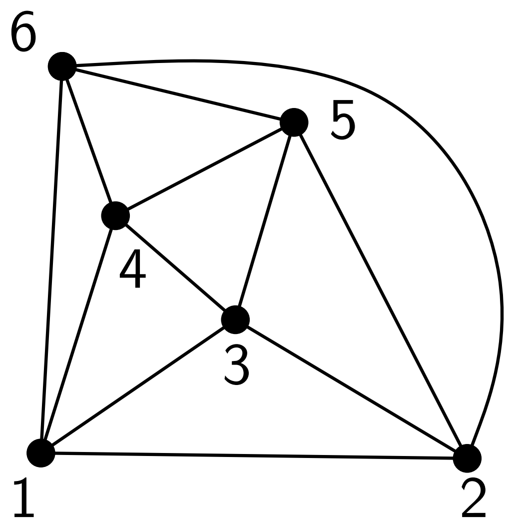

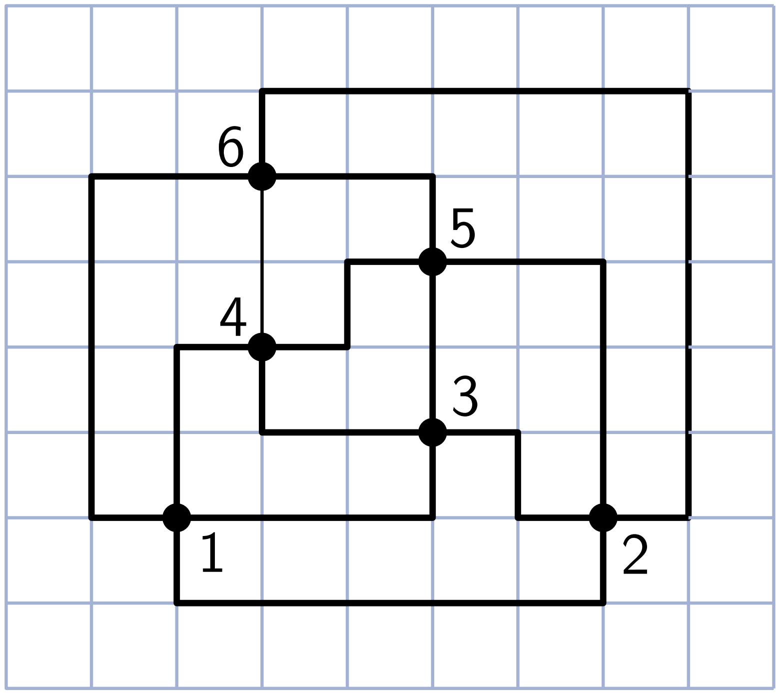

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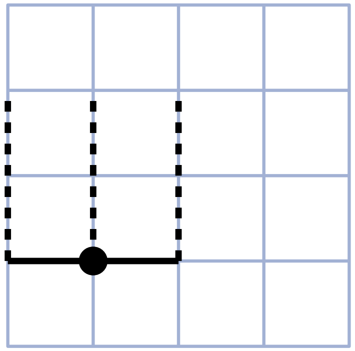

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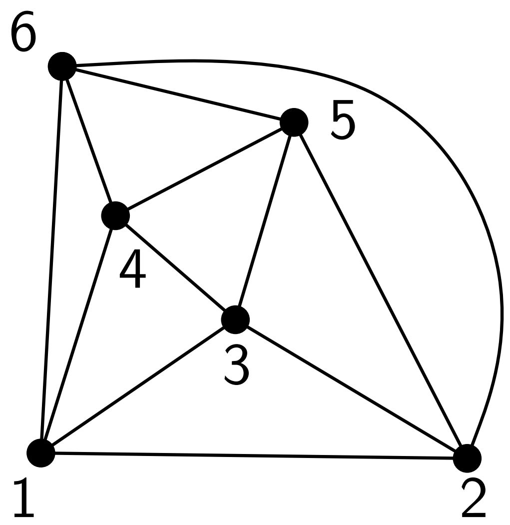

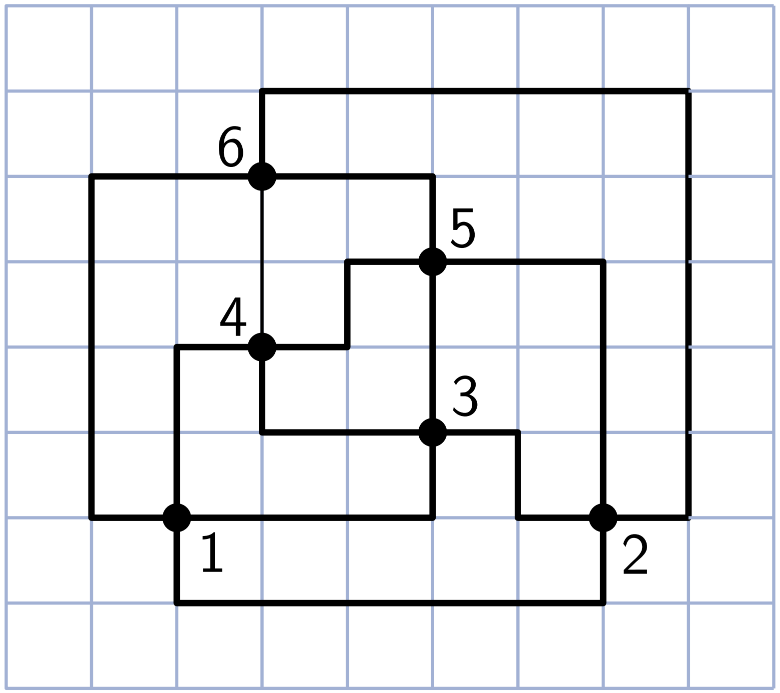

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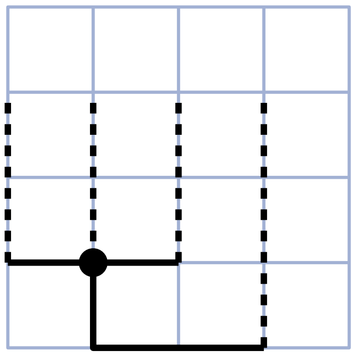

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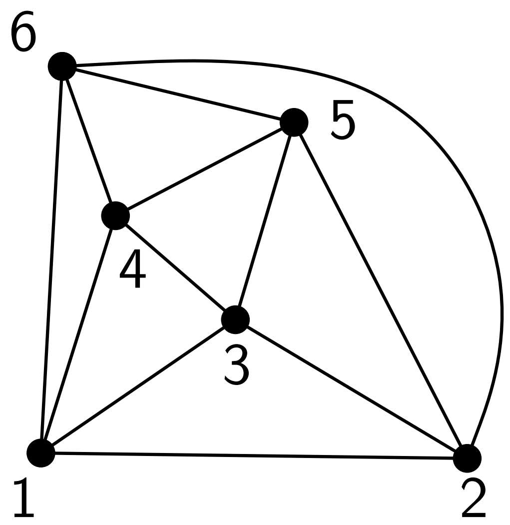

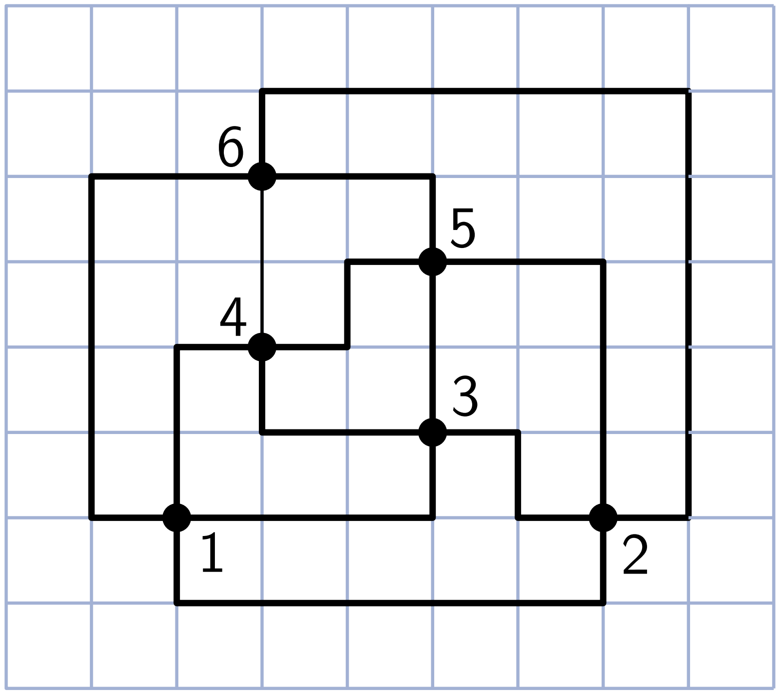

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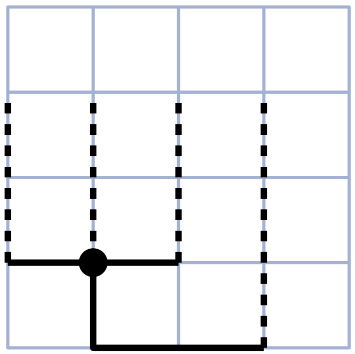
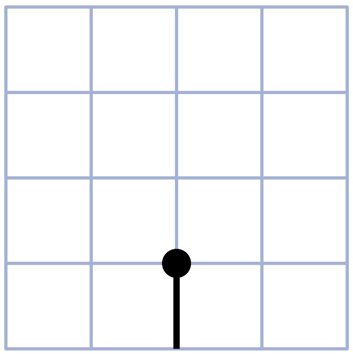


indegree = 1



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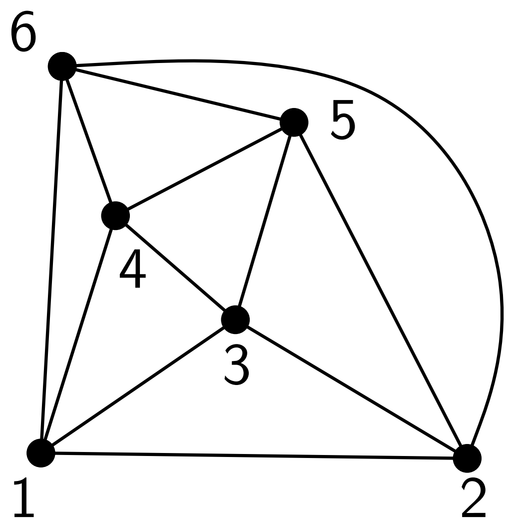
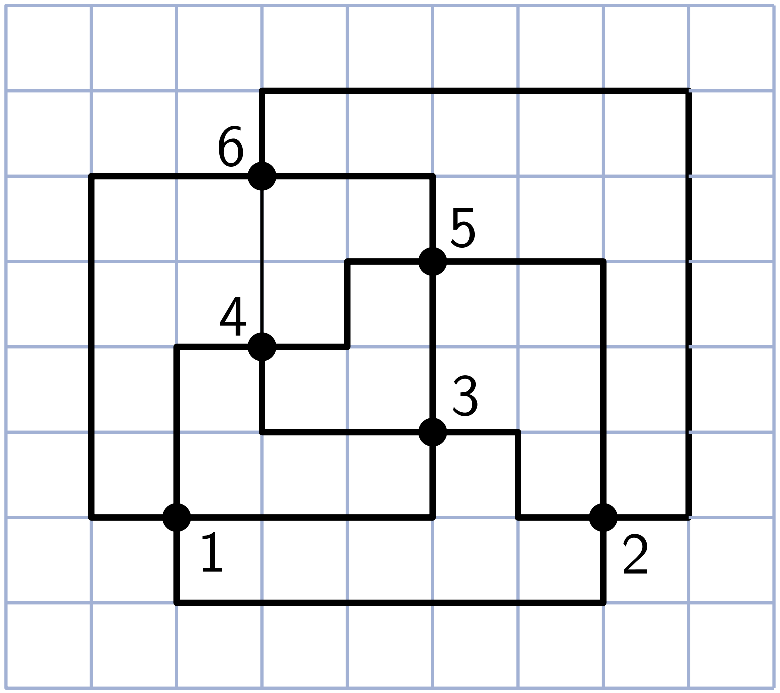
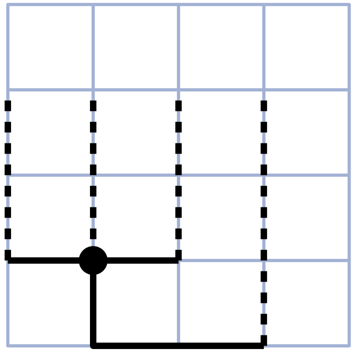
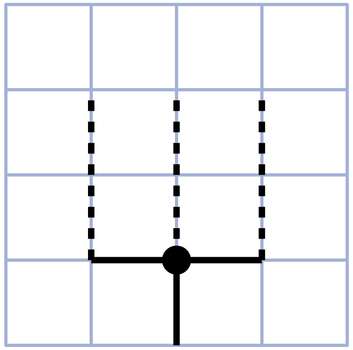




first vertex



indegree = 1



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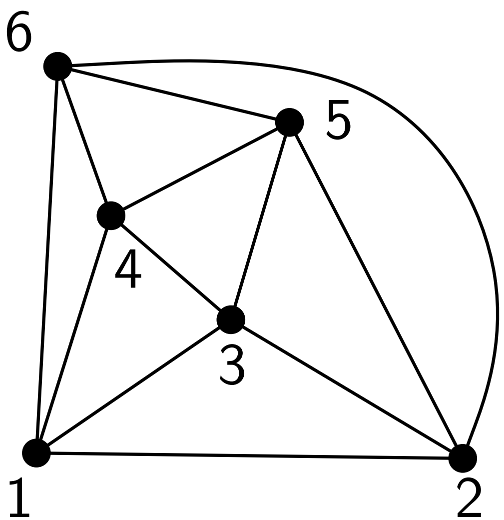
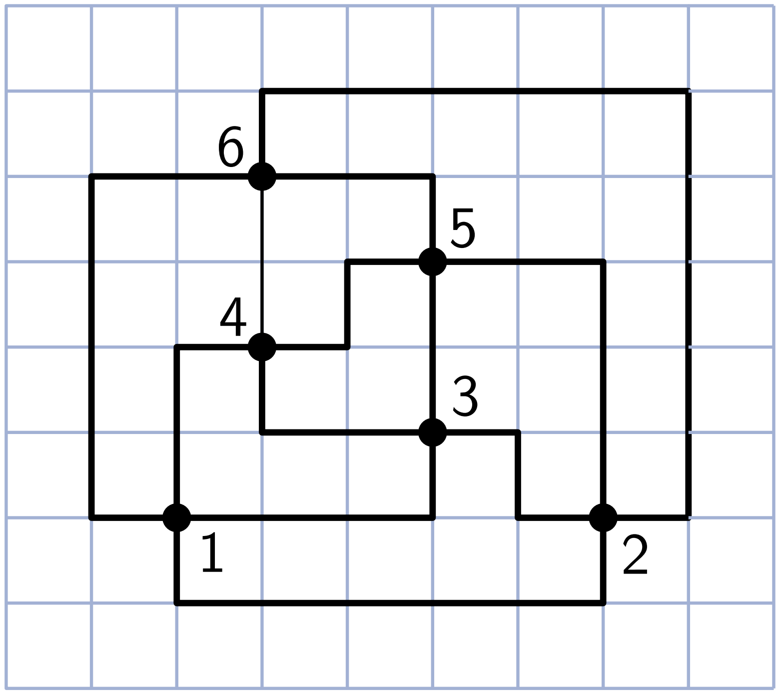
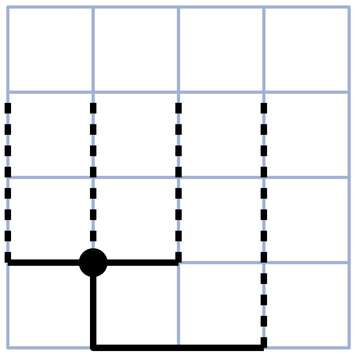
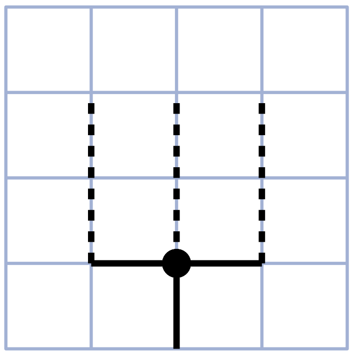




