Karlsruhe Institute of Technology

On Collinearity of Independent Sets – Algorithmic Challenges

Master Thesis of

Almut Demel

At the Department of Informatics
Institute of Theoretical Computer Science

Reviewers:	Prof. Dr. Dorothea Wagner
	Prof. Dr. Peter Sanders
Advisors:	Dr. Tamara Mchedlidze
	Marcel Radermacher, M.Sc.

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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Abstract

A subset $S \subset V(G)$ of vertices of a planar graph G is called *collinear* if G admits a plane straight-line drawing where all the vertices in S lie on a line. The aim of this thesis is to find a lower bound for the maximal number of collinear vertices in graphs of a certain graph class. Our approach is to examine whether independence or a fixed minimal distance between vertices in a subset of vertices is a sufficient condition for collinearity. An independent set of vertices with minimal pairwise distance d is called *d-scattered*. The size of maximum independent sets is linear in the number of vertices in planar graphs [Die17]. However, in general planar graphs the number of collinear vertices is limited by a sublinear upper bound [RV11]. Thus we examine specific graph classes, in particular series-parallel graphs and 4-connected triangulations. For the former we show that any fixed minimal pairwise distance between vertices of an independent set is not a sufficient condition for collinearity. Though we introduce a meaningful subclass of series-parallel graphs, in which 3-scattered sets are always collinear. For 4-connected triangulations we show that independent sets are in general not collinear and present forbidden substructures. We further present approaches to prove collinearity of d-scattered sets and outline the limits of these approaches.

Deutsche Zusammenfassung

Eine Teilmenge $S \subset V(G)$ von Knoten eines planaren Graphen G heißt *kollinear*, wenn G kreuzungsfrei und geradlinig so gezeichnet werden kann, dass die Knoten S auf einer Linie liegen. Das Ziel dieser Arbeit ist es, eine untere Schranke für die maximale Anzahl kollinearer Knoten in Graphen einer bestimmten Graphklasse zu finden. Es wird untersucht, ob Unabhängigkeit oder eine feste minimale paarweise Distanz zwischen Knoten einer Menge ausreicht, um Kollinearität zu garantieren. Knotenmengen mit minimaler paarweiser Distanz d heißen *d-gestreut*. Die Größe maximaler unabhängiger Mengen in planaren Graphen ist linear in der Anzahl Knoten [Die17]. In allgemeinen planaren Graphen unterliegt die Anzahl kollinearer Knoten jedoch einer sublinearen oberen Schranke [RV11]. Daher untersuchen wir Subklassen planarer Graphen, insbesondere serien-parallele Graphen und vierfach-zusammenhängende Triangulierungen. Für erstere zeigen wir, dass kein d existiert, sodass d-gestreute Knotenmengen immer kollinear sind. Wir definieren jedoch eine bedeutende Unterklasse serien-paralleler Graphen, sodass 3-gestreute Knotenmengen in Graphen dieser Klasse kollinear sind. Für vierfach-zusammenhängende Triangulierungen zeigen wir ebenfalls, dass Unabhängigkeit keine hinreichende Bedingung für Kollinearität ist und stellen Teilgraphen vor, die in vierfach-zusammenhängenden Triangulierungen mit unabhängigen kollinearen Knotenmengen nicht vorkommen. Des Weiteren stellen wir zwei Ansätze vor, mit deren Hilfe Kollinearität von d-gestreuten Knotenmengen gezeigt werden könnte, jedoch auch die Grenzen derselben.

Contents

1. Introduction

A subset $S \subseteq V(G)$ of vertices of a planar graph G is called an *alignable* or *collinear set* if G admits a plane straight-line drawing where all the vertices in S lie on a line L. The aim of this thesis is to find a lower bound for the maximal number of collinear vertices in graphs depending on the graph class.

Da Lozzo et al. [DLDF+18] give such a lower bound for planar graphs of treewidth k. They show that every such graph has a collinear set of size in $\Omega(k^2)$. Further they show that every n-vertex 3-connected cubic graph has a collinear set of size at least $\lceil n/4 \rceil$.

For general planar graphs a linear lower bound is not possible since Ravsky et al. [RV11] prove an upper bound on the number of collinear vertices in straight-line drawings of general planar graphs. They construct a class of triangulations $\mathcal{T}$ such that the number of collinear vertices in a drawing of a graph $T \in \mathcal{T}$ is in $O(|V(T)|^{\alpha})$. The parameter α is bounded by the shortness exponent of the dual $\overline{T}$ of T, i.e. the limit inferior of quotients $\log c(\overline{T})/\log |V(\overline{T})|$ over all $T \in \mathcal{T}$, where $c(\overline{T})$ denotes the length of the longest simple cycle in $\overline{T}$. By construction of T, $\overline{T}$ is cubic and 3-connected and thus $\alpha < 0.986$ [GW73]. Therefore, the number of collinear vertices in $T \in \mathcal{T}$ is in $\mathcal{O}(|V(T)|^{0.986})$. According to Da Lozzo et al. [DLDF+18] this sublinear upper bound is still true if the treewidth of the graphs is bounded by 5.

A stronger version of collinear sets are *free collinear sets*. A set $R \subseteq V(G)$ is called *free collinear set* if a total order $<_R$ of R exists such that, given any set of $|R|$ points on a line L, the graph G has a plane straight-line drawing where the vertices in R are mapped to the given points and their order on L matches the order $<_R$ [DLDF+18]. Evidently every free collinear set of a graph G is a collinear set of G. Hence, the size of a free collinear set is a lower bound to the size of a maximal collinear set in the same graph. Dujmovic

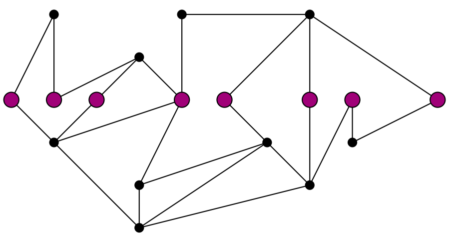


Figure 1.1: A planar graph G with a collinear set S (violet).

et al. [Duj15] prove that every planar graph has a free collinear set of size at least $\sqrt{n/2}$. Thus, the same lower bound applies to collinear sets. For outerplanar graphs this bound improves to $(n+1)/2$ [RV11] and for planar graphs of treewidth at most two even to $n/30$. Da Lozzo et al [DLDF$^+$18] extend the latter result to planar graphs of treewidth at most three by proving a lower bound of $\lceil (n-3)/8) \rceil$ for free collinear sets.

The impact of collinear sets on other graph drawing problems is significantly increased by a recent result by Dujmovic et al. [DFG$^+$18]. They show that a set $S \subseteq V(G)$ is a collinear set if and only if it is a *free set.* A set S is called free if for any set of points X in the plane with $|X| = |S|$, G has a plane straight-line drawing in which the vertices of S are mapped to the points in X. The equivalence of free sets and free collinear sets was known before [BDH$^+$09]. Dujmovic et al. close the gap between collinear and free collinear sets.

Free sets have a wide range of applications to other graph drawing problems, e.g. *untangling* and *n-universal point subsets.* To untangle the (not necessarily crossing-free) straight-line drawing of a planar graph means to change the coordinates of some of its vertices such that in the resulting straight-line drawing no two edges cross [Duj15]. Bose et al. [BDH$^+$09] prove that every planar graph can be untangled while keeping $n^{1/4}$ of its vertices fixed. Although for some graph classes there exist tighter bounds, others can be improved with the new result on collinear and free sets. Bose et al. [BDH$^+$09] as well as Ravsky et al. [RV11] prove that if S is a free collinear set of a planar graph G, every straight-line drawing of G can be untangled while keeping $\Omega(\sqrt{|S|})$ vertices fixed. Together with the above mentioned results on collinear sets, this improves the lower bound for 3-connected cubic graphs from $\Omega(n^{1/4})$ to $\Omega(\sqrt{n})$ and for graphs with treewidth at least k to $\Omega(k)$.

An *n-universal point set* is a set of points P in $\mathbb{R}^2$ such that every planar graph on n vertices admits a plane straight-line embedding on P. Fraysseix et al. [DFPP90] show that a grid of $(2n-3) \times (n-1)$ points in the plane is enough to draw every planar graph. A stronger statement was proven by Cardinal et al. [CHK15]. They show that there exist n-universal point sets for all planar graphs on at most ten vertices. Conversely they show that there are no n-universal point sets of size n for planar graphs on 15 or more vertices. The gap in between 10 and 15 vertices remains to be closed. However, Kurowski [Kur04] proves that for sufficiently large n a n-universal point set has size at least $1.235n$.

Concerning graph sizes which do not admit an n-universal point set of size n, the following question arises. What is the largest natural number σ such that every graph in a collection of σ planar graphs on n vertices admits a plane straight-line drawing on the same set P of n points? Cardinal et al. [CHK15] call this *simultaneous geometric embedding without mapping.* For $n = 35$ they describe a collection of 7393 planar graphs which cannot be drawn on any common point set P of size 35. Thus, $\sigma < 7393$ for $n = 35$.

The problem of *universal point subsets* considers a set of points S of size k such that every planar graph, or every graph of a specific graph class, admits a plane straight-line drawing with k of its vertices represented by the points of S. Angelini et al. [ABE$^+$12] show that there exist universal point subsets of size $\lceil \sqrt{n} \rceil$ for planar graphs on n vertices. Dujmovic et al. [Duj15] show that every set P of at most $\sqrt{n/2}$ points in the plane is a universal point subset for all n-vertex planar graphs. In addition they show that if a graph G has a free set of size k, every set of k points in the plane is a universal point subset for G. The new result by Dujmovic et al. [DFG$^+$18] shows that if S is a collinear set for a graph G, then every set of $|S|$ points in the plane is a universal point subset for G. This is relevant for 3-connected cubic planar graphs, for which the lower bound on universal point subsets is improved from $\Omega(\sqrt{n})$ to size $\lceil n/4 \rceil$.

The proof of the equivalence of collinear and free collinear sets [DFG$^+$18] uses a concept introduced by Da Lozzo et al. [DLDF$^+$18]. An open and simple curve λ is called *good* for a graph G if for each edge e of G, λ either entirely contains e or has at most one point in common with e (if λ passes through an end-vertex of e, that counts as a common point). The curve λ is called *proper* if both end-points are on the outer face of G. A planar graph G has a plane straight-line drawing with vertices $S \subset V(G)$ collinear if and only if G has a proper good curve that passes through the vertices S. This is particularly helpful, since it enables us to examine whether a set of vertices can be aligned without considering their actual coordinates in a drawing. Mchedlidze et al. [MRR18] call proper good curves *pseudolines* and we will use this notion in this thesis. They show that determining whether a pseudoline passing through a given set of vertices exists or not is $\mathcal{NP}$-complete but fixed-parameter tractable.

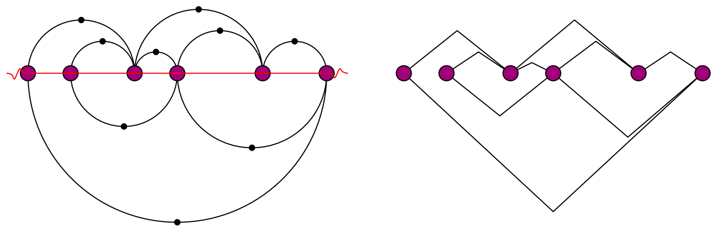
Drawings with bends

Instead of looking at pure straight-line drawings, Kaufmann and Wiese [KW99] consider drawings with a limited number of bends per edge. They show that every planar 4-connected triangulation T admits a plane drawing with at most one bend per edge such that the n vertices of T can be assigned to any n points in the plane. They give an algorithm using the fact that every 4-connected triangulation T admits an external hamiltonian cycle C, i.e. a hamiltonian cycle with an edge on the outer face. The vertices are assigned to the points in the order of C and a bend is introduced to the edge $e \in C$ on the outer face and every edge $E(T) \setminus E(C)$ such that the drawing is crossing free.

General planar triangulations are only 3-connected. Thus, they may contain separating triangles and therefore are not hamiltonian. Consider an edge $e = (u, v)$ of a separating triangle. On either side of e there is a triangular face. One face is bounded by e and the edges (u, w_1) and (v, w_1), the other by e and (u, w_2) and (v, w_2). Kaufmann and Wiese place a dummy vertex z on e and connect it to w_1 and w_2. This decreases the number of separating triangles. They repeat this operation until no further separating triangle is contained in the triangulation. Thus, the triangulation becomes 4-connected. With the above described procedure, they place all vertices on a line, including the dummy vertices. Removing the dummy vertices and replacing them by a bend gives a planar drawing with at most three bends per edge. They further describe a procedure to reduce the number of bends to two per edge.

The above introduced concept of pseudolines gives rise to an alternative proof of the result by Kaufmann and Wiese [KW99], transferring results on *book embeddings* to drawings with bends. In a book embedding the vertices of a graph G are placed on a line called the *spine* of the book. Its edges are drawn on the *pages* of the book. A page can be thought of as a half-plane bounded by the spine where the edges are drawn as circular arcs between their endpoints. A graph admits a *k-page book embedding* if all of its edges can be assigned to k pages and there exists a linear ordering of the vertices on the spine such that no two edges of the same page cross [BK79]. Book embeddings restrict every edge to one page. In *topological* book embeddings the edges are allowed to cross the spine, and thus spread over several pages.

Consider a 2-page topological book embedding of a graph G and identify the spine of the book with a line L on the plane and the two pages with the half-planes on either side of the line. Every edge is cut into curve segments by a crossing of the spine. Insert a dummy vertex in the middle of every curve segment, i.e. if an edge crosses the spine twice insert three dummy vertices. This obtains the graph G'. Then L is a pseudoline with respect to G' passing exactly the vertices $V(G) \subset V(G')$ and crosses edges incident to dummy vertices only. Hence, G' admits a planar straight line drawing with $V(G)$ collinear. Removing the dummy vertices and replacing them by a bend of the respective edge at the same position yields a drawing of G with one bend per edge and an additional bend of



(a) A 2-page book embedding of a graph G.

(b) Drawing of G with one bend per edge and all $V(G)$ collinear.

Figure 1.2: From book embedding to one-bend drawing. The pseudoline L is colored in red.

an edge for every crossing of the spine. Di Giacomo et al. [DGDLW05] prove that every planar graph admits a 2-page topological book embedding with at most one crossing of the spine. Thus, every planar graph admits a drawing with all of its vertices collinear and at most two bends per edge. A 2-page topological book embedding without any crossing of the spine is a 2-page book embedding. See Fig. 1.2 for an example. Bernhart and Kainen [BK79] show, that a planar graph G admits a 2-page book embedding if and only if it is sub-hamiltonian, i.e. it is a subgraph of a planar graph admitting a hamiltonian cyle. Thus sub-hamiltonian planar graphs can be drawn with all vertices on a line and at most one bend per edge. Together with the equality of collinear and free vertex sets [DFG$^+$18], this reaffirms the results by Wiese and Kaufmann. Note that we examine two (sub-)hamiltonian graph classes in Chapters 2 and 3: Every series-parallel graph admits a 2-page book embedding [CLR87] and, as stated before, 4-connected planar triangulations are hamiltonian [Tut56].

1.1 Contribution

We aim to find lower bounds on the number of collinear vertices in specific graph classes. Our approach is to examine properties of a collinear vertex set instead of counting vertices. A collinear set always induces a linear forest, i.e. a induced sequence of paths. In general, we cannot expect a planar graph to have a long induced path since Di Giacomo et al. [GLM16] show that the length of the longest induced path is in $\mathcal{O}(\log(n))$. Thus, we investigate whether there are large independent sets which are collinear. Following from the four color theorem [Die17] and the pigeonhole principle, every planar graph has an independent set of at least one fourth of its vertices. Hence, if independent sets of graphs in a graph class are collinear this implies a linear lower bound on the number of collinear vertices in these graphs. In particular we study independent sets of series-parallel graphs and 4-connected triangulations. For both graph classes we provide a counterexample and thereby show that in general independent sets cannot be aligned in these graph classes. For series-parallel graphs we show that even increasing the pairwise distance of vertices of an independent set to any constant distance is not sufficient for collinearity. We restrict the graph class to series-parallel graphs where the source and sink of every component in parallel composition have distance at least 4. We call this graph class *4-stretched series-parallel graphs* and show that independent sets with vertices of pairwise distance at least three are collinear. For 4-connected triangulations we describe some approaches to prove collinearity of independent sets with higher pairwise distance and the limits of the approaches. We then describe the graph class of 2-sided near-triangulations, a superclass of 4-connected triangulations, and a strongly restricted (not 4-connected) subclass of 2-sided near-triangulations. We prove that independent sets are collinear in the latter.

