Vorlesung Graphenzeichnen

Order in the Underground *or* How to automate drawing metro maps?

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Outline

- Modeling the Metro Map Problem
  - What is a Metro Map?
  - Hard and Soft Constraints
- NP-Hardness: Bad News—Nice Proof
  - Rectilinear vs. Octilinear Drawing
  - Reduction from PLANAR 3-SAT
- MIP Formulation & Experiments
  - Mixed-Integer Programming Formulation
  - Labeling
  - Experiments

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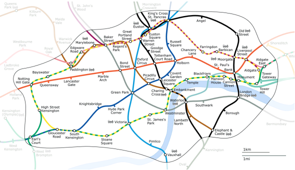
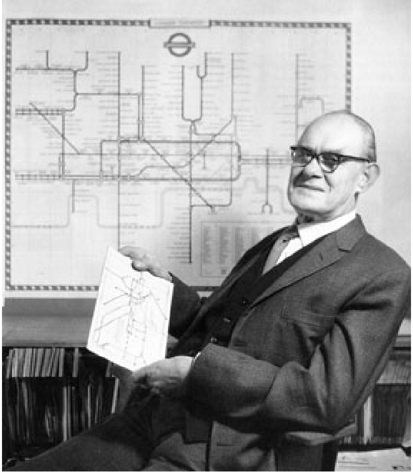
What is a Metro Map?



- schematic diagram for public transport
- visualizes lines and stations
- goal: ease navigation for passengers
  - “How do I get from A to B?”
  - “Where to get off and change trains?”
- distorts geometry and scale
- improves readability
- compromise between
  schematic road map $\leftrightarrow$ abstract graph

Why Automate Drawing Metro Maps?

- current maps designed manually



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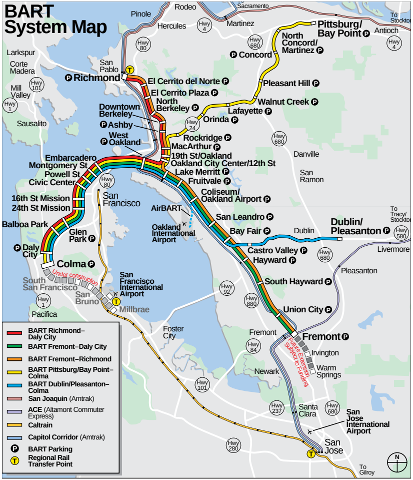

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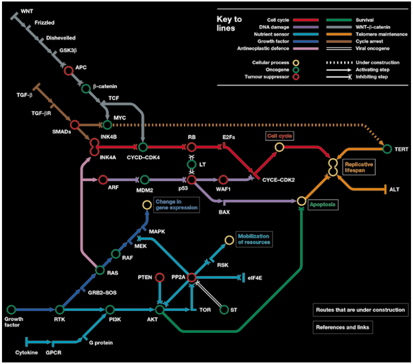
  - metabolic pathways

[Hahn, Weinberg '02]



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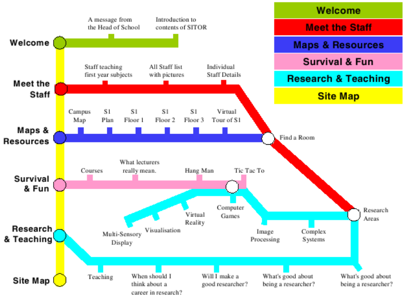

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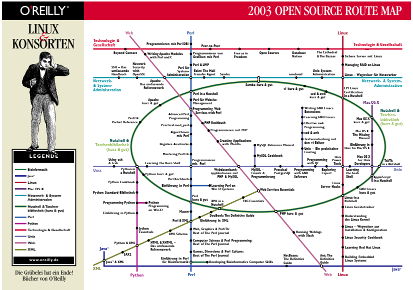

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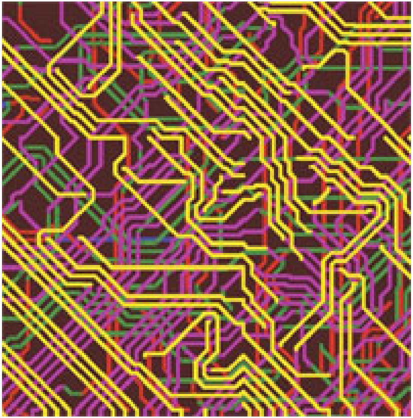
- VLSI: X-architecture



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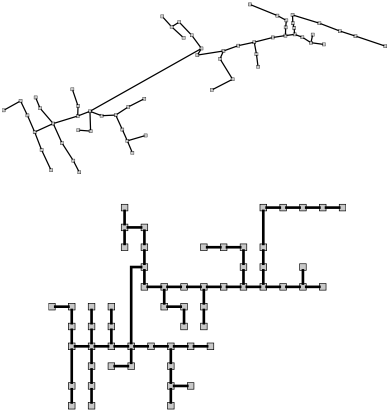
- redrawing sketches [Brandes et al. '03]



More Formally

The Metro Map Problem

Given: planar embedded graph $G = (V, E)$, $V \subset \mathbb{R}^2$,
line cover $\mathcal{L}$ of paths or cycles in G (the metro lines),

Goal: draw G and $\mathcal{L}$ nicely.

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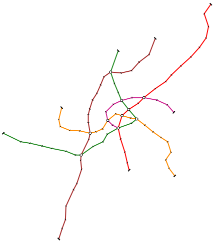
- What is a nice drawing?
- Look at real-world metro maps drawn by graphic designers and model their design principles as
  - *hard* constraints – must be fulfilled,
  - *soft* constraints – should hold as tightly as possible.

Hard Constraints

(H1) preserve embedding of G



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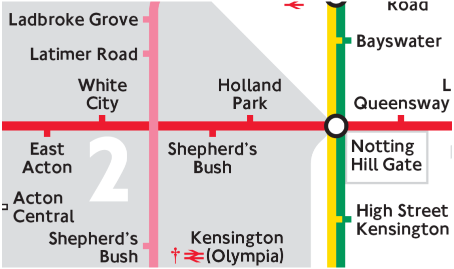
(H4) keep edges d_{min} away from non-incident edges ($\rightarrow$ planar)

Soft Constraints

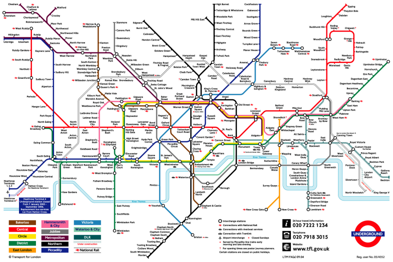
(S1) draw metro lines with few bends



Soft Constraints

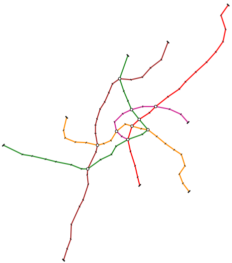
(S1) draw metro lines with few bends

(S2) keep total edge length small



Soft Constraints

- (S1) draw metro lines with few bends
- (S2) keep total edge length small
- (S3) draw each octilinear edge similar to its geographical orientation: keep relative position of adjacent vertices



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RECTILINEARGRAPHDRAWING Decision Problem

Given a planar embedded graph G with max degree 4.
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Our Problem

METROMAPLAYOUT Decision Problem

Given a planar embedded graph G with max degree 8.
Is there a drawing of G that

- preserves the embedding,
- uses straight-line edges,
- is octilinear?

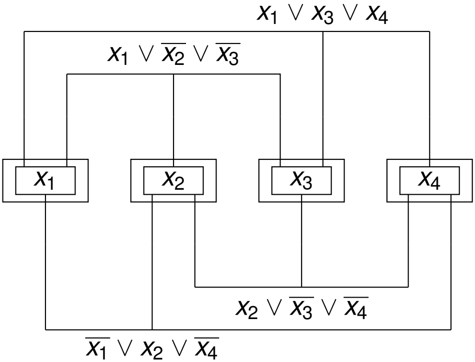
Theorem

METROMAPLAYOUT *is NP-hard.*

Proof.

