Karlsruhe Institute of Technology

# Representations of Graphs by Outside Obstacles

Study Thesis of

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| Time Period: | December 2011 – May 2012 |

**Acknowledgements**
First of all, I have to thank Professor Tokuyama for his warm welcome at Tōhoku University in Sendai, Japan. I felt well-integrated in his lab and a lot of great memories, friendships and a huge pile of worthwile photos remains. I also would like to express my sincere gratitude to Professor Wagner, who established the academic exchange program of Tōhoku University and the KIT together with Professor Tokuyama, and gave me the opportunity to study abroad at one of the top class universities of Japan.

I am also hugely indebted to my first advisor Dr. Marcus Krug for his helpful advice and patience. I have to also thank my second advisor Dr. Ignaz Rutter for taking over the role as advisor, when Marcus changed to a different university and also for helpful comments and corrections. Many thanks to Associate Professor Otachi, who supported me a lot in general graph theoretic questions, and also to my tutor Ōta-san for helping me with bureaucracy and questions on cultural differences during my stay in Japan.

**Statement of Authorship**

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

Karlsruhe, June 1, 2012.

## Abstract

Graphs that are given by an outside-obstacle representation form an interesting superclass of polygon-vertex visibility graphs, which are most studied in the field of visibility problems. We first introduce ray obstacles and compare them to previously defined poylgon and segment obstacle representations (ORs). We define "graph-invariant" maps, which describe possible movements of vertices in a visibility graph, without altering adjacencies. These might be used to continuously transform an outside-OR into a ray OR. We characterize outerplanar graphs that admit a plane outside-OR as chordal outerplane graphs. For general planar graphs we give a set of necessary conditions which we conjecture to also be sufficient.

## Deutsche Zusammenfassung

Graphen, die durch eine Außenhindernis-Darstellung gegeben sind, sind eine interessante Überklasse von Polygon-Knoten-Sichtbarkeitsgraphen, die im Bereich der Sichtbarkeitsprobleme bisher am stärksten untersucht wurden. In der Studienarbeit werden zunächst Strahlhindernisse eingeführt und mit den bereits bekannten Polygon- und Strecken-Hindernis-Darstellungen (HD) verglichen. Da ungeklärt ist, ob jeder Graph mit Polygon-Außenhindernis-Darstellung auch durch eine Strahl-HD dargestellt werden kann, werden „Graph-invariante" Abbildungen definiert, die mögliche Verschiebungen der Knoten eines Sichtbarkeitsgraphen beschreiben, bei der die Adjazenz der Knoten unverändert bleibt. Diese könnten dazu verwendet werden Außenhindernis-Darstellungen stetig in Strahl-HDen zu überführen. Im zweiten Teil der Arbeit, werden Außenhindernisdarstellungen betrachtet, die sich überschneidungsfrei in die Ebene einbetten lassen. Für außenplanare Graphen die mit einer planaren Außenhindernisdarstellung eingebettet werden können, wird eine Charakterisierung als chordale außenplanare Graphen angegeben. Für allgemeine planare Graphen werden einige notwendige Bedingungen gezeigt, von denen vermutet wird, dass sie auch hinreichend sind.

# Contents

# 1 Introduction

Visibility is a very general and real-life notion that gives rise to a large set of visibility problems that naturally occur in many fields, such as computational geometry and geometric graph theory. Therefore, there is a diverse set of applications in robot motion planning, computer vision and computer graphics, to name a few. For example, visibility is used to find Euclidean shortest paths that avoid polygonal obstacles or to determine invisible parts, that need not be rendered in a 3D scene.

The most famous visibility problem is Chvátal's *art gallery problem*: Given a simple polygon (our "art gallery"), we want to place a minimum number of guards (here: vertices) in our gallery, so that every point of the gallery is watched over by a guard. Here, two points (e. g. the guard and a point in the gallery) see each other, if we can draw a straight line connecting the points, without leaving the boundary polygon. A good survey on the topic is given by Shermer [S92].

In the special case, where guards are only allowed at the vertices of the polygon and it suffices to have the vertices watched (both reasonable restrictions to the problem), we compute a *polygon-vertex visibility graph* from the the boundary polygon: As vertex set we choose the vertices of the polygon and we define two vertices to be adjacent, if they can see each other by the definition above. In this setting, it suffices to place guards at a dominating set of the resulting polygon-vertex visibility graph.

This type of visibility graphs is most studied in the field, but although researchers have tried for many years, no combinatorial characterization has been found yet (see Related Works). Therefore, our analysis focuses on a more general class of graphs, where an arbitrary point set within a simple polygon forms our vertex set.

This larger class is equivalent to the graphs with an outside-obstacle representation, recently defined in a paper by Alpert, Koch and Liason [AKL10]: Firstly, an obstacle representation (OR) of a graph $G$ is a set of vertices and polygonal obstacles, where an edge is in $G$ if and only if it does not intersect any obstacle in a straight-line drawing of $G$. An outside-OR of $G$ is an OR that consists of obstacles in the outer face of the drawing of $G$.

Their paper contains a proof showing that outerplanar graphs have an outside-OR, which motivates approaches to extend the result to superclasses of outerplanar graphs. Furthermore, interesting questions arise by formulating additional conditions, e. g. to characterize graphs with an outside-OR that is plane.

The main objective of this thesis is to understand representations of graphs by obstacles in the outside face. An extension of obstacles to not only include polygons, but also segments or ray obstacles, will yield additional insight and gives us a tool to measure the complexity of the obstacles.

## 1.1 Related Work

For the computation of a visibilitiy graph given by points inside a simple polygon, Ben-Moshe et al. [BHKM04] provided a nearly optimal ($O(n + m \log m \log mn + k)$, with $O(m + n)$ space) output-sensitive algorithm, with $k$ being the number of edges in the resulting graph. In the more restricted case of polygon-vertex visiblity graphs, but with general polygons that may contain holes, Pocciola and Vegter [PV96], gave an output-sensitive $O(k + n \log n)$ algorithm using $O(n)$ space.

Alpert, Koch and Liason [AKL10] did not only define obstacle representations, but also a new parameter that can be associated to any graph, its obstacle number, denoted by $\text{obs}(G)$. It is the minimum number of obstacles needed in any obstacle representation. If the OR has the additional restriction of having only convex obstacles, the convex obstacle number is defined analogously. They give a construction for graphs with arbitrary large (convex) obstacle number and analyze certain small graphs which cannot be represented by a single obstacle. They also completely characterize graphs with convex obstacle number one and graphs that use one line segment obstacle. While outerplanar graphs have obstacle number one and an outside-OR, it is still open, whether any graph of obstacle number one can be represented by an outside obstacle.

Subsequent papers extended the results and answered some open questions raised by Alpert et al. To summarize a few results, Pach and Sarıöz [PS11] gave a construction of several small graphs with obstacle number two and showed that there are bipartite graphs with arbitrarily large obstacle number. Mukkamala, Pach, and Pálvölgyi [MPP11] showed that there are graphs on $n$ vertices with obstacle number at least $\Omega(n/\log n)$.

Some progress on the task of computing the obstacle number has been made, at least for the problem with fixed vertex positions: Sarıöz [S11] gave an efficient approximation algorithm that computes an OR from a given graph drawing $D$ with $O(\text{obs}(D) \log \text{obs}(D))$ obstacles. Later, Johnson and Sarıöz [JS11] showed that computing an OR of a plane graph with the minimum number of obstacles is NP-hard, but admits a polynomial-time approximation scheme (PTAS) and is fixed parameter tractable (FPT). Sarıöz also determined the segment obstacle number of paths and cycles on $n$ vertices in [S12] with and without the restriction that segment obstacles may intersect.

Polygon-vertex visibility graphs form an important subclass of graphs with an outside-OR, which were first introduced in 1983 by Avis and ElGindy [AE83]. Ghosh [G97] formulated a set of necessary conditions for the graph class, which was later augmented by Coullard and Lubiw [CL91], but these have not yet been proven to be sufficient.

Everett and Corneil [EC95] also showed, that there is no finite set of forbidden induced subgraphs in polygon-vertex visibility graphs. So far, characterizations have only been achieved for certain restrictions on the polygons of polygon-vertex visibility graphs. For example, according to [S92], Everett and Corneil [EC90, E90] gave a characterization of visibility graphs in spiral and 2-spiral polygons, – polygons which have exactly one chain (two chains, resp.) of reflex vertices – are characterized as interval graphs and perfect graphs, respectively.

A generalization of these graphs are induced subgraphs of polygon-vertex visibility graphs, or “induced visibility graphs” for short. Spinrad [S03] considers this graph class the natural generalization of polygon-vertex visibility graphs, which is hereditary with respect to induced subgraphs. These can also be represented by an outside-OR.

If a graph is given as a pointset inside a simple polygon, Cheng, Chobrak and Sundaram [CCS00] formulated an algorithm to find crossing-free paths in the graph with a time and space complexity of $O(m^2n^2)$, where $m$ is the number of vertices in the polygon and $n$ the cardinality of the pointset. This was improved by Daescu and Luo [DL08], provided that $n = o(m^2/\log m)$. While finding crossing-free spanning trees, one and two-factors in general geometric graphs is NP-complete, as shown by Jansen and Woeginger [JW93], no research has yet been published on the complexity of finding crossing-free spanning trees, etc. in our more restricted class of visibility graphs with an outside-OR.

## 1.2 Outline

In Chapter 2, we introduce basic notions and preliminaries. We begin by defining obstacle representations, turning later to general visibility notions, such as a visibility polygon of a point and a weak visibility polygon of an edge. Readers familiar to the field of visibility problems may skip this section, or consult it for questions of notation.

In Chapter 3 and 4, I present the main findings of my study thesis. I start with defining "graph-invariant" maps, which allow us to move vertices in a graph drawing without changing the graph, by retaining again an outside-OR. This might provide a tool to transform a graph drawing to show that certain restrictions on the obstacles are possible. The aim of this technique was to show, whether any graph which can be represented by polygonal outside-obstacles, may be transformed into a version that needs only rays to block sights between non-adjacent vertices.

Later we characterize the class of outerplanar graphs which can be represented by a plane outside-OR. We further show necessary conditions of general graphs with a plane outside-OR. On the question whether any planar graph can be represented by an outside-OR, we show only partial results.

The last chapter concludes my research and gives pointers to future work that could not be addressed within the scope of my study thesis.

# 2 Preliminaries

## 2.1 Obstacle Representations

In [AKL10], Alpert et al. defined an *obstacle representation* (OR) of a graph $G$ as a straight-line drawing of $G$ in the plane, where an edge is in $G$ if and only if it does not intersect any obstacle. This representation is given by a pair $(V, O)$, where $V$ is a set of points in the plane and $O$ is the set of obstacles. We write $G(V, O)$ for $G$.