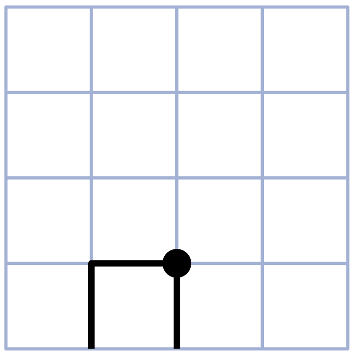
first vertex



indegree = 1

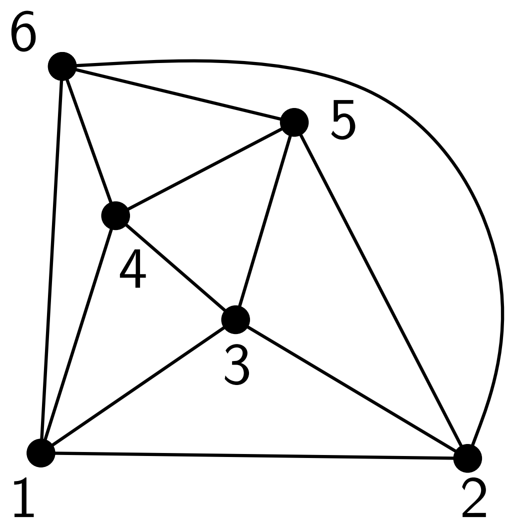


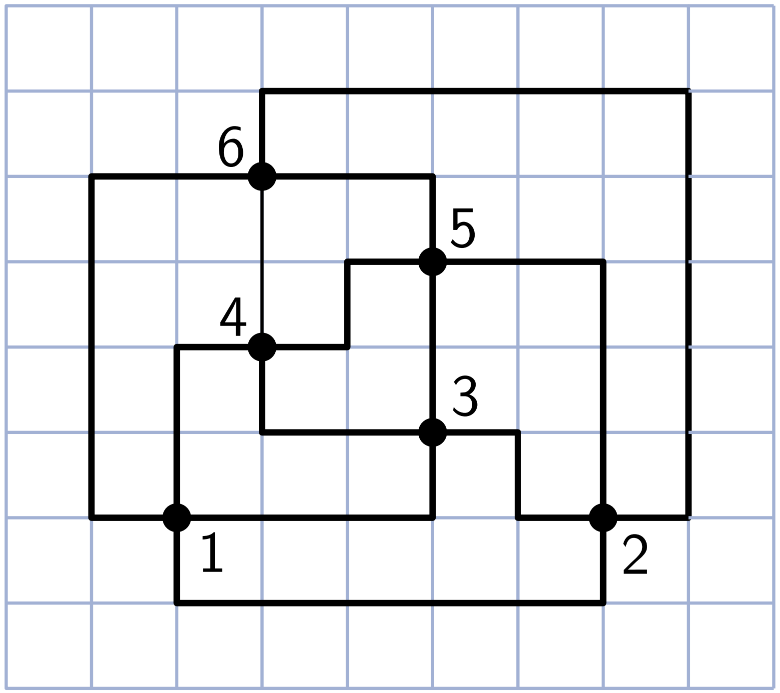
indegree = 2



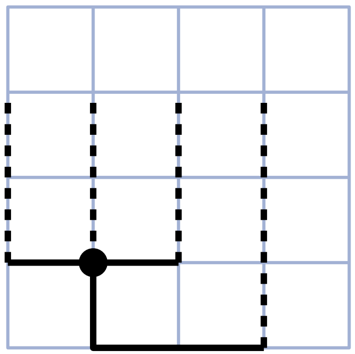
Institut für Theoretische Informatik
Lehrstuhl Algorithmik I

Biedl & Kant Orthogonal Drawing Algorithm

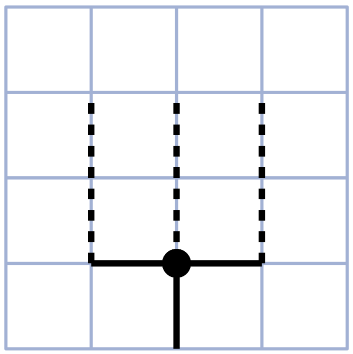




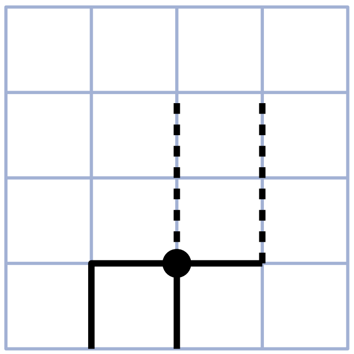
first vertex



indegree = 1



indegree = 2



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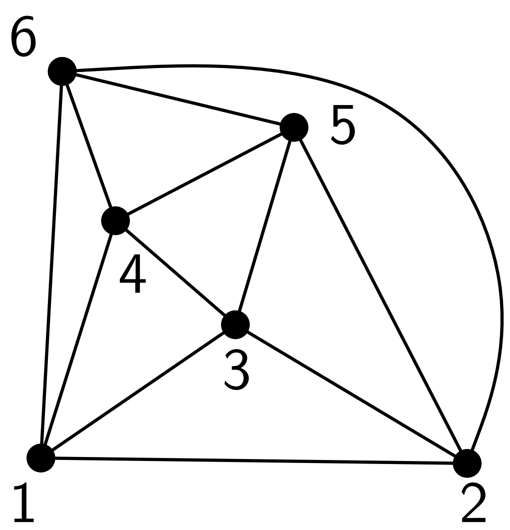

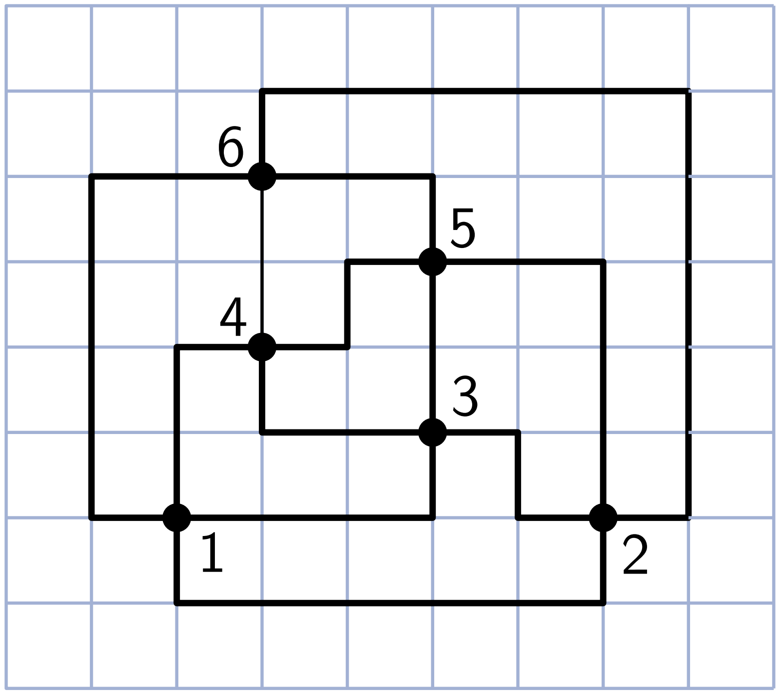

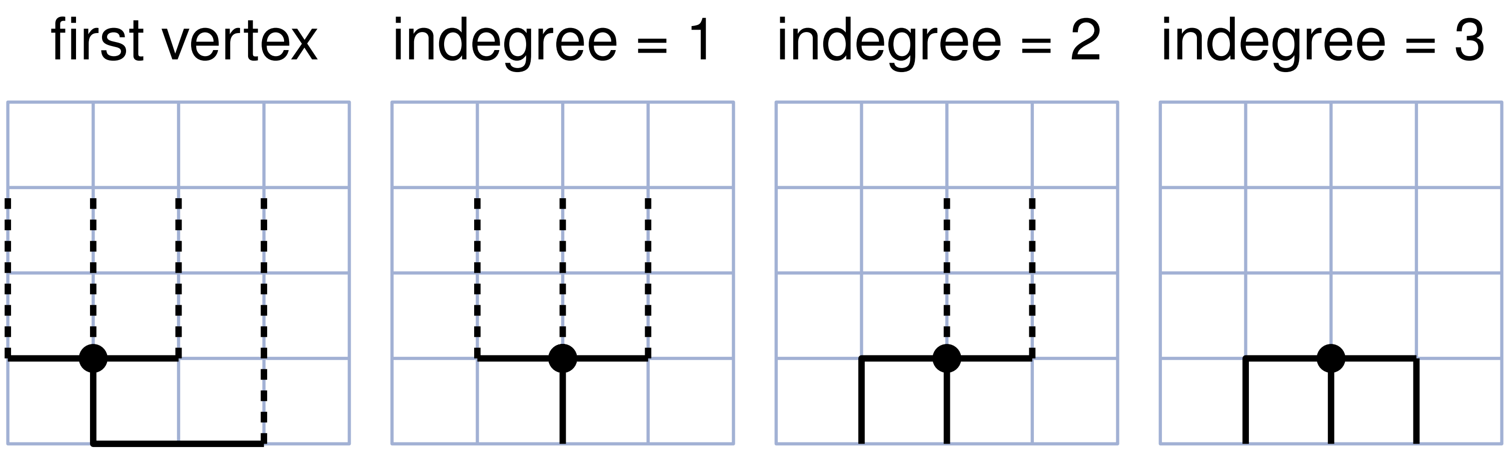

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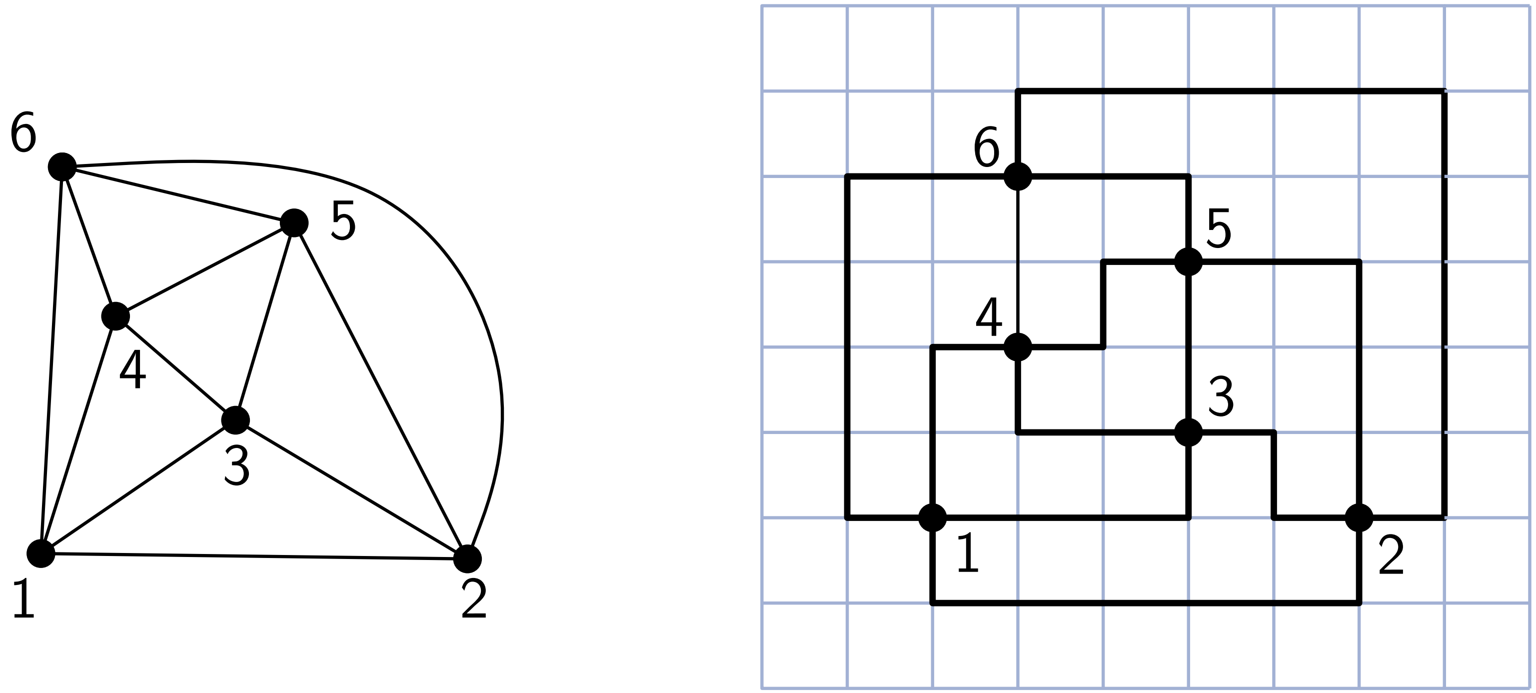
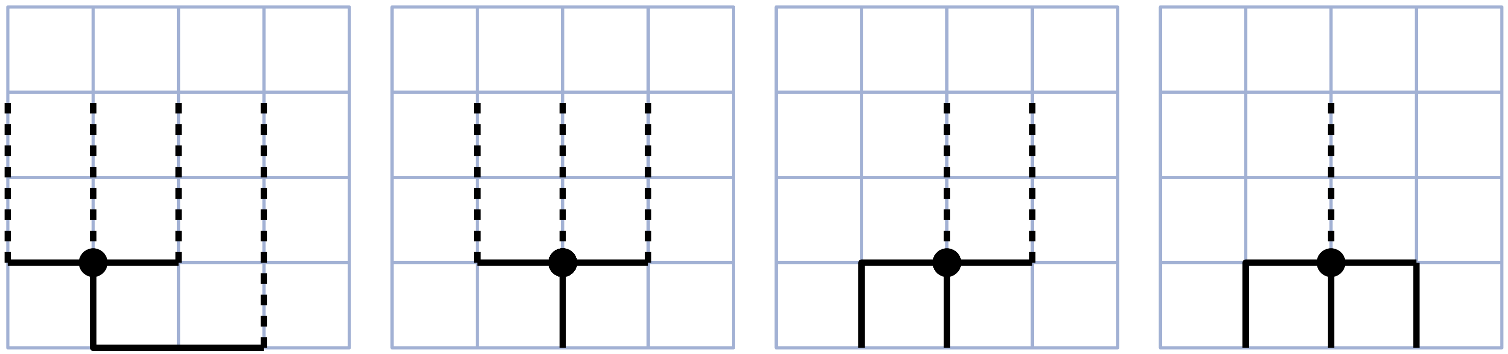