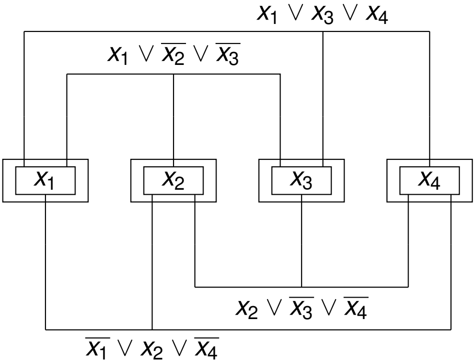
By Reduction from PLANAR 3-SAT to METROMAPLAYOUT. □

Outline of the Reduction



Input: planar 3-SAT formula $\varphi =$
$(x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \dots$

Outline of the Reduction



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G_φ has a metro map drawing $\Leftrightarrow$ φ satisfiable.

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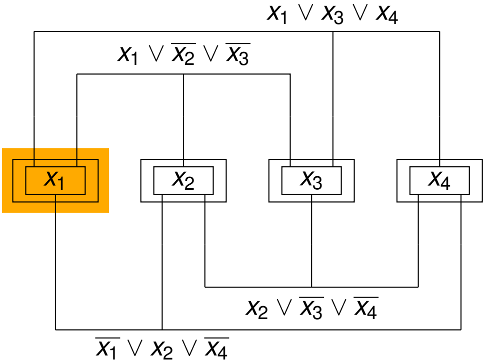

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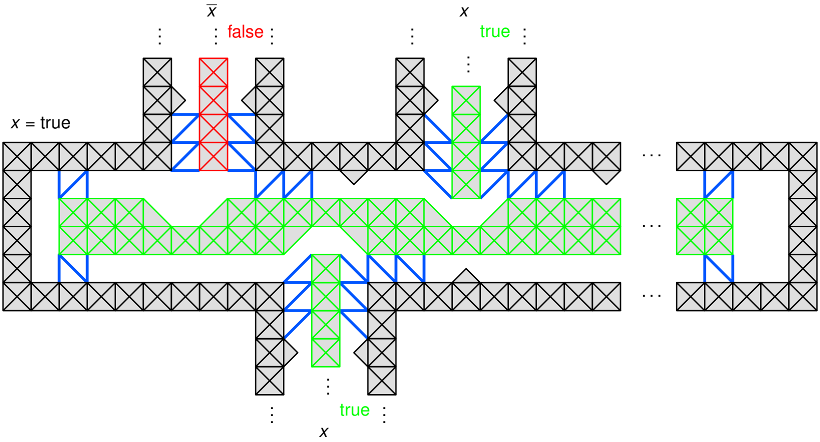

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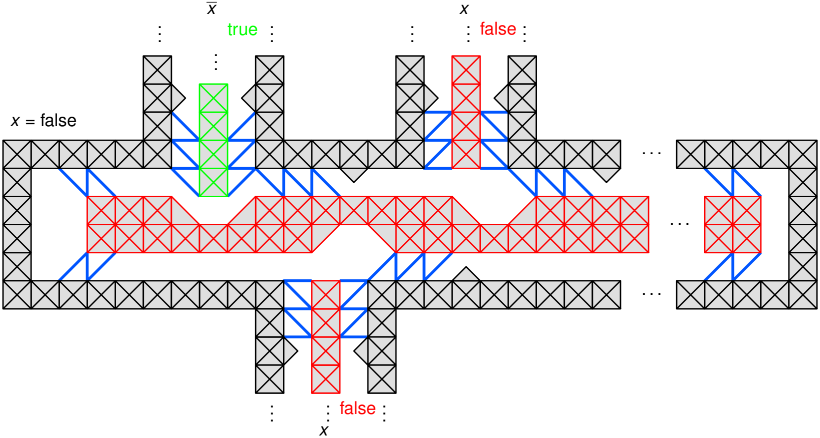

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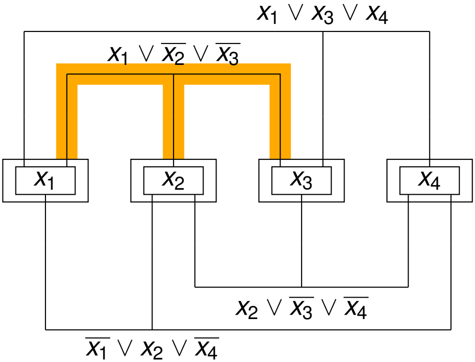

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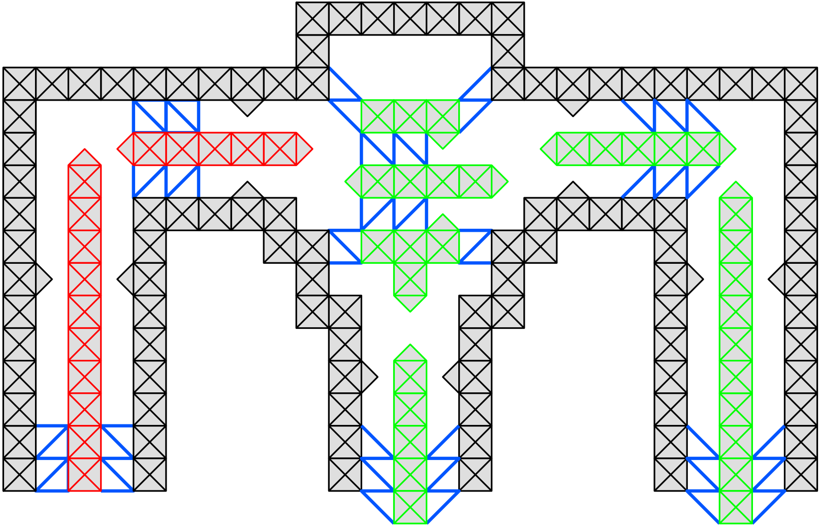

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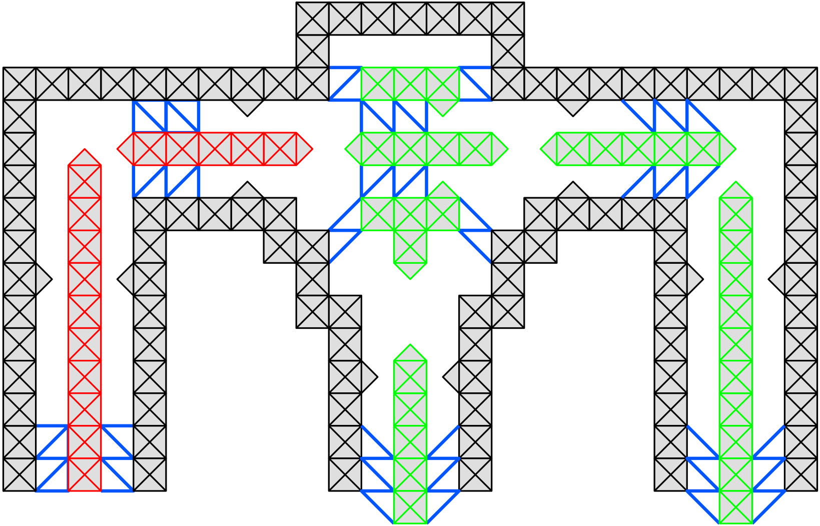

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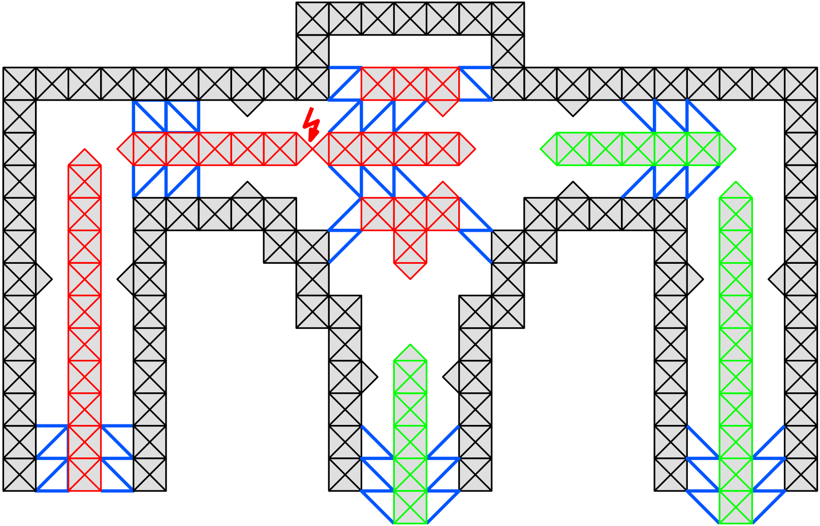