For obstacles, we can choose from several models, resulting in different types of obstacle representations. In [AKL10] two types are defined. A general obstacle, called polygon obstacle here for avoidance of ambiguity, is defined to be a polygon, i. e. a connected union of line segments. For a segment OR, an obstacle is a line segment in the plane. We introduce a third type, so called *ray ORs*, where $O$ is a set of rays, starting at a point and extending infinitely in one direction.

Throughout this document, we assume that points and vertices of obstacles are in *general position*, so no three of them are collinear. For notation, we will denote the segment between points $p_1$ and $p_2$ as $p_1p_2 := \{x \in \mathbb{R}^2 \colon x = \lambda p_1 + (1-\lambda)p_2, \lambda \in [0,1]\}$ and the polygon defined as the union of segments $\bigcup_{i=1}^{n-1} p_ip_{i+1} \cup p_np_1$ as $\mathrm{p}(p_1, \ldots, p_n)$. Furthermore, denote the ray starting at $p_1$ and pointing in direction of $p_2$ as $\mathrm{r}(p_1, p_2) := \{x \in \mathbb{R}^2 \colon x = p_1 + \lambda(p_2 - p_1), \lambda \in \mathbb{R}_{\geq 0}\}$. Let $\mathrm{pointset}(o)$ denote the set of defining points of an obstacle $o$.

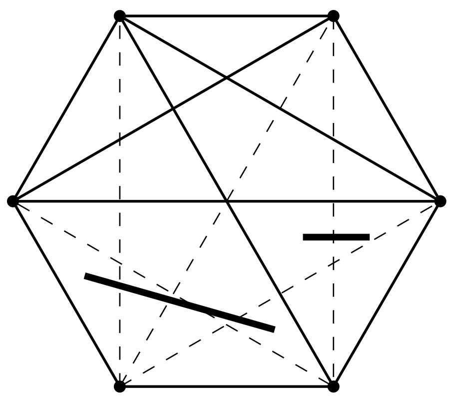

**Figure 2.1.** An example of a segment obstacle representation. The non-edges (dashed) are intersected by segment obstacles (heavy lines).

**Definition 1.** The *obstacle number* of a graph $G$ is the smallest number of obstacles in any obstacle representation of $G$ [AKL10]. The segment obstacle number and ray obstacle number is defined analogously for segment and ray ORs.

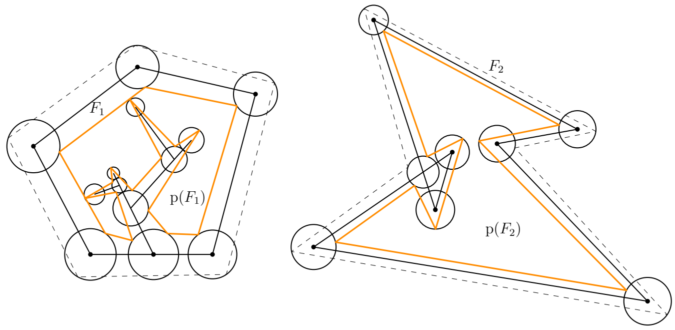


**Figure 2.2.** The face polygon $\mathrm{p}(F_1)$, and the face polygon $\mathrm{p}(F_2)$ of a self-intersecting graph, are displayed in orange. The dashed lines are the polygons corresponding to the outer face.

Let $D_G$ be a drawing of $G$ and consider the plane graph $G(D_G)$ that is constructed from $D_G$ by placing a vertex at any intersection of edges. We call a region of $\mathbb{R}^2 \setminus D_G$, corresponding to a face of $G(D_G)$, a face of $D_G$ or shorter a face of $G$ if no ambiguity is possible.

As points are in general position, given a line segment $pq$, $p, q \in V \cup \mathrm{pointset}(O)$ and a point $r \in V \cup \mathrm{pointset}(O) \setminus \{p, q\}$ arbitrarily, the distance between $\mathrm{span}(pq)$ and $r$ is positive. We set $r_G$ as the minimum of distances of all such points in $(V, O)$.

Each obstacle is contained in exactly one face $f$ of our graph drawing $D_G$. Polygon obstacles can be enlarged to the complete face, so we can assume that each face which is crossed by a non-edge (that is not intersected by polygons in other faces) contains exactly one polygon obstacle. As polygons are closed and faces are open sets, we give a simple construction to shrink (or enlarge in the case of the outside face) the face just a bit and get a closed polygon of a face, denoted by $\mathrm{p}(f)$, which is well-defined. We will call this the *face polygon* of $f$.

The construction is as follows. Draw a circle of radius $r_G$ around any vertices and intersection points on the boundary of $f$, as in Figure 2.2. For each circle determine the point intersection of the angle bisection inside the face and the circle (for intersection points there may be more than one). Exclude points with an angle of $\pi$, as they would unnecessarily increase the complexity of the resulting polygon. By scanning the boundary of the face, connect the points in their order. As the radius of the cirles was small enough, the resulting polygon is simple. Also note that the complexity of the polygon does not increase significantly.

**Definition 2.** As in [AKL10], an *outside-obstacle representation* (outside-OR) of $G$ is an obstacle representation that consists of obstacles in the outer face of the drawing of $G$. Clearly, all rays in a ray OR lie in the outer face.

**Lemma 1.** *1. Graphs with a polygon OR can be represented by a segment OR, and vice versa.*

*2. Every graph with a ray OR has a polygon outside-OR.*

*Proof.* 1. Given a polygon OR $(V, O_p)$ of a graph $G$, we form a segment obstacle set $O_s$ by adding segments $p_i p_{i+1}$ and $p_n p_1$, for each $\text{p}(p_1, \ldots, p_n) \in O_p$. Clearly $(V, O_s)$ is a segment OR of $G$. On the other hand, let $(V, O_s)$ be a segment OR of a graph $G$. For each face $f$ of $D_G$ which contains a segment, construct the polygon $\text{p}(f)$, which fills the face and add it to a set $O_p$. Delete other segments from $O_s$ in this face, to avoid duplication of polygons. Iterate, until $O_s$ is empty. $(V, O_p)$ is now a polygon OR of $G$.

2. Let $f$ be the outer face of $D_G$. $(V, \{\text{p}(f)\})$ is a polygon OR of $G$. □

The missing equivalence of Lemma 1.2 is an open question, which we will look into in section 3.2. The following lemma gives us a way to construct subgraphs of graphs with an OR by intersecting edges with obstacles. It shows, that our graph class is hereditary with respect to subgraph relationship, without having to come up with a new OR from scratch.

**Lemma 2** (removing edges)**.** *Let $(V, O)$ be an obstacle representation of $G$ and $e \in E(G)$ an edge that has an open, non-empty intersection with the border of a face $f$, bordering to $f'$ of $D_G$. Then $G - e$ also has an OR.*

*Proof.* Let $x \in e$ be a point between $f$ and $f'$ and let $U$ be an open set containing $x$, which does not intersect $D_G$ except for $e$. We can place an arbitrarily small segment obstacle in $U$ that intersects only $e$ to obtain an obstacle representation of $G - e$. In particular, if $D_G$ is an outside-OR and $e$ borders to the outside face, $G - e$ also has an outside-OR. See Figure 2.3 for illustration. □

We may be able to construct an OR of $G - e$, without increasing the number of obstacles, by choosing $f$ (and $f'$) along the edge $e$, such that either $f$ or $f'$ already contains an obstacle. Depending on the geometry of the faces, we may be able to change the obstacles in that face, to also intersect with $e$, e. g. if a segment obstacle in either face intersects only one non-edge.

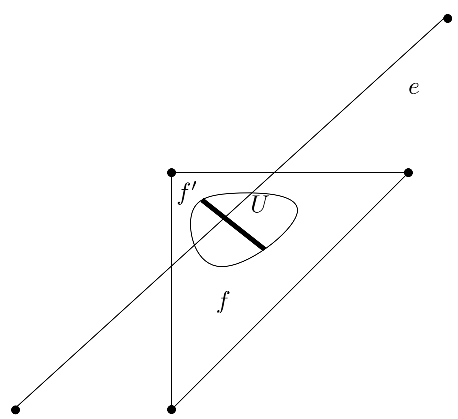


**Figure 2.3.** The heavy line is the additional segment obstacle for $G - e$.

## 2.2 Visibility Polygons of Points and Edges

The definitions and remarks of this section are taken from Ghosh's monograph on Visibility Algorithms in the Plane [G07], which is recommended to anyone with further interest in the field of visibility problems.

**Definition 3.** The *visibility polygon* $\mathrm{vp}(q)$ of a point $q$ in an OR is the set of all points that are visible from $q$, i. e. $\mathrm{vp}(q) = \{p \in \mathbb{R}^2 \colon p \text{ is visible by } q\}$.

Observe that $\mathrm{vp}(q)$ might be unbounded. In the formulation of the art gallery problem, this is the area of sight of a guard, who can see 360 degrees. By definition, $\mathrm{vp}(q)$ is star shaped, with $q$ being in the kernel of $\mathrm{vp}(q)$.

Ghosh [G07] suggests a combination of a preprocessing algorithm of Bhattacharya, Ghosh and Shermer [BGS06] and an algorithm of Lee [L83] or ElGindy and Avis [EA81] to be used to obtain a linear algorithm for computing $\mathrm{vp}(q)$ in a simple polygon $P$. Here, the first algorithm prunes $P$ to remove winding, while the latter one computes $\mathrm{vp}(q)$, if $P$ is non-winding, i. e. the angle subtended at $q$ is less or equal than $2\pi$ while scanning the boundary of $P$. If $P$ is a polygon with $n$ vertices and $h$ holes, Heffernan and Mitchell [HM95] gave an optimal $\Theta(n + h \log h)$ algorithm. Because we can easily compute a bounding box of our graph drawing, we can also use these algorithms in our setting of obstacle representations, in which visibility polygons may be unbounded.

For the visibility of a line segment or an edge, several definitions are possible. For our purposes, the notion of "weak visibility", originally defined by Avis and Toussaint [AT81], fits best. In the context of the art gallery problem it can be seen as a guard who patrols along a line segment in the gallery polygon.

**Definition 4.** Let $P$ be a simple polygon and $pq$ an internal segment in $P$. A point $z \in P$ is said to be *weakly visible* from $pq$, if there exists a point $w \in pq$ (depending on $z$), such that $w$ and $z$ are mutually visible. The set of all points of $P$ weakly visible from $pq$ is called the *weak visibility polygon* of $P$ or $\mathrm{wvp}(P)$ for short. See Figure 2.4 for illustration.