In the remainder of this chapter we give some preliminary definitions and results. In the following Chapter 2 we present our results on series-parallel graphs. Chapter 3 considers 4-connected triangulations and 2-sided near-triangulations. Chapter 4 gives a summary and conclusion to this thesis.

1.2 Preliminaries

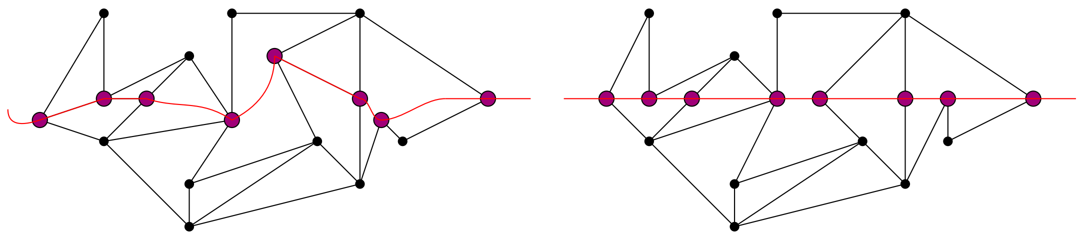
In the following, we introduce concepts and notation used throughout the thesis. The definitions are oriented towards the book on graph theory by Diestel [Die17].

A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq V^2$ and every $e \in E$ is a two-element subset. We call V the *vertices* and E the *edges* of G. A *subgraph* of G is a pair $G_{\text{sub}} = (V_{\text{sub}}, E_{\text{sub}})$ such that $V_{\text{sub}} \subset V$ and $E_{\text{sub}} \subseteq V_{\text{sub}}^2 \cap E$. We say G contains G_{sub}. Furthermore, a subgraph G_{ind} is called *induced*, if $E_{\text{ind}} = V_{\text{ind}}^2 \cap E$. The vertex set V_{ind} induces the subgraph G_{ind}.

The *degree* of a vertex v is the number of incident edges. A *path* $P = v_1v_2 \dots v_n$ is a graph with vertices $V = \{v_1, v_2, \dots, v_n\}$, $|V| \geq 1$, such that $E = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)\}$. The *length* of the path is the number of its edges. Paths are called *simple* if no vertex is repeated. A graph is called *connected* if there exists a path between any pair of vertices. We assume every graph to be connected in this thesis, unless stated otherwise. A *cycle* $C = v_1v_2 \dots v_nv_1$ is a graph with vertices $V = \{v_1, v_2, \dots, v_n\}$, $|V| \geq 1$, such that $E = \{(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n), (v_n, v_1)\}$. A cycle on n vertices has length n. Cycles are called *simple* if v_1 is the only repeated vertex. A graph G is called *hamiltonian* if there exists a subgraph C of G such that $V(C) = V(G)$ and C is a simple cycle. It is called *sub-hamiltonian* if G is the subgraph of a hamiltonian graph. A cycle on 3 vertices is called a *triangle*. For a triangle on the vertices a, b and c we write (a, b, c) instead of $abca$.

A *drawing* Γ of a graph G is a mapping of the vertices V to points and the edges to curves in the plane $\mathbb{R}^2$ such that for every curve representing an edge $e = (u, v)$, its endpoints are exactly the points representing u and v. A (combinatorial) *embedding* of G is the set of all drawings of G with the same cyclic ordering of edges around every vertex. We do not further distinguish between an edge and the curve representing it and between a vertex and the point representing it. The context resolves any ambiguities. Consequently, we do not distinguish between a path or cycle in G and the (closed) curve representing it in Γ. The curve representing an edge without its endpoints is called the *interior* of the edge. The drawing Γ is called *plane* if no two vertices are mapped to the same point and no two edges intersect at their interior. A graph G is called *planar* if it admits a plane drawing. In this thesis we only consider planar graphs. Thus, every time we write "graph", we mean "planar graph". Furthermore, we consider *straight-line* drawings, where every edge is represented by a line segment. By the Fáry-Wagner-Theorem [Fár48] every planar graph admits a plane straight-line drawing.

The *region* R of a (connected) graph G or its drawing Γ is the connected region of $\mathbb{R}^2$ containing exactly the vertices and edges of G. In a plane drawing of G, a *face* is a connected region in $\mathbb{R}^2 \setminus G$. In every drawing of G, there is exactly one unbounded face F_o, the *outer face*. Observe that R is exactly $\mathbb{R}^2 \setminus F_o$. All other faces are *inner faces*. A graph is called *maximally planar* if inserting any additional edge would violate planarity. Maximally planar graphs are also called *triangulations* since every face is bounded by a triangle. We call the cycle bounding a face its *boundary* and the cycle bounding the outer face the boundary of G. The *interior* of G is R without its boundary. The *exterior* of G is its outer face. A vertex is *inside* G if it lies on the interior of G. It is *outside* G if it lies on the outer face. Let C_1 and C_2 be two subgraphs of G. Then C_1 *contains* C_2 in Γ if the region of C_1 contains the region of C_2. We also write that C_1 and C_2 are *nested* if it is



(a) A drawing of the graph G and a pseudoline L. (b) A drawing of the graph G with $V(G)$ collinear.

Figure 1.3: Pseudolines.

irrelevant whether C_1 contains C_2 or vice versa. They *overlap* if their regions have at least one point in common.

Pseudolines

A vertex set $S \subset V(G)$ is called *collinear* or *alignable* if G admits a plane straight-line drawing where all vertices in S lie on a line. In Section 1 the concept of *pseudolines* is introduced. Recall that a curve L is a pseudoline with respect to a drawing of a graph G if L starts and ends at the outer face of G and for each edge $e \in E(G)$, L either entirely contains e or has at most one point in common with e. A planar graph G has a plane straight-line drawing with vertices $S \subset V(G)$ collinear if and only if G admits a drawing such that there exists a pseudoline that passes through the vertices S [MRR18, DLDF$^+$18]. The resulting drawing preserves the embedding. More informal, if L is a pseudoline in a drawing Γ, it can be stretched to a line L' and all vertices and edges of G can be rearranged such that the new drawing Γ' is a plane straight-line drawing with the same embedding as Γ and L' passes through exactly the same vertices, edges and faces of G as L. See Fig. 1.3 for illustration.

Pseudolines are excessively used throughout this thesis. In every proof of collinearity of a vertex set S we construct a pseudoline passing through S. Some additional vocabulary is needed to describe a pseudoline L. We write L *collects* a vertex v if it passes through v. Let L_e be a curve starting and ending at infinity and let O_1 and O_2 be the half-planes on either side of L_e. Let P be a path contained in L_e. The pseudoline L *intersects* P at a point $p \in P$ if there exist points $p_1, p_2 \in L$ such that $p_1 \in O_1$, $p_2 \in O_2$ and $|p - p_1| < \epsilon$, $|p - p_2| < \epsilon$, for any $\epsilon > 0$. L intersects a cycle C at a point $p \in C$ if there exist points $p_1, p_2 \in L$ such that p_1 is inside C, p_2 is outside C and $|p - p_1| < \epsilon$, $|p - p_2| < \epsilon$, for any $\epsilon > 0$. The pseudoline only *touches* a path or cycle if p_1, p_2 are both on O_1 or both on O_2, respectively both are inside or both are outside. Observe that a pseudoline can touch a path or cycle only at a vertex.

Vertex Sets

Consider two vertices $v, u \in V(G)$. Their *distance* $\operatorname{dist}(v, u)$ is the length of the shortest path P with endpoints v and u. Recall that the length of a path is the number of its edges. The neighborhood of a vertex v is the set of vertices $N(v) \subset V(G)$ such that $\operatorname{dist}(v, n) \leq 1$ for every $n \in N(V)$. Note that v itself is part of its neighborhood. The neighborhood $N(A)$ of a vertex set $A \subset V(G)$ is the union of all neighborhoods of the vertices in A. A set $S \subset V(G)$ is called *independent* if $N(s) \cap S = \{s\}$ for every vertex $s \in S$. In Chapter 2 and 3 we examine whether independent sets are collinear in two subclasses of planar graphs. The reason we do not consider general planar graphs is the following observation.

As mentioned before, Ravsky et al. [RV11] constructed a sequence $\mathcal{G}$ of triangulations such that the number of collinear vertices is in $O(|V(G)|^{0.986})$ for every graph $G \in \mathcal{G}$.

Furthermore, every graph $G \in \mathcal{G}$ is planar. Therefore, G has an independent set $S_G \subset V(G)$ of size at least $\lceil |V(G)|/4 \rceil$, following from the four color theorem [Die17] and the pigeonhole principle. Comparing the sublinear upper bound on the number of collinear vertices in G and linear lower bound on the size of the maximum independent set S_G, we infer that S_G cannot be collinear for all but a finite number of graphs in $\mathcal{G}$. Thus we can state the following as a preliminary result.

Corollary 1.1. *There exists a sequence of triangulations $\mathcal{G}$ such that for every graph $G \in \mathcal{G}$ every maximum independent set S_G is not collinear.*

In summation we can say that independence of a set of vertices is not a sufficient condition for collinearity.

Consider an independent set S. S is called *d-scattered* or *distance-d independent*, if $\text{dist}(u, v) \geq d$ for every two vertices $u, v \in S$, $u \neq v$. Similar to independent sets, d-scattered sets are implied by *d-distance colorings* of graphs. A d-distance coloring of a graph G is a coloring of the vertices of G such that any two vertices of the same color have distance at least d. Every color class in such a coloring forms a d-scattered set. The *d-distance chromatic number* $\chi^{(d)}$ of G is the smallest number k such that there exists a d-distance coloring with only k colors. According to Skupień et al. [JS01], given a planar graph G, let $D = \max(8, \Delta(G))$, where $\Delta(G)$ is the maximum degree of G. Then the d-distance chromatic number of G is

$$\chi^{(d)}(G) \leq 6 + \frac{3D+3}{D-2}((D-1)^{d-1} - 1)$$

Again with the pigeonhole principle there is a color class of size at least $\left\lceil \frac{|V(G)|}{\chi^{(d)}(G)} \right\rceil$ in such a coloring. Thus, there is always a d-scattered set of size at least $\left\lceil \frac{|V(G)|}{\chi^{(d)}(G)} \right\rceil$ in G.

Series-parallel Graphs

In Chapter 2, we examine independent and d-scattered sets in *series-parallel* graphs. A graph G is called series-parallel if it contains two vertices s and t and a single edge (s, t) or it consists of two series-parallel graphs G_1, G_2 with sources s_1, s_2 and sinks t_1, t_2 which are combined using one of the following composition operations [BCDB+94].

- *Series composition*: Identify t_1 and s_2, s_1 is the source of G, t_2 is the sink of G.
- *Parallel composition*: Identify s_1, s_2 and set it to be source of G. Identify t_1, t_2 and set it to be sink of G.

Thus, a series-parallel graph is obtained by a sequence of series and parallel compositions of smaller series-parallel graphs. We call every such series-parallel subgraph C of G a *component* of G.

4-connected Triangulations

A graph G is called *connected* if there exists a path between any pair of vertices. G is called *k-connected*, if removing any set $A \subset V(G)$ of size less than k, i.e. $|A| < k$, and all incident edges leaves G connected. If removing a set A', $|A'| \geq k$, disconnects G then A' is a *separating set*.

In Chapter 3 we examine independent sets in 4-connected planar triangulations. Starting with the octahedron graph, all 4-connected planar triangulations can be generated using one of two edge de-contraction operations. For reasons of simplicity we describe the reverse operations here. For the first operation we find a vertex v of degree four, for the second operation of degree five, in a planar 4-connected triangulation T with more than six vertices. In both operations we *contract* one of v's incident edges, i.e. we identify vertex v with one of its neighbors and remove loops and parallel edges. The resulting graph is a planar 4-connected triangulation T' and $|V(T')| = |V(T)| - 1$. We can continuously contract edges in this way and yield a planar 4-connected triangulation until the resulting graph has only six vertices left. This graph on six vertices is the octahedron graph. Brinkmann et al. [BLSVC14] show that every 4-connected triangulation can be generated starting with the octahedron graph and using the reverse of the above described operations.

In Section 3.2 and 3.3 we consider a superclass of 4-connected triangulations. A *near-triangulation* is a planar graph G with a drawing Γ such that every inner face is a triangle. A *2-sided* near-triangulation is a 2-connected near-triangulation T without separating triangles such that going clockwise on its outer face, the vertices are denoted $a_1, a_2, \ldots, a_p, b_q, \ldots, b_2, b_1$, $p \geq 1$ and $q \geq 1$, and such that there is neither a chord (a_i, a_j) nor (b_i, b_j), i.e. an edge (a_i, a_j) or (b_i, b_j) such that $|i - j| > 1$.

2. Collinear sets in series-parallel graphs

One of the first questions posed during the work on this thesis was whether independence or any fixed minimal pairwise distance between vertices of a vertex set in a graph suffices for them to be collinear. As mentioned before, in general planar graphs independence is not enough [RV11]. In this chapter we show that even for series-parallel graphs independence does not suffice (Section 2.1). Recall that a d-scattered set $S \subset V(G)$ is a set of vertices with pairwise distance at least d. In Section 2.2 we show that for any constant number d, there exists a series-parallel graph with a d-scattered set which is not collinear. Section 2.3 defines a meaningful subclass of series-parallel graphs in which every 3-scattered set is collinear.

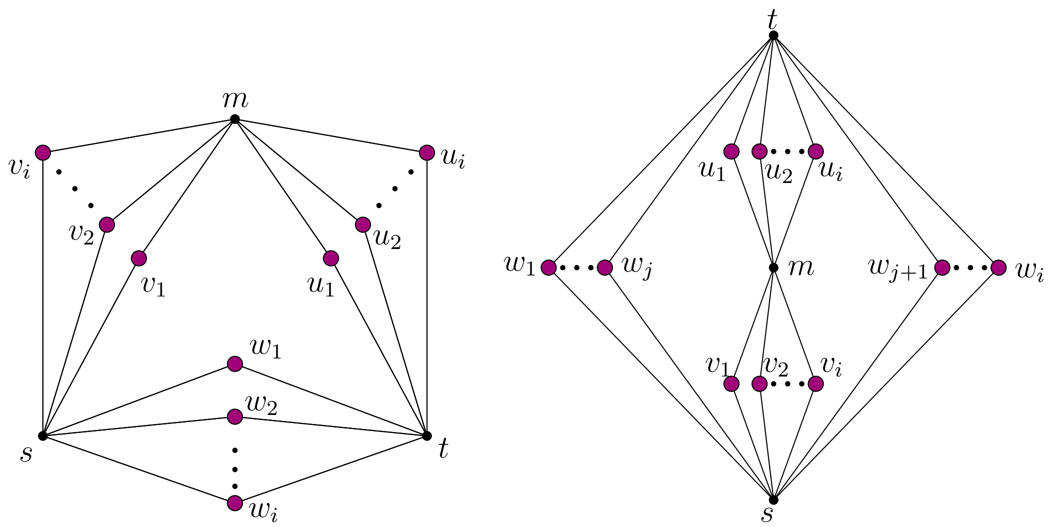
2.1 Counterexample: Independent sets

Since independence of vertices does not suffice for them to be collinear in general planar graphs [RV11], we pose the question whether independence suffices when restricting to series-parallel graphs. However, we construct a series-parallel graph G with a maximum independent set S contradicting this conjecture, see Fig. 2.1.

The series-parallel graph G consists of three components C_u, C_v, C_w. Every component is a parallel composition of $i \in \mathbb{N}$, $i \geq 3$, paths on exactly three vertices each. We obtain G by a series composition C_{uv} of C_u with C_v and a parallel composition of C_{uv} with C_w. We call the source of G s, its sink t, the common vertex of C_u and C_v m. The middle vertices of the components C_u, C_v and C_w are called $u_1, \dots, u_i$, $v_1, \dots, v_i$ and $w_1, \dots, w_i$, respectively. The vertices $\{u_1, \dots, u_i, v_1, \dots, v_i, w_1, \dots, w_i\}$ form the independent set S. Then G is a series-parallel graph with $3(i+1)$ vertices and the independent set S contains $3i$ of them.