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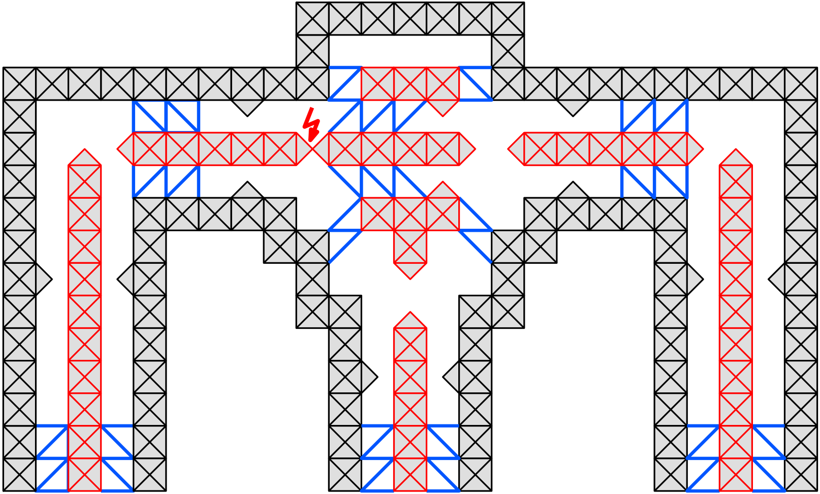

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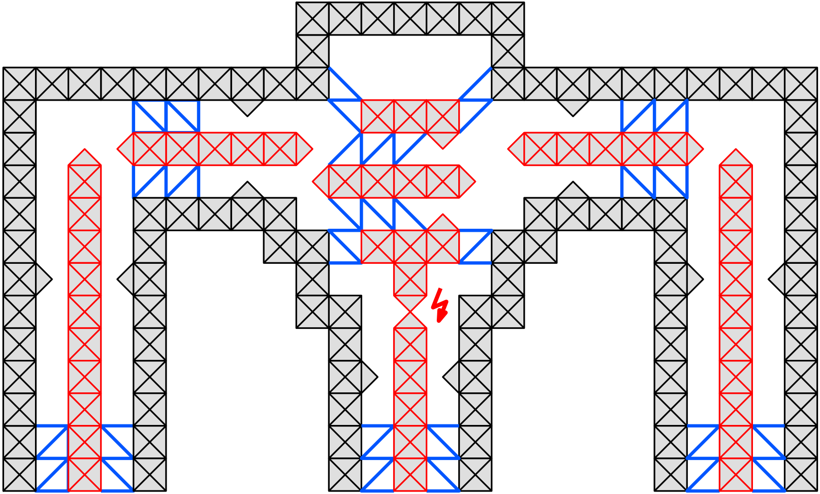

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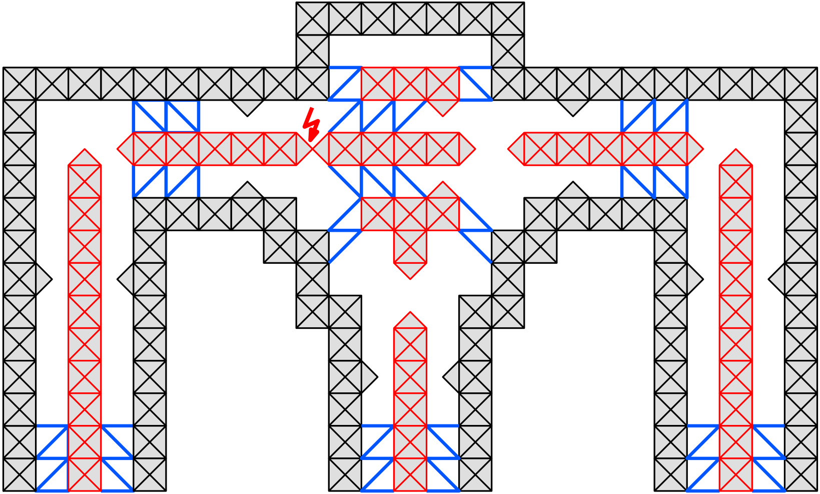

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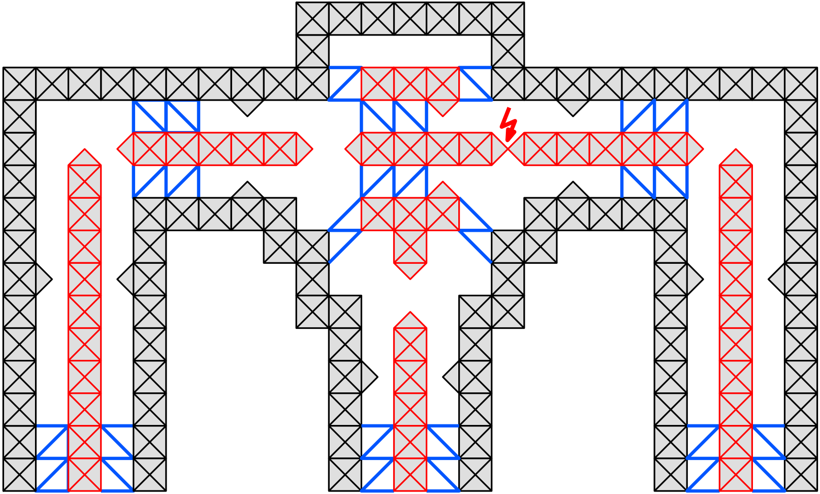
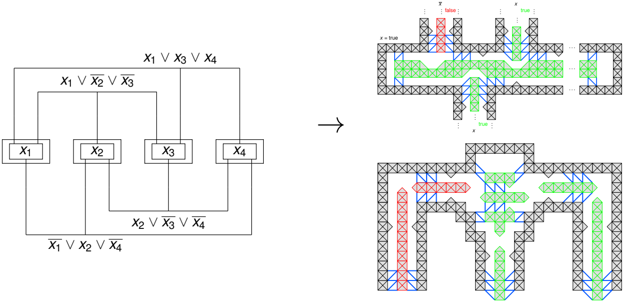


Summary of the Reduction



- Indeed we have:
  - φ satisfiable $\Rightarrow$ corresponding MM drawing of G_φ
  - G_φ has MM drawing $\Rightarrow$ satisfying truth assignment of φ

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- Linear Programming: efficient optimization method for
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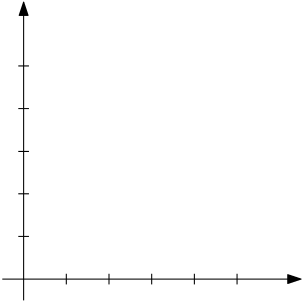
  - linear objective function
  - real-valued variables



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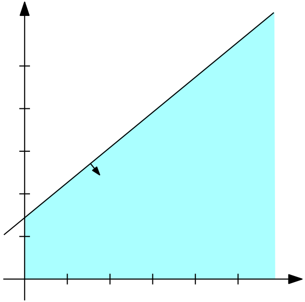

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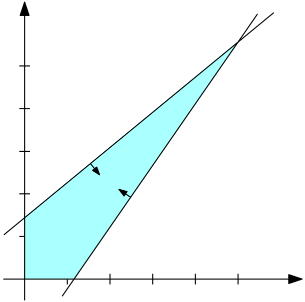

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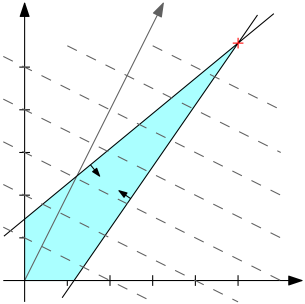

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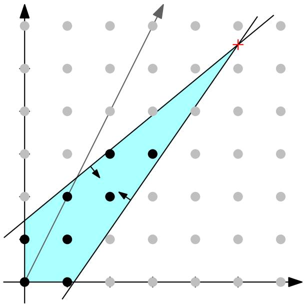

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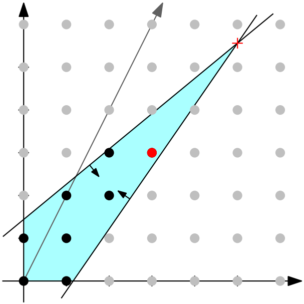