first vertex | indegree = 1 | indegree = 2 | indegree = 3



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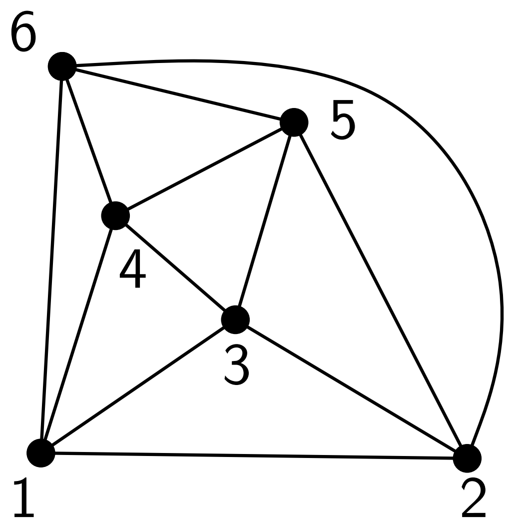

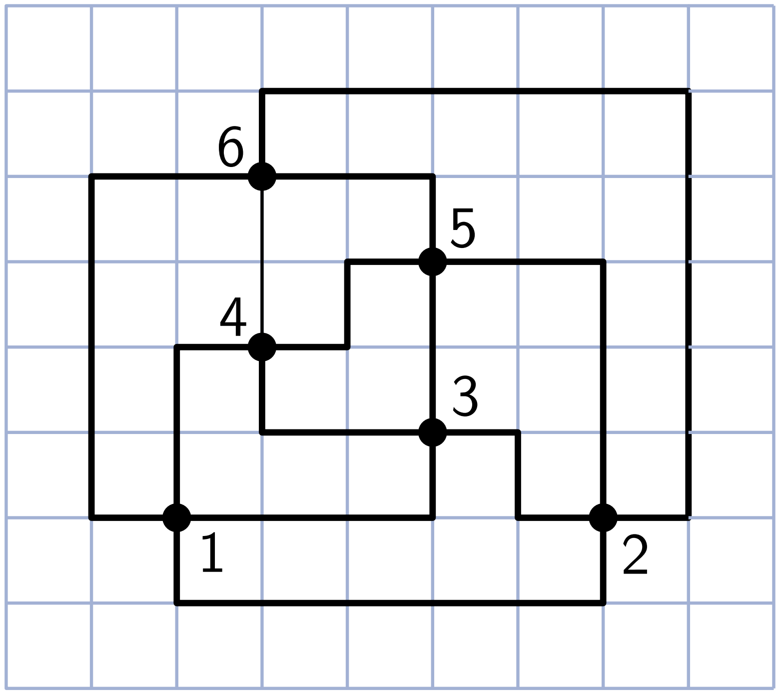

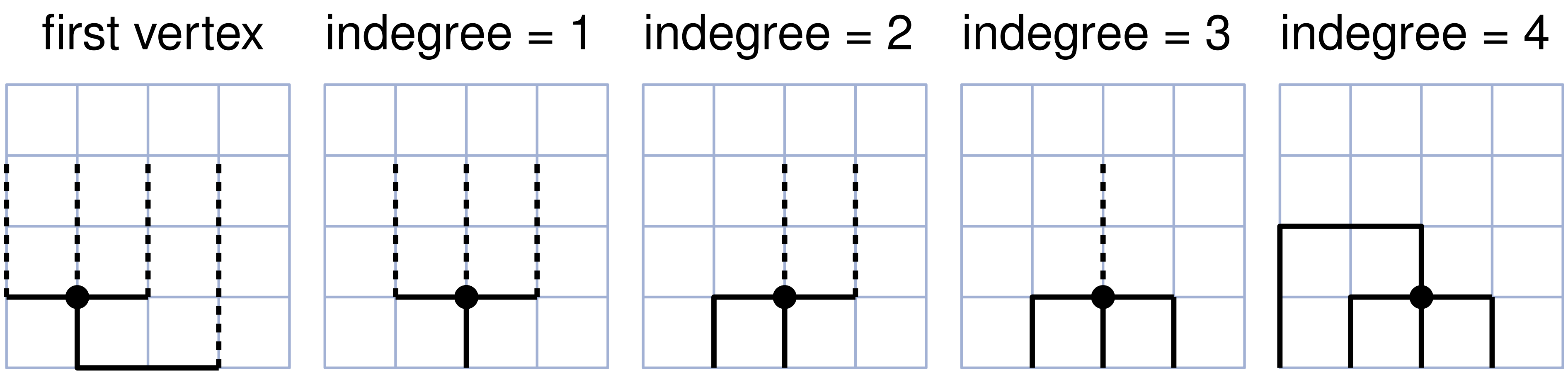

Biedl & Kant Orthogonal Drawing Algorithm

Lemma (Area of Biedl & Kant drawing)

The width is $m - n + 1$ and the height at most $n + 1$.

Biedl & Kant Orthogonal Drawing Algorithm

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The width is $m - n + 1$ and the height at most $n + 1$.

Proof

- Width: At each step we increase the number of columns by $outdeg(v_i) - 1$, if $i > 1$ and $outdeg(v_1)$ for v_1.

Biedl & Kant Orthogonal Drawing Algorithm

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- Height: Every vertex except for v_2 is placed at a new row. Vertex v_n uses one more row if $indeg(v_n) = 4$.

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Lemma (Number of bends in Biedl & Kant drawing)

There are at most $2m - 2n + 4$ bends.

Biedl & Kant Orthogonal Drawing Algorithm

Lemma (Area of Biedl & Kant drawing)

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Lemma (Number of bends in Biedl & Kant drawing)

There are at most $2m - 2n + 4$ bends.

Proof

- Each vertex v_i, $i \neq 1, n$, introduces $indeg(v_i) - 1$ and $outdeg(v_i) - 1$ new bends.

Biedl & Kant Orthogonal Drawing Algorithm

Lemma (Number of bends per edge in Biedl & Kant drawing)

All edges but one bent at most twice. The exceptional edge bent at most three times.

Biedl & Kant Orthogonal Drawing Algorithm

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Proof

- Let (v_i, v_j), $i < j$, $i, j \neq 1, n$. Then $outdeg(v_i)$, $indeg(v_j) \leq 3$. I.e (v_i, v_j) gets at most one bend after placement of v_i and at most one before placement of v_j. Edges outgoing from v_1 can me made 2-bend by using the column below v_1 for the edge (v_1, v_2).

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For planar graphs the algorithm produces a planar drawing.

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Proof

- Let $v_1, \ldots, v_n$ be an st-ordering of G. Let G_i be the graph induced by the vertices $v_1, \ldots, v_i$. We will prove later that if G is planar, vertex v_{i+1} lies on the outer face of G_i.

Biedl & Kant Orthogonal Drawing Algorithm

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Proof (Continuation)

- Let E_i be the edges outgoing from the vertices of G_i in the order they appear in the embedded G.

Biedl & Kant Orthogonal Drawing Algorithm

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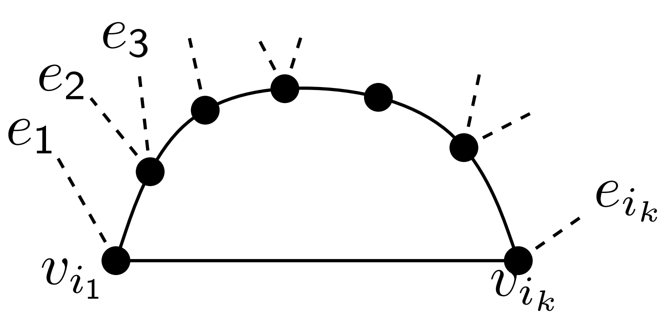
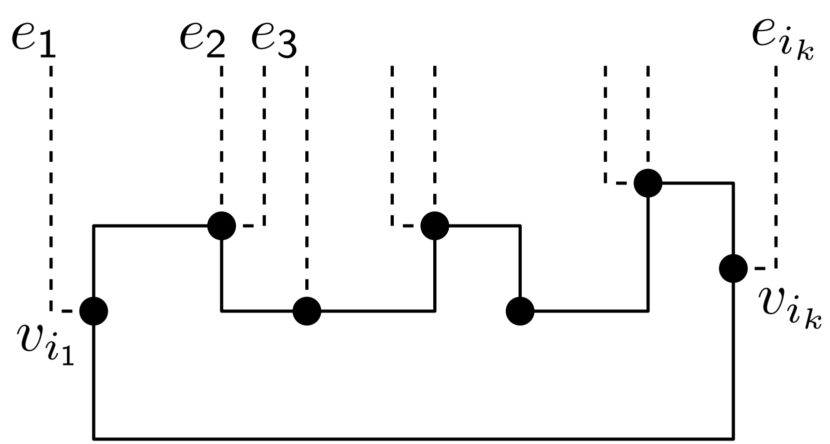
For planar graphs the algorithm produces a planar drawing.

Proof (Continuation)

- Let E_i be the edges outgoing from the vertices of G_i in the order they appear in the embedded G.
- By induction we can show that E_i appear in the same order in the orthogonal drawing of G_i.





Biedl & Kant Orthogonal Drawing Algorithm

Theorem (Biedl & Kant 98)

A biconnected graph G with vertex-degree at most 4 admits an orthogonal drawing on a $(m - n + 1) \times n + 1$ grid, such that each edge, except maybe for one, have at most 2 bends per edge, while the exceptional edge has at most 3 bends. The total number if bends is $2m - 2n + 4$. If G is planar, the the orthogonal drawing is also planar.

Biedl & Kant Orthogonal Drawing Algorithm

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What have we used for the consruction?

- st-ordering $v_1, \ldots, v_n$ of G.
- The following fact: if G is planar, vertex v_{i+1} belongs to the outer face of G_i, where G_i is graph induced by $v_1, \ldots, v_i$.

st-ordering

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How to construct an st-ordering?

- Recall topological ordering for st-graphs.

st-ordering

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- A topological ordering of G' is an st-ordering of G.

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- A topological ordering of G' is an st-ordering of G.
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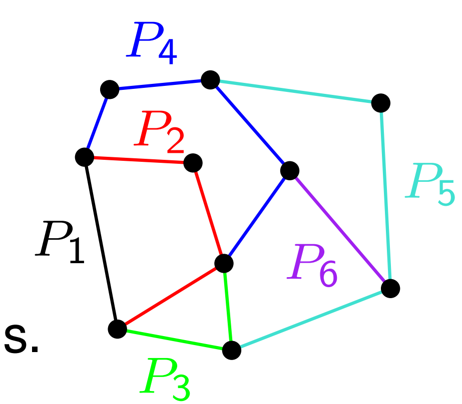
Definition: Ear decomposition

An ear decomposition $D = (P_0, \ldots, P_r)$ of an undirected graph $G = (V, E)$ is a partition of E into an ordered collection of edge disjoint paths $P_0, \ldots, P_r$, such that:

- P_0 is an edge
- $P_0 \cup P_1$ is a simple cycle
- both end-vertices of P_i belong to $P_0 \cup \cdots \cup P_{i-1}$
- no internal vertex of P_i belong to $P_0 \cup \cdots \cup P_{i-1}$

An ear decomposition of open if $P_0, \ldots, P_r$ are simple paths.



st-ordering

Lemma (Ear decomposition)

Let $G = (V, E)$ be a biconnected graph G and let $(s, t) \in E$. G has an open ear decomposition $(P_0, \dots, P_r)$, where $P_0 = (s, t)$.

st-ordering

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Proof

- Let $P_0 = (s, t)$ and P_1 be path between s and t, it exists since G is biconnected.

st-ordering

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Proof

- Let $P_0 = (s, t)$ and P_1 be path between s and t, it exists since G is biconnected.
- Induction hypothesis: $P_0, \ldots, P_i$ are ears.

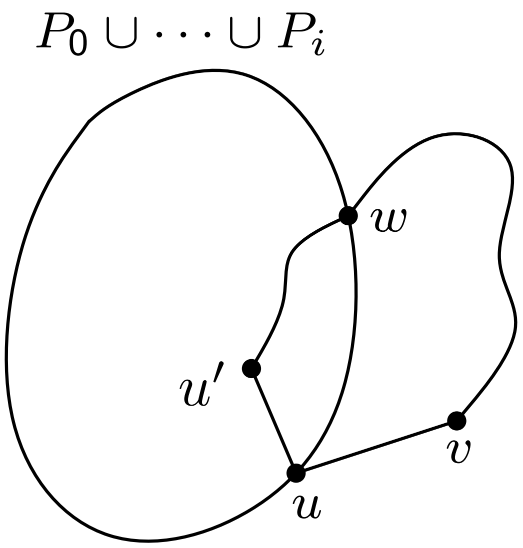
st-ordering

Lemma (Ear decomposition)

Let $G = (V, E)$ be a biconnected graph G and let $(s, t) \in E$. G has an open ear decomposition $(P_0, \dots, P_r)$, where $P_0 = (s, t)$.