To see that the independent set $S \subset V(G)$ is indeed not collinear, we construct a *traversal graph* T for every embedding Γ of G such that there exists a hamiltonian path in T if and only if there exists a pseudoline L collecting the vertices S. Traversal graphs are introduced in [MRR18]. Due to the structure of G we can omit parts of the definition and use a simpler version of traversal graphs. We need the following observations for that.

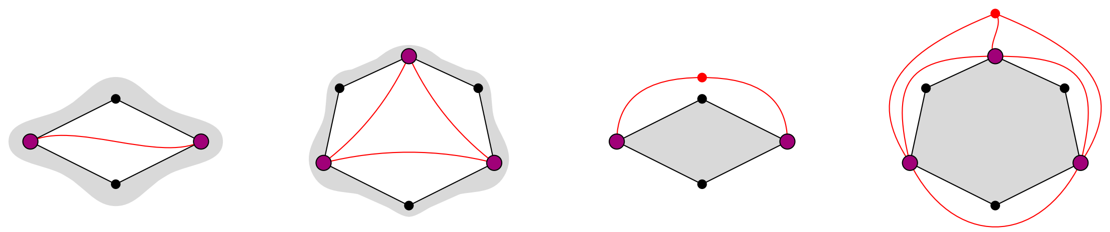
The vertices $S' = V(G) \setminus S = \{s, m, t\}$ form an independent set too. Thus, G is bipartite. Therefore every face in a embedding Γ of G is bounded by a cycle of even length. Since the longest cycle in G has length six and with Euler's formula on planar graphs, there are exactly two faces surrounded by a 6-cycle and all other faces are bounded by a 4-cycle. We call the faces bounded by a 4-cycle *4-faces*, the ones bounded by a 6-cycle *6-faces*.



(a) A 6-cycle bounds the outer face.

(b) A 4-cycle bounds the outer face.

Figure 2.1: The series-parallel graph G and the independent set S (violet).



(a) A 4-face of G.

(b) A 6-face of G.

(c) A 4-face as the outer face of G.

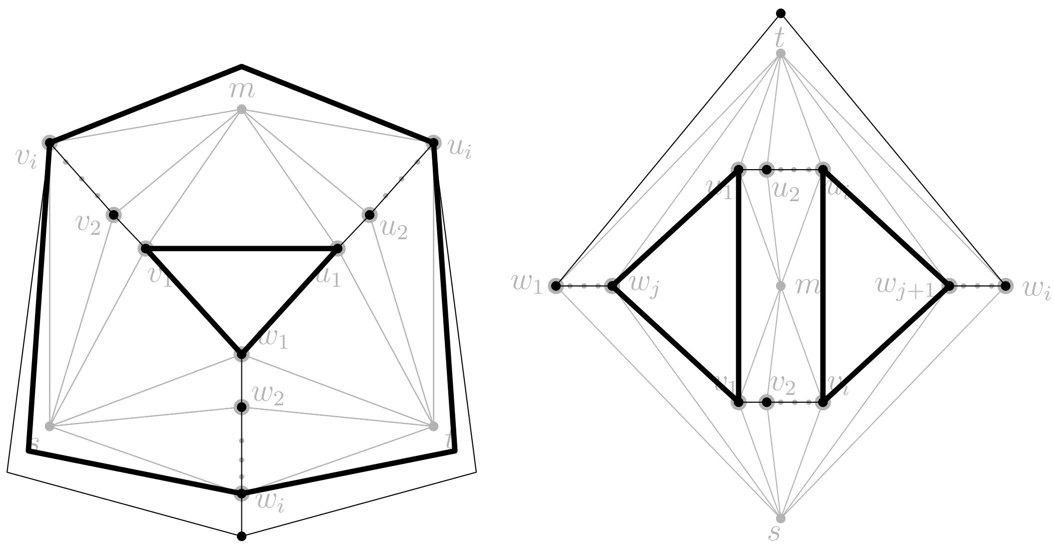
(d) A 6-face as the outer face of G.

Figure 2.2: Faces of G and the corresponding edges of T. Vertices $S \subset V(G)$ are depicted in violet and are also vertices of T, other vertices and edges of T are depicted in red.

Observe that every edge of G is incident to a vertex in S. Thus, a pseudoline L collecting the vertices of S intersects with every edge at one of its endpoints. Hence, L can never cross an edge at its interior.

Consider a fixed embedding Γ of G. The vertices of the traversal graph $T(\Gamma)$ correspond to all points of Γ that have to be collected by L. We omit Γ from $T(\Gamma)$ whenever Γ is clear from the context and just refer to T. Hence for every vertex $v \in S$, there exists a corresponding vertex $a \in V(T)$. The pseudoline L starts and ends at the outer face, thus we add a vertex o corresponding to the outer face f_o. Thus there exists a bijective function $t : S \cup \{f_o\} \to V(T)$. We say a vertex $a \in V(T)$ is *associated* with a face f of G if it corresponds to a vertex $v \in V(G)$ incident to f. The vertex $o \in V(T)$ is always associated with the outer face of G. The edges of T represent direct connections between two points in G without crossing any edge. Thus for two vertices $a_i, a_j \in V(T)$, there exists an edge (a_i, a_j) if and only if a_i and a_j are associated with the same face. See Fig. 2.2 for illustration.

The resulting traversal graph T of an embedding Γ of G has a hamiltonian cycle if and only if S is collinear in Γ. Consider a hamiltonian cycle C in T. We construct a curve L as follows. Let C' be C without the vertex o and its incident edges. For every edge $(a_i, a_j) \in E(C')$, L crosses a face from the vertex $v_i = t^{-1}(a_i)$ to the vertex $v_j = t^{-1}(a_j)$.



(a) Traversal graph for the embedding bounded by a 6-cycle.

(b) Traversalgraph for the embedding bounded by a 4-cycle.

Figure 2.3: Traversal graphs. The triangles are emphasized by thicker lines. The original graph is indicated in grey.

Since C is hamiltonian, L is a simple curve starting and ending at a vertex at the outer face of G and can thus be extended to a pseudoline. Conversely, consider a pseudoline L in the embedding Γ. It starts and ends at the outer face and it does not cross any edge. It collects every vertex S and intersects a face from a vertex v_i to a vertex v_j only if there exists an edge $(t(v_i), t(v_j)) \in T$. Since L is simple, this yields a hamiltonian cycle C in T.

It remains to show that a traversal graph T of any embedding Γ of G is indeed not hamiltonian. We distinguish between two kinds of embeddings of G, depending on the choice of the outer face. Choosing a 6-face as the outer face yields the embedding Γ_{six} shown in Fig. 2.1(a). This embedding is unique. Choosing a 4-face as the outer face yields a family of embeddings $\Gamma_1, \ldots, \Gamma_{\lfloor i \rfloor}$ as shown in Fig. 2.1(b).

The corresponding traversal graphs $\mathcal{T} = \{T_{six}, T_1, \ldots T_{\lfloor i \rfloor}\}$ are depicted in Fig. 2.3. Observe that every $T \in \mathcal{T}$ contains two triangles (a_1, b_1, c_1) and (a_i, b_i, c_i) (resp. (a_i, b_i, c_{i+1})). They are connected to each other via three induced paths $a_1 a_2 \ldots a_i$, $b_1 b_2 \ldots b_j$ and $c_1 c_2 \ldots c_i$ (resp. $c_1 c_2 \ldots c_{i+1}$) on at least three vertices and there is no other path between the triangles. Thus, there exists no simple cycle containing the vertices a_2, b_2 and c_2 and T is not hamiltonian.

Observe that G is not the only series-parallel graph with a non-collinear independent set. We obtain G by a series composition C_{uv} of C_u with C_v and a parallel composition of C_{uv} with C_w. Any series-parallel graph obtained by series compositions of $j \in \mathbb{N}_+$ copies of C_{uv} and a parallel composition of the resulting graph with C_w has a non-collinear independent set. The traversal graph contains two cycles on $2j+1$ vertices, connected via $2j+1$ induced paths on at least three vertices and there exists no other connection. Since $2j+1$ is an odd number, the traversal graph is not hamiltonian. The resulting graph G has $(2j+1)(i+1)$ vertices and the independent set $2ji$ vertices. Thus, we can state the following theorem.

Theorem 2.1. *For every $i, j \in \mathbb{N}, i \geq 3, j \geq 1$, there exists a series-parallel graph G on $(2j+1) * (i+1)$ vertices with an independent set $S \subset V(G)$ of size $2ji$ such that S is not collinear.*

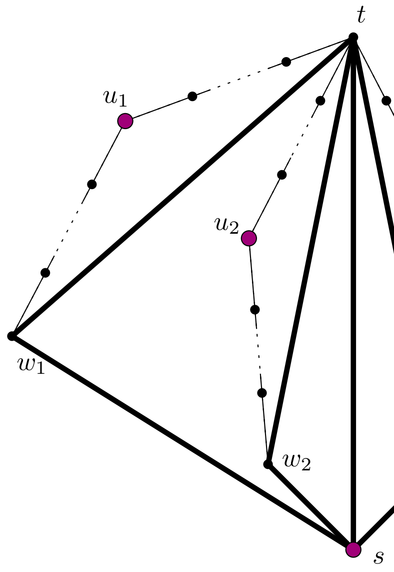


Figure 2.4: A d-nice component $C_{\text{nice}} \in \mathcal{C}_{\text{nice}}$. The distance of u_i to w_i and t is $d-1$.

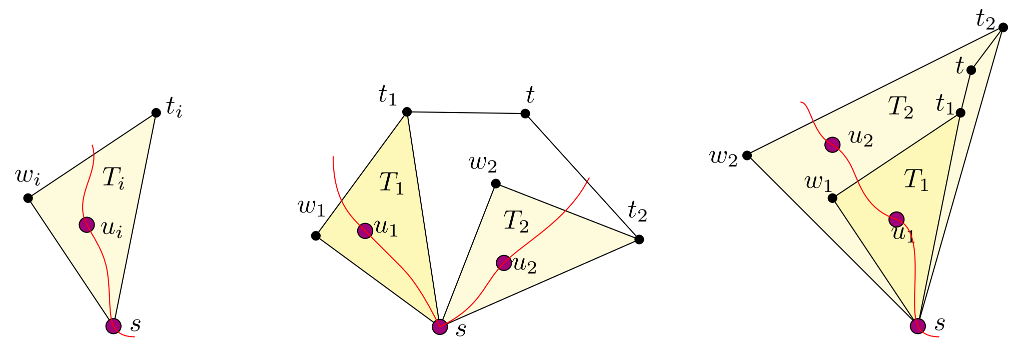
2.2 Counterexample: d-scattered sets

Before proving the main theorem of this section we define two graph classes and prove several lemmas we will use in the proof of the theorem. Consider a parallel composition D_j of a single edge and a path P_j on $2d-1$ vertices. Let w_j and t be the source and sink of this graph and let u_j denote the middle vertex of P_j. Let $\overline{D_j}$ be the series composition of D_j with a single edge (s, w_j) such that s and t are the source and sink of $\overline{D_j}$. The parallel composition of a single edge (s,t) and three copies D_1, D_2 and D_3 of D_j yields the graph C_i depicted in Fig. 2.4. We call C_i on $6d-4$ vertices a *d-nice component* and the family of all d-nice components $\mathcal{C}_{\text{nice}}$. Furthermore, let $\overline{C_i}$ be the series composition of C_i and a single edge (t_i, t_{nice_i}) such that the source s_i of C_i is the source of $\overline{C_i}$ and t_{nice_i} is the sink of C_i. Let $\overline{C_1}$, $\overline{C_2}$ and $\overline{C_3}$ be three copies of $\overline{C_i}$. Let G be the parallel composition of $\overline{C_1}$, $\overline{C_2}$ and $\overline{C_3}$ such that $s = s_1 = s_2 = s_3$ and $t = t_{\text{nice}_1} = t_{\text{nice}_2} = t_{\text{nice}_3}$. We call G a *d-nice graph* and the family of all d-nice graphs, for some $d \geq 2$, we denote by $\mathcal{G}_{\text{nice}}$.

Lemma 2.2. *Let s and t be the source and sink of a d-nice component $C_{\text{nice}} \in \mathcal{C}_{\text{nice}}$. In every planar embedding of C_{nice}, there exists a triangle $T_{\text{nice}} = (s,t,w)$ and a d-scattered set $S \subset V(C_{\text{nice}})$ such that the source $s \in S$ and there exists a vertex $u \in S$ that lies inside the triangle T_{nice}. We call T_{nice} a d-nice triangle.*

Proof. Consider the d-nice component C_{nice} as shown in Fig. 2.4. C_{nice} is a parallel composition of four components, one of them being the edge (s,t). Each of the three other components D_j, $j \in \{1,2,3\}$, contains a vertex w_j incident to both s and t and thus forms a triangle (s,t,w_j), sharing the edge (s,t). Furthermore, the vertices $u_1 \in V(D_1), u_2 \in V(D_2), u_3 \in V(D_3)$ together with the source s form a d-scattered set S.

Since the edge (s,t) and the three components D_j are composed in parallel, in any planar embedding of C_{nice} at least one of the three vertices w_j, $j \in \{1,2,3\}$, lies inside one of the triangles $(s,t,w_i), i \in \{1,2,3\}, i \neq j$. Along with w_j the inner vertices of any component composed in parallel to the edge (w_j,t) lie inside the same triangle (s,t,w_i). Otherwise an edge of this component would cross the boundary of the triangle. The path $P_j = w_j \ldots u_j \ldots t$ is such a component, i.e. the vertex $u_j \in S$ lies inside the triangle $T_{\text{nice}} := (s,t,w_i)$. $\square$



(a) Scheme of a d-nice triangle T_i.

(b) d-nice triangles T_1 and T_2 next to each other.

(c) d-nice triangles T_1 and T_2 nested in each other.

Figure 2.5: Scheme of d-nice triangles T_i and pseudolines (red) collecting the vertices of the d-scattered set S (violet).

Lemma 2.3. *Let $G \in \mathcal{G}_{\text{nice}}$ be a d-nice series-parallel graph. Then there exists a d-scattered set S such that the source s of G is in S and in every embedding of G there are three vertices $u_1, u_2, u_3 \in S$ lying inside three d-nice triangles T_1, T_2 and T_3 only sharing the vertex s.*

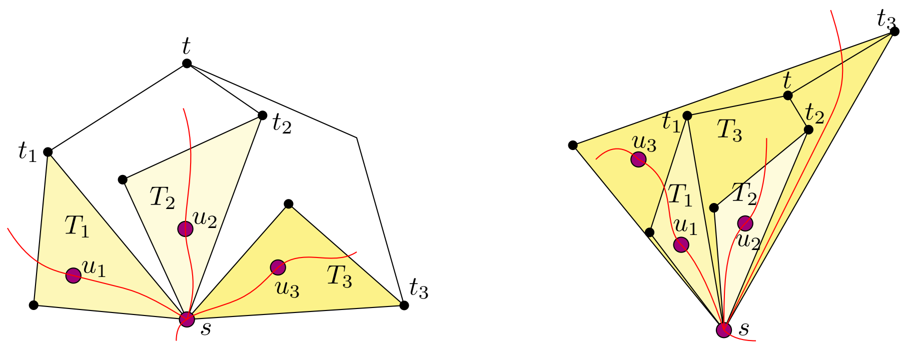
Proof. Let C_i, $i \in \{1,2,3\}$, be the three d-nice components contained in G. From Lemma 2.2 we know that there exists a d-scattered set $S_i \subset V(C_i)$ such that in every planar embedding of C_i, there exists a d-nice triangle $T_i = (s_i, t_i, w_i)$ and a d-scattered set $S_i \subset V(C_i)$ such that the source s_i is in S_i and another vertex $u_i \in S_i$ lies inside the triangle T_i. Since $s = s_1 = s_2 = s_3$ is the source of G and the triangles do not share any other vertex, it remains to show that $S = S_1 \cup S_2 \cup S_3$ is a d-scattered set in G.

Consider C_i. The vertices s and t_i are the source and sink of C_i and thus every path from a vertex $v_i \in V(C_i)$ to a vertex $v_j \in V(G) \setminus V(C_i)$ contains s or t_i. Additionally, $(s, t_i) \in E(C_i)$ and therefore every shortest path between two vertices $v_i \in C_i$ and $v'_i \in C_i$ lies completely in C_i. Otherwise it would pass s and t_i and using the edge (s, t_i) would shorten the path.