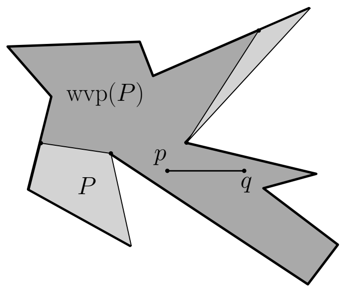


**Figure 2.4.** The weak visibility polygon (dark gray) of $pq$ in $P$.

For the question of computing $\mathrm{wvp}(pq)$, let $P$ be a simple polygon and $pq$ an internal segment in $P$. Guibas et al. [GHLST87] gave an algorithm to compute $\mathrm{wvp}(pq)$ in $O(n)$ time, using a given triangulation of $P$.

There is a characterization of weak visibility using Euclidean shortest paths. The *Euclidean shortest path* from $s$ to $t$ in a polygon $P$ is a length-minimal path connecting $s$ and $t$, which lies completely in $P$. It is denoted by $\mathrm{SP}(s,t)$. Properties of $\mathrm{SP}(s,t)$ are summarized by Ghosh [G07] as follows:

1. $\mathrm{SP}(s,t)$ is not self-intersecting
2. Let $\mathrm{SP}(s,t) = (s,\dots,u,\dots,v,\dots,t)$. Then, $v \in \mathrm{SP}(u,t)$ and $u \in \mathrm{SP}(s,v)$.
3. $\mathrm{SP}(s,t)$ turns only at vertices of $P$.
4. If $P$ is simple, then $\mathrm{SP}(s,t)$ is a unique path in $P$.

In the following, a vertex $p$ of a polygon is *convex* if the interior angle at $p$ is less than $\pi$. A *convex edge* is an edge of two convex vertices.

**Lemma 3.** *[G07, Lemma 3.2.7] Let $v_k v_{k+1}$ be a convex edge of a simple polgon $P$. A vertex $v_i$ of $P$ is visible from some point of $v_k v_{k+1}$ if and only if $\mathrm{SP}(v_k, v_i)$ makes a left turn at every vertex in the path and $\mathrm{SP}(v_{k+1}, v_i)$ makes a right turn at every vertex in the path. See Figure 2.5 for illustration.*

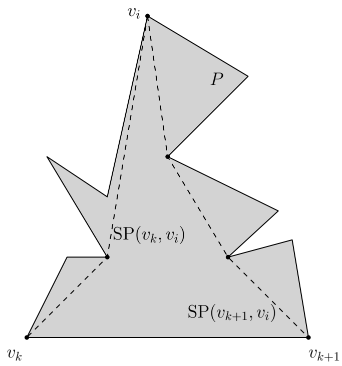


**Figure 2.5.** $v_k v_{k+1}$ is a convex edge in $P$ and the shortest paths make left and right turns according to Lemma 3.

**Theorem 1.** *[G07, Theorem 3.2.13] Let $v_k v_{k+1}$ be a convex edge of a simple polygon $P$. The following statements are equivalent:*

1. *$P$ is weakly visible from $v_k v_{k+1}$.*
2. *For any two vertices $v_i$ and $v_j$ of $P$, where $v_k v_{k+1}$ belongs to $\mathrm{bd}(v_j, v_i)$, $\mathrm{SP}(v_i, v_j)$ passes only through vertices of $\mathrm{bd}(v_i, v_j)$.*
3. *For any two vertices $v_i$ and $v_j$ of $P$, where $v_k v_{k+1}$ belongs to $\mathrm{bd}(v_j, v_i)$, $\mathrm{SP}(v_i, v_j)$ makes a right turn at every vertex in the path.*
4. *For any vertex $v_i$ of $P$, $\mathrm{SP}(v_{k+1}, v_i)$ makes a right turn at every vertex in the path and $\mathrm{SP}(v_k, v_i)$ makes a left turn at every vertex in the path.*

# 3 Ray Obstacle Representations

In this chapter, we study the class of graphs with a ray OR. The motivating question of the chapter is, whether graphs with an outside-OR can also be represented by ray obstacles. For an illustratory example, given a graph with a spiral chain of segment obstacles in the outside face, can we “straighten” it to use linear ray obstacles, as in Figure 3.1. Therefore, we introduce “graph-invariant” regions, in which we can freely move a vertex, without changing the incidences of the graph.

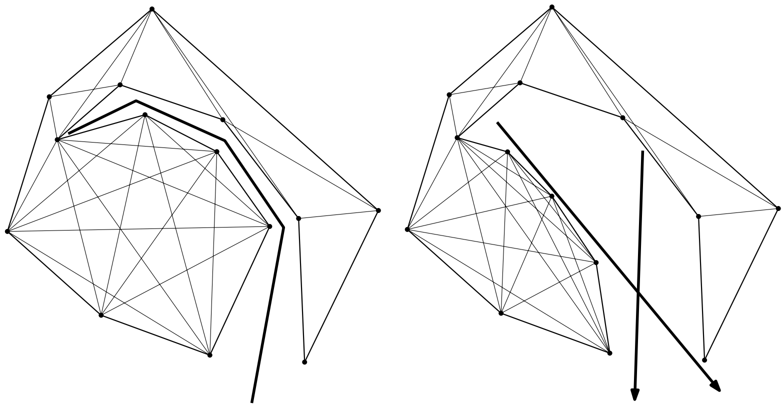

**Figure 3.1.** The graph is given by a segment outside-OR (left) and a ray OR (right), obstacles are represented by heavy lines.

## 3.1 Graph-invariant Maps on Obstacle Representations

**Definition 5.** Let $G$ be a graph with an OR $(V, O)$.

- Let $v \in V$, then $v_{(x,y)}$ denotes $v$ with new position $(x, y) \in \mathbb{R}^2$. Then $F_{v\to(x,y)}$ is the corresponding map of ORs, which maps $v$ to $v_{(x,y)}$ and leaves everything else fixed. Also $G_{v\to(x,y)} := G(F_{v\to(x,y)}((V, O))$.
- Let $o = \mathrm{p}(p_1, \dots, p_n) \in O$, then $o_{i\colon (x,y)}$ denotes $o$ with new position of $p_i$ at $(x, y) \in \mathbb{R}^2$. Analogously, $F_{o\to i\colon (x,y)}$ is the map of ORs, which maps $o$ to $o_{i\colon (x,y)}$ and leaves everything else fixed. Also $G_{o\to i\colon (x,y)} := G(F_{o\to i\colon (x,y)}((V, O))$. In the case of segment or ray obstacles, $i = 1, 2$ correspond to the first and second component of the obstacle's definition.

**Definition 6** (vertex invariance region)**.** Let $G$ be a graph with an OR $(V, O)$, $v \in V$. We define the *invariance region* $I(v)$ of a vertex $v \in V$ under translation as the set $M \subseteq \mathbb{R}^2$, such that $uv \in E(G) \Leftrightarrow uv \in E(G_{v \to (x,y)})$ for all $(x, y) \in M$ and $u \in V \setminus \{v\}$.

Additionaly the *strong (or outside-preserving) invariance region* $I_s(v)$ of $v$ is defined to be the set $M \subseteq I(v)$, such that obstacles that are outside in $(V, O)$, remain outside in $F_{v \to (x,y)}((V, O))$.

**Definition 7** (obstacle invariance region)**.** Let $G$ be a graph with an OR $(V, O)$, $o \in O$, $i \in \mathbb{N}$. We define the *invariance region* $I^i(o)$ of an obstacle $o$ under translation of the $i$th component as the set $M \subseteq \mathbb{R}^2$, such that $E(G) = E(G_{o \to i\colon (x,y)})$ for all $(x, y) \in M$. Analog to the previous definition, the strong invariance region $I_s^i(o)$ has the additional property that $o$ remains outside if it was outside in $(V, O)$.

Let $G$ be a graph with a OR $(V, O)$, and $v \in V$. The invariance region of $v$ is the area, in which we can move $v$ without changing the adjacencies of the graph. These may change if we move a vertex out of the visibility polygon of an adjacent vertex, or if we move it into a visibility polygon of a previously non-adjacent vertex. Hence,

$$I(v) = \left( \bigcap_{u \in N(v)} \mathrm{vp}(u) \right) \setminus \left( \bigcup_{u \in V \setminus N(v)} \mathrm{vp}(u) \right).$$

**Lemma 4.** *Let $G$ be a graph with an OR $(V, O)$, and $uv$, $xy$ two crossing edges in $G$. Let $w \in \{u, v, x, y\}$ and $p\colon [0, 1] \to I(w)$ be a continuous path in $I(w)$, starting at $p(0) = w$. If $p$ crosses the edge that does not contain $w$, then $w$ is adjacent to both vertices of this edge. See Figure 3.2 for illustration.*

*Proof.* Assume w.l.o.g. $w = v$. For any point $p(a)$, $a \in [0, 1]$ on the path, the line segment $up(a)$ does not cross any obstacle, as otherwise $p(a)$ would not be in $I(v)$. Furthermore, if a vertex is visible from $p(a)$, then $v$ is adjacent to it. In a crossing position $p((0, 1]) \cap xy$ an obstacle which blocks sight to a vertex of $xy$ cannot lie between $x$ and $y$, as they are adjacent. Therefore, $v$ is adjacent to $x$ and $y$. ☐

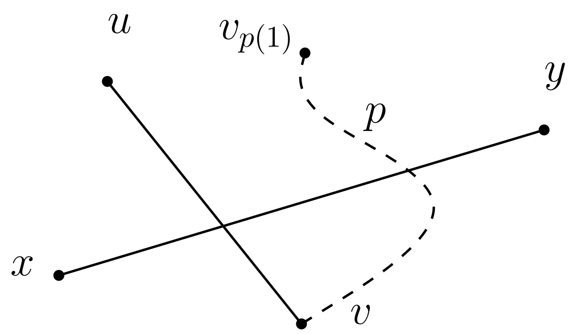


**Figure 3.2.** Vertex $v$ is moved along continuous path $p$ in $I(v)$ and crosses the edge $xy$. Therefore, it will be adjacent to $x$ and $y$ (not in the picture).

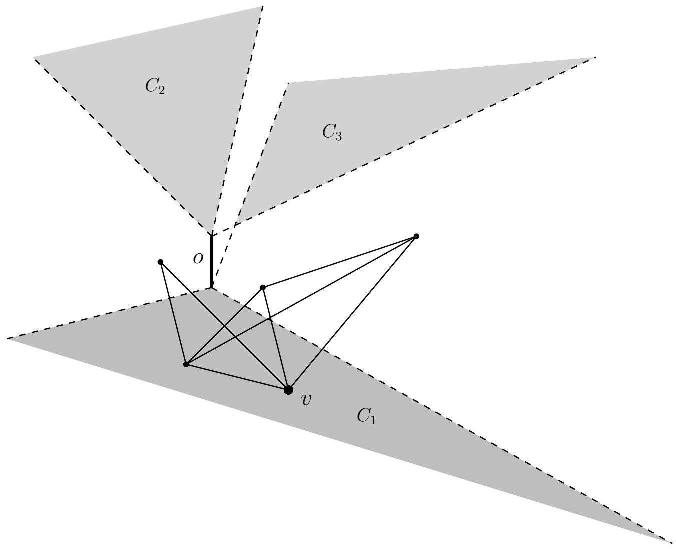


**Figure 3.3.** The graph-invariant region $I(v)$ in grey. The connected component $C_1$ is $I_s(v)$, as $o$ is only outside for positions of $v$ in $C_1$.