Proof

- Let $P_0 = (s, t)$ and P_1 be path between s and t, it exists since G is biconnected.
- Induction hypothesis: $P_0, \dots, P_i$ are ears.
- Let (u, v) be an edge in G such that $u \in P_0 \cup \dots \cup P_i$ and $v \notin P_0 \cup \dots \cup P_i$. Let $(u, u') \in P_0 \cup \dots \cup P_i$. Let P be a path between v and u', disjoint from path $u' - u - v$. P exists since G is biconnected.



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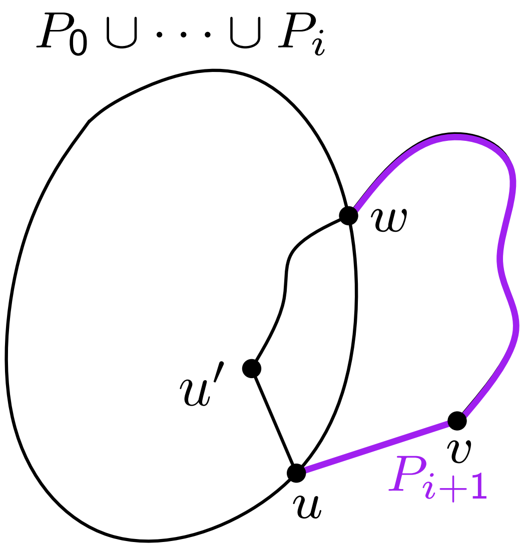

st-ordering

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Let $G = (V, E)$ be a biconnected graph G and let $(s, t) \in E$. There is an orientation G' of G which represents an st-graph. G' is called st-orientation of G.

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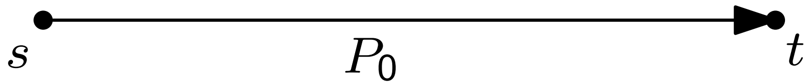

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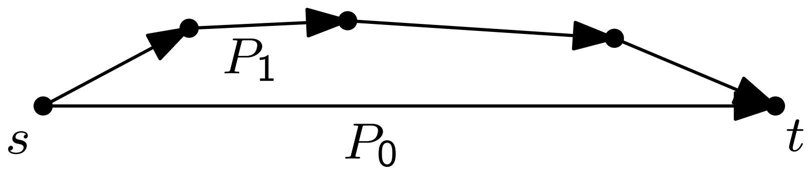

st-ordering

Karlsruhe Institute of Technology

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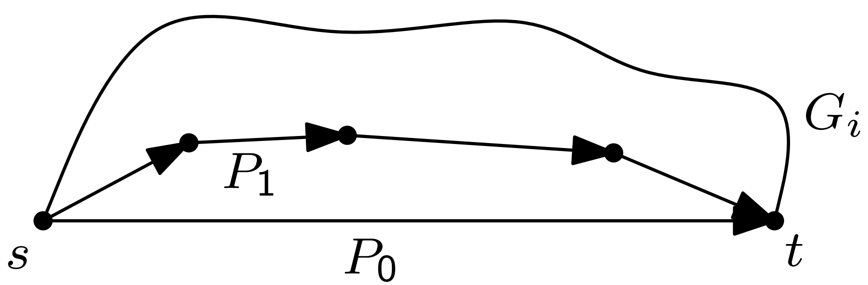

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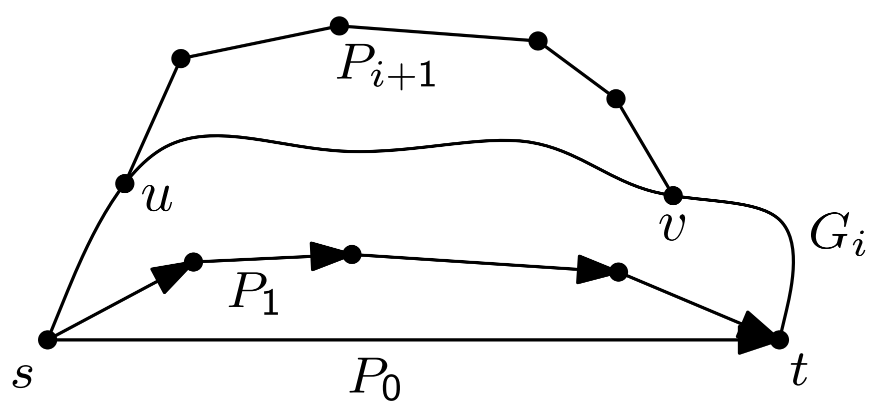

st-ordering

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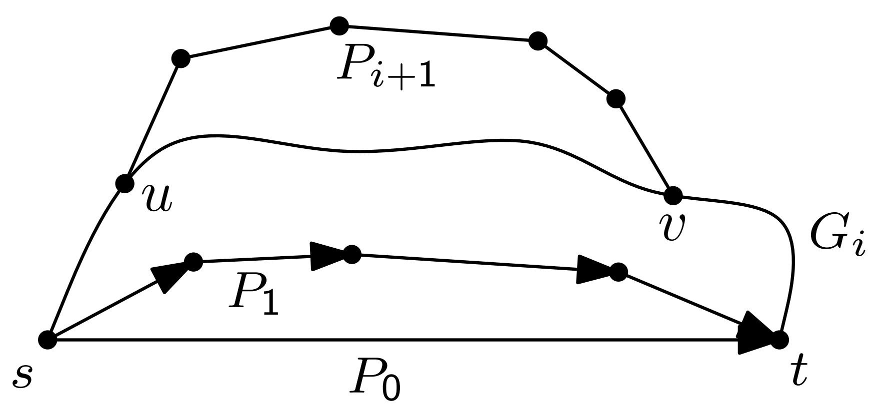

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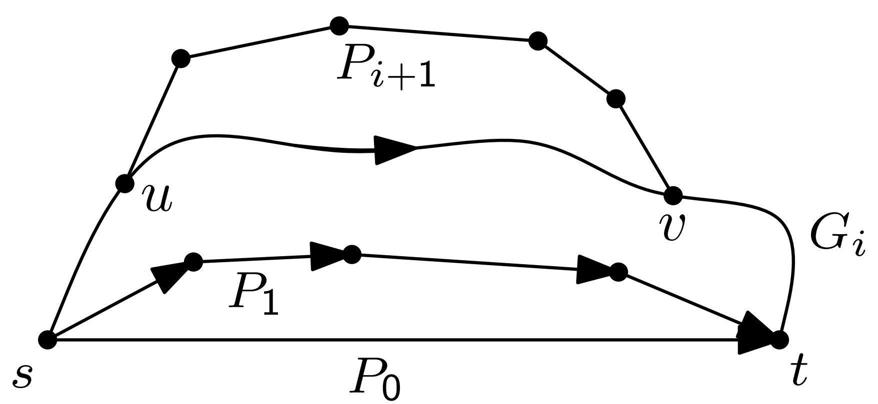

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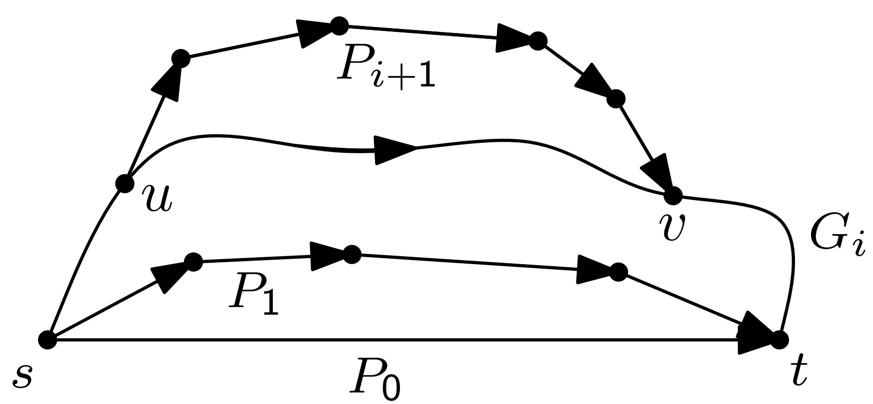

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st-ordering

- Recall that if G is biconnected graph and G' is an st-orientation of G, then a topological ordering of G' is an st-ordering of G.

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- For G_1, let $P_1 = \{u_1^1, \ldots, u_{r_1}^1\}$, here $u_1^1 = s$ and $u_{i_1}^1 = t$. The sequence $L = \{u_1^1, \ldots, u_{r_1}^1\}$ is an st-ordering of G_1.

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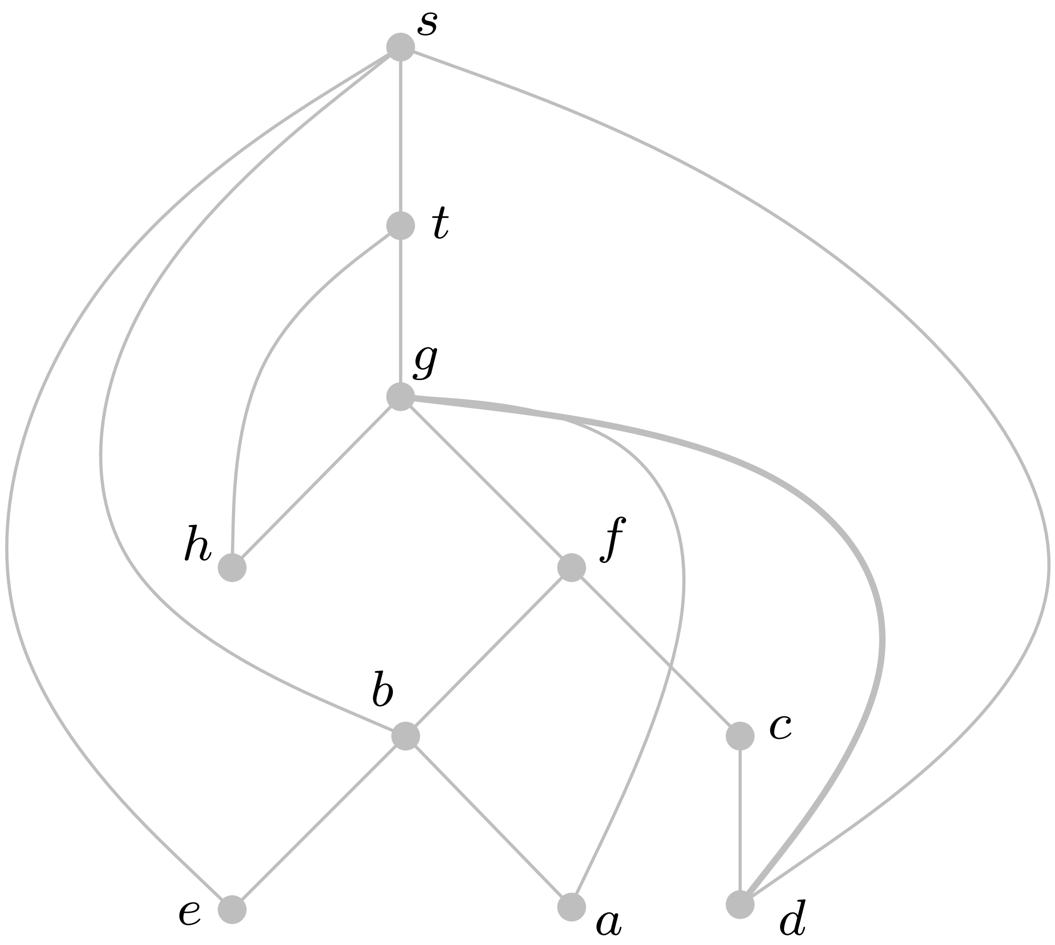
- Assume that L contains an st-ordering of G_i and let ear $P_{i+1} = \{u_1^{i+1}, \ldots, u_{r_{i+1}}^{i+1}\}$. We insert vertices $u_1^{i+1}, \ldots, u_{r_{i+1}}^{i+1}$ to L after vertex u_1^{i+1}. Let G'_{i+1} be an st-orientation of G_i as constructed in the previous proof. L is a topological ordering of G'_{i+1} and therefore an st-ordering of G_i.

st-ordering

Algorithm: st-ordering (example)



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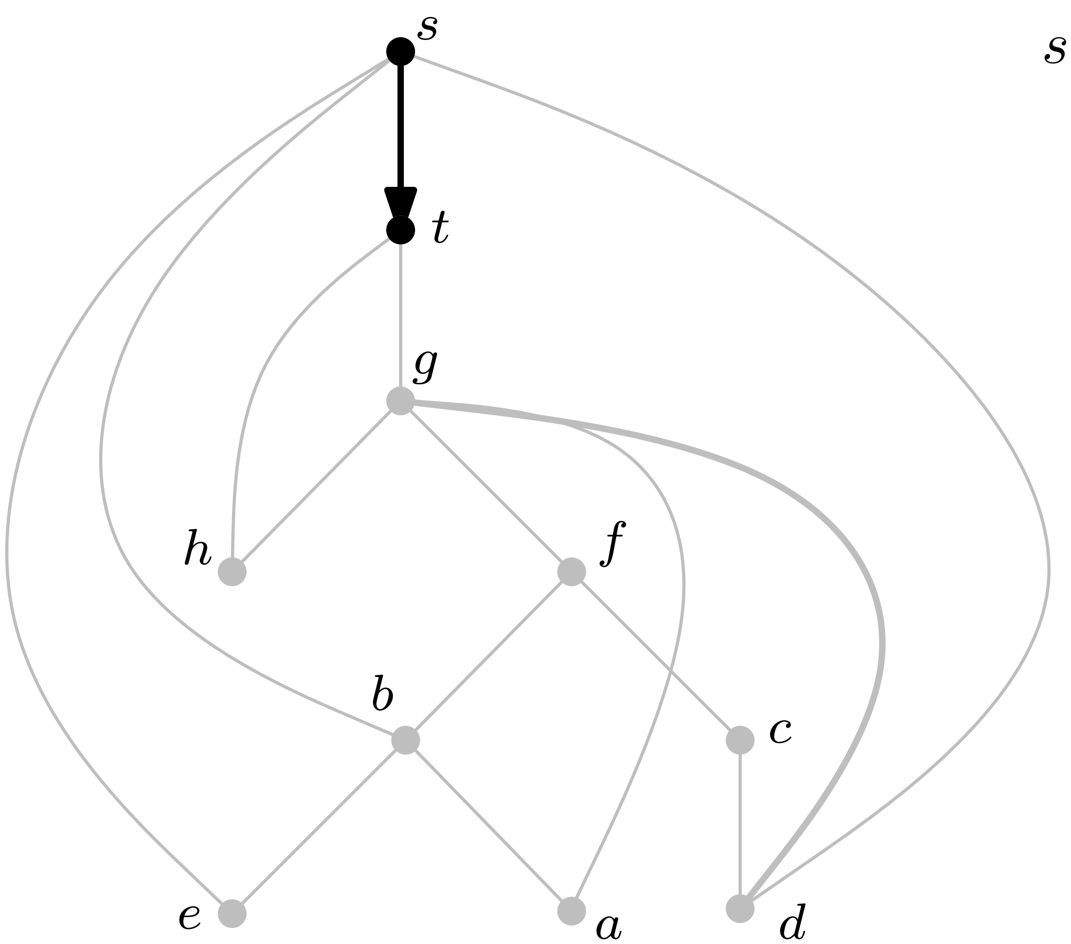