Now consider a vertex $x_i \in S_i$. Since S_i is d-scattered in C_i, it has distance at least d to every other vertex in S_i. In particular if $x_i \neq s$, then every shortest path between the vertices s and x_i has length at least d as $s \in S_1 \cap S_2 \cap S_3$. Denote this path by P_{x_i}. It remains to show that x_i has distance at least d to all vertices $S \setminus S_i$.

Assume there exists a vertex $x_j \in S_j$, $j \in \{1,2,3\}$ and $j \neq i$, such that the distance e between x_j and x_i is less than d. Then there exists a path P from x_i to x_j of length $e < d$. Since G is a parallel composition of $\overline{C_1}$, $\overline{C_2}$ and $\overline{C_3}$, P contains either s or t. If it contains s, then it contains P_{x_i}, which is a contradiction since the length of P_{x_i} is at least $d > e$. Otherwise if P contains t, then there exists a path P' of length at most $\lfloor \frac{e}{2} \rfloor$ from t to x_i or to x_j. The path P' does not contain s. Thus, it contains t_i and t_j and there exists a path P'' of length at most $\lfloor \frac{e}{2} \rfloor - 1$ between t_i and u_i or between t_j and u_j. Since s is also a neighbor of t_i and t_j, this means that there exists a path of length at most $\lfloor \frac{e}{2} \rfloor$ from s to u_i or u_j, which is a contradiction as P_{u_i} and P_{u_j} are the shortest paths from s to u_i and u_j and have length at least $d > e$. Therefore all vertices in $S = S_1 \cup S_2 \cup S_3$ have pairwise distance at least d and S is a d-scattered set. □

For the upcoming Lemma 2.4 we use the notion of a region of a graph. Recall that the region of a graph with a given embedding is the complement of the outer face.



(a) T_1, T_2 and T_3 overlapping at vertex s only.

(b) T_3 contains T_1 and T_2.

Figure 2.6: The series-parallel graph G and the vertices of a d-scattered set (violet). A pseudoline (red) collecting the vertices of the d-scattered set.

Lemma 2.4. *In every plane embedding of a d-nice graph G with source s and sink t, the regions of two of the three contained d-nice components C_1, C_2 and C_3 only overlap at the source s. The region of the third d-nice component*

1. *either only overlaps at s too (see Fig. 2.6(a)) or*
2. *both of the other triangles are contained in the d-nice triangle of it (see Fig. 2.6(b)).*

Proof. To show this we derive all possible embeddings of G. In particular we distinguish between different mutual positions of the three d-nice triangles $T_i = (s, t_i, w_i) \subset \overline{C_i}$, $i \in \{1, 2, 3\}$, which have a vertex $u_i \in S$ embedded inside. Since G contains three copies of the d-nice component C_{nice} sharing the vertex s and from Lemma 2.3 we know that these three triangles T_1, T_2 and T_3 exist, all sharing the vertex s and they are all connected with the vertex t via edges (t_1, t), (t_2, t) and (t_3, t).

First consider the mutual position of two of the three triangles, without loss of generality T_1 and T_2. Either they overlap at the vertex s only (see Fig. 2.5(b)), or, without loss of generality, T_1 is nested inside T_2 (see Fig. 2.5(c)). Since s is their only common vertex, partial overlapping of the triangles would yield an edge crossing.

Now consider the complementary position of the third triangle T_3. If T_1 is nested inside T_2, then the vertex t is also placed inside T_2 and outside T_1 since $t_1 \in V(T_1)$ and $t_2 \in V(T_2)$ are adjacent to t and otherwise the edges (t_1, t) and (t_2, t) would cross the boundary of the triangle T_2, T_1 respectively. The vertex $t_3 \in V(T_3)$ is also adjacent to t and thus T_3 is nested inside T_2 and overlaps with T_1 at s only. This yields the second case. If T_1 and T_2 only overlap at the vertex s, the vertex t is placed on the outer face. Therefore T_3 cannot be nested inside T_1 or T_2 since the edge (t_3, t) would cross the triangle's boundary. Neither can only one of T_1 or T_2 be nested inside T_3 because switching the numbering of T_3 with said triangle would yield the same situation as before. Hence T_3 either overlaps with T_1 and T_2 at s only (first case) or T_1 and T_2 are both nested inside T_3 (second case). □

We now use Lemmas 2.3 and 2.4 to proof the main theorem.

Theorem 2.5. *For every natural number d there exists a series-parallel graph G and a d-scattered set $S \subset V(G)$ such that S is not collinear.*

Proof. Let $G \in \mathcal{G}_{\text{nice}}$ be a d-nice series-parallel graph and let C_1, C_2 and C_3 be its d-nice components. From Lemma 2.3 we know that there exists a d-scattered set $S \subset V(G)$

such that the source s of G is in S and in every embedding of G there are three vertices $u_1, u_2, u_3 \in S$ lying inside three d-nice triangles T_i only sharing the vertex s.

For the sake of a contradiction, assume that S is collinear and let L be a pseudoline collecting $S = S_1 \cup S_2 \cup S_3$, $S_i = S \cap V(C_i)$, $i \in \{1, 2, 3\}$. First consider the partial route of L to collect the vertex $u_i \in S_i$ inside the d-nice triangle T_i, $i \in \{1, 2, 3\}$. For illustration see Fig. 2.5(a). L has to enter and exit T_i to collect u_i, i.e. cross its boundary at least twice. Since L also collects s, it does not cross the interior of the edges (s, t_i) and (s, w_i) and it does not cross at the vertices w_i and t_i. Hence it crosses the boundary of T_i at s and at the interior of the edge (w_i, t_i) due to the lack of another possible crossing point.

From Lemma 2.4 we know that in every plane embedding of G, the regions of two of the three contained d-nice components only overlap at the source s. Without loss of generality let C_1 and C_2 be those components. Now consider the route of L to collect the vertices of $u_1 \in S_1$ and $u_2 \in S_2$ inside the respective triangles. L crosses the boundary of T_1 and T_2 exactly at the vertex s and the interior of the edges (w_1, t_1) and (w_2, t_2) to collect the vertices u_1 and u_2. As s can only be collected once, L enters T_1 at the edge (w_1, t_1), leaves T_1 and directly enters T_2 via s and leaves T_2 at the edge (w_2, t_2) (or reversed). So L always collects s between collecting u_1 and u_2.

We use the restrictions on L to contradict its existence. We distinguish between possible embeddings of the third component C_3.

Case 1: C_1, C_2 and C_3 overlap at vertex s only (Fig. 2.6(a)).

Since T_3 does not overlap with T_1 and T_2 at its interior, L collects the vertex u_3 inside T_3 either before or after collecting the vertices u_1, s and u_2. To do so it crosses the boundary of T_3 at s and at the edge (w_3, t_3). This is a contradiction since s cannot be collected twice.

Case 2: C_1 and C_2 are nested inside T_3 (Fig. 2.6(b)).

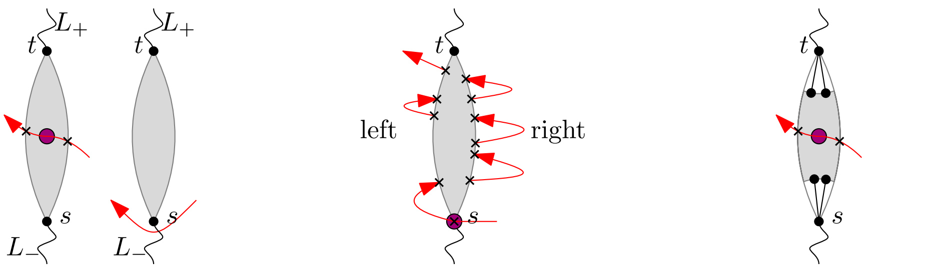
L can cross the boundary of T_3 only twice. Thus, it enters and leaves via s and the edge (w_3, t_3) and collects the vertices u_1 and u_2 in between. Since T_1 and T_2 are nested inside T_3 and L always collects s between collecting u_1 and u_2, s is collected twice and L violates pseudoline properties.

From Lemma 2.4 we know that there is no other possible embedding of G, concluding the proof. □

2.3 3-scattered sets in 4-stretched series-parallel graphs

Increasing the pairwise distance between vertices of an independent set does not suffice for S to be collinear in series-parallel graphs in general. Thus we restrict the graph class of series-parallel graphs to a subclass. Let G be a series-parallel graph. Consider the series and parallel compositions obtaining G. Then G is called *c-stretched* if the distance between the source and sink is at least c in every parallel composition. There is no restriction on the series composition. Observe that every series-parallel graph is 1-stretched. We show that in a 4-stretched series-parallel graph G every 3-scattered set $S \subset V(G)$ is collinear.

Before we proof the main theorem of this section, we need some additional definitions. Let C be a component of a series-parallel graph G. Let Γ be a drawing of G and $\Gamma(C)$ be the drawing of C in Γ. Throughout this section we assume that for every component C of G, the source s and sink t of C are on the outer face in the drawing $\Gamma(C)$. A series-parallel graph always admits such a drawing. Let L_+ be a simple curve connecting the sink t of C to infinity without intersecting $\Gamma(C)$. Let L_- be a simple curve connecting the source s of C to infinity without intersecting $\Gamma(C)$ or L_+. Then L_+ and L_- divide the outer face of $\Gamma(C)$ into two half-planes. We call one of the half-planes the right outer face, the other



(a) Invariant C.

(b) Invariants D, E and F.

(c) Invariant G.

Figure 2.7: Invariants C to G. The pseudoline is colored in red. Entry- and exitpoints are marked with a cross. The vertices of S are colored violet.

one left outer face. Further we call the path B_r from s to t with all vertices on the right outer face the right boundary of C, the left boundary B_l is defined symmetrically. We do not linguistically distinguish between a vertex or edge and the point or line segment representing it in Γ. Further we do not distinguish between B_r (resp. B_l) and the polyline representing it.

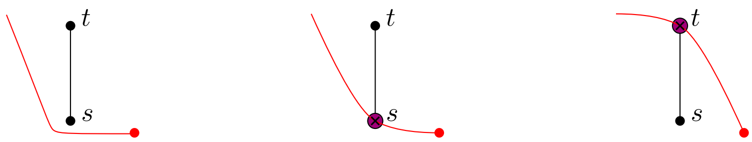
Let L be a pseudoline with respect to C. Then both endpoints of L are on the outer face of C. In this section we explicitly give L a direction and say it starts at the starting point and ends at the endpoint. We say L *enters* C at a point $p_\leftarrow$ if L intersects the boundary of C at $p_\leftarrow$ and for $\epsilon > 0$ there exists a point p_b on the outer face of C such that $|p_\leftarrow - p_b| < \epsilon$ and p_b is before $p_\leftarrow$ on L. L enters from the right (resp. left) if p_b is on the right (resp. left) outer face. We call $p_\leftarrow$ an *entrypoint*. We say L *exits* or *leaves* C at a point $p_\rightarrow$ if L intersects the boundary of C at $p_\rightarrow$ and for $\epsilon > 0$ there exists a point p_a on the outer face of C such that $|p_\rightarrow - p_a| < \epsilon$ and p_a is after $p_\rightarrow$ on L. L leaves to the right (resp. left) if p_a is on the right (resp. left) outer face. We call $p_\rightarrow$ an exitpoint. All entry- and exitpoints are *intersection points*. Further let p, q be two points on L. Then Lp is the curve segment from the starting point of L to p and pL is the curve segment of L from p to its endpoint. pLq is the curve segment of L from p to q. Let p be the crossing point of two curves L_1 and L_2. Then $L_1pL_2 = L_1p \cup pL_2$.

Theorem 2.6. *Let G be a 4-stretched series-parallel graph and let $S \subset V(G)$ be a 3-scattered set. Then S is collinear.*

Proof. Let Γ be a drawing of G. We inductively construct a pseudoline L for every component C of G collecting all vertices $S \cap V(C)$. Let $p_<$ be the first entrypoint and $p_>$ be the last exitpoint point of L with C. Let L_+ and L_- be the curves separating the right and left outer face.

We preserve the following invariants.

- Invariant A: L is a pseudoline with respect to C.
- Invariant B: L collects all vertices $V(C) \cap S$.
- Invariant C: If $V(C) \cap S = \emptyset$, L does not intersect with C. Otherwise L does not intersect with L_+ or L_- except for their endpoints. See Fig. 2.7(a) for illustration.
- Invariant D: L starts on the right outer face and ends on the left outer face. See Fig. 2.7(b) for illustration.
- Invariant E: The intersection points on the right boundary (resp. left boundary) are ordered along the right boundary (resp. left boundary) from s to t in the same order as they appear on L. See Fig. 2.7(b) for illustration.



(a) $s,t \notin S$ (b) $s \in S, t \notin S$ (c) $s \notin S, t \in S$

Figure 2.8: Induction base. The pseudoline is colored in red. Entry- and exitpoints are marked with a cross. The vertices of S are colored in violet.

- Invariant F: If $s \in S$ (resp. $t \in S$), then s (resp. t) is an intersection point on the right as well as on the left boundary. Furthermore, if $S \cap (V(C) \setminus \{s,t\}) \neq \emptyset$, then there exists at least one entrypoint on the right and one exitpoint on the left unequal to s and t. See Fig. 2.7(b) for illustration.
- Invariant G: If $(\{s\} \cup N(s)) \cap S = \emptyset$ (resp. $(\{t\} \cup N(t)) \cap S = \emptyset$), then none of the edges $(s,n), n \in N(s)$ (resp. $(t,n), n \in N(t)$), is intersected by L. See Fig. 2.7(c) for illustration.

Note that Invariants C and E mean that the intersection points on either boundary are alternately entry- and exitpoints. Together with D, the alternation starts with the first entrypoint $p_<$ and ends with an entrypoint on the right boundary. On the left boundary it starts with an exitpoint and ends with the last exitpoint $p_>$. Invariants E and F combined state that if $s \in S$ (resp. $t \in S$), then s (resp. t) is the first entrypoint on the right as well as the first exitpoint on the left.

Induction base:

Let C be the series-parallel graph on two vertices, s and t and either $s \in S$, $t \in S$ or $s,t \notin S$. Then there exists a pseudoline L from the right to the left outer face complying with the invariants. See Fig. 2.8 for illustration.

Induction step:

Let C be a component of G with at least three vertices. Then C is a composition of two series-parallel graphs C_1 and C_2. Further, there exist pseudolines L_1 and L_2 for C_1, respectively C_2, complying with the invariants. We distinguish whether C is a series (Case 1) or a parallel composition (Case 2) of C_1 and C_2.

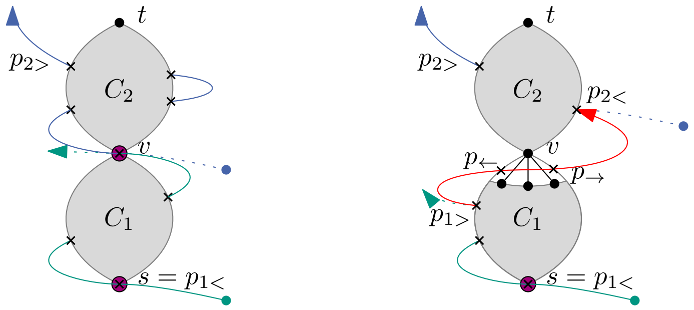
Case 1: C is a series composition of C_1 and C_2, see Fig. 2.9.

Let s_1, t_1 be the source and sink of C_1 and let s_2, t_2 be the source and sink of C_2. Without loss of generality let $v = t_1 = s_2$ be the common vertex of C_1 and C_2 in C. Then $s = s_1$ and $t = t_2$ are the source and sink of C. Furthermore, the right (resp. left) boundary of C is the union of the right (resp. left) boundaries of C_1 and C_2.

If $V(C_1) \cap S = \emptyset$, we leave out L_1, and set $L = L_2$. Otherwise if $V(C_2) \cap S = \emptyset$, we leave out L_2, and set $L = L_1$. Then L trivially conforms with the invariants. Otherwise we further distinguish whether $v \in S$ (Case 1a) or $v \notin S$ (Case 1b).