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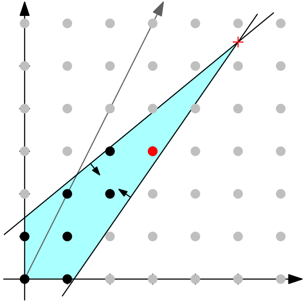


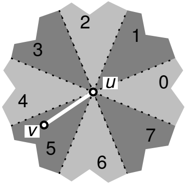
Theorem (GD'05 / TVCG'11)

The metro map layout problem can be formulated as a MIP s.th.

hard constraints $\rightarrow$ *linear constraints*

soft constraints $\rightarrow$ *objective function*

Definitions: Sectors and Coordinates



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– for each vtx. u partition plane into sectors 0–7

- here: $\mathrm{sec}(u, v) = 5$ (input)

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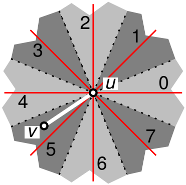

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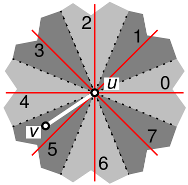

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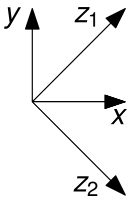


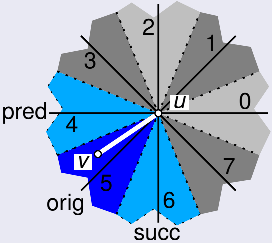
Coordinates

assign z_1- and z_2-coordinates to each vertex v:

- $z_1(v) = x(v) + y(v)$
- $z_2(v) = x(v) - y(v)$

Octilinearity and Relative Position (H2, H3, S3)

Goal



Draw edge *uv*

- octilinearly
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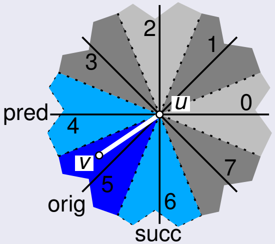

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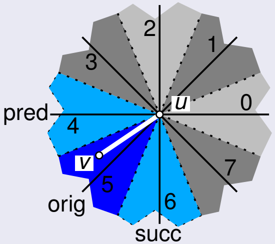

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Binary Variables

$$\alpha_{\text{pred}}(u,v) + \alpha_{\text{orig}}(u,v) + \alpha_{\text{succ}}(u,v) = 1$$

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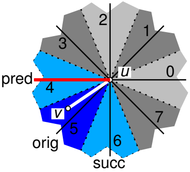

Predecessor Sector

$$
\begin{aligned}
y(u) - y(v) &\leq M(1 - \alpha_{\text{pred}}(u, v)) \\
-y(u) + y(v) &\leq M(1 - \alpha_{\text{pred}}(u, v)) \\
x(u) - x(v) &\geq -M(1 - \alpha_{\text{pred}}(u, v)) + \ell_{uv}
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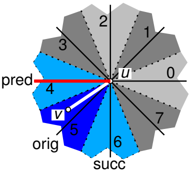

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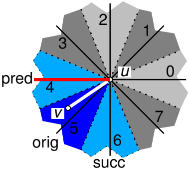
How does this work?

Octilinearity and Relative Position



Predecessor Sector

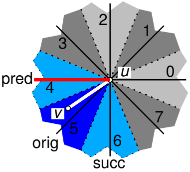
$$\begin{aligned} y(u) - y(v) &\leq M(1 - \alpha_{\text{pred}}(u, v)) \\ -y(u) + y(v) &\leq M(1 - \alpha_{\text{pred}}(u, v)) \\ x(u) - x(v) &\geq -M(1 - \alpha_{\text{pred}}(u, v)) + \ell_{uv} \end{aligned}$$

How does this work?

Case 1: $\alpha_{\text{pred}}(u, v) = 0$

$$\begin{aligned} y(u) - y(v) &\leq M \\ -y(u) + y(v) &\leq M \\ x(u) - x(v) &\geq \ell_{uv} - M \end{aligned}$$

Octilinearity and Relative Position



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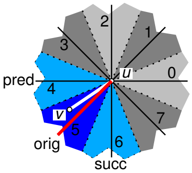
How does this work?

Case 2: $\alpha_{\text{pred}}(u,v) = 1$

$$\begin{aligned} y(u) - y(v) &\leq 0 \\ -y(u) + y(v) &\leq 0 \\ x(u) - x(v) &\geq \ell_{uv} \end{aligned}$$

Octilinearity and Relative Position (H2, H3, S3)



Original Sector

$$\begin{aligned} z_2(u) - z_2(v) &\leq M(1 - \alpha_{\text{orig}}(u,v)) \\ -z_2(u) + z_2(v) &\leq M(1 - \alpha_{\text{orig}}(u,v)) \\ z_1(u) - z_1(v) &\geq -M(1 - \alpha_{\text{orig}}(u,v)) + 2\ell_{uv} \end{aligned}$$

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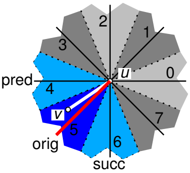

Original Sector

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z_2(u) - z_2(v) &\leq M(1 - \alpha_{orig}(u,v)) \\
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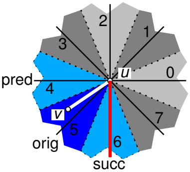


Successor Sector

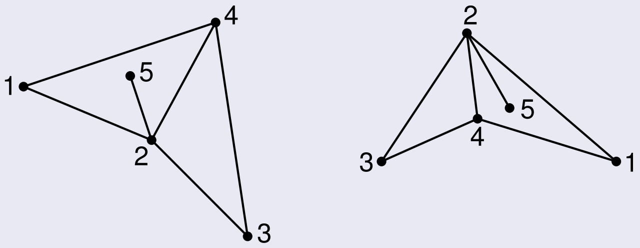
$$
\begin{aligned}
x(u) - x(v) &\leq M(1 - \alpha_{succ}(u,v)) \\
-x(u) + x(v) &\leq M(1 - \alpha_{succ}(u,v)) \\
y(u) - y(v) &\geq -M(1 - \alpha_{succ}(u,v)) + \ell_{uv}
\end{aligned}
$$

Preserving the Embedding (H1)

Definition

Two planar drawings of G have the same *embedding* if the induced orderings on the neighbors of each vertex are equal.

Same Embedding



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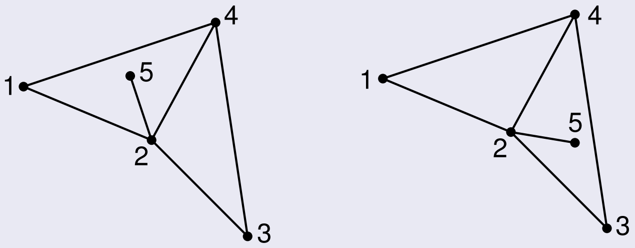
Different Embeddings



Preserving the Embedding (H1)

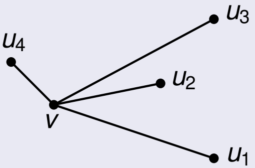
Constraints (Example)

- $N(v) = \{u_1, u_2, u_3, u_4\}$
- circular input order: $u_1 < u_2 < u_3 < u_4 < u_1$

All but one of the following inequalities must hold

$$\mathrm{dir}(v, u_1) < \mathrm{dir}(v, u_2) < \mathrm{dir}(v, u_3) < \mathrm{dir}(v, u_4) < \mathrm{dir}(v, u_1)$$

Input



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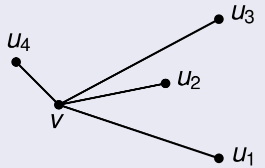
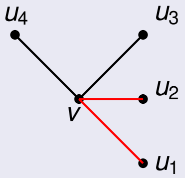
Input



Output

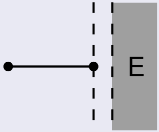


Planarity (H4)

Observation

For octilinear, straight edge e_1 non-incident edge e_2 must be placed $d_{\min}$ to the

- east, northeast, north, northwest, west, southwest, south, or southeast



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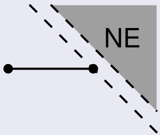