**Proposition 1.** *Let $G$ be a graph with a segment outside-OR $(V, O)$ and $v \in V$. Then $I(v)$ decomposes into connected components $C_1, \dots, C_k$. If a point $p \in C_i, i \in \{1, \dots, k\}$ satisfies $p \in I_s(v)$ then $C_i \subseteq I_s(v)$. Put differently, satisfying the property of being outside preserving is constant if restricted to a connected component.*

*Proof.* Let $C$ be a connected component of $I(v)$, $v \in C$ and $v' = v_{(x,y)}$ with $(x, y) \in C$. Assume that there is an obstacle $o \in O$ that will no longer be in the outside face in $G' := F_{v \to v'}(G)$. Let $f$ be the face containing $o$ in $D_{G'}$. There are at least three edges on the boundary of $f$, one of them incident and one non-incident to $v'$. Let $\tilde{G} = G - v$ be the graph of $(V \setminus \{v\}, O)$. Then $o$ is outside of $D_{\tilde{G}}$, as the drawing of $\tilde{G}$ is the fixed part while $v$ is moved in $G$. Let $p\colon [0, 1] \to C$ be a continuous path in $C$, going from $p(0) = v$ to $p(1) = v'$.

First, consider the case that $o$ is enclosed in a region of $D_{G'}$ (not necessarily the face $o$ is contained in) bounded by two edges incident to $v'$ and necessary edges of $D_{\tilde{G}}$, as in Figure 3.4, left. Because $D_{\tilde{G}}$ does not change, it is clear that there is an intermediate configuration on $p$ such that a point of $o$ lies on either of the two bounding edges incident to $v$. But this is not possible, as the edge would be intersected by $o$.

Therefore consider the second case, as in Figure 3.4, right. Then, there is exactly one incident edge $e = uv'$ on $f$. Let $R_{p,u} := \{x \in \mathbb{R}^2 : x \in up(a), a \in [0, 1]\}$, $u \in N(v)$ be the area that is touched by the edge $uv$ when $v$ traverses $p$. We show that any point that is not in the outside face of $G$ and not in $R_{p,u}$, is also not outside in $G'$. For this, let $q \in \mathbb{R}^2$ be an inner point in $G$ but not contained in $R_{p,u}$. We can safely assume that $q$ is not inside in $\tilde{G}$, but in a face that has an edge $e \in E(\tilde{G})$ on its border.

Assume that $q$ is outside in $G'$. Then, either i) $v$ crossed an edge $e$ on the border of $f$ or ii) a vertex $z$ of an edge $e$ on $f$ is in $R_{p,u}$. There is no other possibility to “open” a face,

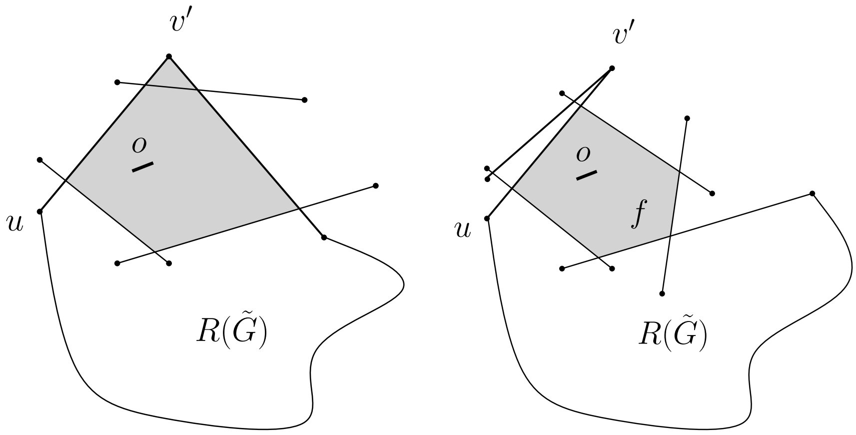


**Figure 3.4.** Case 1 on the left: $o$ is enclosed in $D_{\tilde{G}}$ and a triangle with tip $v'$, and case 2 on the right: $o$ is enclosed in $D_{\tilde{G}}$ and a single edge incident to $v'$.

because the topology of $f$ will not change, if no point lies on an edge in an intermediary position.

For i), it follows by Lemma 4 that $v$ is adjacent to both vertices of $e = xy$, and hence the first case applies, as $q$ is enclosed in a region bounded by $uv$ and $ux$ or $uy$ and $D_{\tilde{G}}$. For ii), it follows by Lemma 4 that $u$ is adjacent to $z$ and therefore $q$ is inside in $D_{\tilde{G}}$, a contradiction.

Therefore, a point on $o$ that is outside in $G'$ but inside in $G$ has to be in $R_{p,u}$. In the intermediate configuration corresponding to a point in $R_{p,u}$, $v$ does not see $u$. $\square$

In the next section, we try to find a way to transform a graph with an outside-OR into a graph with a ray OR. For ease of analysis, we restrict ourselves to continuous movements of vertices only. To justify this a bit, I guessed that there is a simple criterion, when $I_s(v)$ is connected. However, 2-connectedness of the graph does not imply connectedness of $I_s(v)$ as can be seen in Figure 3.5. Therefore, there might be non-continuous transformations of the graph that we did not include in our analysis.

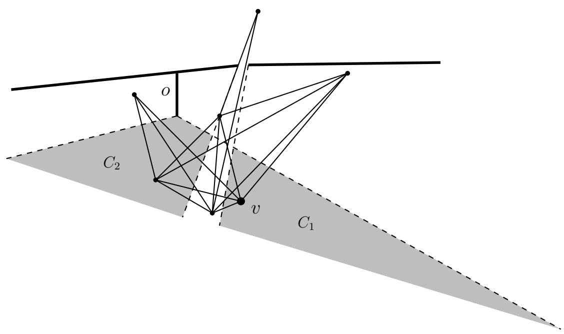


**Figure 3.5.** Although the graph is 2-connected, it does not have a connected $I_s(v)$.

## 3.2 Ray Obstacle Representations

At first, we present a condition of a given drawing of $G$ which is equivalent to the existence of a ray OR. For this, we use weak visibility polygons as defined in Chapter 2.

Let $G$ be a graph with a polygon or segment outside obstacle representation $(V, O)$ and $D_G$ its drawing. In this section, we will try to find an answer to the question, whether every graph representable by a polygon or segment outside-OR, can also be represented by a ray OR.

**Lemma 5.** *Let $P_i$ be the polygon defined $v_i v_{i+1}$ and the path from $v_i$ to $v_{i+1}$ in $G$ that uses only edges on the boundary of $D_G$ which are not on the boundary of the convex hull of $V$. There is a ray OR $(V, O')$ of a given drawing $D_G$, if and only if for any non-edge $e$, there is an $i$ such that $e$ has an open intersection with $P_i$ and with the weak visibility polygon of $v_i v_{i+1}$ in $P_i$.*

*Proof.* We show the proof by assuming $(V, O)$ to be a segment outside-OR. There are no non-edges and therefore w.l.o.g. no segment obstacles outside the convex hull of $V$, so it suffices to look at the $P_i$ defined above. Therefore, fix an $i$ and assume the condition holds for $P_i$. Let $e$ be a non-edge of $G$ with an open intersection with $P_i$. Let $z \in e \cap \mathrm{wvp}(P_i)$, which exists due to our condition. By definition of weak visibility we find a $w \in v_i v_{i+1}$, such that $z$ and $w$ see each other. Introduce a ray obstacle $r_e = \mathrm{r}(z, w)$, which intersects $e$. Repeat this for all non-edges which are not already intersected by previously inserted rays.

Now, let $(V, O)$ be a ray OR of $G$ and assume there is a non-edge $e$ which does not intersect $\mathrm{wvp}(P_i)$ for any $i$. Hence, there is no straight line starting from an arbitrary point $z \in e \cap P_i$ to a point of $v_i v_{i+1}$. A ray obstacle to intersect $e$ cannot be in $P_i$. □

Remark that the number of ray obstacles may increase, as we can have segment obstacles which intersect more than one non-edge and cannot be replaced by a single ray obstacle.

The property can be strengthened a bit by assuming that $P_i = \mathrm{wvp}(P_i)$ for all non-edges $v_i v_{i+1}$ on boundary of the convex hull of $V$. For drawings with this stricter property the outer boundary of our graph is called *weakly externally visible polygon*, as defined in [G07, Section 3.7]. As all pairs of non-adjacent vertices of $P_i$ are non-edges in $G$, this strengthening of our condition concerns only previously non-weakly-visible parts of single triangles or convex parts of the boundary polygon consisting of intersecting edges.

Our criteria developed above becomes more useful by the characterization of weak visibility polygons with Euclidean shortest paths, as seen in chapter 2. We will look at the situation that a vertex of $P$ is not in the weak visibility polygon of $v_k v_{k+1}$, therefore a drawing which might not be represented by ray obstacles without moving vertices.

**Lemma 6** (cf. Lemma 3)**.** *Let $v_k v_{k+1}$ be a convex edge of a simple polygon $P$ and $v_i$ be a vertex that is not weakly visible from $v_k v_{k+1}$. Then there is a vertex $v_r$, such that $\mathrm{SP}(v_k, v_i)$ makes a right turn at $v_r$ or there is vertex $v_l$ such that $\mathrm{SP}(v_{k+1}, v_i)$ makes a left turn at $v_l$.*

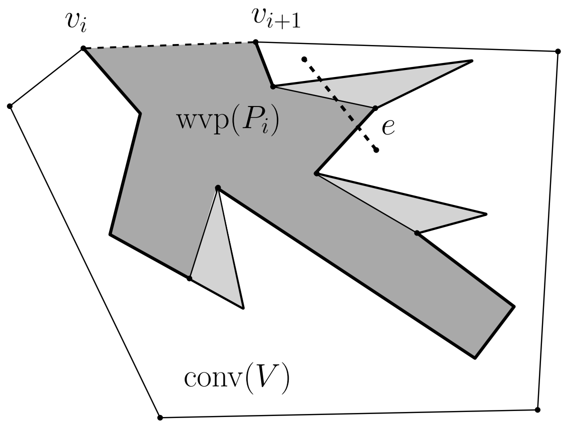


**Figure 3.6.** The non-edge $e$ has an open intersection with $P_i$ and wvp($P_i$) and thus our condition is valid.

Looking (w.l.o.g.) first at $sp_0 := \mathrm{SP}(v_{k+1}, v_i)$ only on $\mathrm{bd}(v_{k+1}, v_i)$, i. e. the convex hull of $\mathrm{bd}(v_{k+1}, v_i)$, as we ignore $\mathrm{bd}(v_k, v_i)$, the path on the boundary between $v_k$ and $v_i$. In the situation of Lemma 6, $\mathrm{bd}(v_k, v_i)$ crosses $sp_0$. We will therefore try to transform $\mathrm{bd}(v_k, v_i)$ to eliminate these intersections. Later, we change the roles of $\mathrm{bd}(v_k, v_i)$ and $\mathrm{bd}(v_{k+1}, v_i)$.