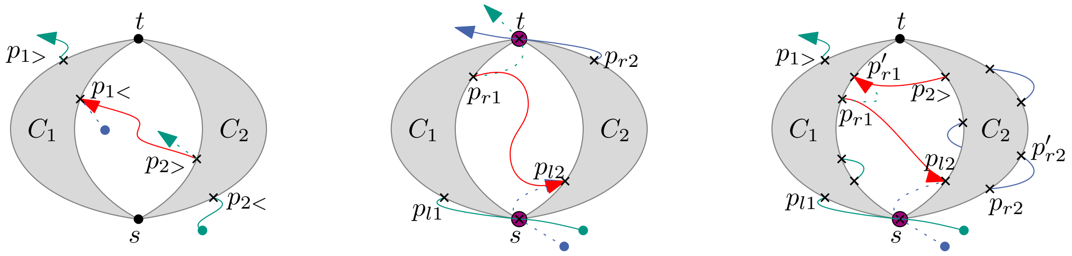
Case 1a: $v \in S$, see Fig. 2.9(a).

The vertex $v \in S$ is the last entry- and exitpoint of L_1 and the first entry- and exitpoint of L_2 (Invariants E and F). Let $L = L_1 v L_2$. Then v is an entrypoint on the right and an exitpoint on the left. Thus, L preserves Invariant F. L trivially preserves the other invariants.



(a) Case 1a: $v \in S$ (b) Case 1b: $v \notin S$

Figure 2.9: Case 1: Series composition of C_1 and C_2. The pseudoline L_1 is colored in green, L_2 in blue and L_{cross} in red. Entry- and exitpoints are marked with a cross. The vertices of S are colored in violet.



(a) Case 2a: $s, t \notin S$ (b) Case 2b: $s, t \in S$ (c) Case 2c: $s \in S, t \notin S$

Figure 2.10: Case 2: Parallel composition of C_1 and C_2. The pseudoline L_1 is colored in green, L_2 in blue and L_{cross}, L_{low} and L_{up} in red. Entry- and exitpoints are marked with a cross. The vertices of S are colored in violet.

Case 1b: $v \notin S$, see Fig. 2.9(b).

Since S is 3-scattered, either $N(v) \cap V(C_1) \cap S = \emptyset$ or $N(v) \cap V(C_2) \cap S = \emptyset$. Without loss of generality let $N(v) \cap V(C_1) \cap S = \emptyset$. Invariant G states that L_1 crosses none of the edges $(v, n), n \in N(v) \cap V(C_1)$. Furthermore, the last exitpoint of C_1, $p_{1>}$, is on the left outer face and the first entrypoint of C_2, $p_{2<}$, is on the right outer face (Invariant D). Let L_{cross} be a pseudoline from $p_{1>}$ to $p_{2<}$ crossing exactly the edges $(v, n), n \in N(v) \cap V(C_1)$. This yields a new entrypoint $p_{\leftarrow}$ and a new exitpoint $p_{\rightarrow}$ on the left and right boundary of C. Set $L = L_1 p_{1>} L_{\text{cross}} p_{2<} L_2$. Then L trivially preserves Invariants A, B, C, D and F. It preserves Invariant E since L collects all intersection points of L_1, then $p_{\leftarrow}$, then $p_{\rightarrow}$ and all intersection points of L_2 at last. Thus it collects all intersection points in order of appearance on the respective boundary. Invariant G is preserved since L crosses the same edges as L_1 and L_2 together and the edges $(v, n), n \in N(v) \cap V(C_1)$, are neither incident to s nor to t (C is 4-stretched).

Case 2: C is a parallel composition of C_1 and C_2, see Fig. 2.10.

Let s_1, t_1 be the source and sink of C_1 and let s_2, t_2 be the source and sink of C_2. Then $s = s_1 = s_2$ and $t = t_1 = t_2$ are the source and sink of C. Without loss of generality let C_1 be on the right outer face of C_2. Then the left boundary of C_1 is the left boundary of C. The right boundary of C_2 is the right boundary of C. The left boundary of C_1 and the right boundary of C_2 bound an inner face f_{internal} of C. We distinguish whether s and t are both in S (Case 2a), both not in S (Case 2b) or one of them is in S (Case 2c).

(a) Case 2a: $S \cap V(C) = \emptyset$ (b) Case 2c: $S \cap V(C) = \{s\}$

Figure 2.11: Case 2: Creation of entry- and exitpoints.

Case 2a: $s, t \notin S$, see Fig. 2.10(a).

If $V(C) \cap S = \emptyset$, set $L = L_1$. It does not intersect with C (Invariant C) and thus trivially preserves all invariants. Otherwise examine the pseudolines $L_i, i \in \{1, 2\}$, first. If the set $S \cap V(C_i)$ is empty, then L_i does not intersect with C_i (Invariant C). Then re-route L_i such that it enters C_i a point p_{ri} and leaves at a point p_{li} such that L_i does not intersect with any vertex or any edge $(s_i, n_s), n_s \in N(s_i)$ or $(t_i, n_t), n_t \in N(t_i)$. This is possible since $\text{dist}(s_i, t_i) \geq 4$. See Fig. 2.11(a) for illustration. Then C_1 and C_2 each have an entrypoint on the right boundary and an exitpoint on the left boundary. If the set $S \cap V(C_i)$ is not empty, these intersection points exist by Invariant F.

Furthermore, the last exitpoint $p_{2>}$ of L_2 is on the left boundary of C_2 and the first entry point $p_{1<}$ of L_1 is on the right boundary of C_1 (Invariants C and D). Thus, in $\Gamma(C)$ $p_{2>}$ and $p_{1<}$ are both at the inner face f_{internal}. Hence, there exists a curve L_{cross} from $p_{2>}$ to $p_{1<}$ without crossing any vertex, edge, or L_1 and L_2. Then let $L = L_2 p_{2>} L_{\text{cross}} p_{1<} L_1$ and L preserves the invariants.

Case 2b: $s, t \in S$, see Fig. 2.10(b).

Invariant E states that s_i, t_i, $i \in \{1, 2\}$, are both an entrypoint on the right boundary and an exitpoint on the left boundary. Thus, L_i leaves at s_i to the left outer face and enters at t_i from the right outer face. In between it intersects C_i from left to right since it cannot intersect L_+ and L_- (Invariant C). Thus, there exist points p_{li} and p_{ri}, the first entrypoint (resp. last exitpoint) after s (resp. before t) on L_i on the left (resp. right) boundary.

Regarding $\Gamma(C)$, p_{r1} and p_{l2} are both at the inner face f_{internal}. Hence, there exists a curve L_{cross} from p_{r1} to p_{l2} without crossing any vertex, edge, or L_1 and L_2. Then $L = L_1 p_{r1} L_{\text{cross}} p_{l2} L_2$ preserves the invariants.

Case 2c: $s \in S, t \notin S$ or $s \notin S, t \in S$, see Fig. 2.10(c).

Without loss of generality let $s \in S, t \notin S$. In the case $s \notin S, t \in S$ the construction of L is symmetric.

If $V(C_i) \cap S = \{s_i\}$, then s_i is an entrypoint on the right and an exitpoint the left boundary (Invariant F). Then re-route L_i such that after collecting s_i, L_i intersects C_i from the left to the right outer face and from the right to the left outer face without intersecting itself, any vertex, any edge twice or any edge $(s_i, n_s), n_s \in N(s_i)$ or $(t_i, n_t), n_t \in N(t_i)$. This is possible since $\text{dist}(s_i, t_i) \geq 4$. See Fig. 2.11(b) for illustration. This creates entry- and exitpoints p_{li}, p_{ri}, p'_{ri} and $p_{i>}$ in this order on L_i such that p_{ri}, p'_{ri} are on the right boundary and $p_{li}, p_{i>}$ are on the left boundary.

If $V(C_i) \cap S \supsetneq \{s_i\}$, then p_{li} is the first entrypoint of L_i after s, exitpoint p_{ri} and entrypoint p'_{ri} are consecutive on the right boundary of C and $p_{i>}$ is the last exitpoint of L_i. Entrypoint p'_{ri} and exitpoint $p_{i>}$ exist by Invariant F. The existence of p_{li} and p_{ri} follows from Invariant C and the existence of p'_{ri} and $p_{i>}$.

Then p_{r1}, p_{l2}, $p_{2>}$ and p'_{r1} are on the inner face f_{internal} of C in $\Gamma(C)$ and there exist curves L_{low} from p_{r1} to p_{l2} and L_{up} from $p_{2>}$ to p'_{r1} such that $L_{\text{up}}, L_{\text{low}}$ do not intersect each other or any vertex or edge. Further they only intersect L_1 and L_2 at their endpoints. Then $L = L_1 p_{r1} L_{\text{low}} p_{l2} L_2 p_{2>} L_{\text{up}} p'_{r1} L_1$ preserves the invariants. □

Observe, that to be 3-scattered is a necessary condition on S, otherwise Case 1b could not be resolved. On the other hand, only Case 2a and Case 2c of the induction step require the graph G to be 4-stretched. In both cases 3-stretched would be enough to preserve all invariants except Invariant G. Invariant G is singly used in Case 1b to ensure that the pseudoline L_{cross} does not interfere with L_1 or L_2. For simplicity both C_1 and C_2 are required to comply with Invariant G, although one of them complying would be enough. Thus for instance 7-scattered sets in 3-stretched graphs are collinear. The proof requires adjustment of Case 2a, Case 2c and Invariant G.

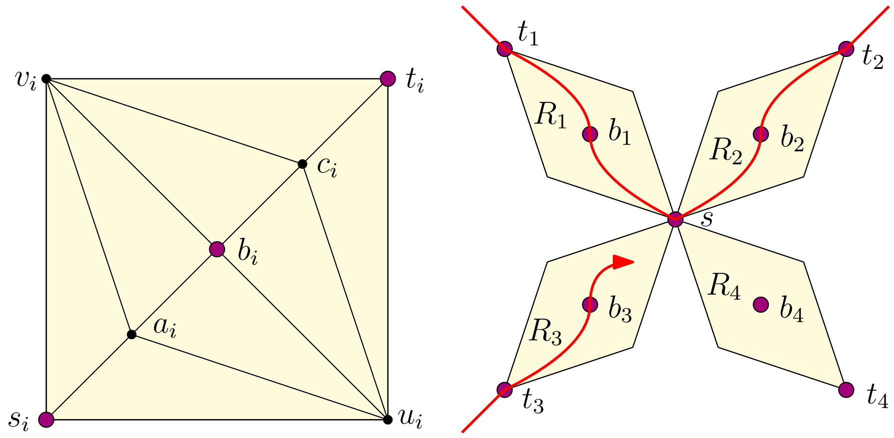
3. Collinear sets in 4-connected triangulations

Considering the counterexample for d-scattered sets in series-parallel graphs in Chapter 2, we ask whether independent sets are collinear if the graph does not contain such a series-parallel graph as a subgraph. The counterexample for d-scattered sets in series-parallel graphs contains several triangles such that removing the vertices of a triangle would disconnect the graph. Ravsky et al. [RV11] construct a family of graphs with a maximum collinear set – and use separating triangles to prove that the size of the maximum collinear set is sub-linear. In a 4-connected graph such triangles are forbidden. A graph G is 4-connected if removing any set of less than four vertices and their incident edges does not disconnect the graph. This leads to the question whether independent sets in 4-connected planar graphs are collinear, in particular in maximally planar 4-connected graphs, i.e. 4-connected triangulations. In Section 3.1 we answer this question in the negative. For d-scattered sets in 4-connected triangulations the question is still open, though we describe some approaches on how to prove their collinearity and describe their limits in Section 3.2. One of the approaches uses the decomposition of 2-sided near-triangulations, a superclass to 4-connected triangulations. In Section 3.3 we prove that if a 2-sided near-triangulation can be decomposed using only one of two possible decomposition operations, an independent set is collinear in this graph.

3.1 Counterexample: independent sets in 4-connected triangulations

Similar to the counterexample in 2.2 we construct a graph containing several identical components as subgraphs and an independent set S. A component C is depicted in Fig. 3.1(a). It consists of a path $P = sabct$ and two vertices u and v, each directly connected to every vertex in P. We call the 4-cycle $sutvs$ the cycle F. Note that in every embedding of C all faces are triangles except the one bounded by F and that C does not contain any separating triangles. However, C is not 4-connected.

Let $\mathcal{T}_{4\text{nice}}$ be a family of planar 4-connected triangulations such that every $T \in \mathcal{T}_{4\text{nice}}$ contains four copies of C, namely C_1, C_2, C_3 and C_4, and for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, $V(C_i) \cap V(C_j) = \{s_i\} = \{s_j\} = \{s\}$ and $(t_i, t_j) \notin E(T)$. We show that for every $T \in \mathcal{T}_{4\text{nice}}$ there exists an independent set S such that S is not collinear. The following lemma serves as preparation. Note that Lemma 3.1 and 3.2 do not consider a concrete graph of the



(a) A component C_i.

(b) Four components C_1, C_2, C_3, C_4, only overlapping at the vertex s.

Figure 3.1: Components C_i. Vertices of the independent set S are depicted in violet, regions R_i in light yellow and the pseudoline L in red.

family $\mathcal{T}_{4\text{nice}}$ but only use its properties: 4-connectivity and the structure of the contained components. We show the existence of a graph $T \in \mathcal{T}_{4\text{nice}}$ in Theorem 3.3.

Let Γ be a drawing of T and let $\Gamma[C_i]$ be the drawing of C_i in Γ. Let R_i be the region of $\Gamma[C_i]$, i.e. the complement of its the outer face.

Lemma 3.1. *Let Γ be a plane drawing of a graph $T \in \mathcal{T}_{4\text{nice}}$ with drawings $\Gamma[C_i]$ of the components C_i in Γ, $i \in \{1, 2, 3, 4\}$, and their regions R_i. Then the outer face of at least three of the four drawings $\Gamma[C_i]$ is bounded by the 4-cycle F_i containing the vertex $s = s_1 = s_2 = s_3 = s_4$ and their regions R_i only overlap at the vertex s.*

Proof. Since T is 4-connected and its drawing is plane, in every triangle $D \in T$ there is either no vertex $x \in V(T) \setminus V(D)$ embedded inside D or all vertices $V(T) \setminus V(D)$ are embedded inside D, i.e. D bounds the outer face of T in Γ. Otherwise D would be separating. This holds in particular for all triangles in the drawing $\Gamma[C_i]$ of a component C_i. In a component C_i only one face is bounded by the 4-cycle F_i, which contains the vertex s. All other faces are bounded by triangles.

Assume that there are two components C_i and C_j in T, $i \neq j$, such that their outer faces in $\Gamma[C_i]$ and $\Gamma[C_j]$ are both bounded by a triangle D_i, D_j respectively. As shown above this means that D_i and D_j both bound the outer face of T in Γ. Since $D_i \neq D_j$ and the outer face of T is unique, this is a contradiction. Thus, at least three of the four drawings $\Gamma[C_1], \Gamma[C_2], \Gamma[C_3], \Gamma[C_4]$ are bounded by a 4-cycle, without loss of generality $\Gamma[C_1], \Gamma[C_2], \Gamma[C_3]$.

Furthermore, since all inner faces of $\Gamma[C_i]$, $i \in \{1, 2, 3\}$, are triangles, no vertex $x \in V(T) \setminus V(C_i)$ is embedded inside an inner face of $\Gamma[C_i]$. Hence, no vertex $x \in V(T) \setminus V(C_i)$ is embedded in the region R_i. Since the only common vertex of C_1, C_2, C_3 is s, their regions only overlap at s. □

Lemma 3.2. *Every graph $T \in \mathcal{T}_{4\text{nice}}$ has an independent set $S \subset V(T)$ such that S is not collinear.*

Proof. Let S be an independent set of T such that $\{s, b_1, t_1, b_2, t_2, b_3, t_3, b_4, t_4\} \subset S$. For the sake of a contradiction assume that S is collinear and let L be a pseudoline collecting

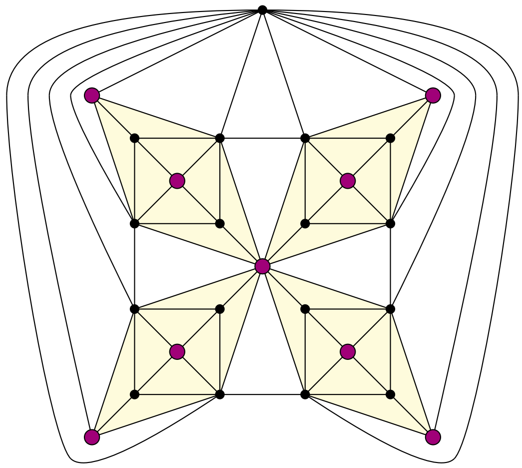


Figure 3.2: A graph $T \in \mathcal{T}_{4\text{nice}}$ with an independent set S. S is depicted in violet, the regions of the four components C_1, C_2, C_3, C_4 are highlighted in yellow.