s, t

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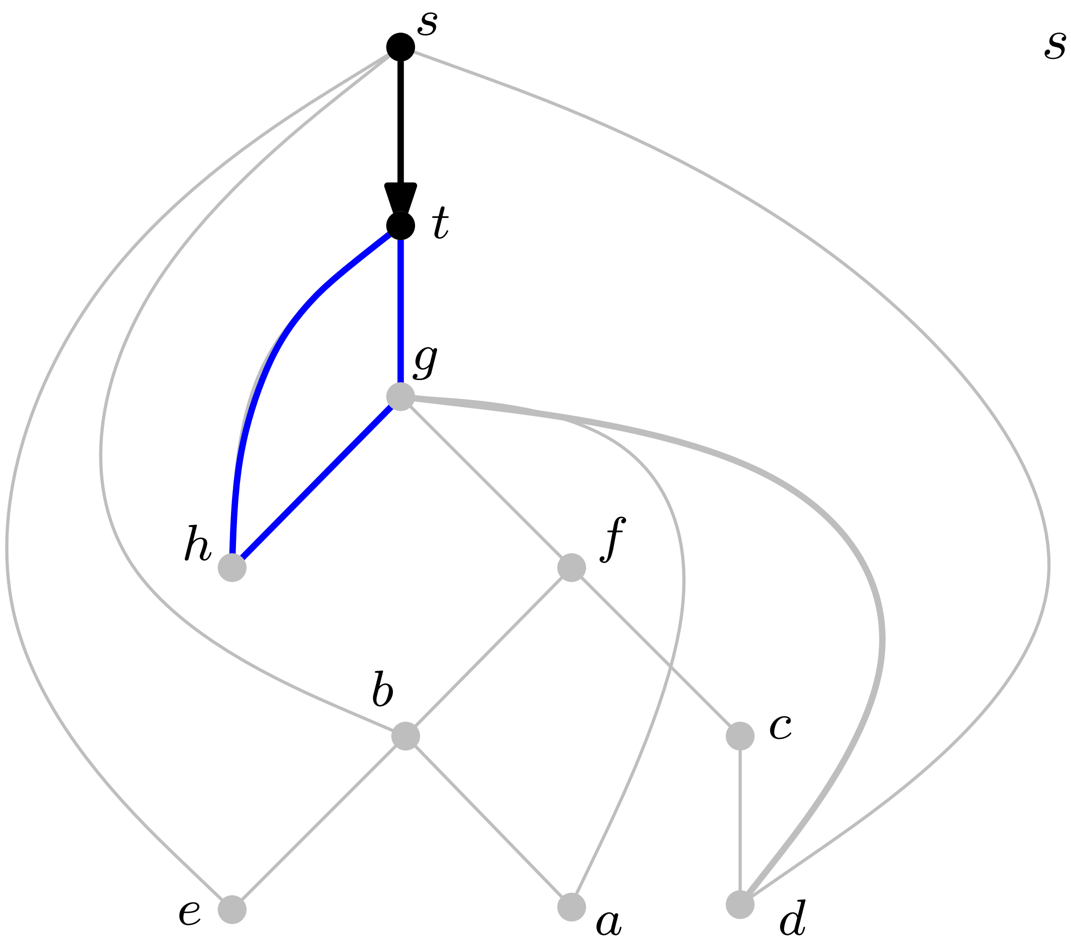

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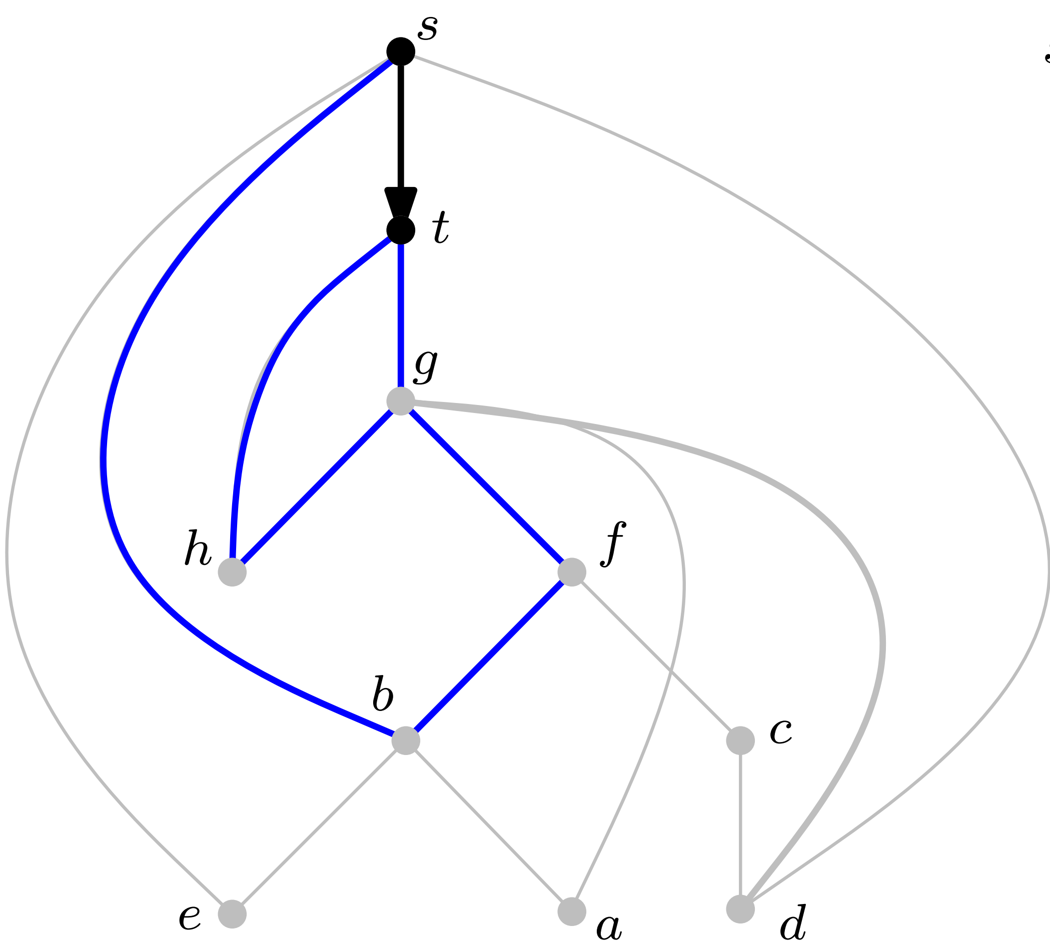

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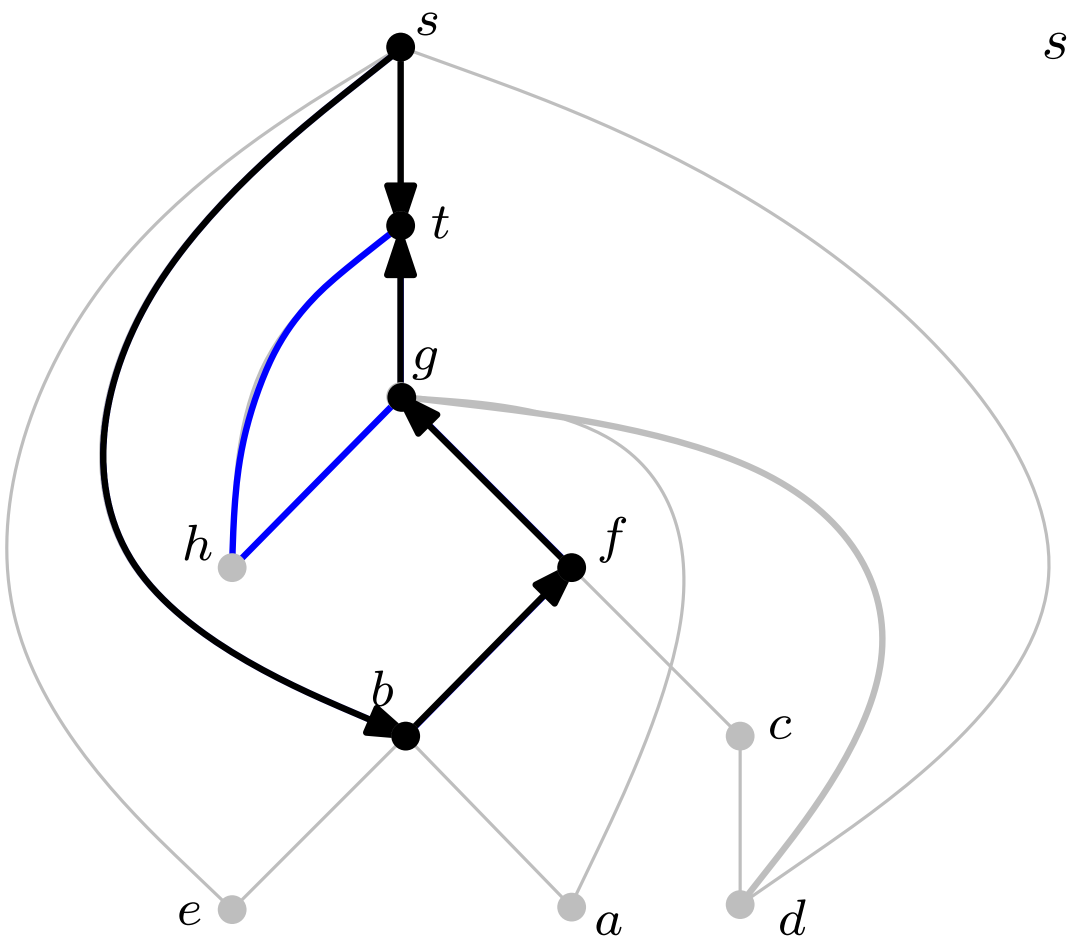

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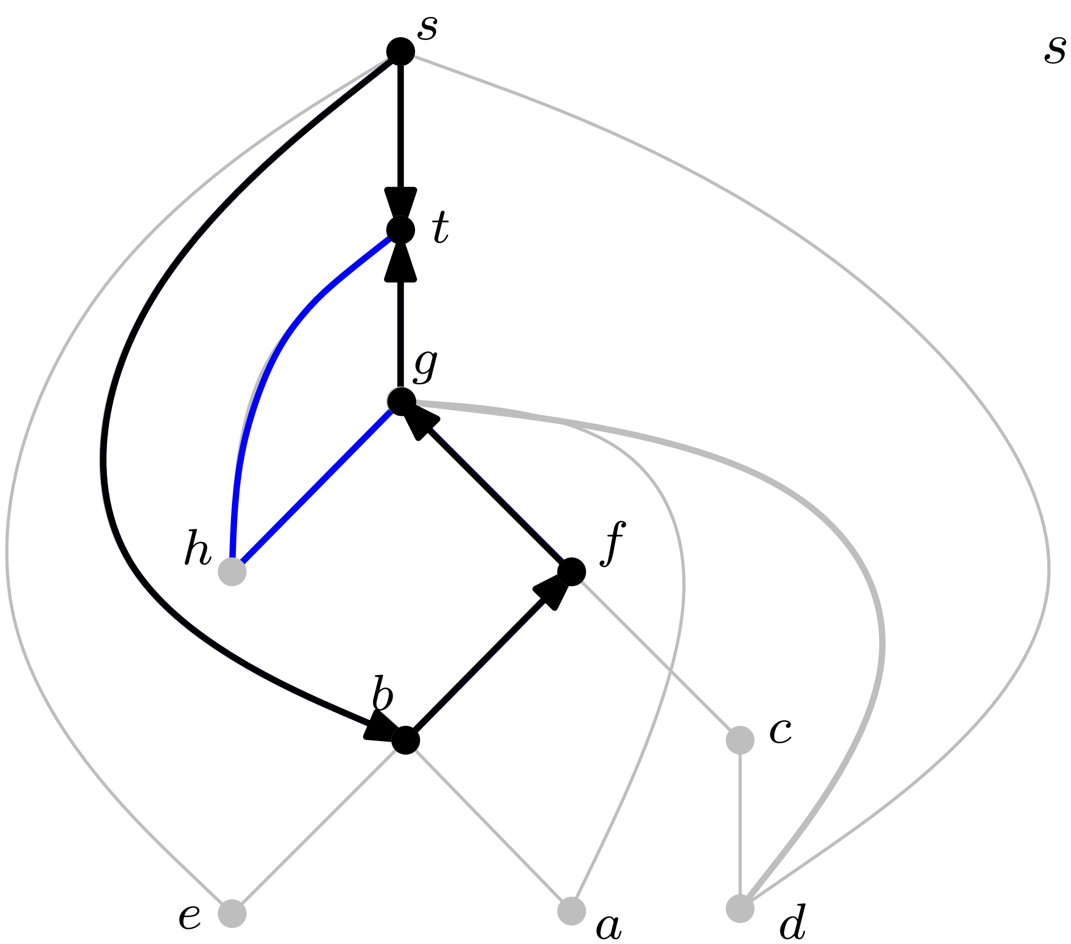