To transform the vertices on the boundary, we may have to move vertices outside of their invariance regions and adjust other vertices whose adjacencies would be altered by the move afterwards. See Figure 3.7, for an illustration of the delta in visibility polygons. Any vertices in this region have to be moved of this region, possibly starting a process of subsequent moves. The difficulty of this technique for our main question is that we have to show that this chain of moves is acyclic. This is not obvious and still open.

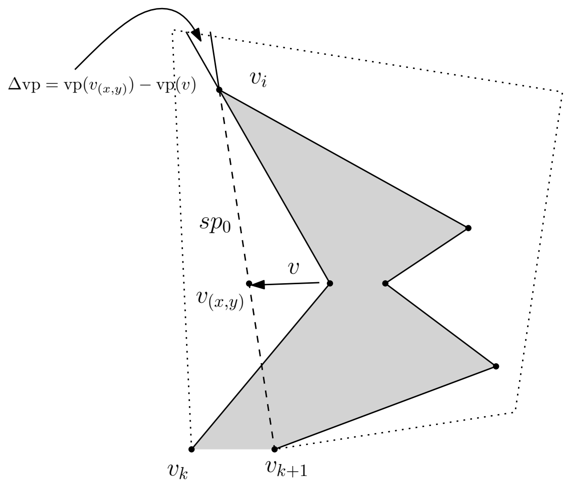


**Figure 3.7.** The dashed line $sp_0$ crosses the left part of the boundary. Therefore moving $v$ transformes the boundary of the outer face into an externaly visible polygon, with possibly non-empty $\Delta\,\text{vp}(v)$.

# 4 Plane Outside-Obstacle Representations

An obstacle representation (of one of the three types defined above) is called *plane*, if the drawing $D_G$ of $(V, O)$, without the obstacles, is a plane graph. In this section, we try to characterize the class of graphs with a plane outside-OR. The first proposition allows us to restrict our analysis to a certain type of *chordal plane* graphs, which we define as plane graphs which are chordal, with the restriction that chords have to be contained inside their cycle.

**Definition 8.** A chordal graph $G$ is called *inner-chordal*, if every cycle of length larger than three, has an inner chord, i. e. a chord that is contained in the region enclosed by the cycle. See Figure 4.1 for illustration.

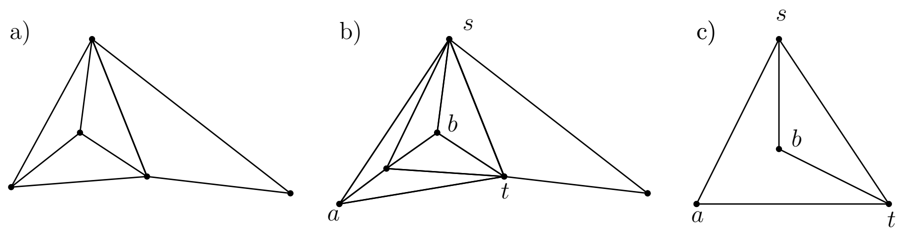


**Figure 4.1.** Example of a) an inner-chordal graph and b), c) chordal graphs with an outside chord $st$ of cycle $a, s, b, t$.

**Proposition 2.** *Graphs with a plane outside-OR are inner-chordal plane graphs.*

*Proof.* Let $G$ be a graph with a plane outside-OR $D_G$. Assume that $G$ is not chordal plane, i. e. that it has a cycle $C_n$ of length $n \geq 4$ without a chord. Because of the existence of a plane outside-OR, all non-edges have an open intersection with the outer face. In particular, this holds for all pairs of vertices of distance two on $C_n$. Consider a chain of neigbouring vertices $v_1, \ldots, v_4 \in C_n$. Here, the non-edges $\{v_1, v_3\}$ and $\{v_2, v_4\}$ have an open intersection with the outside face and the edges do not cross, therefore $v_2$, $v_3$ is the same side of $\{v_1, v_3\}$ and $\{v_2, v_4\}$, respectively. Thus, the chain is in convex position, see Figure 4.2. As it was chosen arbitrarily, $C_n$ is a convex polygon and all non-edges are completely contained in the inner face, a contradiction.

Assume that all chords of a cycle $C_n$ of length $n \geq 4$ are embedded outside of $C_n$. We create a subgraph of $G$ by intersecting all outer chords of $C_n$ with an outside obstacle (cf. Lemma 2), without intersecting $C_n$ itself. The resulting graph is also a graph with a plane outside-OR, but is not chordal, a contradiction. Therefore, our graph class is also inner-chordal. □

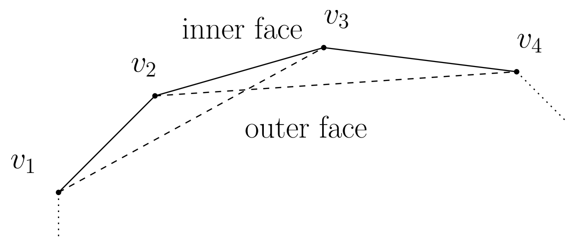


**Figure 4.2.** Four neighboring vertices form a convex chain, as non-edges (dashed) have to be in the outside face.

For our characterization, it suffices to restrict our graph class to biconnected graphs, as the different 2-connected components can be embedded separately and then put together by arbitrary scaling and rotation. Note that the graph generated by contracting the 2-connected components to single vertices, is acyclic, as otherwise there would be a second path, contradicting a connectivity of less than 2.

Any chordal outerplane graph is also inner-chordal, as it is inner-triangulated. Remark that inner-chordal graphs are hereditary to removing outside edges (cf. Lemma 2), as we cannot remove necessary chords with outside obstacles, without intersecting the cycle of the chord. As our analysis will look at both, plane and outerplane graphs, we give a characterization of inner-chordal graphs via a construction method. If we find a way to embed these graphs with a plane outside-OR during every step of the construction, we would be able to show a characterization of the class.

For the proof of the following lemma, we use a well-known characterization of chordal graphs. Let $G = (V, E)$ be a graph. $G$ is chordal, if and only if it has a *perfect elimination ordering*, i.e. there is an ordering $\{v_1, \ldots, v_n\}$ of $V$, such that $v_i$ is simplicial in the subgraph induced by $\{v_1, \ldots, v_i\}$. A vertex $v$ is *simplicial* if the subgraph induced by $v$ and its neighbors, is a clique. Note that we first show a construction for chordal plane graphs and then change it to a construction for inner-chordal plane graphs.

**Lemma 7** (Chordal plane graphs)**.** *Biconnected chordal plane graphs are exactly the graphs which are given by the following construction. We start with a single triangle $G_1 = (V_1, E_1)$ with $V_1 = \{v_1, v_2, v_3\}$, and $E_1 = \mathcal{P}(V_1)$. Given a graph $G_i = (V_i, E_i)$ at construction step $i$, we obtain $G_{i+1}$ by either*

1. *placing the tip $w$ of a new triangle that shares an existing edge $e = \{u, v\} \in E_i$, so that $\{u, w\}$ and $\{v, w\}$ do not cross any edges in $E_i$. Then, $V_{i+1} = V_i \cup \{w\}$ and $E_{i+1} = E_i \cup \{u, w\} \cup \{v, w\}$, or*

2. *replacing a triangle with a plane 4-clique by adding a center vertex $c$ in a given triangle $\{u, v, w\}$ and connect it to the surrounding vertices. Then $V_{i+1} = V_i \cup \{c\}$ and $E_{i+1} = E_i \cup \{u, c\} \cup \{v, c\} \cup \{w, c\}$.*

*Proof.* We first show that graphs generated by this construction are biconnected chordal plane graphs. For the induction start, the triangle embedded in general position is a biconnected chordal plane. Let $G_i$ be a biconnected chordal plane graph. As we apply the first rule, by construction, $G_{i+1}$ is plane and also no $n$-cycle of cardinality larger than three not having a chord is generated. Because $w$ is on a cycle, it is also 2-connected and as we do not remove any edges, $G_{i+1}$ is biconnected. If we apply the second rule, $G_{i+1}$ is also plane and and no larger cycle is generated.

For the other direction, we have to show that given any chordal plane graph $G$, there is a way to construct it using the rules above. Hence, let $G = (V, E)$ be chordal plane and $\{v_1, \ldots, v_n\}$ a perfect elimination ordering of $G$. $G$ does not contain any simplicial vertices of degree four, because this would result in a 5-clique, which is not planar. Also, as $G$ is biconnected, only the penultimate simplicial vertex has degree one, and will not be looked at here, because our construction starts with a triangle. Thus, in order to construct $G$ from the ordering, we start with the triangle of $v_1, v_2, v_3$ and continue with vertex $v_4$ as below.