S. From Lemma 3.1 we know that the outer faces of at least three of the four drawings $\Gamma[C_i]$, $i \in \{1, 2, 3, 4\}$, are bounded by a 4-cycle $F_i = sv_it_iu_is$. Without loss of generality let the three drawings be $\Gamma[C_1], \Gamma[C_2], \Gamma[C_3]$.

Let R_i be the region of $\Gamma[C_i]$. All vertices of C_i are contained in R_i, in particular the vertex $b_i \in S$. To collect a vertex b_i L crosses F_i at least twice. Since $s, t_i \in S$, none of the edges of F_i can be crossed by L. Hence, L crosses at the vertices s and t_i due to the lack of another possible crossing point.

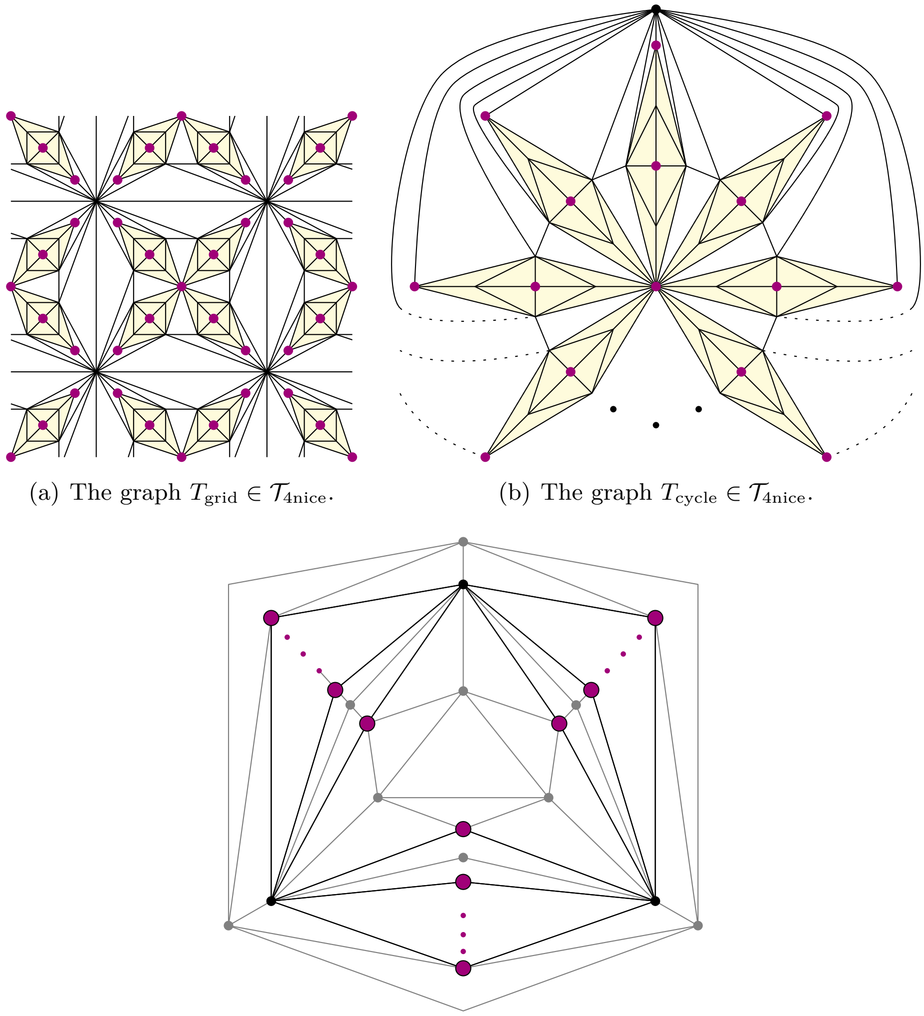
Consider the route of L to collect the vertices b_1 and b_2 inside the respective region. See Fig. 3.1(b) for illustration. To enter and leave the regions R_1 and R_2, L crosses F_1 at s and t_1 and F_2 at s and t_2. As s can only be collected once, L enters R_1 at t_1, leaves R_1 at s and directly enters R_2 via s and leaves R_2 at t_2 (or reversed). So L always collects s between collecting t_1 and t_2 without leaving the region $R_1 \cup R_2$.

We use the restriction on L to contradict its existence. Since R_3 only overlaps with R_1 and R_2 at s (Lemma 3.1), L collects the vertex b_3 inside R_3 either before or after collecting the vertices t_1, s and t_2. To do so it crosses F_3 at s and at t_3. This is a contradiction since s cannot be collected twice. □

Theorem 3.3. *There exists a 4-connected triangulation T with an independent set $S \subset V(T)$ such that S is not collinear.*

Proof. Consider the graph T depicted in Fig. 3.2. It is 4-connected and since $T \in \mathcal{T}_{4\text{nice}}$, it has an independent set S, which is not collinear. S is even a maximum independent set. □

Observe that T is not the only 4-connected triangulation with a non-collinear independent set S. Let G be the subgraph of T induced by the vertices $V(C_1) \cup V(C_2) \cup V(C_3) \cup V(C_4)$. The non-collinear set $S \subset V(T)$ is an independent set in G as well. Consider a grid of vertices and edges bounding every cell by a 4-cycle. Insert a copy of G in every cell of the grid and triangulate the graph without creating a separating triangle. This obtains an arbitrarily large 4-connected triangulation $T_{\text{grid}} \in \mathcal{T}_{4\text{nice}}$. The independent set $S_{\text{grid}} \subset V(T_{\text{grid}})$ is the union of all independent sets S of the copies of G. See Fig. 3.3(a) for illustration. Since every $S \subset S_{\text{grid}}$ is not collinear, S_{grid} is not collinear.



(a) The graph $T_{\text{grid}} \in \mathcal{T}_{\text{4nice}}$.

(b) The graph $T_{\text{cycle}} \in \mathcal{T}_{\text{4nice}}$.

(c) The graph $T_{\text{sp}} \in \mathcal{T}_{\text{4nice}}$. The subgraph G_{sp} is colored black.

Figure 3.3: Large 4-connected triangulations with a non-collinear independent set.

The independent set $S_{\text{grid}} \subset V(T_{\text{grid}})$ is not the largest independent set in T_{grid}. For an arbitrarily large 4-connected triangulation with a non-collinear maximum independent set we compose not only four, but $i \geq 4$ copies $C_1, \ldots C_i$ of the component C in Fig. 3.1(a). Let all of the components overlap at the vertex $s = s_j \in V(C_j), j \in \{1, \ldots i\}$. We insert a vertex on the outer face and triangulate the resulting graph without a separating triangle and without edges between the vertices t_j and t_k, $j, k \in \{1, \ldots i\}$. The resulting graph T_{cycle} with $6i + 2$ vertices has a maximum independent set $S_{\text{cycle}} = \{\, s, b_1, \ldots, b_i, t_1, \ldots, t_i\}$ of size $2i + 1$ (see Fig. 3.3(b) for T_{cycle} and Fig. 3.1(a) for the labeling of the vertices).

For yet another 4-connected planar triangulation with a non-collinear independent set consider the counterexample for independent sets in series-parallel graphs in Section 2.1. The graph G_{sp} (see Fig. 2.1) has $3(i + 1)$ vertices, $6i$ edges and thus $3i - 1$ faces, $i \geq 3$. We insert a vertex into every face bounded by a 4-cycle and connect it to every vertex of the bounding cycle. We insert 3 vertices in the faces bounded by a 6-cycle and triangulate the faces. The resulting graph is a planar 4-connected triangulation T_{sp} with $6(i + 1)$ vertices. The independent set $S_{\text{sp}} \subset V(G_{\text{sp}})$ of size $3i$ is independent in T_{sp} as well. Since G_{sp} is a subgraph of T_{sp}, S_{sp} cannot be aligned in T_{sp} either. Observe that T_{sp} also contains three copies of the component C (see Fig. 3.1(a)), but they do not overlap at the vertex s.

3.2 Attempts for d-scattered sets

Independence is not a sufficient condition for collinearity of a vertex set in 4-connected triangulations. As in Chapter 2 we increase the pairwise distance between any two vertices in an independent set we want to align. Hence, we consider d-scattered sets for some $s \geq 3$. In this section we sketch two approaches to prove collinearity of d-scattered sets. We do not go into detail, but outline the limits of the approaches and where further restrictions might be useful.

The first approach in Section 3.2.1 uses canonical ordering to iterate over the vertices of a 4-connected triangulation. The second approach uses the fact, that 4-connected triangulations are a subclass of 2-sided near-triangulations and as such they can be decomposed into smaller 2-sided near-triangulations (see Section 3.2.2).

In the preliminaries we introduced a recursive definition of 4-connected triangulations using edge (de-)contractions. It did not prove useful to decide whether an independent or d-scattered set is collinear since the distance between vertices is altered when contracting edges. However, we suspect it might be useful when counting (non-independent) collinear vertices in 4-connected triangulations.

3.2.1 (2,2)-canonical ordering

Throughout this section we use *closed* pseudolines instead of pseudolines. A closed pseudoline L_c is a closed and simple curve which contains a pseudoline L_o such that the curve segment $L_c \setminus L_o$ lies completely on the outer face of G. Conversely to L_c being closed, we call L_o an *open* pseudoline.

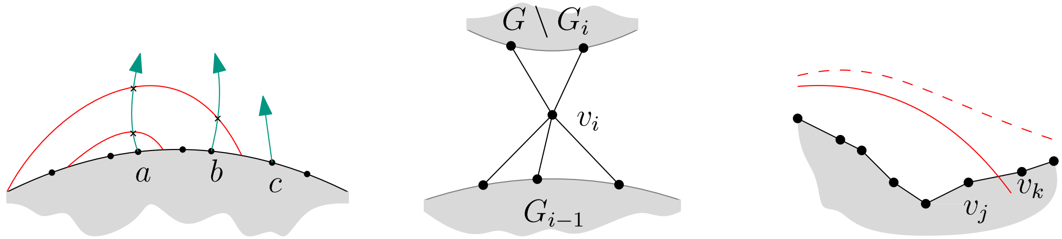
Let C denote the cycle bounding the outer face of a graph G in a drawing Γ of G. Let $l_\infty \in \mathcal{L}_\infty(x)$ be a curve connecting a vertex $x \in C$ with infinity such that $\Gamma \cap l_\infty = \{x\}$. We say a closed pseudoline L *covers* the vertex x if every curve $l_\infty \in \mathcal{L}_\infty(x)$ crosses L at least once in its interior. L covers x c times if every curve $l_\infty \in \mathcal{L}_\infty(x)$ crosses L at least c times in its interior. The vertex x is uncovered if there exists a curve $l_\infty \in \mathcal{L}_\infty(x)$ not crossing L at in interior. Note that x does not count as a crossing point. See Fig. 3.4(a) for illustration.

A useful tool when dealing with a triangulation is *canonical ordering*, i.e. an ordering of its vertices $v_1, v_2, v_3, \ldots, v_n$ such that v_1, v_2 and v_n bound the outer face and for every subgraph $G_{i-1} \subseteq G$, $4 \leq i \leq n$, induced by the vertices $v_1, \ldots, v_{i-1}$, the following requirements are met:

- G_{i-1} is 2-connected and the boundary of its outer face is a cycle C_{i-1} containing the edge (v_1, v_2).
- Vertex v_i is on the outer face of G_{i-1} and its neighbors in G_{i-1} form an (at least 2-element) subinterval of the path $C_{i-1} \setminus (v_1, v_2)$.

Such an ordering exists for every planar triangulation [DFPP90]. For 4-connected triangulations this ordering can be refined such that every vertex v_i, $3 \leq i \leq n-2$, has at least two neighbors in $G \setminus G_i$ [KH97]. See Fig. 3.4(b) for illustration. In other words, in a *refined canonical ordering* not only G_{i-1} is 2-connected, but $G \setminus G_{i-1}$ is 2-connected as well [BD16].

This concept is generalized to (r, s)-*canonical* orderings in [BD16]. Let G be a plane triangulation. A vertex partition $V_1 \cup \cdots \cup V_L$ is an (r, s)-canonical ordering if the vertices v_1, v_2 and v_n bound the outer face, $v_1 \in V_1$ and $v_n \in V_L$ and for every $1 < i < L$ the graph G_i is r-connected and $G \setminus G_i$ is s-connected. They also show that a $(3, 1)$-canonical ordering exists for every 4-connected triangulation such that every V_i, $1 < i < L$, is either

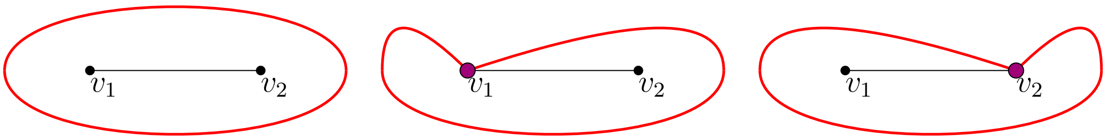


(a) Vertex a is covered twice, vertex b is covered once and vertex c is uncovered.

(b) A vertex v_i in $(2,2)$-canonical ordering.

(c) Invariant D: The vertices are in canonical ordering on the y-axis. Thus, $j < k$ and v_j is covered once more than v_k.

Figure 3.4: Some basic definitions.



(a) $v_1, v_2 \notin S$.

(b) $v_1 \in S$.

(c) $v_2 \in S$.

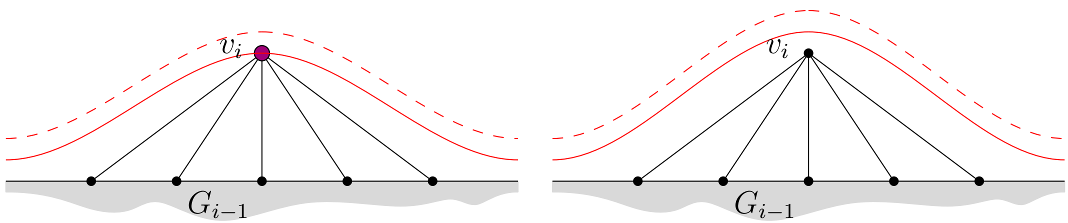
Figure 3.5: Base case. The pseudoline L (red) collects a vertex v_1 or v_2 if it is in S (violet).

a single vertex or an induced path of which every vertex has degree three in $V_1 \cup \cdots \cup V_i$. The above introduced refined canonical ordering is a $(2,2)$-canonical ordering.

Our attempt is to construct a pseudoline L for the graph G collecting a d-scattered set $S \subset V(G)$ inductively. We start with the graph G_2 and let L collect the vertices $V(G_2) \cap S$. We iterate over all graphs G_i in the sequence $G_2, \ldots, G_n$ and adjust the route of L such that it collects $V(G_i) \cap S$. In every step of the iteration we try to preserve the following invariants with respect to the graph G_i.

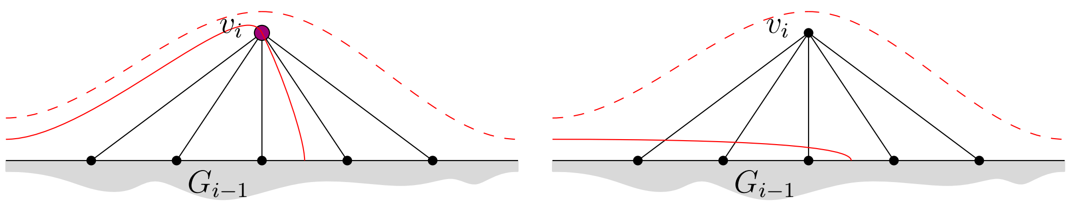
- Invariant A: L is a pseudoline.
- Invariant B: L collects all vertices $S \cap V(G_i)$.
- Invariant C: L covers every vertex on the boundary of G_i at most twice.
- Invariant D: For every edge (v_j, v_k), $j < k$, on the boundary of G_i, it holds that if L intersects (v_j, v_k) at its interior, v_j is covered once more than v_k by L. See Fig. 3.4(c) for illustration.