$s, \underline{b}, \underline{f}, \underline{g}, t$

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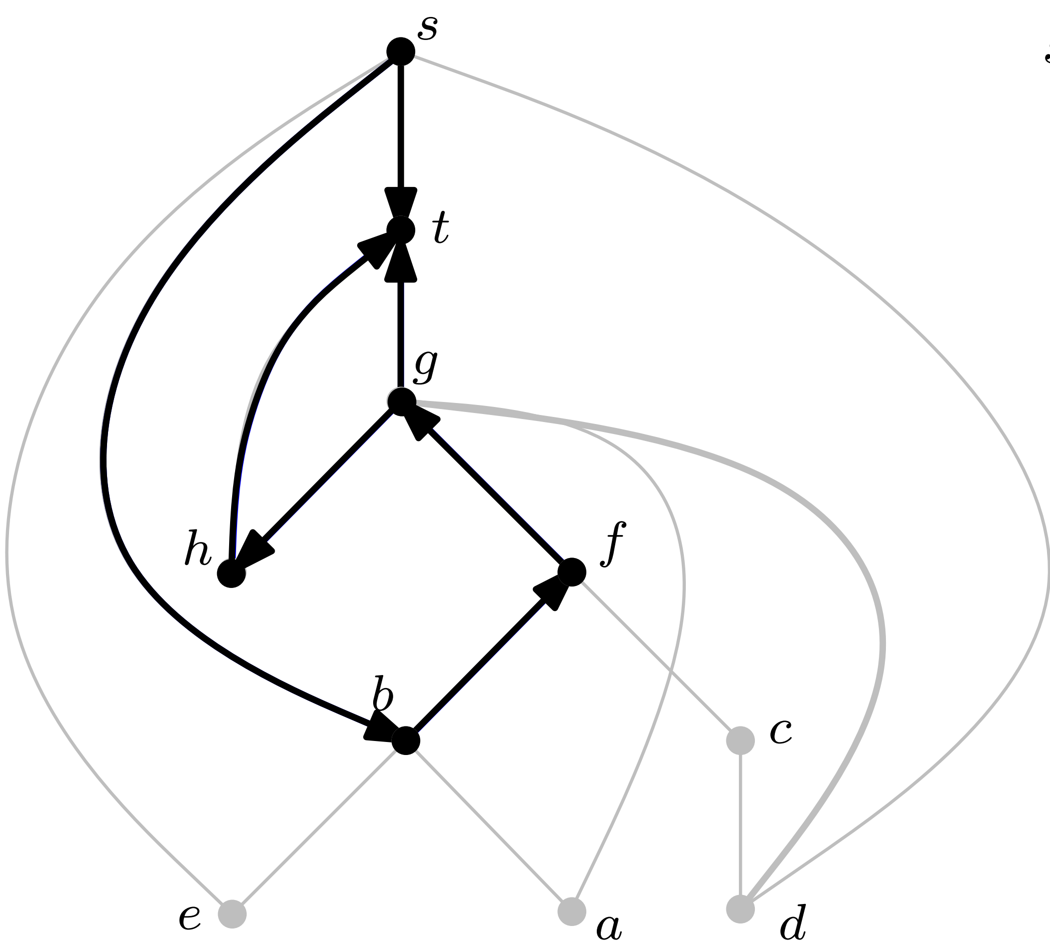

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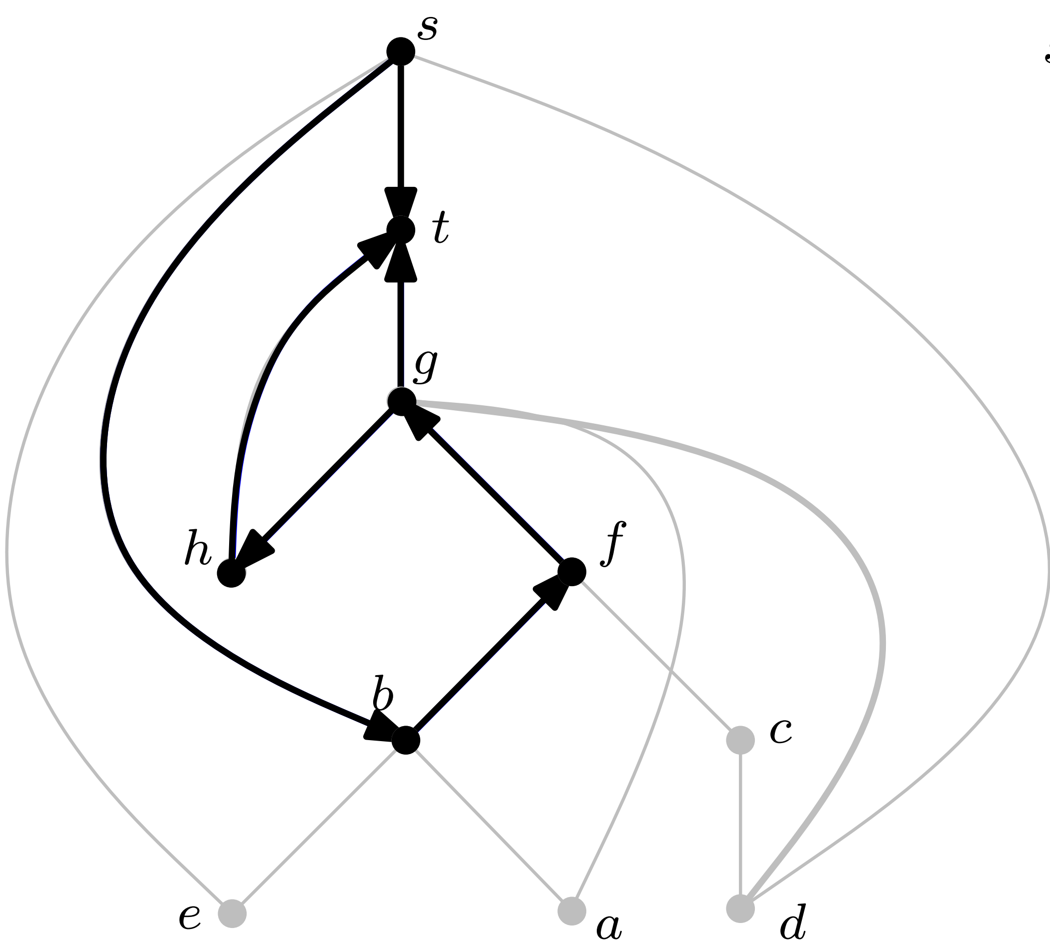

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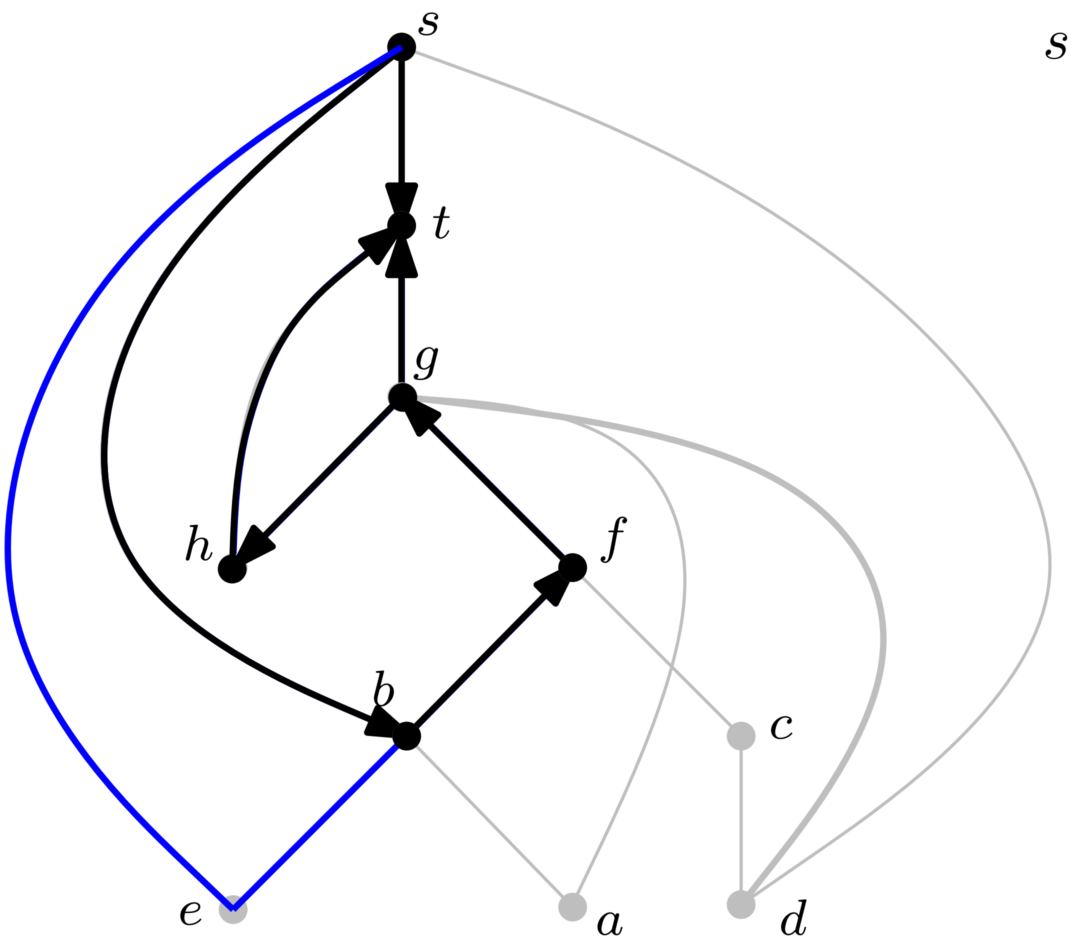

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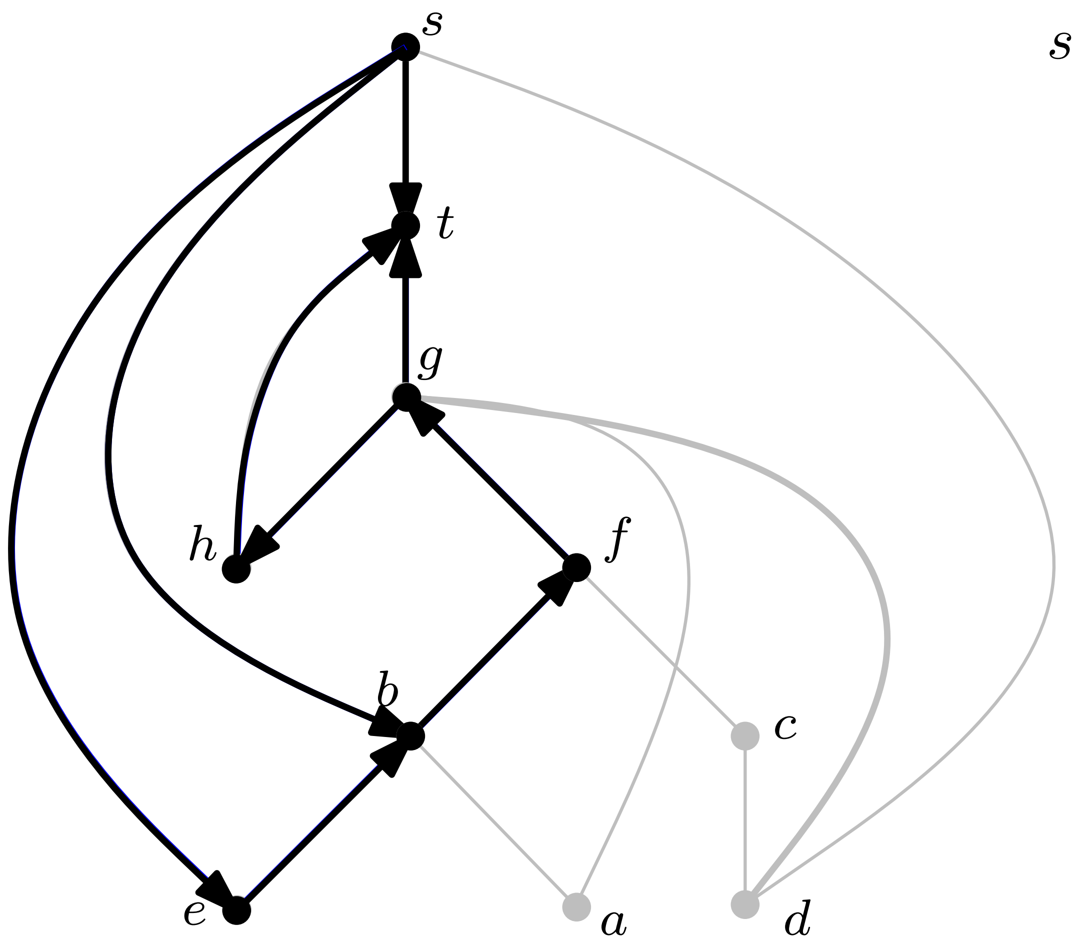

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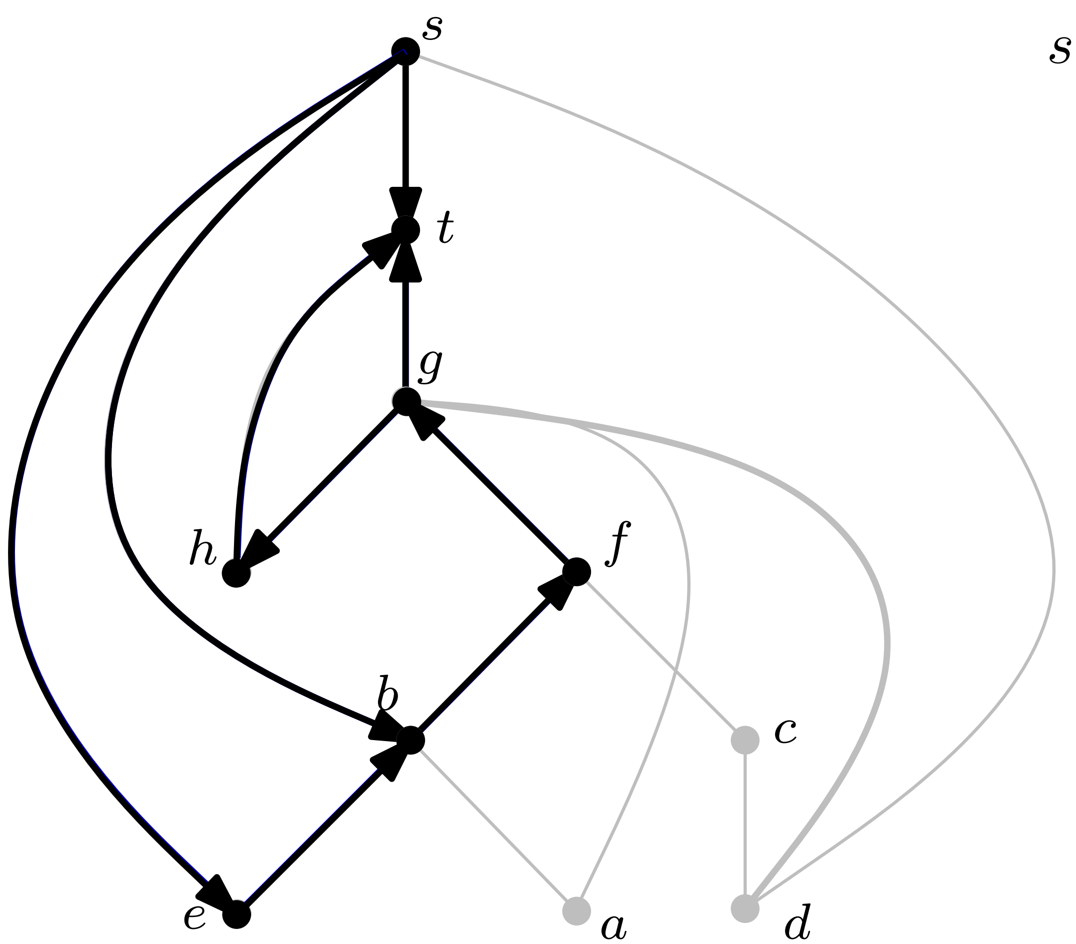