We have already seen that all simplicial vertices in our ordering starting at $v_4$ are of degree two and three. By chordality, simplicial vertices of degree two have to be vertices of triangles, and simlicial vertices of degree three are inner vertices of plane 4-cliques. Our perfect elimination ordering gives us a way to embed the graph by rules 1 and 2. If $v_i$ is a simplicial vertex of degree 2, apply rule 1 and connect it to its neighborhood accordingly. If it is of degree 3, apply rule 2 and place it inside the triangle forming its neighborhood. □

**Proposition 3** (Inner-chordal plane graphs)**.** *Biconnected inner-chordal plane graphs are exactly the graphs which are given by the following construction. We start with a single triangle $G_1 = (V_1, E_1)$ with $V_1 = \{v_1, v_2, v_3\}$, and $E_1 = \mathcal{P}(V_1)$. Given a graph $G_i = (V_i, E_i)$ at construction step $i$, we obtain $G_{i+1}$ by either*

1′. *placing the tip $w$ of a new triangle that shares an existing* outside *edge $e = \{u, v\} \in E_i$, so that $\{u, w\}$ and $\{v, w\}$ do not cross any edges in $E_i$.* Additionally, $w$ has to be outside. *Then, $V_{i+1} = V_i \cup \{w\}$ and $E_{i+1} = E_i \cup \{u, w\} \cup \{v, w\}$, or*

2′. *replacing a triangle* that was created by rule 1′ *with a plane 4-clique by adding a center vertex $c$ in a given triangle $\{u, v, w\}$ and connect it to the surrounding vertices. Then $V_{i+1} = V_i \cup \{c\}$ and $E_{i+1} = E_i \cup \{u, c\} \cup \{v, c\} \cup \{w, c\}$.*

*For outerplane embeddings, only the first rule is used. This construction of biconnected inner-chordal plane graphs can be conducted such that first only rule 1′ and afterwards only rule 2′ is used. Note that this results in the same graphs as replacing triangles by plane 4-cliques in a chordal outerplane graph.*

*Proof.* We first show that the generated graphs are biconnected inner-chordal plane graphs. Rule 1′ can be implemented by rule 1, with extra caution on where to put $w$, and rule 2′ can be implemented by rule 2, with extra caution to not use it on triangles that have been generated by rule 2 itself, as this would create a stacked triangle configuration as in Figure 4.1b). Therefore, the resulting graphs are biconnected chordal plane. It remains to show inner-chordality. The starting triangle is obviously inner-chordal. Let $G_i$ be a biconnected inner-chordal plane graph. As we apply rule 1′, in $G_{i+1}$ attaching a triangle, any newly generated $n$-cycle have to use both vertices of the outer edge, which is an inner chord. As rule $2'$ can only be used on triangles where all vertices are outside vertices, adding a center vertex inside the triangle to create a plane 4-clique does not create an outer chord in $G_{i+1}$. For outerplane embeddings this holds as well, because applying rule $1'$ does not change outerplanarity.

For the other direction, we have to show that given any inner-chordal (outer)plane graph $G$, there is a way to embed it using the rules above. For the easier case, let $G$ be a chordal outerplane graph with a weak dual $T$. Because of biconnectivity, $T$ is a tree. Along $T$ by application of rule $1'$, $G$ can be generated.

Now let $G = (V, E)$ be inner-chordal plane and $\{v_1, \dots, v_n\}$ a perfect elimination ordering of $G$. As in the proof of the preceeding lemma, all simplicial vertices in our ordering starting at $v_4$ are of degree 2 and 3. By inner-chordality, simplicial vertices of degree 2 have to be outside vertices of triangles, and simlicial vertices of degree three are inner vertices. Our perfect elimination ordering gives us a way to embed the graph by rules $1'$ and $2'$. If $v_i$ is a simplicial vertex of degree 2, apply rule $1'$, place it outside and connect it to its neighborhood accordingly. If it is of degree 3, apply rule $2'$ and place it inside the triangle forming its neighborhood.

Because we cannot attach any triangle with rule $1'$ to inner edges of 4-cliques generated by rule $2'$, we can find a perfect elimination ordering, such that first only simplicial degree 2-vertices and afterwards degree three-vertices are listed. Hence, we can first construct the chordal outerplane skeleton of $G$ with rule $1'$ and afterwards apply rule $2'$ to generate plane 4-cliques. □

The following corollary gives our final form of the construction, which will be used in the proofs of Section 4.2.

**Corollary 1.** *Note that instead of rule $2'$ in Proposition 3 we could also introduce an alternative rule $2''$, yielding the same graphs, if we additionally allow to use a plane 4-clique as starting point $G_1$.*

$2''$. *Placing the tip $w$ of a new plane 4-clique with center $c_w$ that shares an existing outside edge $e = \{u, v\} \in E_i$, so that $\{u, w\}$ and $\{v, w\}$ do not cross any edges in $E_i$. Additionally, $w$ has to be outside. Then, $V_{i+1} = V_i \cup \{w, c_w\}$ and $E_{i+1} = E_i \cup \{u, w\} \cup \{v, w\} \cup \{u, c_w\} \cup \{v, c_w\} \cup \{w, c_w\}$.*

## 4.1 Embedding Outerplanar Graphs with a Plane Outside-OR

Our characterization of graphs with a plane outside-OR will start with a restriction to the simpler class of outerplanar graphs. Their description will yield an easy characterization, which can be extended to get necessary conditions for the full class.

The class of outerplanar graphs with a plane outside-OR is equivalent to the outerplanar graphs with an outerplane outside-OR, therefore it suffices to look at outerplane outside-ORs only. This is true, because of chordality and the fact that an inner vertex sees at least three surrounding vertices. This would create a clique with cardinality of at least 4, contradicting outerplanarity of the graph.

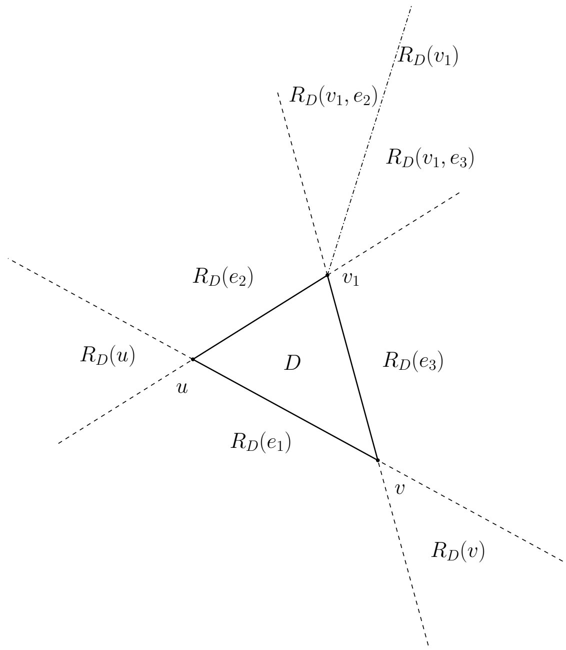


**Figure 4.3.** The regions of a triangle $D$.

As discussed before, we now look in detail at outerplanar graphs with a plane outside-OR. Obviously, the triangle is such a graph. Given a biconnected chordal outerplane graph $G_i$ with a plane outside-OR, is there a way to place the tip $w$ of a new triangle, sharing an edge $e$, such that the resulting obstacle representation is still an outside-OR?

A necessary condition is that the non-edge between $w$ and the opposite tip of the neigboring triangle, has an (open) intersection with the outside face of our drawing and hence can be intersected by an outside obstacle of the graph.

Before we proceed, we introduce some notation to describe possible positions for the tip of our new triangle. See Figure 4.3 for an illustration of the following definitions.

**Definition 9.** Let $D$ be a triangle in the plane, with edges $e_i$ $(i = 1, 2, 3)$. Let $l(D)$ be the set of lines $l(e_i) = \text{span}(e_i)$ defined by the edges of $D$. As points are in general position, these lines partition $\mathbb{R}^2 - (\bigcup_{i=1}^{3} l(e_i))$ into seven open regions. Let $R$ be such a region and define a mapping $\sigma_R\colon l(D) \to \{+, -\}$ which maps $l \in l(D)$ to $+$, if $R$ lies on the side of $l$ containing $D$ and $-$ otherwise. We call $\sigma_R$ a signature of $R$ with respect to $D$.

For a vertex $x \in V(D)$, define $R_D(x)$ to be the region with $\sigma_R(l(e)) = -$ for incident edges $e \in E(D)$ of $x$. For an edge $e \in E(D)$, define $R_D(e)$ to be the region with $\sigma_R(l(e)) = -$ and $\sigma_R(l(\tilde{e})) = +$ for all $\tilde{e} \in E(D) \setminus \{e\}$. For the inside of $D$, called $R(D)$, it is $\sigma_R(l(\tilde{e})) = +$ for all $e \in E(D)$ by definition of $\sigma_R$. This defines seven disjoint regions. The region with $\sigma_R(l(\tilde{e})) = -$ for all $e \in E(D)$ is empty.

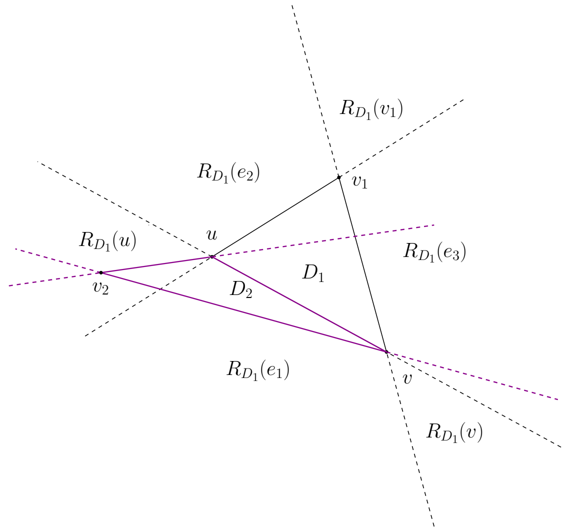


**Figure 4.4.** Embedding a new triangle $D_2$ attached to $D_1$.

**Definition 10.** Let $D$ be a triangle and $e_1, e_2 \in E(D)$ incident to a vertex $x \in V(D)$. We partition the region $R_D(x)$ into two regions, called $R_D(x, e_i)$ by deleting the angle bisection of $e_1$ and $e_2$ from $R_D(x)$, whereas $R_D(x, e_i)$ is on the side of $e_i$, $i = 1, 2$.

**Lemma 8** (region symmetry)**.** *Let $D_1$, $D_2$ be triangles in the plane with vertex sets $V(D_1) = \{v_1, u, v\}$ and $V(D_2) = \{v_2, u, v\}$. $D_1$ and $D_2$ share an edge $e = \{u, v\}$. Then $v_2 \in R_{D_1}(w) \Leftrightarrow v_1 \in R_{D_2}(w)$ for $w \in e$ and $v_i \in R_{D_j}(v_j) \Leftrightarrow v_j \in D_i$ for $i, j \in \{1, 2\}, i \neq j$. This means, that our definition of the regions of two neighbouring triangles exhibits a certain symmetry.*

*Proof.* Let $w \in e \in E(D_1) \cap E(D_2)$ and $v_2 \in R_{D_1}(w)$. Observe that $v_1$ is on the side of $l(e)$ which does not contain $D_2$. Furthermore, $v_1$ is on the side of $l(\{w, v_2\})$ not containing $D_2$, because $l(\{w, v_2\})$ and $l(\{w, v_1\})$ cross at $w$ and $v_2$ is on the side of $l(\{w, v_1\})$ not containing $D_1$. By definition of $R_{D_2}(w)$ and symmetry the first equivalence holds.