We start with the graph G_2 as the base case. We construct L as shown in Fig. 3.5. The pseudoline L is a closed curve around G_2, collecting v_1 or v_2 if necessary. It trivially conforms with the invariants.



(a) $v_i \in S$. (b) $v_i \notin S$.

Figure 3.6: Case 1. The pseudoline L (red) collects the vertex v_i if it is in S (violet). The dashed red line depicts the optional second layer of L.



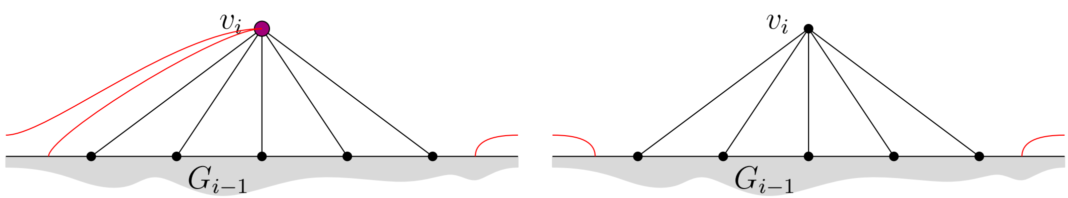
(a) $v_i \in S$. (b) $v_i \notin S$.

Figure 3.7: Case 2. The pseudoline L (red) collects the vertex v_i if it is in S (violet). The dashed red line depicts the optional second layer of L.

Let $N(v_i)$ the neighborhood of v_i and let $N_{i-1}(v_i) = N(v_i) \cap V(G_{i-1})$ be the neighborhood of v_i in the subgraph G_{i-1}. Considering the graph G_{i-1} and its boundary we distinguish how often L covers the neighbors $N_{i-1}(v_i)$ in G_{i-1}. We yield the following cases.

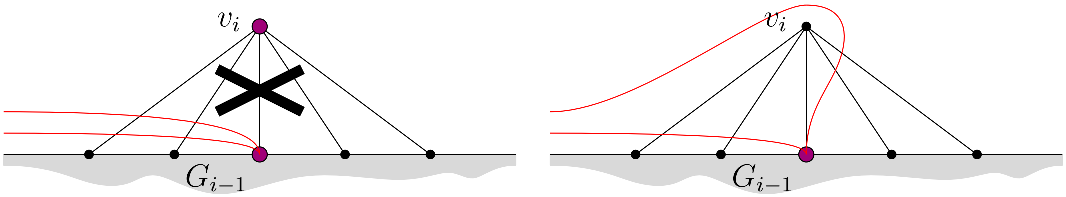
- Case 1: L covers all $N_{i-1}(v_i)$ once or all $N_{i-1}(v_i)$ twice, see Fig. 3.6.
- Case 2: L covers parts of $N_{i-1}(v_i)$ twice, the rest once, or L covers parts of $N_{i-1}(v_i)$ once, the rest not at all, see Fig. 3.7.
- Case 3: L does not cover $N_{i-1}(v_i)$ at all, see Fig. 3.8.
- Case 4: L covers parts of $N_{i-1}(v_i)$ twice, the rest not at all, see Fig. 3.9.
- Case 5: L covers parts of $N_{i-1}(v_i)$ once, parts twice and some vertices of $N_{i-1}(v_i)$ not at all, see Fig. 3.10.

For every case, we further distinguish, whether $v_i \in S$ (Case a) or $v_i \notin S$ (Case b). So for example in Case 3b L does not cover $N_{i-1}(v_i)$ and $v_i \notin S$. Beforehand note that in the Cases 2 and 4, whenever there is a vertex $v_j \in N_{i-1}(v_i)$ covered $c \in \{0, 1, 2\}$ times in G_{i-1}, there exists a path P_j induced by the vertices $N_{i-1}(v_i)$ from v_j to the rightmost or the



(a) $v_i \in S$. (b) $v_i \notin S$.

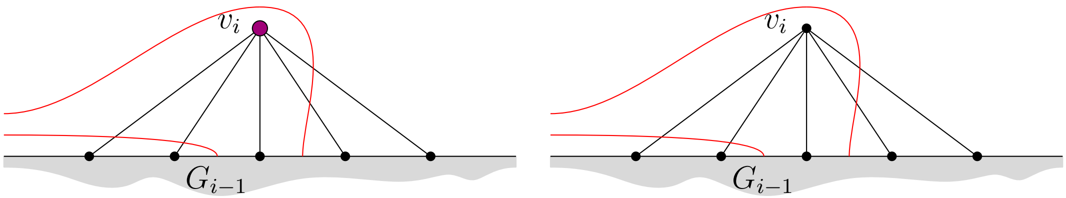
Figure 3.8: Case 3. The pseudoline L (red) collects the vertex v_i if it is in S (violet).



(a) $v_i \in S$. This case does not occur.

(b) $v_i \notin S$.

Figure 3.9: Case 4. The pseudoline L (red) crosses all edges incident to v_i.



(a) $v_i \in S$. This case cannot be resolved.

(b) $v_i \notin S$. This case cannot be resolved

Figure 3.10: Case 5.

leftmost neighbor of v_i such that all vertices on P_j are covered c times. This follows from Invariant D and from the fact that v_j has at least one neighbor $v_k \neq v_i$ in $G \setminus G_j$ such that $k > j$.

In the Cases 1a, 1b, 2a and 3b we place the vertex v_i such that none of its incident edges intersect with L at their interior. For illustration see Fig. 3.6, 3.7(a), 3.8(b). In Case 2b we place v_i such that it is covered as few times as possible (see Fig. 3.7(b)). Case 3a is the reason why some vertices on the boundary of G_i are covered more than once. All neighbors of v_i are uncovered and $v_i \in S$ has to be collected by L. Thus we go along the boundary of G_{i-1} to the first crossing point p of L with the boundary of G_{i-1} and re-route L from p to v_i. Hence, all vertices on the boundary of G_{i-1} between p and v_i are covered twice (see Fig. 3.7(a)). This yields Case 4; Some neighbors of v_i are covered twice, the rest not at all. Then there exists a vertex $v_k \in N_{i-1}(v_i)$ in S (obtained in Case 3a). Thus $v_i \notin S$, omitting Case 4a (see Fig. 3.9(a)). In Case 4b we place v_i between the two layers of L such that every incident edge crosses L once and v_i is covered once (see Fig. 3.9(b)).

So far in all cases the invariants can be preserved, even for independent sets. In Cases 5a and 5b consider the path P_{i-1} induced by the vertices $N_{i-1}(v_i)$ on the boundary of G_{i-1}. Starting at one endpoint of the path the first vertices are covered twice, then some once covered vertices follow and the rest is uncovered. This follows again from Invariant D and the fact that every $v_j \in N_{i-1}(v_i)$ has at least one neighbor $v_k \neq v_i$ such that $k > j$. In Case 5b ($v_i \notin S$) v_i is placed such that it is covered once and its incident edges cross L at most once. This violates Invariant D since v_i is covered once and some of its neighbors are uncovered. Placing v_i at another position results in two crossings of L with the same edge, thereby violating pseudoline properties and Invariant A. In Case 5a ($v_i \in S$) no matter where we place v_i, L crosses an incident edge at least twice (possibly at v_i) and thereby violates pseudoline properties and Invariant A.

The proposed invariants cannot be preserved. Thus, consider changing or omitting the invariants. Invariant A and B are necessary to ensure that L is a pseudoline collecting all vertices $S \subset V(G)$. Changing Invariant C to covering the boundary at most once would make it impossible to place v_i in Case 3a. Allowing three or more covers of a vertex on the boundary would yield an additional case, where some vertices $N_{i-1}(v_i)$ are uncovered, covered once, twice or three times each. Thus v_i cannot be placed without violating pseudoline properties. Invariant D ensures that whenever there is a vertex $v_j \in N_{i-1}(v_i)$ covered $c \in \{0, 1, 2\}$ times in G_{i-1}, there exists a path P_j induced by the vertices $N_{i-1}(v_i)$ from v_j to the rightmost or the leftmost neighbor of v_i such that all vertices on P_j are covered c times. Omitting Invariant D would make it impossible to place v_i in Case 2a without violating Invariant A.

Batches

Increasing the pairwise distance between vertices in S, new invariants and different strategies to place v_i might help to prove collinearity of a set S. To find these, we wish for a broader view. We propose iterating over a sequence of *batches* of vertices instead of single vertices, while keeping the canonical ordering of the vertices as before. Let $\mathcal{B}$ be a division of the vertices $V(G)$ into a sequence of batches $B_1, \ldots, B_m$. Naturally, $B_1 \cup \cdots \cup B_m = V(G)$ and no two batches have a common vertex. Further, the following requirements with respect to the canonical ordering of the vertices $v_1, \ldots, v_n$ appear to be innate.

- For two vertices $v_i \in B_k$ and $v_j \in B_l$ we have $k \leq l$ if $i < j$.
- In every batch $B_k = \{v_i \ldots v_{i+r}\}$, $k < m$: $B_k \cap S = \{v_{i+r}\}$.
- The last batch $B_m = \{v_{n-s} \ldots v_n\}$ might not have a vertex in S: $B_m \cap S = \emptyset$ or $B_k \cap S = \{v_n\}$
- Every vertex $v_i \in B_k$, $v_i \notin S$, has a neighbor $v_j \in B_k$ such that $j > i$ or $v_i = v_n$.

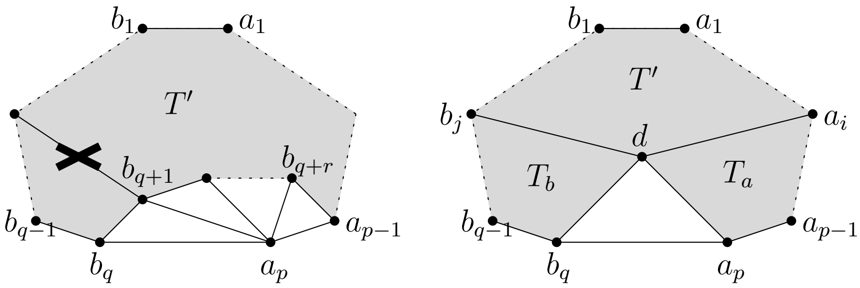
Not every canonical ordering admits such a division of the vertices, but every graph admitting a canonical ordering also admits a canonical ordering meeting the requirements: Consider a canonical ordering $v_1, \ldots, v_n$ of the vertices of G. Subdivide the sequence, such that the last vertex of every part is in S. This division meets the first two requirements, but possibly not the third. We re-order the vertices as follows. Consider a vertex $v_i \notin S$ in a batch $B_k = \{v_l, \ldots, v_{l+r}\}$, such that v_i has no neighbor $v_j \in B_k$ with $j > i$. Then we switch the vertices $v_{i+1}, \ldots, v_{l+r}$ with v_i in the canonical ordering. In the new canonical ordering $v'_1, \ldots v'_n$, all vertices have the same position, but $v'_{l+r} = v_i$ and $v'_i = v_{i+1}, \ldots, v'_{l+r-1} = v_{l+r}$. Also, $v'_{l+r} = v_i$ is moved to batch B_{k+1}. We repeat this, until all vertices meet the third requirement.

This subdivision of the vertex set $V(G)$ into batches may be useful to prove collinearity of d-scattered sets in 4-connected triangulations, where $d \geq 3$ or even more. We leave this to future work.

3.2.2 Decomposition as 2-sided near-triangulation

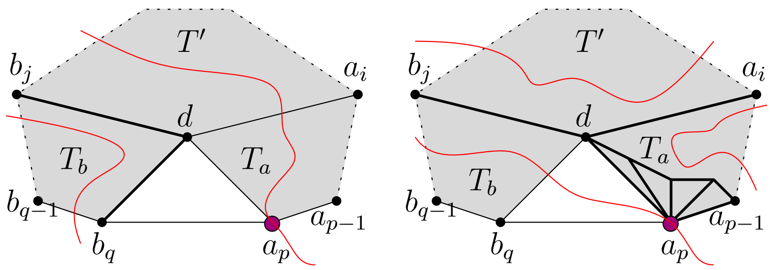
Gonçalves et al. [GIP18] present another way to decompose 4-connected triangulations. They introduce a decomposition of the graph class of 2-sided near-triangulations, a superclass of 4-connected triangulations. Recall that a near-triangulation is a plane graph G such that every inner face is a triangle. A 2-sided near-triangulation is a 2-connected near-triangulation T without separating triangles such that going clockwise on its outer face the vertices are denoted $a_1, a_2, \ldots, a_p, b_q, \ldots, b_2, b_1$, with $p \geq 1$ and $q \geq 1$, and such that there is neither a chord (a_i, a_j) nor (b_i, b_j) (that is an edge (a_i, a_j) or (b_i, b_j) such that $|i - j| > 1$). To decompose a 2-sided near-triangulation Gonçalves et al. define the following operations.

- *a_p-removal* (see Fig. 3.11(a)): This operation applies if $p > 1$, a_p has no neighbor b_i with $i < q$ and none of the inner neighbors of a_p has a neighbor b_i with $i < q$.



(a) a_p-removal. (b) Cutting.

Figure 3.11: Operations to decompose a 2-sided near triangulation.



(a) T_a yields Case 1. (b) T_b yields Case 2.

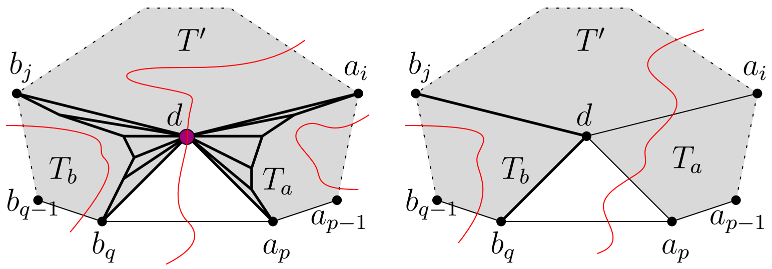
Figure 3.12: Case 1.

This operation consists in removing a_p from T and in denoting $b_{q+1}, \dots, b_{q+r}$ the new vertices on the outer face in anti-clockwise order. This yields a 2-sided near-triangulation T'.

- *b_q-removal*: This operation applies if $q > 1$, b_q has no neighbor a_i with $i < p$ and none of the inner neighbors of b_q has a neighbor a_i with $i < p$. This operation consists in removing b_q from T and in denoting $a_{p+1}, \dots, a_{p+r}$ the new vertices on the outer face in clockwise order. This yields a 2-sided near-triangulation T'. This operation is strictly symmetric to the previous one.
- *cutting* (see Fig. 3.11(b)): This operation applies if $p > 1$, $q > 1$ and the unique common neighbor of a_p and b_q, denoted d, has a neighbor a_i with $i < p$ and a neighbor b_j with $j < q$. This operation consists in cutting T into three 2-sided near-triangulations T', T_a and T_b.
  - T' is the 2-sided near-triangulation contained in the cycle formed by vertices $a_1, \dots, a_i, d, b_j, \dots, b_1$ and the vertex d is renamed a_{i+1}.
  - T_a (resp. T_b) is the 2-sided near-triangulation contained in the cycle formed by the vertices $a_i, \dots, a_p, d$ (resp. $d, b_q, \dots, b_j$), where the vertex d is denoted b_1 (resp. a_1).

Note that when applying the cutting operation, the vertex d could also be renamed b_{j+1}. We use the decomposition operations to pose requirements on a pseudoline L collecting an independent set S.

Consider the 2-sided near-triangulation T and an independent set $S \subset V(G)$ such that S is collinear. For the cutting-operation we propose the following recursion. Consider a pseudoline L collecting the vertices of S. The cutting-operation decomposes T into T', T_a and T_b. This yields requirements on L with respect to the subgraphs T', T_a and T_b. We



(a) T' yields Case 1. (b) T_a yields Case 2.

Figure 3.13: Case 2.

describe these by requirements on separate pseudolines L', L_a and L_b for the respective subgraph. In particular L', L_a and L_b cannot intersect with specific edges of T', T_a and T_b, respectively. We call these edges *walls.* A pseudoline L cannot intersect a wall $e \in W(L)$ neither at its interior nor its endpoints. Note that after a cutting operation T_a and T_b have either less than four vertices or they are further decomposed by a_p- or b_q-removal. The cutting-operation requires $q > 1$ and $p > 1$, but $q = 1$ in T_a and $p = 1$ in T_b.