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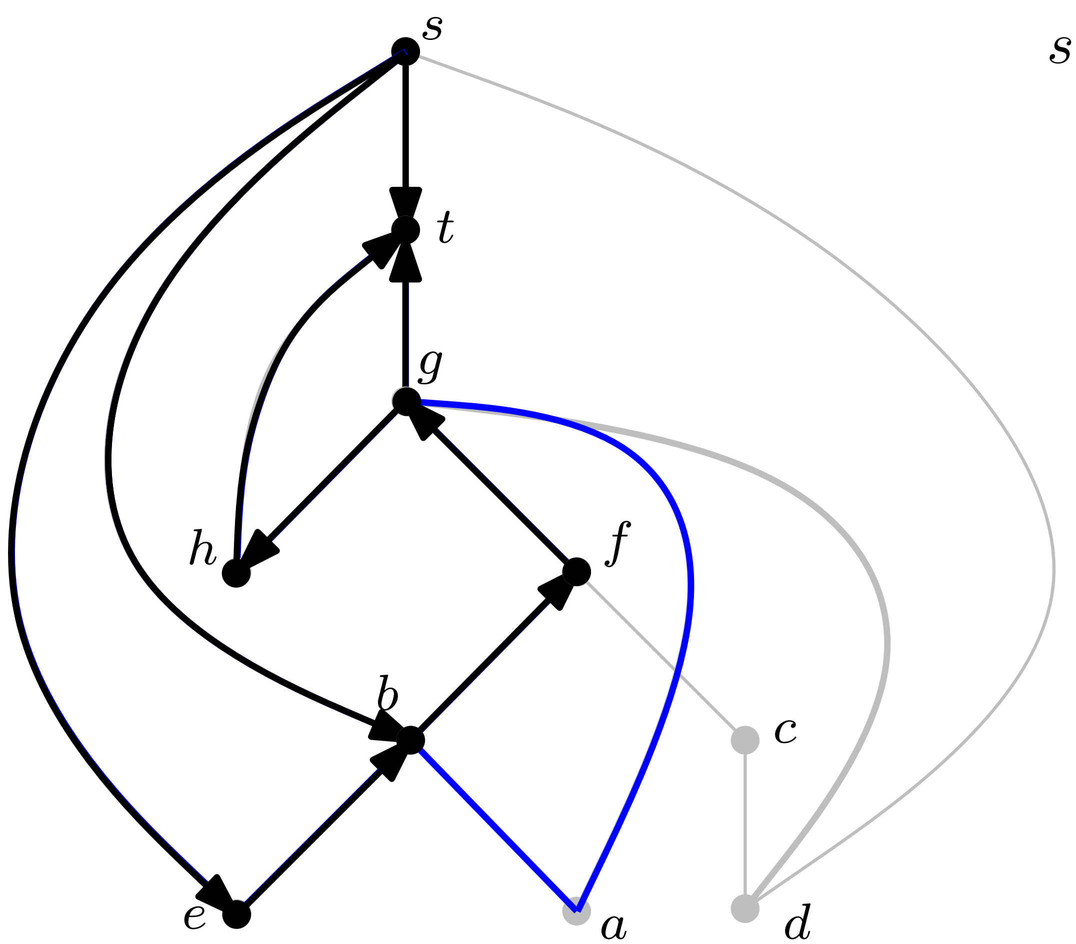

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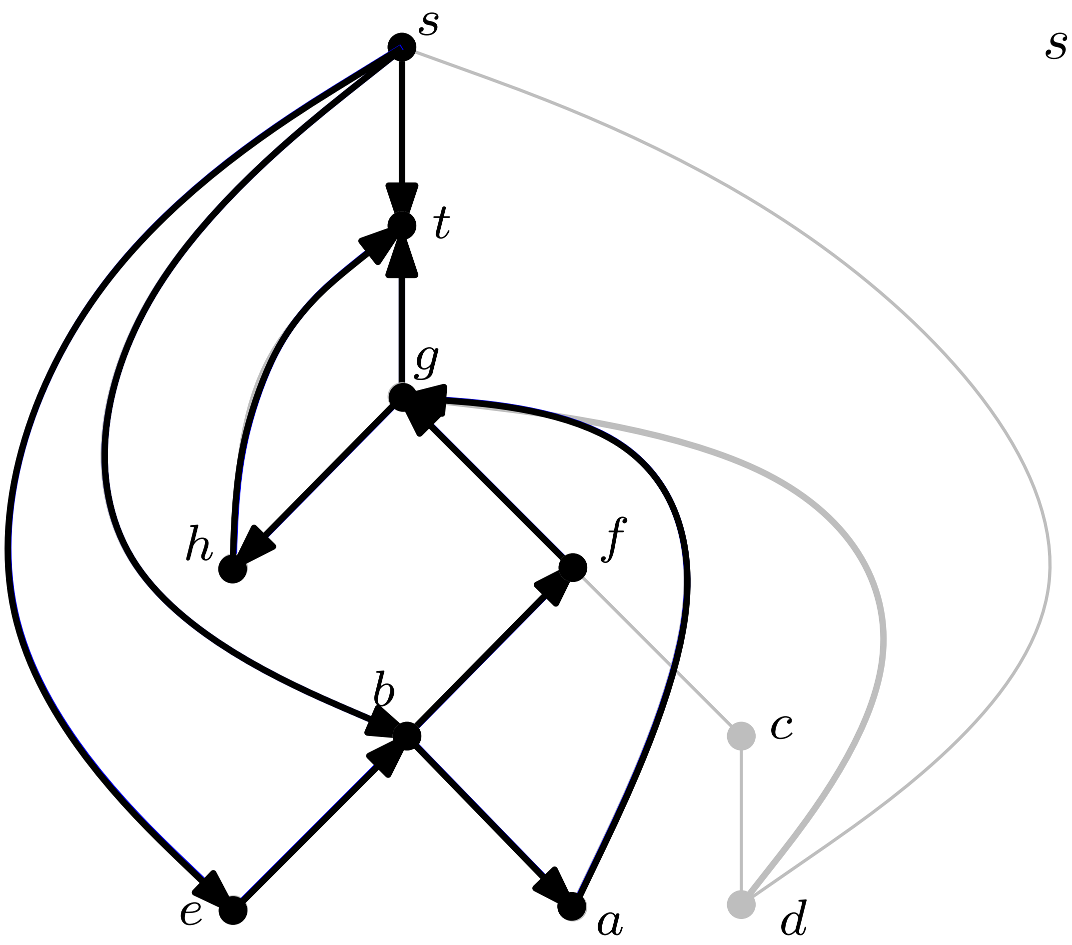

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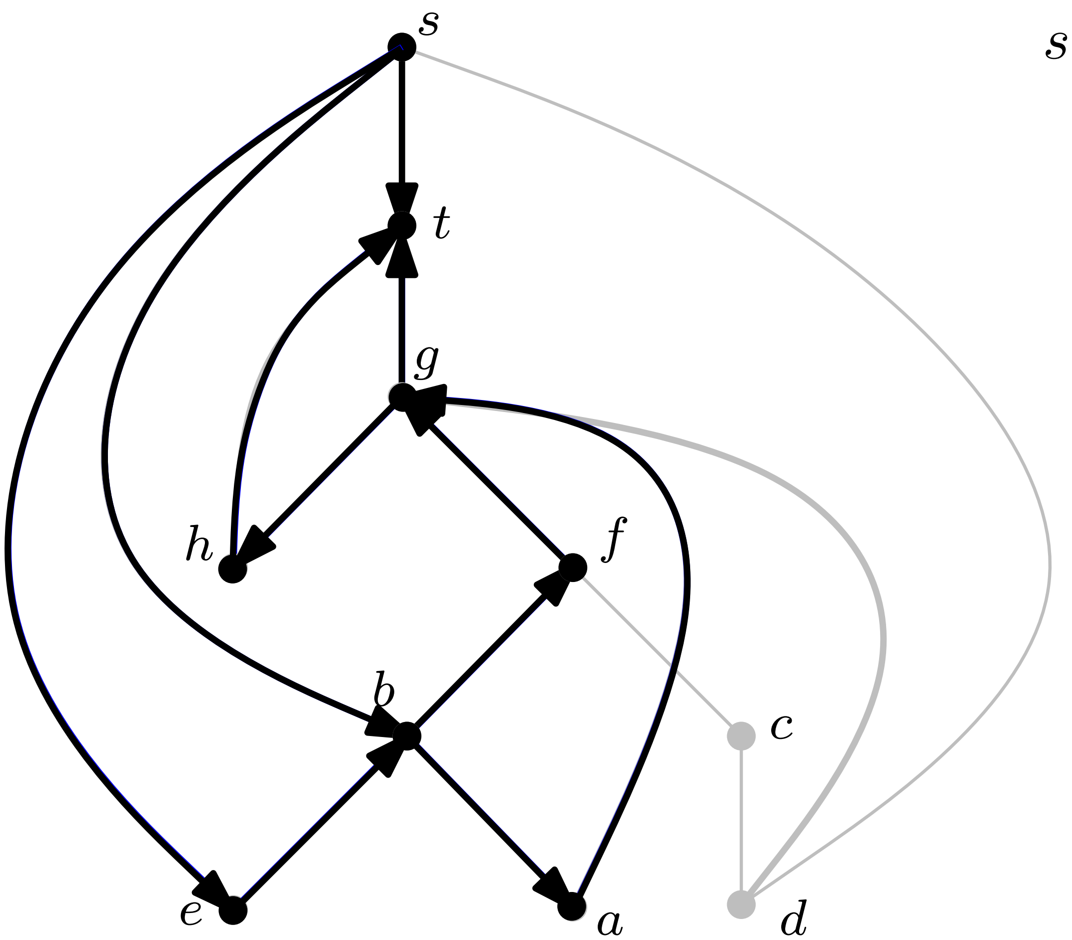

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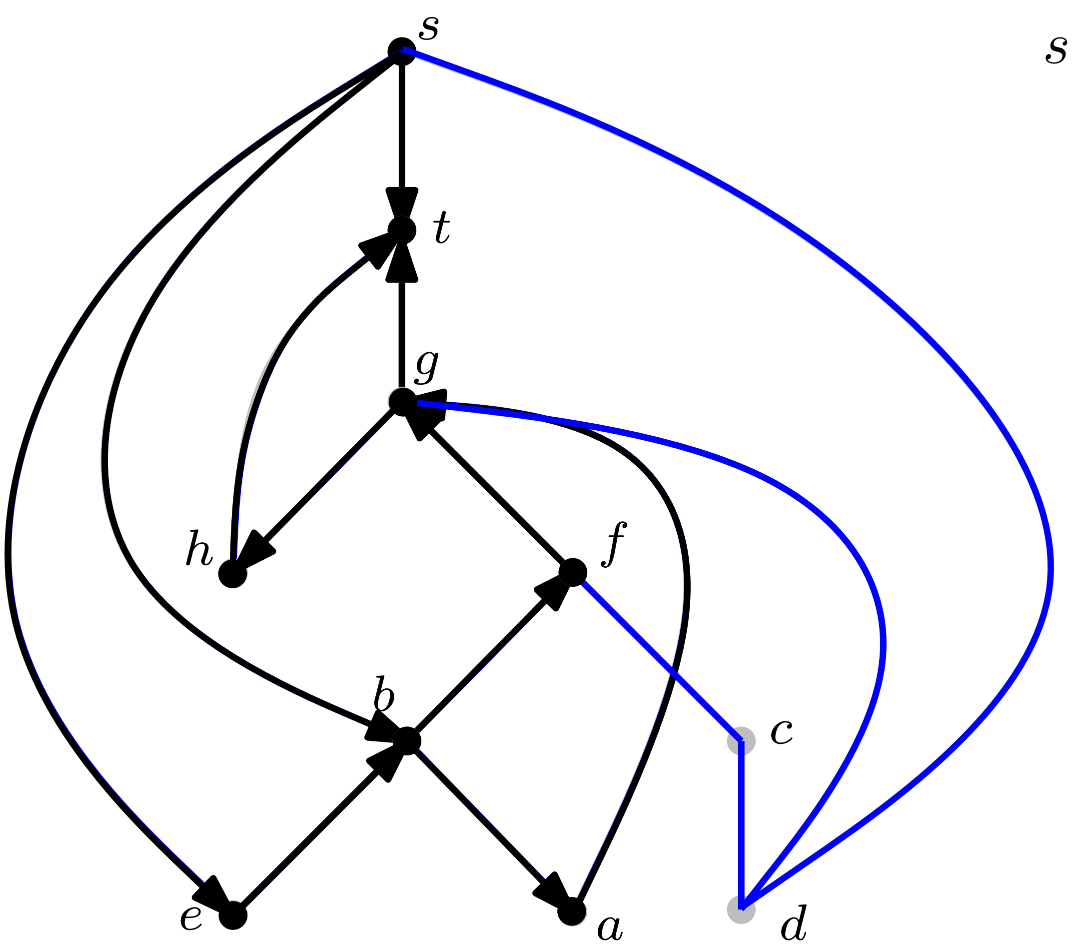