Let w.l.o.g. be $v_2 \in R_{D_1}(v_1)$. Then $v_1$ is on the side of $l(\{u, v_2\})$ and $l(\{v, v_2\})$ containing $D_2$. Furthermore $v_1$ is on the same side of $l(e)$ as $D_2$. By symmetry the second equivalence follows. □

**Lemma 9.** *Let $D_1$, $D_2$ be triangles in the plane with vertex sets $V(D_1) = \{v_1, u, v\}$ and $V(D_2) = \{v_2, u, v\}$. $D_1$ and $D_2$ share an edge $e = \{u, v\}$ and assume that $v_1, v_2$ is not contained in $D_2, D_1$, respectively. Then the following statements are equivalent:*

1. *The drawing of $D_1$ and $D_2$ is a plane outside-OR.*
2. $v_2 \in R_{D_1}(u) \cup R_{D_1}(v)$.
3. $v_1 \in R_{D_2}(u) \cup R_{D_2}(v)$.

*In particular, if we construct the graph starting from $D_1$, then $v_2$ can be chosen to be in $R_{D_1}(u, e) \cup R_{D_1}(v, e)$.*

*Proof.* We first show "1 ⇔ 2" by verifing or falsifying the property for each of the seven regions defined above. For ease of notation call the edges $e_1 = \{u, v\}, e_2 = \{u, v_1\}, e_3 = \{v, v_1\}$ (see Figure 4.4 for an illustration). By assumption, $v_2$ is not inside $D_1$ and neither in $R_{D_1}(v_1)$, as then $v_1$ would be inside $D_2$. If $v_2 \in R_D(e_2)$ then $\{v, v_2\}$ crosses $e_2$, because $v$ and $v_2$ are on opposite sides of $l(e_2)$, but inside the cone of $l(e_1)$ and $l(e_3)$ containing $D_1$ and thus cannot cross $l(e_i)$ to the left or right of $e_i$. This is a contradiction to the planarity of the embedding. The case $v_2 \in R_D(e_3)$ follows by symmetry. If $v_2 \in R_D(e_1)$, then $\{v_1, v_2\}$ crosses $e_1$, so $v_1$ and $v_2$ are not adjacent. This non-edge is completely contained inside both triangles so there is no outside obstacle which can intersect it, a contradiction.

The remaining case is $v_2 \in R_{D_1}(u) \cup R_{D_1}(v)$. By symmetry we can assume w.l.o.g. $v_2 \in R_{D_1}(u)$. The edge $\{v_1, v_2\}$ does not cross $l(e_2)$ and lies completely on the side of $l(e_2)$ opposite to the triangle, thus on the outside face. Both, the graph with $\{v_1, v_2\}$ and without it, have a plane outside-OR.

The equivalence "2 ⇔ 3" follows directly from the preceding Lemma 8. □

Given that we have a construction of any biconnected chordal outerplane graph, such that we can add a new triangle sharing an edge with a triangle of a given graph, i. e. $R_{D_1}(u, e) \cup R_{D_1}(v, e)$ contains enough (outer) space to contain the tip $v_2$, and that all non-edges intersect the outside face, we will have a complete characterization of outerplanar graphs with a plane outside-OR. We will show this in Proposition 4. Before, we argue that it suffices to look at maximal outerplanar graphs in the following Lemma 10.

**Lemma 10.** *Chordal outerplanar graphs are obtained from a maximal outerplanar supergraph by subsequent deletion of outer edges. Thus, if a maximal outerplanar graph has an outside-OR, then so has any chordal outerplanar subgraph.*

*Proof.* Assume that there is an inner edge, that needs to be removed, in order to get a chordal outerplanar graph. This inner edge divides two faces with each at least two other edges on the border. Removal of the edge would generate a cycle of length at least 4 without a chord as a boundary of an inside face. This contradicts the definition of chordality in outerplane graphs. □

**Proposition 4.** *Outerplanar graphs with a plane outside-OR are exactly the chordal outerplanar graphs.*

*Proof.* Proposition 2 shows the first direction. For the opposite direction, because of Lemma 10. it suffices to show that maximal outerplanar graphs have a plane outside-OR. Maximal outerplanar graphs are constructed as biconnected chordal outerplane graphs with rule $1'$ of Proposition 3. It suffices to show that we can embed the new triangle as in Lemma 9, to retain a plane outside-OR for the graph. We do this by giving a construction in which the following invariant is preserved: for each edge $e = \{u, v\}$ on exactly one triangle $D$, bordering to the unbounded face, one of $R_D(u,e), R_D(v,e)$ does not contain any vertices or edges of $G$.

Our proof is by induction on the number $n$ of vertices. For $n = 3$ the invariant holds trivially, as all regions are empty. Furthermore, $G$ has a plane outside-OR. Let $G$ be a maximal outerplane graph of order $n$ and assume that the induction hypothesis holds for $G$. Let $e = \{u, v\}$ be an edge of $G$ on the unbounded face on a triangle and let w.l.o.g. $R(u,e) \cap G = \emptyset$.

We embed a new triangle $\tilde{D}$ sharing the edge $e$ by placing a new vertex $y$ in $R_D(u,e)$ such that $R_{\tilde{D}}(y) \subset R_D(u,e)$. This holds if we place $y$ on the line through $v$ which is parallel to the angle bisection of $D$ at $u$, see Figure 4.5 for illustration. As $y \in R_D(u,e)$, our embedding is a plane outside-OR if we add an obstacle intersecting the non-edge of $y$ and the opposite vertex of $D$.

It remains to show that if there is an edge $e' = \{u, w\} \in D' \neq D$ on the outside face (for which there might be an intersection of $R_{D'}(u,e')$ with our new triangle) $R_{D'}(w,e')$ does not contain any edges or vertices of $G$. This holds true, because by our construction, all triangles in the sequence of edges $(e_i)_{i=0,\dots,k}$, starting from the edge $e_0 \in D = D_0$, $e_0 \neq e$ incident to $u$, to $e_k = e' \in D_k$ added to $D_{i-1}$ were embedded so that the region $R_{D_i}(x, e_i)$ $(x \in e_i \setminus \{u\})$ allowed further embedding of $D_{i+1}$, and hence also for $e' = e_k$ in $R_{D_k}(w, e_k)$. ☐

## 4.2 Embedding Graphs with a Plane Outside-OR

For the larger class of general graphs with a plane outside-OR, we allow a plane 4-clique in our construction. Again, we first introduce notation for the different regions around a plane 4-clique.

**Definition 11.** Let $\Delta$ be a plane 4-clique with center $c$, $x \in V(\Delta) \setminus \{c\}$ and $e_1, e_2$ incident edges of $x$ in $\Delta$. Let $D_i$ be the inner triangle of $\Delta$ sharing edge $e_i$. We define $R_c(x, e_1) := R_{D_2}(x)$ and $R_c(x, e_2) := R_{D_1}(x)$. *Note the change of indices.*

In order to attach a 4-clique to another 4-clique, analog to Lemma 9, certain conditions have to be obeyed.

**Lemma 11.** *Let $\Delta_1$, $\Delta_2$ be plane 4-cliques with vertex sets $V(\Delta_1) = \{v_1, u, v, c_1\}$ and $V(\Delta_2) = \{v_2, u, v, c_2\}$ and center $c_1$, $c_2$, respectively. $\Delta_1$ and $\Delta_2$ share an edge $e = \{u, v\}$ and assume that $v_1, v_2$ is not contained in $\Delta_2, \Delta_1$, respectively. Then the following statements are equivalent (cf. Figure 4.6 for notation):*

1. *The drawing of $\Delta_1$ and $\Delta_2$ is a plane outside-OR.*
2. *$c_2, v_2 \in R_{c_1}(u,e) \cup R_{c_1}(v,e)$ and $c_2$ is inside $\Delta_2$.*
3. *$c_1, v_1 \in R_{c_2}(u,e) \cup R_{c_2}(v,e)$ and $c_1$ is inside $\Delta_1$.*

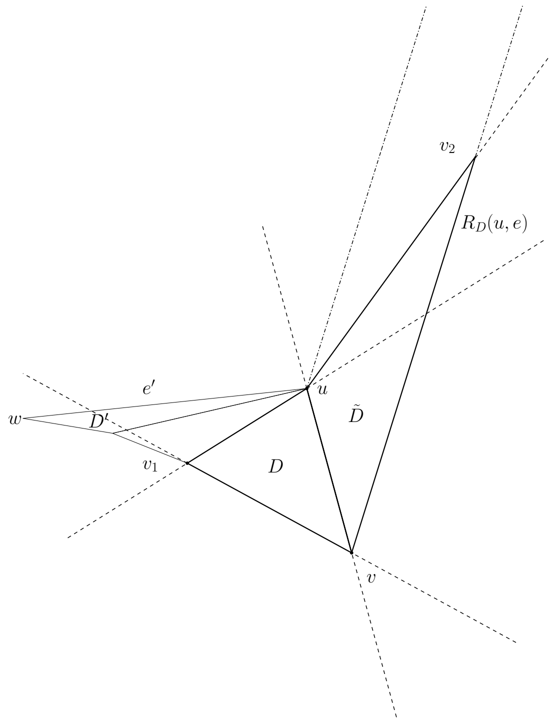


**Figure 4.5.** The construction in the proof of Proposition 4 with added triangle $\tilde{D}$, here the dash-dotted lines are parallel.

*Proof.* Let $D$ be the triangle with $V(D) = \{c_1, u, v\}$, sharing the edge $e = \{u, v\}$ and consider $D_2$, the triangle of $\Delta_2$ without the center point $c_2$. To be a plane outside-OR, we have to place $v_2$ in $R_D(u) \cup R_D(v) = R_{c_1}(u, e) \cup R_{c_1}(v, e)$ by Lemma 9. This region is disconnected, but as $\{u, v, c_2\}$ form a triangle sharing edge $e$, $c_2$ has to be in the same connected componend in order to avoid crossing the boundary of $D_2$. The equivalence of 2) and 3) follows from symmetry by Lemma 8. □

Of course, instead of $\Delta_2$, we can also add only a triangle $D_2$ to $\Delta_1$.