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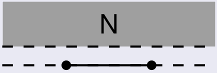

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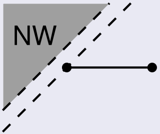

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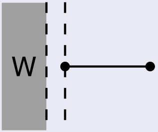

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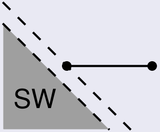

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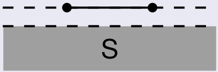

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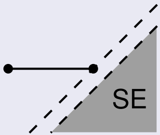

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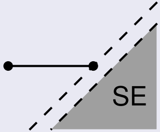

Constraints

- model as MIP with binary variables
  $\alpha_E + \alpha_{NE} + \alpha_N + \alpha_{NW} + \alpha_W + \alpha_{SW} + \alpha_S + \alpha_{SE} \geq 1$
- required for each pair of non-incident edges

Objective Function

Objective Function

- corresponds to soft constraints (S1)–(S3)
- weighted sum of individual cost functions

minimize $\lambda_{\text{bends}}\, \text{cost}_{\text{bends}} + \lambda_{\text{length}}\, \text{cost}_{\text{length}} + \lambda_{\text{relpos}}\, \text{cost}_{\text{relpos}}$

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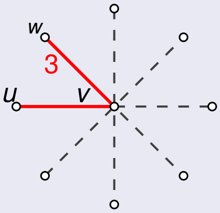
Line Bends (S1)



Edges uv and vw on a metro line $L \in \mathcal{L}$

- draw as straight as possible
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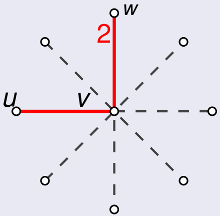

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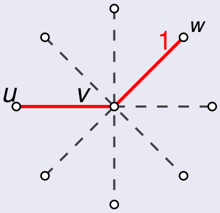

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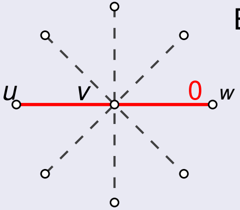

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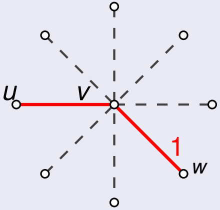

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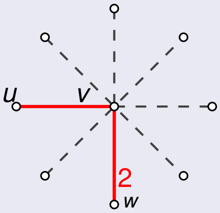

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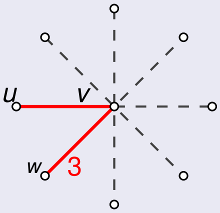

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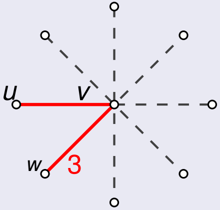

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$$\text{cost}_{\text{bends}} = \sum_{L \in \mathcal{L}} \sum_{uv, vw \in L} \text{bend}(u, v, w)$$

Objective Function

Total Edge Length (S2)

$$\text{cost}_{\text{length}} = \sum_{uv \in E} \text{length}(\overline{uv})$$

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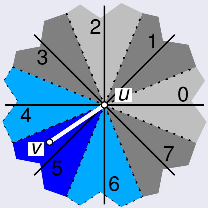

- only three directions possible

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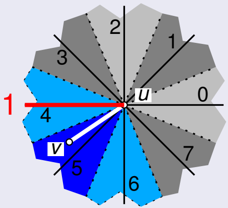

- only three directions possible
- charge 1 if edge deviates from original sector

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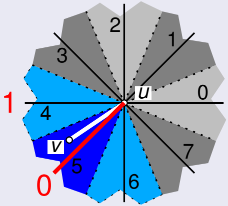

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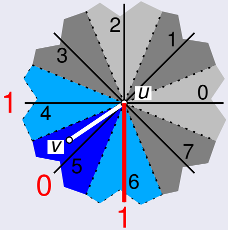

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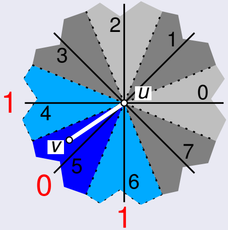

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$$\text{cost}_{\text{relpos}} = \sum_{uv \in E} \text{relpos}(uv)$$

Effects of the Soft Constraints

Objective Function

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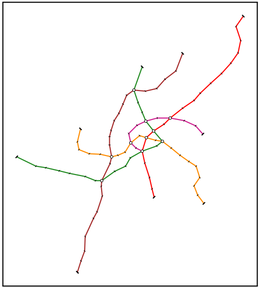

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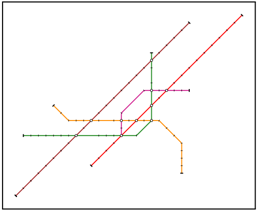

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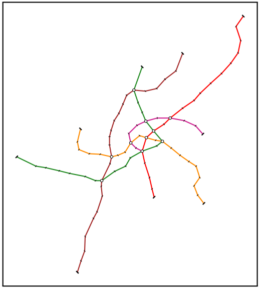

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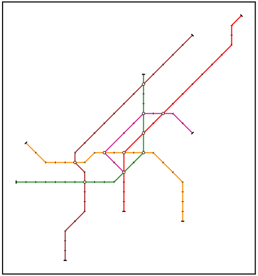

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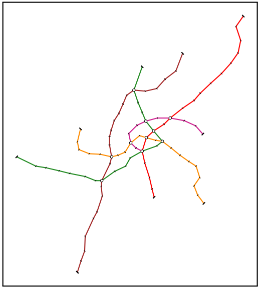

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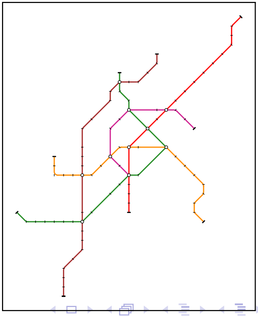

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- hard constraints:
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- models METROMAPLAYOUT as MIP

Summary of the MIP

- hard constraints:
  - octilinearity
  - minimum edge length
  - (partially) relative position
  - preservation of embedding
  - planarity
- soft constraints:

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- in total $O(|V|^2)$ constraints and variables

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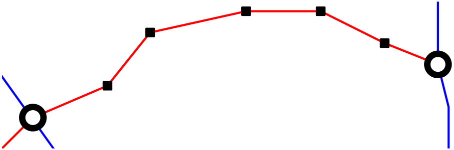
Speed-Up Techniques: Reduce Graph Size



- metro graphs have many degree-2 vertices
- want to optimize line straightness

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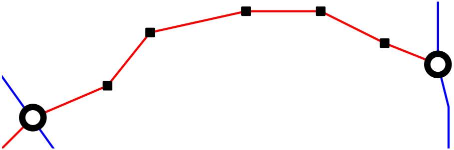

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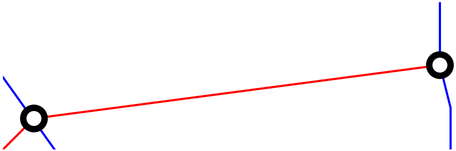

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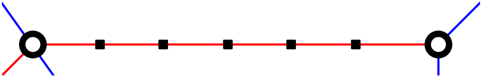

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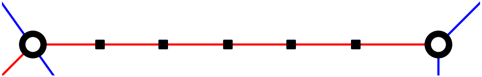

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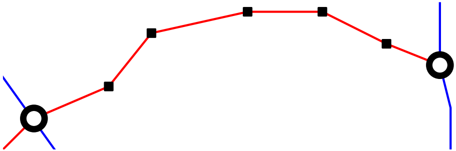

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Idea 2 keep two *joints*

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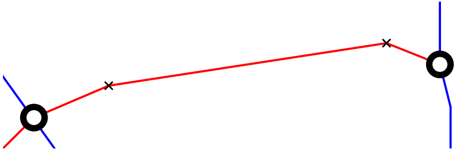

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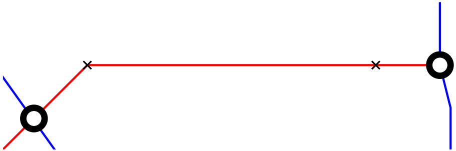