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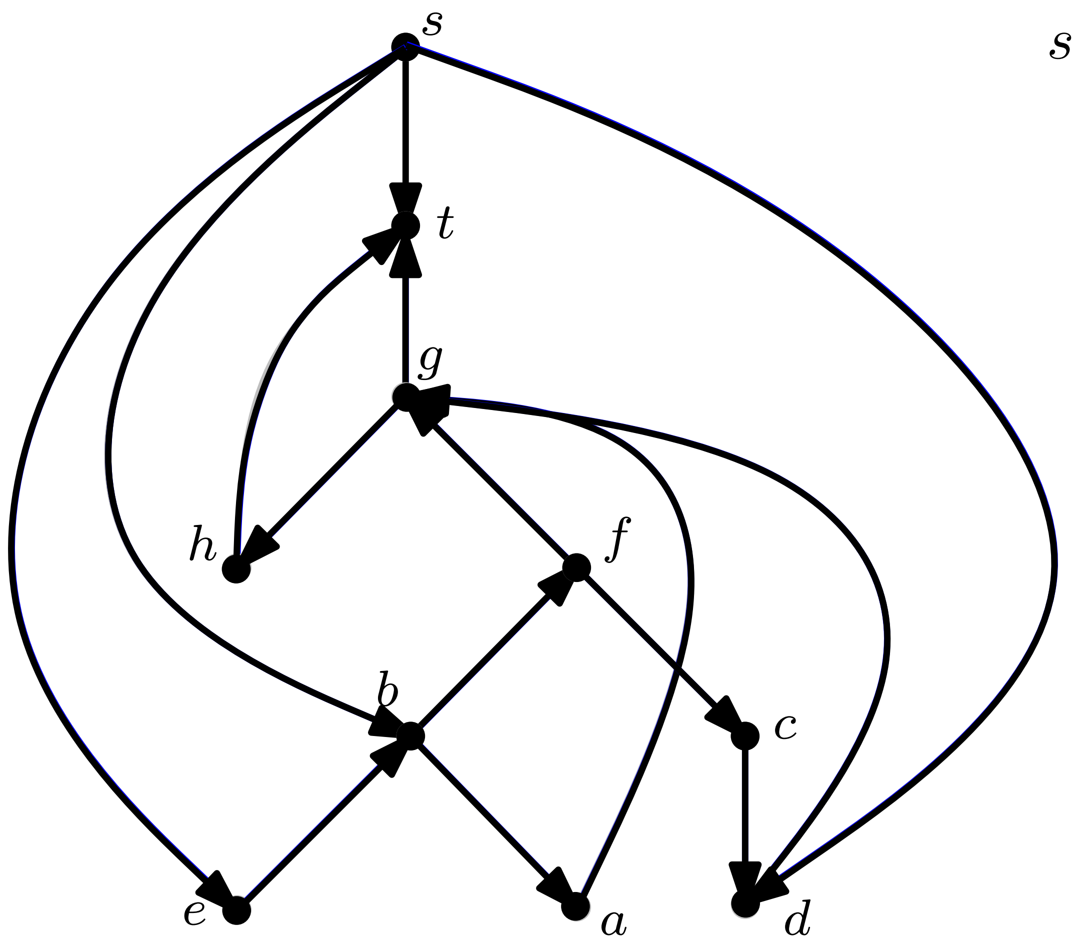

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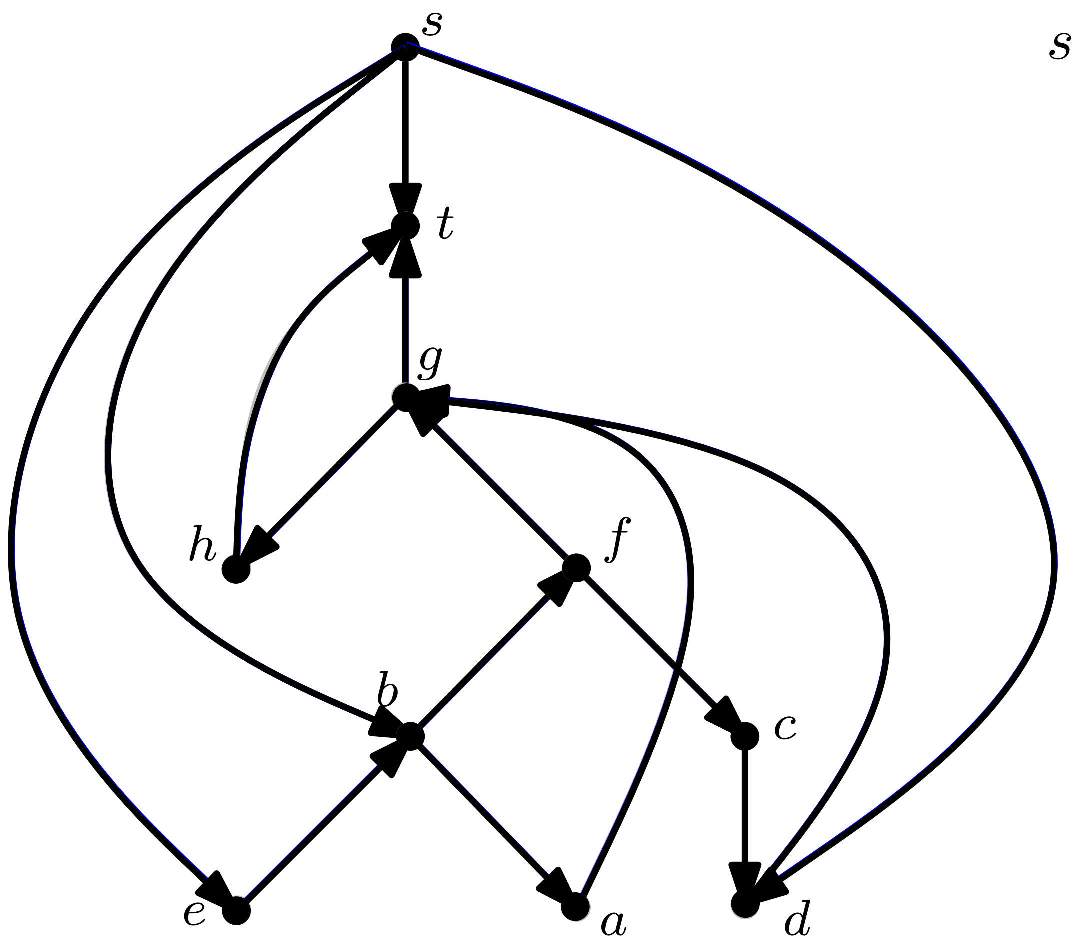

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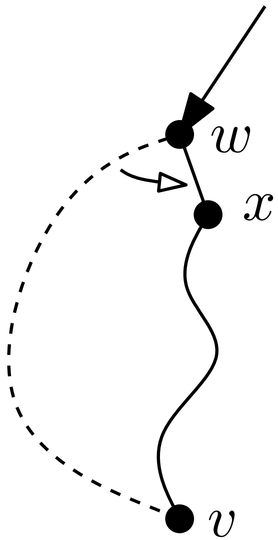
st-ordering

Algorithm st-ordering

Data: Undirected biconnected graph $G = (V, E)$, edge $\{s, t\} \in E$
Result: List L of nodes representing an st-ordering of G)

dfs(vertex v) begin

  $i \leftarrow i + 1$; $DFS[v] \leftarrow i$;
  while *there exists non-enumerated* $e = \{v, w\}$ **do**
    $DFS[e] \leftarrow DFS[v]$;
    if w *not enumerated* **then**
      $CHILDEDGE[v] \leftarrow e$; $PARENT[w] \leftarrow v$;
      $dfs(w)$;
    else
      $\{w, x\} \leftarrow CHILDEDGE[w]$; $D[\{w, x\}] \leftarrow D[\{w, x\}] \cup \{e\}$;
      if $x \in L$ **then** $process\_ears(w \rightarrow x)$;



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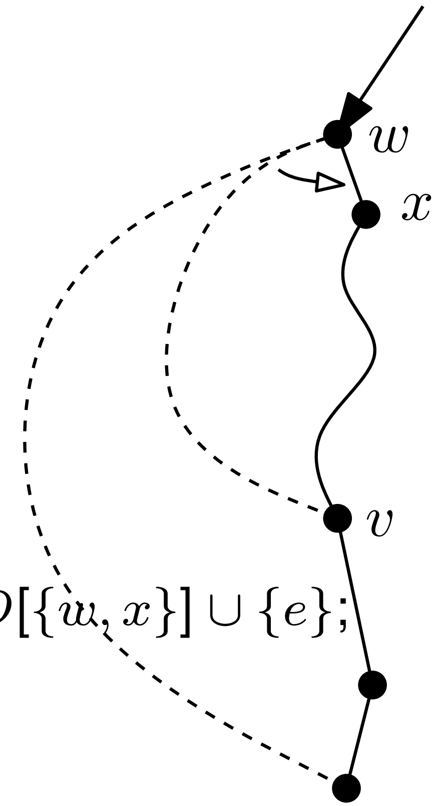

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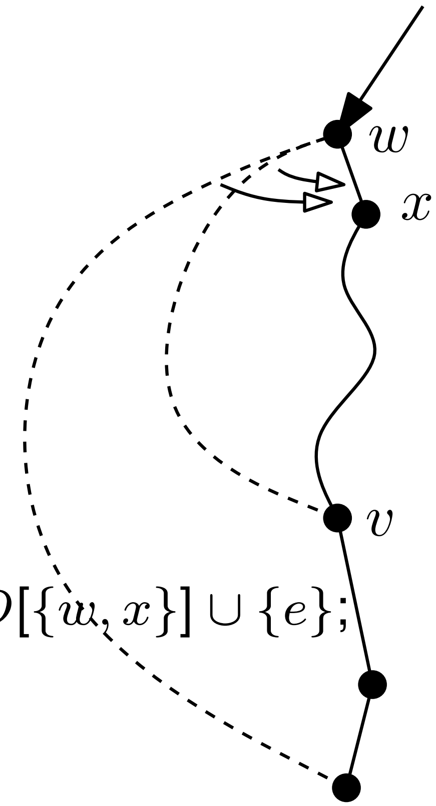

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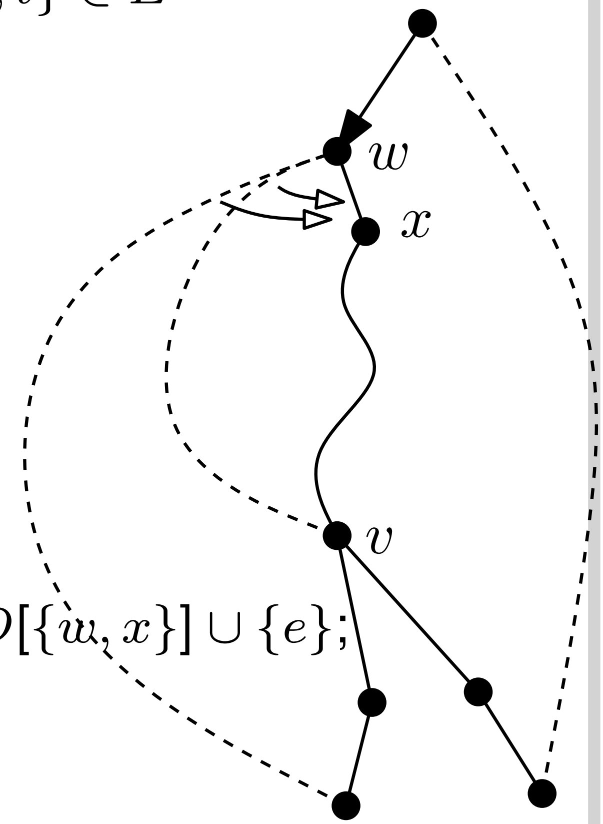

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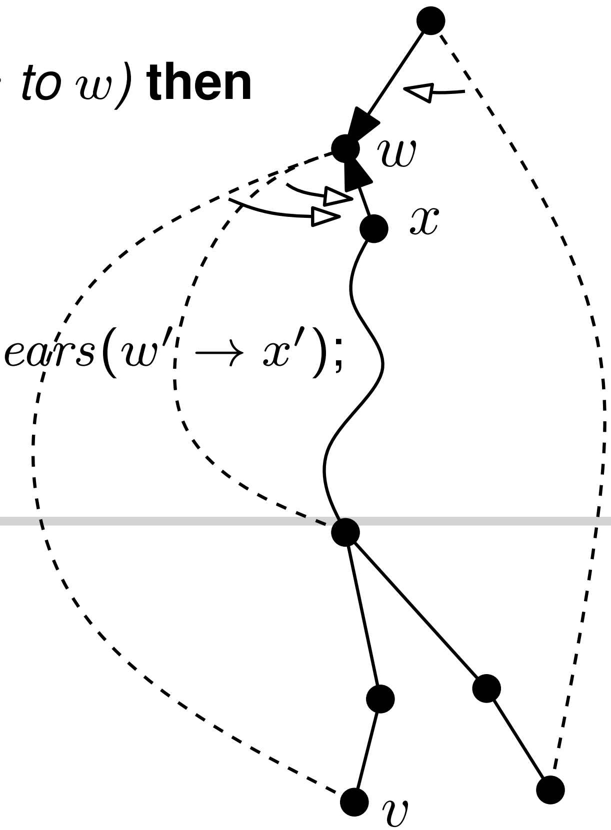
Function *process_ears*

process_ears(tree edge $w \to x$) begin

- **foreach** $v \hookrightarrow w \in D[w \to x]$ **do**
  - $u \leftarrow v$;
  - **while** $u \notin L$ **do** $u \leftarrow PARENT[u]$;
  - $P \leftarrow (u \xrightarrow{*} v \hookrightarrow w)$;
  - **if** $w \to x$ *is oriented from* w *to* x *(resp.from* x *to* w*)* **then**
    - orient P from w to u (resp. from u to w);
    - paste the inner nodes of P to L before (resp. after) u ;
  - **foreach** *tree edge* $w' \to x'$ *of* P **do** $process\_ears(w' \to x')$;
- $D[\{w, x\}] \leftarrow \emptyset$;



st-ordering

Theorem (Correctness and time complexity)

The described algorithm produces an st-ordering of a given biconnected graph in $O(n)$ time.

Proof

- Correctness can be proven by induction on ears. Notice that a new ear is added when function process_ears is called. Notice that after adition of an ear and its orientation, we have a biconnected st-graph and its topological ordering.

st-ordering

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The described algorithm produces an st-ordering of a given biconnected graph in $O(n)$ time.

Proof

- Correctness can be proven by induction on ears. Notice that a new ear is added when function process_ears is called. Notice that after adition of an ear and its orientation, we have a biconnected st-graph and its topological ordering.

Lemma (Necessary for planarity of orthogonal drawing)

Let G be a plane graph and edge (s, t) on the boudary of G. Let $v_1, \ldots, v_n$ be an st-ordering of G. If G_i is the graph induced by the vertices $v_1, \ldots, v_i$ then vertex v_{i+1} lies on the outer face of G_i. (Exersize sheet 4)