**Corollary 2.** *Let $\Delta_1$ be a plane 4-clique and $D_2$ be triangle, with vertex sets $V(\Delta_1) = \{v_1, u, v, c_1\}$ and $V(D_2) = \{v_2, u, v\}$ and center $c_1$. $\Delta_1$ and $D_2$ share an edge $e = \{u, v\}$ and assume that $v_1, v_2$ is not contained in $D_2, \Delta_1$, respectively. Then the following statements are equivalent:*

1. *The drawing of $\Delta_1$ and $D_2$ is a plane outside-OR.*
2. *$v_2 \in R_{c_1}(u, e) \cup R_{c_1}(v, e)$.*
3. *$c_1, v_1 \in R_{D_2}(u) \cup R_{D_2}(v)$ and $c_1$ is inside $\Delta_1$.*

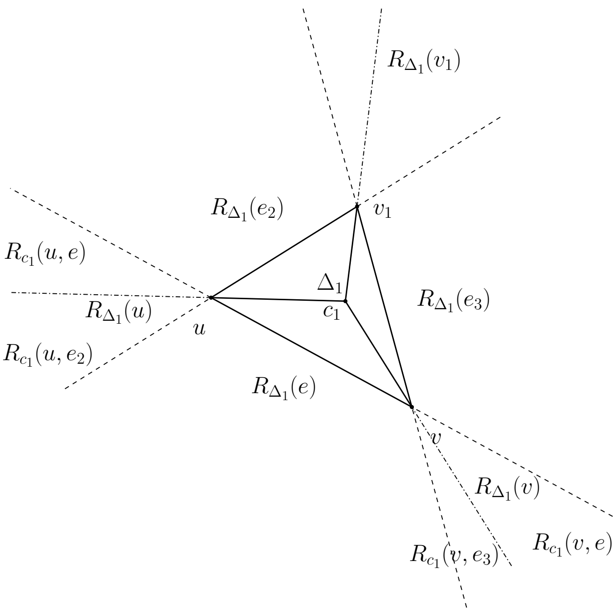


**Figure 4.6.** The regions of a plane 4-clique $\Delta_1$ with center $c_1$.

In the following we try to characterize graphs with plane outside-OR as chordal outerplane graphs, in which we can replace triangles with plane 4-cliques, provided that certain conditions hold. We introduce the following notation: Let $G$ be an inner-chordal plane graph, then $T(G)$ denotes the weak dual of $G$ after removing the inner vertices. In $T(G)$, nodes which had its inner vertex removed, are marked, they represent the plane 4-cliques. $T(G)$ is a tree with maximal degree 3, as after removal of inner vertices, $G$ is chordal outerplane. For a given subset $C$ of vertices in a graph, we denote $\overline{C} := C \cup N(C)$ as the vertices of $C$, together with their neighbors.

**Lemma 12.** *Let $G$ be an inner-chordal plane graph. Let $C_i$, $i \in \{1, \ldots, n\}$ be the connected components of marked vertices in $T(G)$. If $G$ has a plane outside-OR, the following holds:*

1. *$\overline{C_j}$ does contain at most one degree 3-vertex for any $j \in \{1, \ldots, n\}$.*
2. *If $\overline{C_j}, \overline{C_i}$ have degree 3-vertices, $\overline{C_j}$ is disjoint $\overline{C_i}$ for $j \neq i \in \{1, \ldots, n\}$.*

*Proof.* We first note that, besides mirroring, there is no other possibility to embed a plane 4-clique that shares edges with three other plane 4-cliques, than that of Figure 4.7 left, also if the outer 4-cliques are replaced by triangles. Given this configuration and one of the outside cliques $\Delta$, observe that both of their outside edges contain only one convex vertex (at the tip $w$ of $\Delta$), the other vertices have an outside angle of less than $\pi$. Therefore, if we attach a plane 4-clique to one of these edges, assume $uw$, the tip $w$ reduces its outside angle to less than $\pi$ and no other plane 4-clique or triangle can be attached to $\Delta$. As the outside angle at $u$ was further reduced, the newly attached

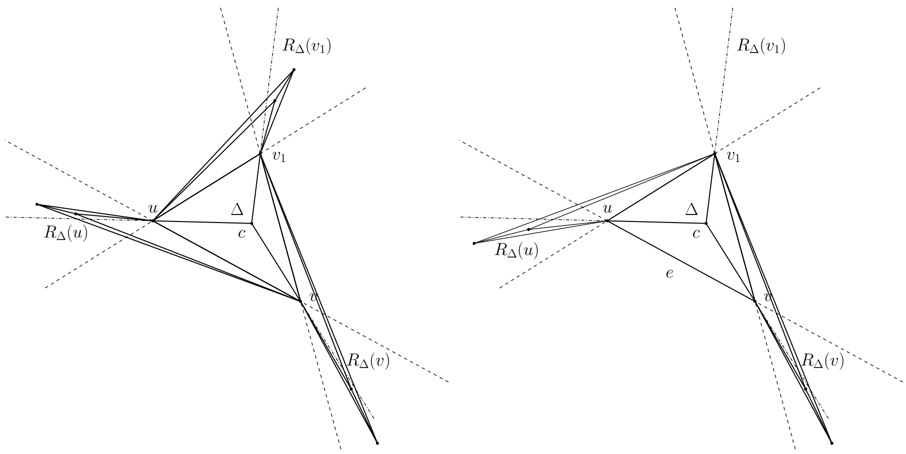


**Figure 4.7.** Two examples of a graph with a plane outside-OR. On the right side, no additional triangle can be attached to the edge $e$.

4-clique retain the invariant of having only one convex outside vertex, as $u$ and $w$ are not convex. By induction, if we only attach plane 4-cliques, we cannot create another 4-clique that is surrounded by three other 4-cliques, or maximally one 4-clique and two triangles.

For the second property, see Figure 4.8 and note that on the left side, the connected components are disjoint and embeddable, while on the right side, they intersect and the configuration can not be embedded. In both cases, the first property holds. This is because the triangle in the intersection of the components has contradictory conditions of placing the tip, from both components. □

Due to time constraints, we can as yet, only formulate the sufficiency of our necessary conditions as the the following conjecture. We have not yet come up with a good invariant of an embedding which would ensure that when adding a new triangle or 4-clique, we have "enough space" to do so while obeying the conditions of Lemma 9, 11 and Corallary 2.

**Conjecture 1.** *Graphs with a plane outside-OR are exactly the inner-chordal plane graphs which satisfy the necessary condition of Lemma 12.*

## 4.3 Planar Graphs with an Outside-OR

In [AKL10] the question is posed, whether planar graphs do always have an outside-OR. As for outerplanar graphs, this is true, we were guessing, that planar graphs with an outside-OR have a treewidth of 2, or at least bounded treewidth. In the following we show, that this is wrong. See Figure 4.9 for a forbidden minor of tree-width 3, which can be represented by an outside-OR. We may be able to show an even stronger result that grid graphs can be represented with an outside-OR. Two-dimensional *grid graphs* are defined as the product of two path graphs. However, due to time constraints on the thesis, our possiby correct construction and formal proof might be given afterwards.

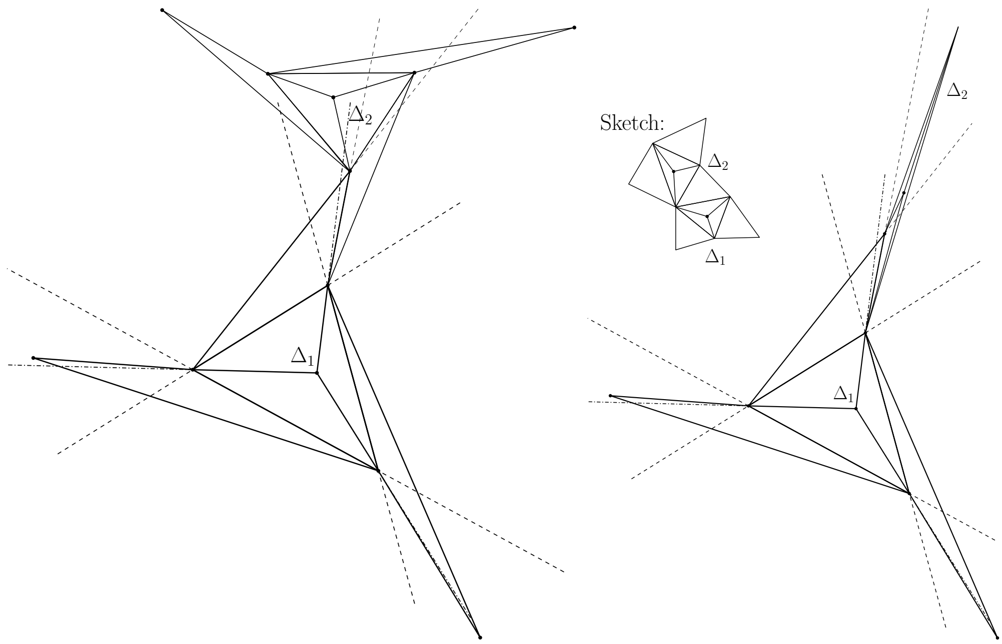


**Figure 4.8.** Left: Two degree three nodes, but not in the same connected component, right: this configuration is not embeddable via a plane outside-OR, because at the vertex shared by $\Delta_1$ and $\Delta_2$ the outside angle is less than $\pi$.

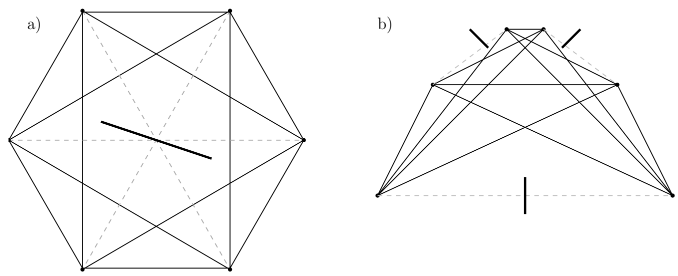


**Figure 4.9.** The octahedral graph as a) a plane 1-segment OR, b) a 3-segment outside-OR.

# 5 Conclusion

Throughout this thesis, several results about graphs with a representation by outside obstacles are shown. The newly introduced type of ray obstacles was compared to previously defined obstacle representations, though it remains open, whether any polygon outside-OR can be transformed into a ray OR. Furthermore, the outerplanar graphs with a plane outside-OR were characterized in terms of chordal outerplane graphs. This result might be extended to a characterization of the general class, but within the scope of this thesis only a set of necessary conditions could be shown so far.

## 5.1 Future Work

While it is still unknown whether a full combinatorial characterization of polygon-vertex visibility graphs exists, the same question can be asked for induced visibility graphs and our even larger class of graphs represented by outside obstacles. As far as I know, it is also open whether induced visibility graphs form a proper subclass of our graphs with a polygon outside-OR. This could be answered negative, if via graph-invariant vertex moves, an inner vertex can be moved outside, without moving outer vertices inside. Without the additional condition of retaining outsideness of outer vertices, one could show that graphs with an outside-OR are hereditary with respect to induced subgraph relationship.

It would be also interesting to analyze segment obstacle number and to obtain some results on this in relation to ray obstacle number and polygon complexity. Also concerning obstacle numbers, it would be interesting to adopt an algorithm of [S11] to approximate the segment obstacle number of a given graph drawing. Throughout this thesis, obstacles were allowed to intersect. In addition to the question of plane obstacle representations, we could as well ask about non-crossing obstacles.

While there are theoretical results on the problem of "point set embeddability" for outerplanar and planar graph, it would be interesting to look at this problem for graphs with a (plane) outside-OR.

For our specific embedding of graphs with a plane outside-OR, it would be nice to analyze properties, like the resolution or the segment obstacle number. There are probably ways to improve the embedding concerning these properties.

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