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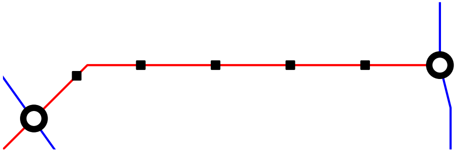

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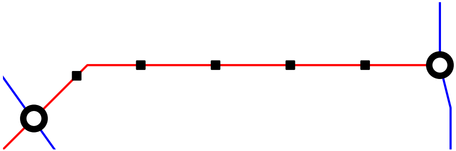

- metro graphs have many degree-2 vertices
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Idea 1 collapse all degree-2 vertices
- low flexibility

Idea 2 keep two *joints*
- higher flexibility
- more similar to input

Speed-Up Techniques: Reduce MIP Size

- $O(|V|^2)$ planarity constraints (for each pair of edges...)
- in practice 95–99% of constraints

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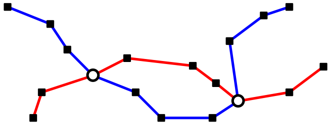

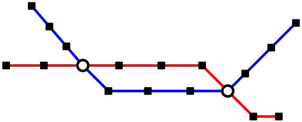

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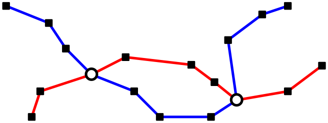
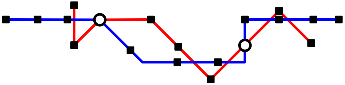




Speed-Up Techniques: Callback Functions

- MIP optimizer CPLEX offers advanced callback functions
- add required planarity constraints on the fly

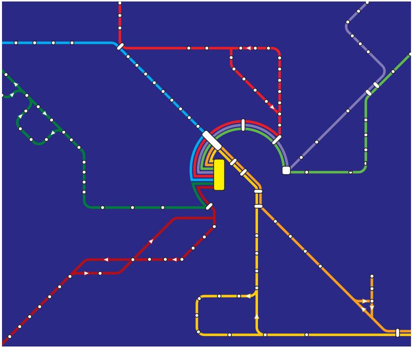
Algorithm

1. start solving MIP without planarity constraints
2. for each new solution
    1. interrupt CPLEX
    2. if solution is not planar
        - add planarity constraints for intersecting edges
        - reject solution

       else
        - accept solution
    3. continue solving the MIP (until optimal)

Labeling

- unlabeled metro map of little use in practice



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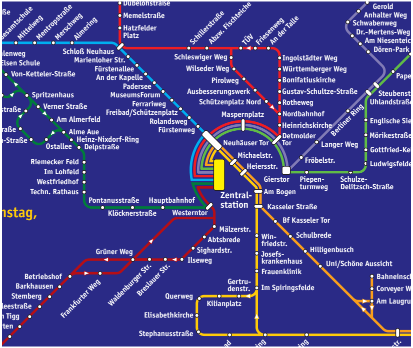
- labels
  - occupy space
  - may not overlap



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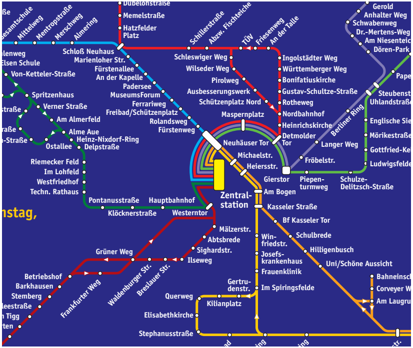

Labeling

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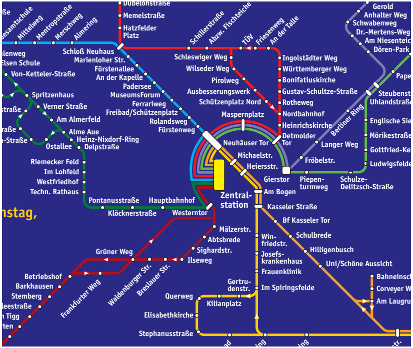
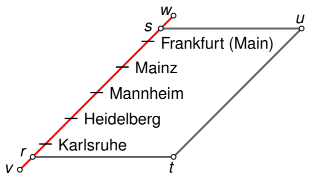
- labels
  - occupy space
  - may not overlap
- static edge labeling is NP-hard [Tollis, Kakoulis '01]
- combine layout and labeling for better results



Modeling Labels

Model labels as special metro lines:

- put all labels between each pair of interchange stations into one parallelogram,



Modeling Labels

Model labels as special metro lines:

- put all labels between each pair of interchange stations into one parallelogram,
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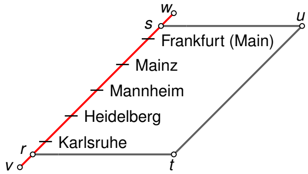

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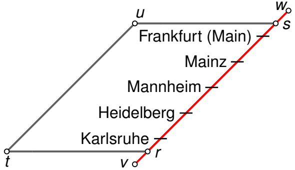

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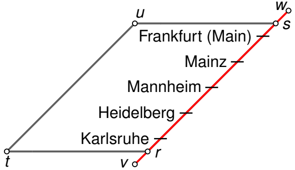

Modeling Labels

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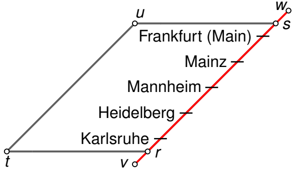
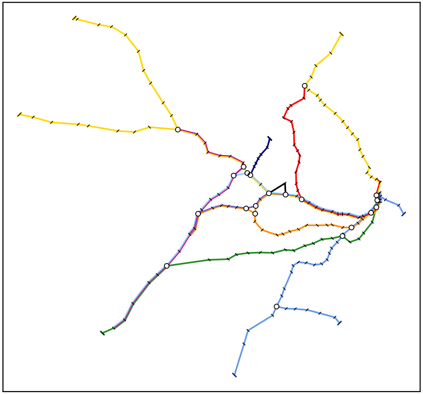
- allow parallelograms to change sides,
- bad news: a **lot** more planarity constraints
- good news: callback method helps



Results – Sydney unlabeled

Input



Input	$\lvert V \rvert$	$\lvert E \rvert$	fcs.	$\lvert \mathcal{L} \rvert$
full	174	183	11	10
reduced	88	97		

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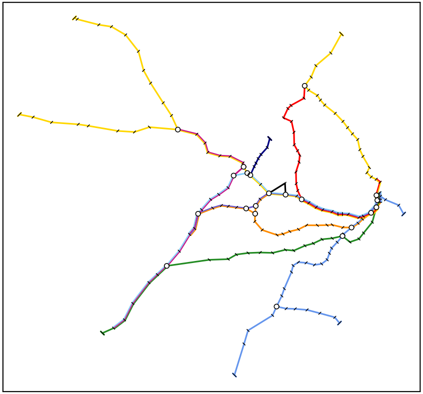

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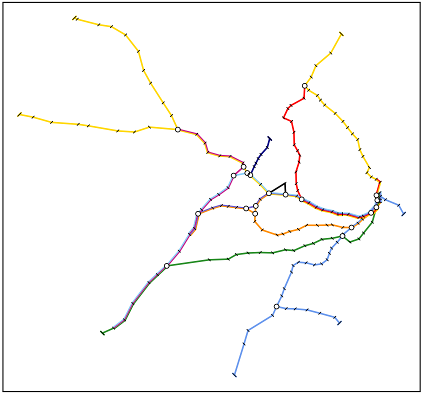
↓

MIP	constr.	var.
full	152,194	37,802
callback	3,529	4,834
skipped	3,034	1,642

Results – Sydney unlabeled

Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
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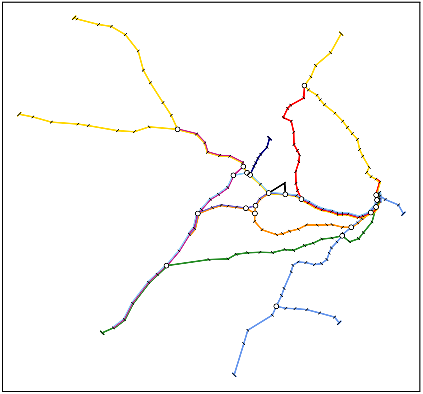
$\downarrow$