We distinguish the following cases for L with respect to T.

- Case 1: L crosses the boundary of T at the vertex a_p. The case when L crosses the boundary at b_q is symmetric.
- Case 2: L crosses the boundary of T at the interior of the edge (a_p, b_q).
- Case 3: $(a_p, b_q) \in W(L)$.

For every case we list the possible cases for the subgraphs T', T_a and T_b.

Case 1: L crosses the boundary of T at the vertex a_p.

- If T_a yields Case 1, then $(d, b_q) \in W(L_b)$, yielding Case 3 for further decomposition of T_b (see Fig. 3.12(a)).
- If T_b yields Case 2, then all edges induced by $a_p \cup N(a_p) \subset V(T_a)$ are walls. This yields Case 3 for T_a (see Fig. 3.12(b)).

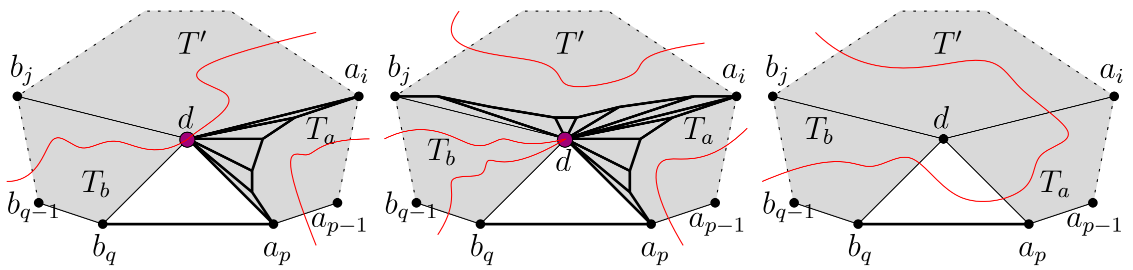
In both cases if the common edge of $\{e_a\} = T_a \cap T'$ or $\{e_b\} = T_b \cap T'$ is not intersected by L_a, resp. L_b, then $e_a \in W$, resp. $e_b \in W$. This yields Case 2 or 3 for T'.

Case 2: L crosses the boundary of T at the interior of the edge (a_p, b_q).

- If $d \in S$, then T', T_a or T_b yields Case 1. Then all edges induced by $d \cup N(d)$ are walls in the remaining two subgraphs. This yields Case 3 for the remaining two subgraphs (see Fig. 3.13(a)).
- If $d \notin S$, then T_a yields Case 2 and T_b yields Case 3 or vice versa. If the common edge of $\{e_a\} = T_a \cap T'$ or $\{e_b\} = T_b \cap T'$ is not intersected by L_a, resp. L_b, then $e_a \in W$, resp. $e_b \in W$. This yields Case 2 or 3 for T' (see Fig. 3.13(b)).

Case 3: $(a_p, b_q) \in W(L)$.

- If $d \in S$, then either two of the subgraphs T', T_a and T_b are in Case 1 and in the third all edges induced by $d \cup N(d)$ are walls (see Fig. 3.14(a)). Or d is collected in one of the subgraphs without intersecting its boundary; thus; yielding walls in both of the other subgraphs (see Fig. 3.14(b)).



(a) T' & T_b yield Case 1. (b) L collects d from inside T_b. (c) Both T_a and T_b yield Case 2.

Figure 3.14: Case 3.

- If $d \notin S$, then T_a and T_b are either both in Case 2 or both in Case 3. The case of T' depends on the pseudolines L_a and L_b and whether they cross the common edges or not (see Fig. 3.14(c)).

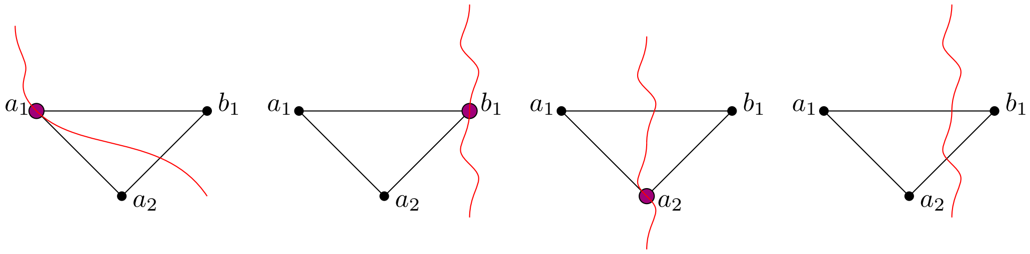
The first missing piece in this case distinction is a case where a_p or b_q are collected from inside T'. The second is a way to guarantee that there exist pseudolines L', L_a and L_b, meeting the requirements in all three cases. The third piece is to expand these cases to the a_p- and b_q-removal-operation. It remains to show that this can be achieved by restricting S to d-scattered sets, with $d \geq 3$. Observe, that restricting S to d-scattered sets would implicitly avoid substructures of T in combination with S as shown in the counterexample in Section 3.1.

3.3 Independent sets in restricted 2-sided near-triangulations

In the previous section we presented an approach to construct a pseudoline for a 4-connected triangulation G, using the fact that 4-connected triangulations are a subclass of 2-sided near-triangulations. Due to that fact, we also know that independent sets in 2-sided near-triangulations in general are not collinear. However, if a 2-sided near-triangulation can be decomposed such that cutting operations are used only and no a_p- or b_q-removal operations are applied, independent sets can always be aligned. We call this graph class *restricted 2-sided near-triangulations*. Recall the definition of the cutting-operation. The cutting-operation applies if $p > 1$, $q > 1$ and the unique common neighbor of a_p and b_q, denoted d, has a neighbor a_i with $i < p$ and a neighbor b_j with $j < q$. The operation cuts T into three 2-sided near-triangulations T', T_a and T_b.

- T' is the 2-sided near-triangulation contained in the cycle formed by vertices $a_1, \ldots, a_i, d, b_j, \ldots, b_1$, and the vertex d is renamed a_{i+1} or b_{j+1}.
- T_a (resp. T_b) is the 2-sided near-triangulation contained in the cycle formed by vertices $a_i, \ldots, a_p, d$ (resp. $d, b_q, \ldots, b_j$), where the vertex d is denoted b_1 (resp. a_1).

Let T be a restricted 2-sided near-triangulation on n vertices. First we observe that restricting possible decompositions to the cutting operation means that the subgraphs T_a and T_b of T are both triangles. As mentioned in the previous section, after a cutting operation T_a and T_b have either less than four vertices or they are further decomposed by a_p- or b_q-removal. The cutting-operation requires $q > 1$ and $p > 1$, but $q = 1$ in T_a and $p = 1$ in T_b. With the restriction to the cutting operation T_a and T_b have at most three vertices. By the definition of the cutting operation T_a (resp. T_b) has at least three vertices, namely d, a_p and a_{p-1} (resp. d, b_q and b_{q-1}). Therefore T_a and T_b are both triangles. For the subgraph T' this means that it has exactly $|T| - 2$ vertices. Thus, we define the



(a) Case 1: $a_1 \in S$. (b) Case 2: $b_1 \in S$. (c) Case 3: $a_2 \in S$. (d) Case 4: $T_0 \cap S = \emptyset$.

Figure 3.15: Four cases of the induction base.

decomposition sequence $T_0, \ldots, T_k$ where $T_k = T$, $T_{i-1} = T_i'$, $i \in \{1, \ldots, k\}$ and T_0 is a restricted 2-sided near-triangulation that is not further decomposed. We further observe that T_0 has exactly three vertices. T_0 has at most three vertices since otherwise it would be decomposed. It has at least two vertices since otherwise T_1 would have less than four vertices – and would not be decomposed. If T_0 has two vertices, then T_1 has four vertices, which is not possible since there exists no 2-sided near-triangulation on four vertices such that the cutting operation can be applied. Hence, T_0 has exactly three vertices – and every restricted 2-sided near-triangulation has an odd number of vertices.

Lemma 3.4. *Independent sets in restricted 2-sided near-triangulations can always be aligned.*

Proof. Let T be a restricted 2-sided near triangulation with an independent set $S \subset V(T)$. We prove the claim by induction on the decomposition sequence $T_0, \ldots, T_k$, $T = T_k$. We show that for every $T_i \in \{T_0, \ldots, T_k\}$ there exists a pseudoline L_i preserving the following invariants.

- Invariant A: L is a pseudoline.
- Invariant B: The pseudoline L_i intersects the boundary of T_i at the edge (a_1, b_1). We call this intersection point p. If one of the endpoints a_1 or b_1 is in the independent set S, then $p = a_1$ or $p = b_1$, respectively.
- Invariant C: L_i intersects the boundary of T_i at the edge (a_p, b_q). We call this intersection point p_i. If one of the endpoints a_p or b_q is in the independent set S, then $p_i = a_p$ or $p_i = b_q$, respectively.
- Invariant D: L_i does not intersect the boundary of T_i at any other point. However, it may touch the boundary from inside T_i, i.e. it can collect a vertex on the boundary, but not intersect the boundary.
- Invariant E: L_i collects all vertices $S \cap V(T_i)$.

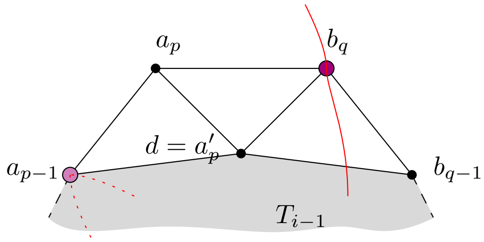
Induction base:

We assume that T_0 has three vertices a_1, b_1 and a_2. The cases with T_0 has the vertices a_1, b_1 and b_2 are symmetric. We construct the pseudoline L_0 as follows.

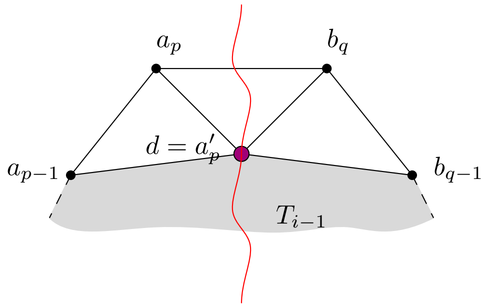
Case 1: $a_1 \in S$. Let $p = a_1$ and p_0 be in the interior of the edge (a_2, b_1). See Fig. 3.15(a) for illustration.

Case 2: $b_1 \in S$. Let $p = p_0 = b_1$. See Fig. 3.15(b) for illustration.

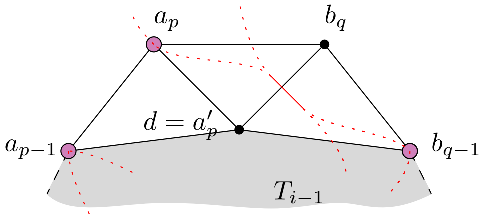
Case 3: $a_2 \in S$. Let p lie in the interior of the edge (a_1, b_1) and let $p_0 = a_2$. See Fig. 3.15(c) for illustration.



(a) Case 1: $b_q \in S$.



(b) Case 2: $d = a'_p \in S$.



(c) Case 3: $a'_p, b_q \notin S$.

Figure 3.16: Three cases of the induction step. The pseudoline L_i is denoted in red. For alternative routes of L_i it is dotted. The independent set S is denoted in violet; if a vertex may or may not be in S it is denoted in light violet.

Case 4: $T_0 \cap S = \emptyset$. Let p lie in the interior of the edge (a_1, b_1) and let p_0 lie in the interior of the edge (a_2, b_1). See Fig. 3.15(d) for illustration.

L_0 conforms to all invariants in all four cases.

Induction step:

We assume that the common neighbor d of the vertices a_p and b_q in T_i is renamed a_p in T_{i-1}, thus we call it a'_p. We distinguish the following cases.

- Case 1: $b_q \in S$
- Case 2: $d = a'_p \in S$
- Case 3: $b_q, a'_p \notin S$

In the case d is renamed b_q all cases are symmetric.

By induction we know that there exists a pseudoline L_{i-1} such that L_{i-1} conforms with all invariants with respect to T_{i-1}. For Invariant C, with respect to the naming of vertices in T_i, this means that the intersection point p_{i-1} lies in the interior of the edge (a'_p, b_{q-1}) or if one of the endpoints a'_p or b_{q-1} is in the independent set S, then $p = a'_p$, respectively $p = b_{q-1}$.

Case 1 ($b_q \in S$): Since $b_q \in S$, it follows $a'_p, b_{q-1} \notin S$. Therefore, p_{i-1} is in the interior of the edge (a'_p, b_{q-1}). Thus, we construct a pseudoline L_i, which follows the route of L_{i-1} to p_{i-1}. From p_{i-1} it enters the face (a'_p, b_{q-1}, b_q) and intersects the boundary of T_i at the vertex b_q, hence $p_i = b_q$. For illustration see Fig. 3.16(a).

Case 2 ($d = a'_p \in S$): Since $a'_p \in S$, it follows $p_{i-1} = a'_p$. Thus, we construct a pseudoline L_i, which follows the route of L_{i-1} to the intersection point p_{i-1}. From p_{i-1} it enters the face (a'_p, a_p, b_q) and intersects the boundary of T_i in the interior of the edge (a_p, b_q), hence p_i lies in the interior of (a_p, b_q). For illustration see Fig. 3.16(b).

Case 3 $(b_q, a'_p \notin S)$: Since $a'_p \notin S$, it follows p_{i-1} lies in the interior of the edge (a'_p, b_{q-1}) or $p_{i-1} = b_{q-1}$. Thus, we construct a pseudoline L_i following the route of L_{i-1} to the intersection point p_{i-1}. From p_{i-1} it enters the face (a'_p, b_{q-1}, b_q) without intersecting the boundary of T_i, intersects the edge (a'_p, b_q) and intersects the boundary of T_i at the vertex a_p, resp. the edge (a_p, b_q). Thus $p_i = a_p$ or p_i lies in the interior of (a_p, b_q). For illustration see Fig. 3.16(c).

In all three cases L_i preserves Invariants A and B as L_{i-1} does so. It preserves Invariant C by construction of p_i. Furthermore, it does not intersect the boundary of T_i at any point between p and p_{i-1} as well as between p_{i-1} and p_i. Thus, Invariant D is preserved. Invariant E is preserved since L_i collects all vertices which are collected by L_{i-1} as well as a_p or b_q, if $a_p \in S$, respectively $b_q \in S$. □

4. Conclusion

Independent sets are in generally not collinear in neither series-parallel graphs, nor 4-connected triangulations. For series-parallel graphs, we show that there exist arbitrarily large series-parallel graphs with non-collinear independent sets. We further show that any fixed minimal pairwise distance between vertices of an independent set does not suffice for them to be collinear. We present a family of graphs $\mathcal{G}_{\text{nice}}$ such that for every $d \in \mathbb{N}$ there exists a graph $G \in \mathcal{G}_{\text{nice}}$ with a non-collinear d-scattered subset of its vertices. On the other hand, we show that 3-scattered sets in 4-stretched series-parallel graphs are collinear. In 3-stretched series-parallel graphs, 7-scattered sets are collinear. Whether 6-scattered sets or vertex sets with smaller pairwise distance are collinear in 3-stretched series-parallel graphs is still open. Furthermore, the question whether d-scattered sets are collinear in 2-stretched series-parallel graphs is still open for any d.

We present a graph G with an independent set S in Chapter 3. If G is contained in a 4-connected triangulation T and $S_T \supset S$ is an independent set in T, then S_T is not collinear. G is not the only forbidden substructure in a 4-connected triangulation with collinear independent set. We present an extension of the counterexample for independent sets in series-parallel graphs, which is a 4-connected triangulation with a non-collinear independent set. It contains a similar but different substructure from G.

For d-scattered sets we sketch two approaches to prove collinearity. The limits of the approaches are outlined and where further restrictions might be useful. In particular we use a refined canonical ordering to iteratively construct a pseudoline collecting the vertices of a d-scattered set, but cannot preserve the proposed invariants. However, we present a tool to possibly find more suitable invariants in the future. Instead of iterating over single vertices in a canonical ordering, we propose iterating over batches of vertices. Hopefully, this enables to prove collinearity of d-scattered sets for some d – or gives insights to find forbidden substructures which make d-scattered sets in 4-connected triangulations non-collinear.

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