MIP	constr.	var.
full	152,194	37,802
callback*	3,529	4,834
skipped	3,034	1,642

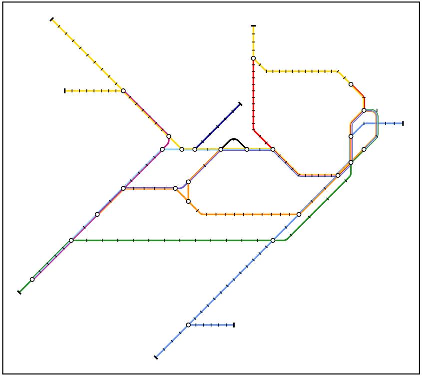
*) 23 minutes w/o proof of opt.
constr. of 3 edge pairs added

Results – Sydney unlabeled

Input

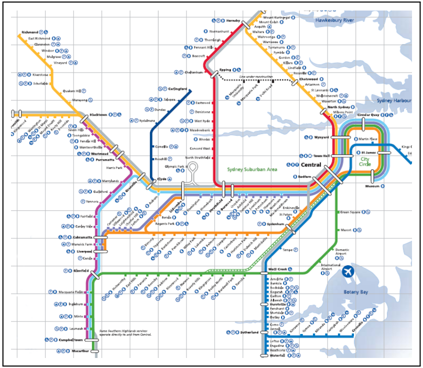


Output (23 min.)



Results – Sydney unlabeled

Official map



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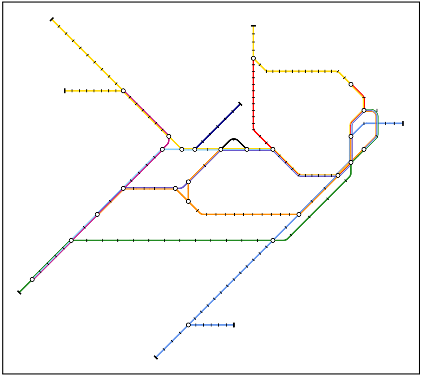
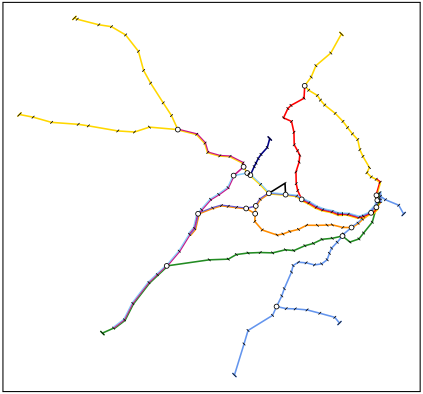


Results – Sydney labeled

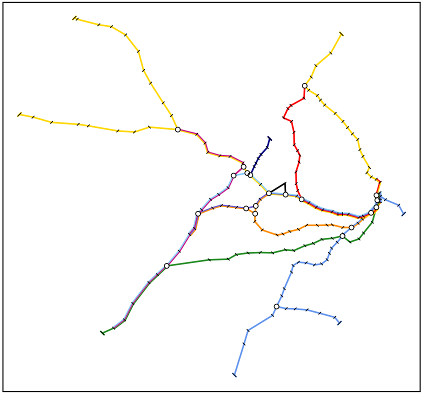
Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
full	174	183	11	10
reduced	88	97		
labeled	242	270	30	

Results – Sydney labeled

Input



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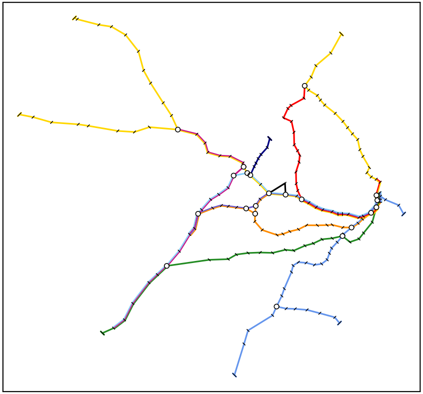
↓

MIP	constr.	var.
full	1,191,406	290,137
callback	21,988	92,681
skipped	6,838	2,969

Results – Sydney labeled

Input



Input	$\|V\|$	$\|E\|$	fcs.	$\|\mathcal{L}\|$
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labeled	242	270	30	

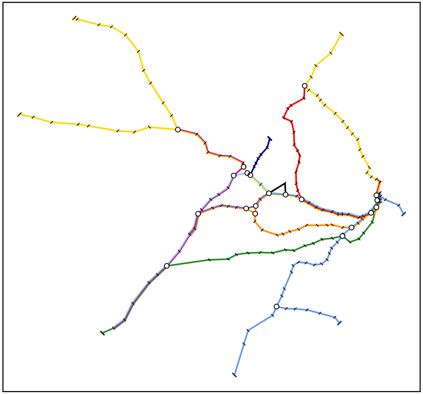
↓

MIP	constr.	var.
full	1,191,406	290,137
callback*	21,988	92,681
skipped	6,838	2,969

*) 10:30 hours w/o proof of opt.
add constr. of 123 edge pairs

Results – Sydney labeled

Input



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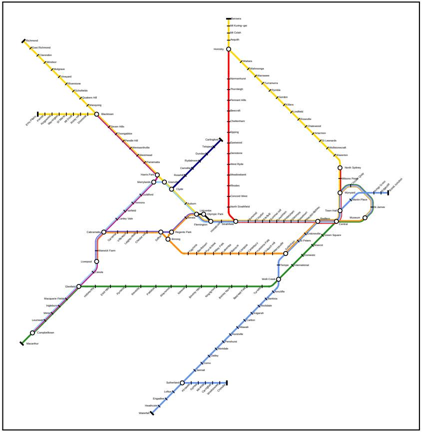

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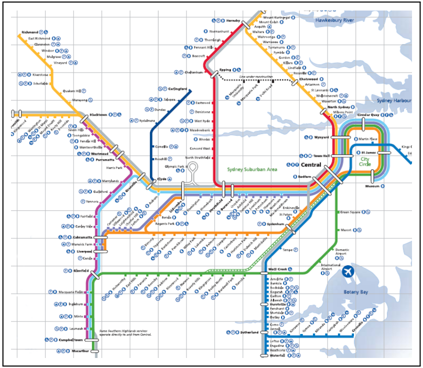

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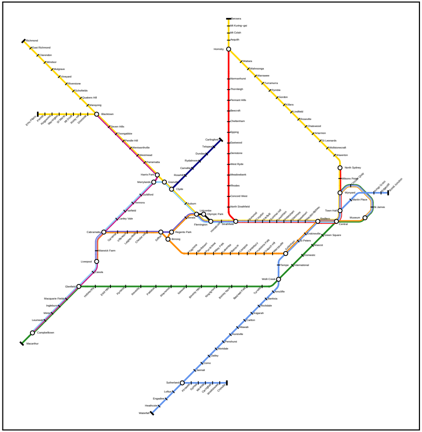

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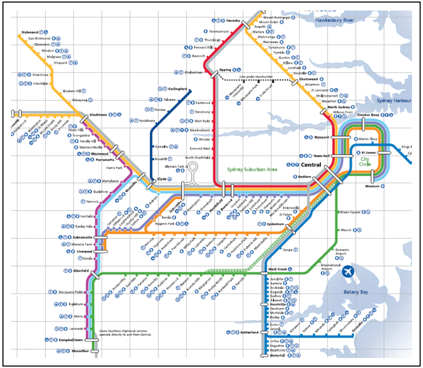
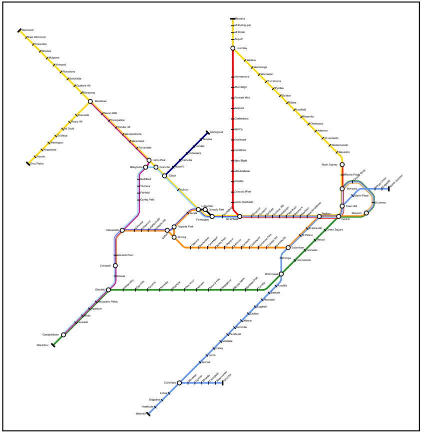
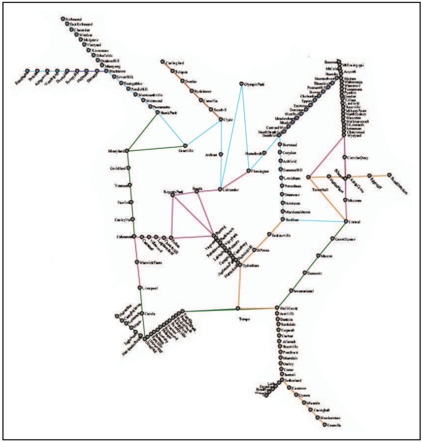


Output (1:40 hrs.)

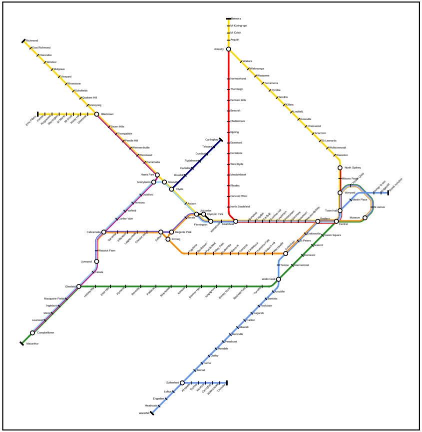


Sydney: Related Work

[Hong et al. GD'04] (7.6 sec.)

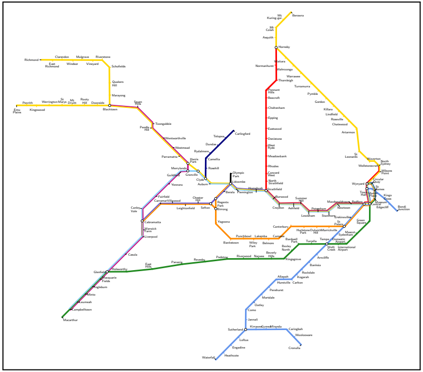


Our output (10:30 hrs.)



Sydney: Related Work

[Stott, Rodgers TVCG'10] (2 hrs.)



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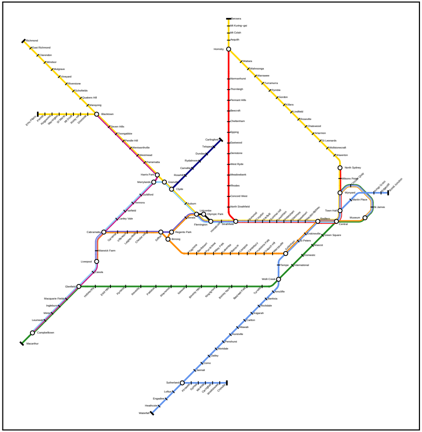
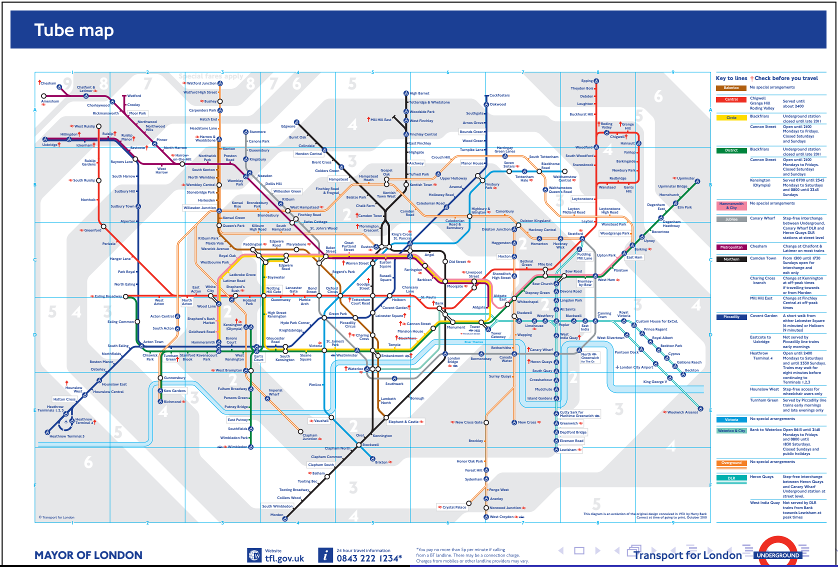


Large Example: London



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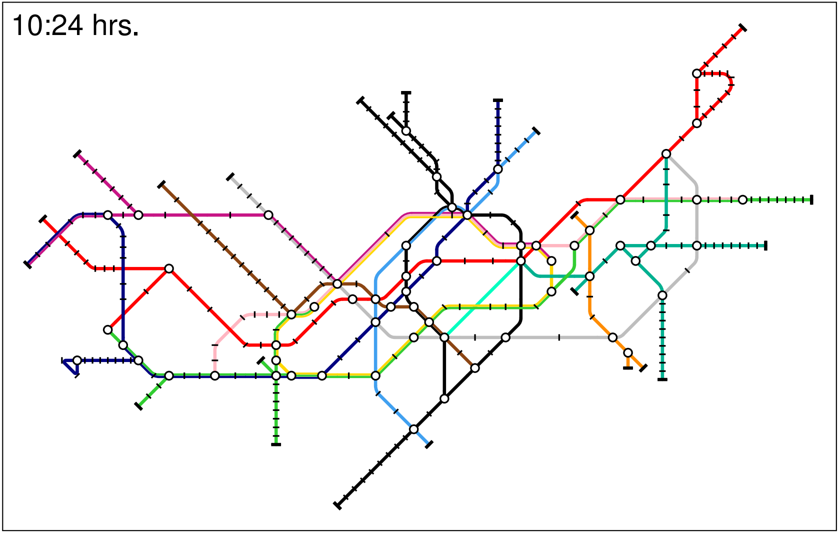

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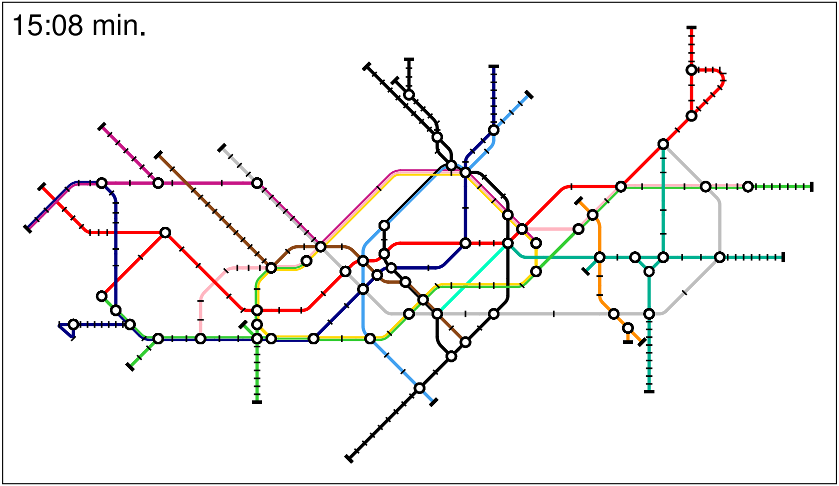
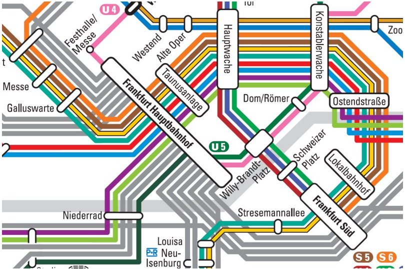
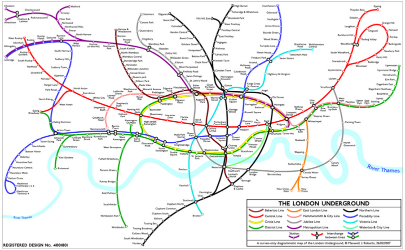


Are we done yet?





- more user interaction
- how to handle large stations and many parallel lines?
- formulate global aesthetics like symmetry and balance
- octilinear maps are not necessarily always best – how to compute curved metro maps?

Summary

- METROMAPLAYOUT is NP-hard
- formulation of hard and soft constraints as MIP
- combined layout and labeling
- MIP size & runtime reductions
- high-quality results
- MIP can schematize *any* kind of graph sketch

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For more info see:

M. Nöllenburg and A. Wolff. *Drawing and labeling high-quality metro maps by mixed-integer programming.* IEEE Trans. Visualization and Computer Graphics 17(5):626–641, 2011.